A generalization of Weierstrass' $\wp$-function to quasi-abelian varieties

理工学教育部
数理・ヒューマンシステム科学専攻
学籍番号　30971003
濽江　厚子

Contents

Preface

For a given lattice Γ Weierstrass constructed a doubly periodic meromorphic function on $\mathbb{C}$ with period Γ. We call it Weierstrass' $\wp$-function. It was generalized on abelian varieties by Zappa in 1983. Let $T^n = \mathbb{C}^n/\Gamma$ be an n-dimentional complex torus. Γ-invariant $\overline{\partial}$-closed $(0, n-1)$-forms on $\mathbb{C}^n/\Gamma$ are considered as representatives of classes in $H^{n-1}(T^n \setminus \{0\}, \mathcal{O})$. First, Zappa constructed a Γ-invariant $\overline{\partial}$-closed $(0, n-1)$-form $\wp^i$, and then gave a $\overline{\partial}$-closed $(n-1, n-1)$-form $\wp^{ij}$ on $T^n \setminus \{0\}$ with the following property:
If T^n is an abelian variety and Θ is a divisor on T^n defined by a theta function θ, then we have

$$\int_\Theta \wp^{ij}(z-p) = -\frac{\partial^2}{\partial z_i \partial z_j} \log \theta \Big|_p + \text{constant}.$$

In the case of one variable, this is just the relation between Weierstrass' $\wp$-function and a theta function.

The purpose of this paper is to give a further generalization of Zappa's result. We show that we can construct a similar $(n-1, n-1)$-form $\wp^{ij}$ even for a non-compact quasi-abelian variety.

This paper consists of three chapters. In Chapter 1, we explain in detail a part of the theory of Andreotti-Norguet which is the basis of our argument. We think that our proofs are more explicit and comprehensible than the original ones.

In Chapter 2, we state Zappa's result. Several lemmas and propositions are stated in more general setting in order to use them later.

In Chapter 3, Weierstrass' $\wp$-function is generalized on quasi-abelian varieties. Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group. We can construct a Γ-invariant $\overline{\partial}$-closed $(n-1, n-1)$-form $\wp^{ij}$ on X in the same manner as in the case of complex tori. But when we consider a positive divisor Θ on X, we do not know in general the convergence of the integral of $\wp^{ij}$ on Θ, because Θ is not compact. Even if it converges, we are not able to apply the same argument as in the case of abelian varieties. We suppose that X is a quasi-abelian variety of kind 0 with the standard compactification $\overline{X}$. We further assume that a positive divisor Θ on X is holomorphically extendable to $\overline{X}$. Then we can prove that $\wp^{ij}$ is integrable on Θ and a similar equation as the case of abelian varieties is obtained (Theorem 3.7 in Chapter 3). To prove it, we first take a family of relatively compact subdomains in X. We give formulas on these subdomains. As the limit of them we obtain the expected formula.

Acknowledgement
The author would like to thank Professor Yukitaka Abe for giving her constructive comments and warm encouragement.

Chapter 1

Cohomology Groups of a Punctured Polydisc

1.1 Cohomology groups

We give a proof of Lemma 2 in [5] in this section. This lemma is essentially Proposition 31.1 in [6]. The proof given here is more explicit than that in [6].

Let $D = \{\zeta \in \mathbb{C} \ ; \ |\zeta| < 1\}$ be the unit disc in $\mathbb{C}$. Consider a polydisc of n-dimention

$$D^n := D \times \cdots \times D.$$

Let $z = (z_1, \ldots, z_n)$ be coordinates of $\mathbb{C}^n$.

Lemma 1.1 ([5]).

a) $H^i(D^n \setminus \{0\}, \ \mathcal{O}) = 0$ *for* $i \neq 0, \ n-1$.

b) $H^0(D^n \setminus \{0\}, \ \mathcal{O}) \cong H^0(D^n, \mathcal{O})$ *if* $n \geqq 2$.

c) $H^{n-1}(D^n \setminus \{0\}, \ \mathcal{O}) \cong \left\{ \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^{-\alpha} \ ; \ \lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|c_\alpha|} = 0 \right\}.$

Proof. If $n \geqq 2$, then (b) is clear, because $\{0\}$ is an analytic set of D^n with codimention more than 1 .

We show a). For $i = 1, \ldots, n$ we set $U_i := \{z \in D^n; z_i \neq 0\}$. Then $\mathfrak{U} := \{U_i; i = 1, \ldots, n\}$ is a Stein covering of $D^n \setminus \{0\}$. Therefore we get

$$H^q(D^n \setminus \{0\}, \mathcal{O}) \cong H^q(\mathfrak{U}, \mathcal{O})$$

by Leray's theorem. For any $(i_0, \ldots, i_q) \in \{1, \ldots, n\}^{q+1}$, we set

$$U_{i_0 \ldots i_q} := U_{i_0} \cap \cdots \cap U_{i_q}.$$

Since $\mathfrak{U}$ consists of n open sets, it is obvious that $H^n(\mathfrak{U}, \mathcal{O}) = 0$. For any $j \in \{1, \ldots, n\}$ and $(i_0, \ldots, i_q) \in \{1, \ldots, n\}^{q+1}$, we define a map

$$\varphi_j^{i_0 \ldots i_q} : \Gamma(U_{j i_0 \ldots i_q}, \mathcal{O}) \to \Gamma(U_{i_0 \ldots i_q}, \mathcal{O})$$

by the following way:
When $j = i_\mu$ for some $\mu = 0, \ldots, q$ we set $\varphi_j^{i_0 \ldots i_q} = id$. When $j \neq i_\mu$ for all $\mu = 0, \ldots, q$, we take R with $0 < R < 1$ such that $|z_j| < R$ for any $z = (z_1, \ldots, z_n) \in U_{j i_0 \ldots i_q}$. Let $C_j(R)$ be a circle with center 0 and radius R. For any $f \in \Gamma(U_{j i_0 \ldots i_q}, \mathcal{O})$ we define

$$(\varphi_j^{i_0 \ldots i_q} f)(z) := \frac{1}{2\pi\sqrt{-1}} \int_{C_j(R)} \frac{f(z_1, \ldots z_{j-1}, t_j, z_{j+1}, \ldots, z_n)}{t_j - z_j} dt_j.$$

Then we have $\varphi_j^{i_0 \ldots i_q} f \in \Gamma(U_{i_0 \ldots i_q}, \mathcal{O})$.

If $z_j \neq 0$, then we take r such that $0 < r < |z_j| < R < 1$ and denote by $C_j(r)$ a circle with center 0 and radius r. By Cauchy's integral theorem we can represent $f \in \Gamma(U_{j i_0 \ldots i_q}, \mathcal{O})$ as

$$\begin{aligned} f(z) = & \frac{1}{2\pi\sqrt{-1}} \int_{C_j(R)} \frac{f(z_1, \ldots z_{j-1}, t_j, z_{j+1}, \ldots, z_n)}{t_j - z_j} dt_j \\ & - \frac{1}{2\pi\sqrt{-1}} \int_{C_j(r)} \frac{f(z_1, \ldots z_{j-1}, t_j, z_{j+1}, \ldots, z_n)}{t_j - z_j} dt_j. \end{aligned}$$

Therefore we note that

$$(\varphi_j^{i_0 \ldots i_q} f)(z) = \frac{1}{2\pi\sqrt{-1}} \int_{C_j(r)} \frac{f(z_1, \ldots z_{j-1}, t_j, z_{j+1}, \ldots, z_n)}{t_j - z_j} dt_j + f(z)$$

for $z = (z_1, \ldots, z_n)$ with $z_j \neq 0$.

Let $r^{i_0 \ldots i_q}_{i_0 \ldots \hat{i_\mu} \ldots i_q}$ be the restriction map from functions on $U_{i_0 \ldots \hat{i_\mu} \ldots i_q}$ to those on $U_{i_0 \ldots i_q}$, where $\hat{i_\mu}$ means that i_μ shall be omitted. Then it follows that

$$\begin{cases} r^{i_0 \ldots i_q}_{i_0 \ldots \hat{i_\mu} \ldots i_q} \varphi_j^{i_0 \ldots \hat{i_\mu} \cdots_q} = \varphi_j^{i_0 \ldots i_q}, & \\ \varphi_j^{i_0 \ldots i_q} r^{j i_0 \ldots i_q}_{i_0 \ldots i_q} = id & \text{on } \Gamma(U_{i_0 \ldots i_q}, \mathcal{O}). \end{cases}$$

For any $j \in \{1, \ldots, n\}$, we define a map

$$k_j : C^q(\mathfrak{U}, \mathcal{O}) \to C^{q-1}(\mathfrak{U}, \mathcal{O})$$

as follows:
For any $f = (f_{i_0 \ldots i_q}) \in C^q(\mathfrak{U}, \mathcal{O})$ we set

$$(k_j f)_{i_0 \ldots i_{q-1}} := \varphi_j^{i_0 \ldots i_{q-1}} f_{j i_0 \ldots i_{q-1}}.$$

Let δ be the usual coboundary operater. We take any $f \in C^q(\mathfrak{U}, \mathcal{O})$. For any multiindex $(i_0, \ldots, i_q) \in \{1, \ldots, n\}^{q+1}$ we calculate

$$(f - \delta k_j f - k_j \delta f)_{i_0 \ldots i_q} = f_{i_0 \ldots i_q} - (\delta k_j f)_{i_0 \ldots i_q} - (k_j \delta f)_{i_0 \ldots i_q}.$$

Since

$$\begin{aligned}(\delta k_j f)_{i_0\ldots i_q} &= \sum_{\mu=0}^{q}(-1)^{\mu}(k_j f)_{i_0\ldots\widehat{i_\mu}\ldots i_q}\\ &= \sum_{\mu=0}^{q}(-1)^{\mu}\varphi_j^{i_0\ldots\widehat{i_\mu}\ldots i_q} f_{ji_0\ldots\widehat{i_\mu}\ldots i_q}\end{aligned}$$

and

$$\begin{aligned}(k_j\delta f)_{i_0\ldots i_q} &= \varphi_j^{i_0\ldots i_q}(\delta f)_{ji_0\ldots i_q}\\ &= \varphi_j^{i_0\ldots i_q}\left(f_{i_0\ldots i_q} - \sum_{\mu=0}^{q}(-1)^{\mu}\varphi_j^{i_0\ldots i_q} f_{ji_0\ldots\widehat{i_\mu}\ldots i_q}\right)\\ &= \varphi_j^{i_0\ldots i_q} f_{i_0\ldots i_q} - \sum_{\mu=0}^{q}(-1)^{\mu}\varphi_j^{i_0\ldots i_q} f_{ji_0\ldots\widehat{i_\mu}\ldots i_q},\end{aligned}$$

we have

$$\begin{aligned}&f_{i_0\ldots i_q} - (\delta k_j f)_{i_0\ldots i_q} - (k_j\delta f)_{i_0\ldots i_q} =\\ &\qquad f_{i_0\ldots i_q} - \varphi_j^{i_0\ldots i_q} f_{i_0\ldots i_q} + \sum_{\mu=0}^{q}(-1)^{\mu}\varphi_j^{i_0\ldots\widehat{i_\mu}\ldots i_q} f_{ji_0\ldots\widehat{i_\mu}\ldots i_q}\\ &\qquad\qquad - \sum_{\mu=0}^{q}(-1)^{\mu}\varphi_j^{i_0\ldots i_q} f_{ji_0\ldots\widehat{i_\mu}\ldots i_q}.\end{aligned}$$

When $j \notin \{i_0,\ldots,i_q\}$ we have $\varphi_j^{i_0\ldots i_q} f_{i_0\ldots i_q} = f_{i_0\ldots i_q}$. Since

$$\varphi_j^{i_0\ldots i_q} f_{ji_0\ldots\widehat{i_\mu}\ldots i_q} = r^{i_0\ldots i_q}_{i_0\ldots\widehat{i_\mu}\ldots i_q}\varphi_j^{i_0\ldots\widehat{i_\mu}\ldots i_q} f_{ji_0\ldots\widehat{i_\mu}\ldots i_q},$$

we have in general

$$\begin{aligned}&f_{i_0\ldots i_q} - (\delta k_j f)_{i_0\ldots i_q} - (k_j\delta f)_{i_0\ldots i_q}\\ &\qquad = \begin{cases}0 & \text{if } j\notin\{i_0,\ldots,i_q\}\\ (id - r^{i_0}_{i_0\ldots\widehat{i_{\mu_j}}\ldots i_q}\varphi_j^{i_0\ldots\widehat{i_{\mu_j}}\ldots i_q})f_{i_0\ldots i_q} & \text{if } j = i_\mu \text{ for some } \mu,\end{cases}\end{aligned}$$

where μ_j is μ with $j = i_\mu$. For any $j \in \{1,\ldots,n\}$ we define a map

$$\Phi_j : C^q(\mathfrak{U},\mathcal{O}) \to C^q(\mathfrak{U},\mathcal{O})$$

by

$$\Phi_j f := f - \delta k_j f - k_j\delta f$$

for any $f \in C^q(\mathfrak{U},\mathcal{O})$. It is obvious that Φ_j maps an element in $Z^q(\mathfrak{U},\mathcal{O})$ to $Z^q(\mathfrak{U},\mathcal{O})$. Then we define a map

$$\Phi : Z^q(\mathfrak{U},\mathcal{O}) \to Z^q(\mathfrak{U},\mathcal{O})$$

by $\Phi := \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_n$. If $1 \leqq q \leqq n-2$, then for any $(i_0, \dots, i_q) \in \{1, \dots, n\}^{q+1}$ there exists $j \in \{1, \dots, n\}$ such that $j \notin \{i_0, \dots, i_q\}$. Hence we have $\Phi f = 0$ for any $f \in Z^q(\mathfrak{U}, \mathcal{O})$. This means that $f \in B^q(\mathfrak{U}, \mathcal{O})$. Therefore we obtain

$$H^q(\mathfrak{U}, \mathcal{O}) = 0 \quad \text{for } 1 \leqq q \leqq n-2.$$

Next we show c). First we note

$$Z^{n-1}(\mathfrak{U}, \mathcal{O}) = \Gamma(U_{12\cdots n}, \mathcal{O}).$$

Any $f \in \Gamma(U_{12\cdots n}, \mathcal{O})$ has the following Laurent expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^{-\alpha} + \sum_{\mu=1}^{n} (-1)^{n-1} \sum_{n_\mu = 0}^{\infty} g_{n_\mu}(z_1, \dots, \widehat{z_\mu}, \dots, z_n) z_\mu^{n_\mu},$$

where $g_{n_\mu}(z_1, \dots, \widehat{z_\mu}, \dots, z_n)$ is a holomorphic function in $(z_1, \dots, z_{\mu-1}, z_{\mu+1}, \dots, z_n) \in (D^*)^{n-1}$ and coefficients c_α satisfy

$$\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|c_\alpha|} = 0.$$

Let

$$g_{12\cdots\widehat{\mu}\cdots n} := \sum_{\mu=1}^{n} (-1)^{n-1} \sum_{n_\mu=0}^{\infty} g_{n_\mu}(z_1, \dots, \widehat{z_\mu}, \dots, z_n) z_\mu^{n_\mu}.$$

Then we have an element $g = (g_{12\cdots\widehat{\mu}\cdots n})$ in $C^{n-2}(\mathfrak{U}, \mathcal{O})$. Therefore we have

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^{-\alpha} + \delta g,$$

which shows the isomorphism in c). $\square$

1.2 Generalized Martinelli formula

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we define a $(0, n-1)$-form

$$\psi_\alpha := \left(\sum_{1 \leqq j \leqq n} z_j^{\alpha_j} \overline{z_j}^{\alpha_j} \right)^{-n} \sum_{1 \leqq j \leqq n} (-1)^{j-1} \overline{z_j}^{\alpha_j} \bigwedge_{\substack{1 \leqq k \leqq n \\ k \neq j}} d(\overline{z_k}^{\alpha_k}).$$

Letting $dz := \bigwedge_{1 \leqq j \leqq n} dz_j$, we consider an $(n, n-1)$-form $K_\alpha^{(n)} := dz \wedge \psi_\alpha$. For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we set

$$\alpha + \mathbf{1} := (\alpha_1 + 1, \dots, \alpha_n + 1) \text{ and } \alpha' := (\alpha_2, \dots, \alpha_n),$$

where $\mathbf{1} = (1, \dots, 1)$. We also set $\alpha! := \prod_{1 \leqq j \leqq n} \alpha_j!$.

Let $B = \{z \in \mathbb{C}^n; \sum_{1 \leqq j \leqq n} z_j \overline{z}_j < 1\}$ be the unit ball of $\mathbb{C}^n$. We denote by $S = \partial B$ the boundary of B. Let

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

The equation in the following proposition is called the generalized Martinelli formula.

Proposition 1.2 (Proposition 1 in [4])**.** *We have*

$$\int_S f K_{\alpha+\mathbf{1}}^{(n)} = \frac{(2\pi\sqrt{-1})^n}{(n-1)!} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0)$$

for any holomorphic function f on a neighbourhood of $\overline{B}$, where B has the direction in which $\left(\frac{\sqrt{-1}}{2}\right)^n dz \wedge d\overline{z}$ is positive and the direction of S is compatible with the formula of Stokes.

Proof. If $n = 1$, then it is just the integral formula of Cauchy. We prove the statement by induction on n. To prove it we temporarily take the direction of B in which $\bigwedge_{1 \leqq j \leqq n} \left(\frac{\sqrt{-1}}{2} dz_j \wedge d\overline{z_j}\right)$ is positive. Moreover, we assume that the direction of S is compatible with Stokes's formula. We denote

$$\theta_\alpha := \frac{1}{n-1} \frac{1}{z_1^{\alpha_1}} \left(\sum_{1 \leqq j \leqq n} z_j^{\alpha_j} \overline{z_j}^{\alpha_j} \right)^{1-n} \sum_{2 \leqq j \leqq n} (-1)^j \overline{z_j}^{\alpha_j} \bigwedge_{\substack{2 \leqq k \leqq n \\ k \neq j}} d(\overline{z_k}^{\alpha_k}),$$

$$L_\alpha := (-1)^n dz \wedge \theta_\alpha.$$

We can check

$$\psi_\alpha = \overline{\partial} \theta_\alpha, \ K_\alpha^{(n)} = \overline{\partial} L_\alpha = d L_\alpha$$

at any point with $z_1 \neq 0$ by straight calculation. For a multiindex $\beta = (\alpha_1 + 1, \alpha_2, \ldots, \alpha_n)$ we have

$$
\begin{aligned}
\int_S f K_\beta^{(n)} &= \lim_{\varepsilon\to 0}\int_{S\cap\{|z_1|>\varepsilon\}} d(fL_\beta)\\
&= \lim_{\varepsilon\to 0}\int_{S\cap\{|z_1|=\varepsilon\}} fL_\beta\\
&= \frac{(-1)^{n-1}}{n-1}\lim_{\varepsilon\to 0}\int_{-S\cap\{|z_1|=\varepsilon\}} f \bigwedge_{1\leqq j\leqq n} dz_j\\
&\qquad \bigwedge \frac{1}{z_1^{\alpha+1}}\left(\sum_{1\leqq j\leqq n} z_j^{\beta_j}\overline{z_j}^{\beta_j}\right)^{1-n}\sum_{2\leqq j\leqq n}(-1)^j\overline{z_j}^{\alpha_j}\bigwedge_{\substack{2\leqq k\leqq n\\ k\neq j}} d(\overline{z_k}^{\alpha_k})\\
&= \frac{(-1)^{n-1}}{n-1}\lim_{\varepsilon\to 0}\int_{-S\cap\{|z_1|=\varepsilon\}} f\frac{dz_1}{z_1^{\alpha_1+1}}\wedge\left(\bigwedge_{2\leqq j\leqq n} dz_j\right)\\
&\qquad \bigwedge\frac{\sum_{2\leqq j\leqq n}(-1)^j\overline{z_j}^{\alpha_j}\bigwedge_{\substack{2\leqq k\leqq n\\ k\neq j}} d(\overline{z_k}^{\alpha_k})}{\left(\varepsilon^{2(\alpha_1+1)}+\sum_{2\leqq j\leqq n} z_j^{\alpha_j}\overline{z_j}^{\alpha_j}\right)^{n-1}}\\
&= \frac{(-1)^{n-1}}{n-1}\frac{2\pi\sqrt{-1}}{\alpha_1!}\lim_{\varepsilon\to 0}\int_{S\cap\{z_1=0\}}\frac{\partial^{\alpha_1} f}{\partial z_1^{\alpha_1}}\left(\bigwedge_{2\leqq j\leqq n} dz_j\right)\\
&\qquad \bigwedge\frac{\sum_{2\leqq j\leqq n}(-1)^j\overline{z_j}^{\alpha_j}\bigwedge_{\substack{2\leqq k\leqq n\\ k\neq j}} d(\overline{z_k}^{\alpha_k})}{\left(\varepsilon^{2(\alpha_1+1)}+\sum_{2\leqq j\leqq n} z_j^{\alpha_j}\overline{z_j}^{\alpha_j}\right)^{n-1}}\\
&= \frac{(-1)^{n-1}}{n-1}\frac{2\pi\sqrt{-1}}{\alpha_1!}\int_{S\cap\{z_1=0\}}\frac{\partial^{\alpha_1} f}{\partial z_1^{\alpha_1}} K_{\alpha'}^{(n-1)}.
\end{aligned}
$$

By the assumption of induction we have

$$
\int_{S\cap\{z_1=0\}}\frac{\partial^{\alpha_1} f}{\partial z_1^{\alpha_1}} K_{\alpha'+\mathbf{1}'}^{(n-1)} = \frac{(2\pi\sqrt{-1})^{n-1}}{(n-2)!}\frac{1}{\alpha'!}\frac{\partial^{|\alpha'|}}{\partial z_2^{\alpha'_2}\cdots\partial z_n^{\alpha'_n}}\left(\frac{\partial^{\alpha_1} f}{\partial z_1^{\alpha_1}}\right)(0).
$$

We note that the relation between the original direction of B and one given here to prove the statement is as follows:

$$
\left(\frac{\sqrt{-1}}{2}\right)^n\omega\wedge\overline{\omega} = (-1)^{\frac{n(n-1)}{2}}\bigwedge_{1\leqq j\leqq n}\left(\frac{\sqrt{-1}}{2}dz_j\bigwedge d\overline{z_j}\right).
$$

Using the above result, we obtain the integral $\displaystyle\int_S fK_{\alpha+\mathbf{1}}^{(n)}$ in the original direction as follows:

$$\int_S fK^{(n)}_{\alpha+\mathbf{1}} = (-1)^{\frac{n(n-1)}{2}} \frac{(-1)^{n-1}}{n-1} \frac{2\pi\sqrt{-1}}{\alpha_1!} \frac{(2\pi\sqrt{-1})^{n-1}}{(n-2)!} \frac{1}{\alpha'!} \frac{\partial^{|\alpha'|}}{\partial z_2^{\alpha'_2} \cdots \partial z_n^{\alpha'_n}} \left(\frac{\partial^{\alpha_1} f}{\partial z_1^{\alpha_1}} \right)(0)$$
$$= \frac{(2\pi\sqrt{-1})^n}{(n-1)!} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0).$$

Then the proof is completed. □

1.3 Dolbeault isomorphism

We have seen cohomology groups of a punctured polydisc in Section 1.1. Lemma 1.1 c) shows that $H^{n-1}(D^n \setminus \{0\}, \mathcal{O})$ is a Fréchet space generated by cohomology classes

$$z_1^{-\alpha_1-1} \cdots z_n^{-\alpha_n-1} \in \Gamma(\bigcap_{i=1}^n U_i, \mathcal{O}), \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n.$$

In this section, we study $\overline{\partial}$-closed $(0, n-1)$-forms corresponding to cohomology classes $z_1^{-\alpha-1} \cdots z_n^{-\alpha_n-1}$ by the Dolbeault isomorphism.

Lemma 1.3 (Lemma 4 in [5]). *a) By the Dolbeault isomorphism the cohomology class of* $z_1^{-\alpha_1-1} \cdots z_n^{-\alpha_n-1}$ $((\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n)$ *corresponds to the* $\overline{\partial}$*-closed* $(0, n-1)$*-form (up to a sign)*

$$(n-1)!\psi_{\alpha+\mathbf{1}} =$$
$$(n-1)! \frac{\sum_{k=1}^n (-1)^k \overline{z_k}^{\alpha_k+1} d(\overline{z_1}^{\alpha_1+1}) \wedge \cdots \wedge \widehat{d(\overline{z_k}^{\alpha_k+1})} \wedge \cdots \wedge d(\overline{z_n}^{\alpha_n+1})}{\left(\sum_{j=1}^n |z_j^{\alpha_j+1}|^2\right)^n}.$$

b) The Dolbeault representative so chosen is such that to the class $(z_1 \cdots z_n)^{-1} \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^{-\alpha}$ *with* $\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{c_\alpha} = 0$*, corresponds to the* $\overline{\partial}$*-closed* $(0, n-1)$*-form expressed by absolutely convergent series*

$$(n-1)! \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \psi_{\alpha+1} \text{ on } \mathbb{C}^n \setminus \{0\}.$$

Proof. a) Let $\varepsilon^{(a,b)}$ be the sheaf of germs of C^∞ (a, b)-forms. We particularly set $\varepsilon = \varepsilon^{(0,0)}$ when $(a, b) = (0, 0)$. Since $\varepsilon^{(a,b)}$ is a fine sheaf, it follows that

$$H^q(D^n \setminus \{0\}, \varepsilon^{(a,b)}) = 0 \text{ for any } q \geqq 1.$$

There exists $\varphi = (\varphi_{1\cdots\widehat{i}\cdots n}) \in C^{n-2}(\mathfrak{U}, \varepsilon)$ such that $z^{-\alpha-1} = \delta\varphi$, i.e.

$$z^{-\alpha-1} = \sum_{i=1}^n (-1)^{i-1} \varphi_{1\cdots\widehat{i}\cdots n},$$

for we can consider $z^{-\alpha-1} \in C^{n-1}(\mathfrak{U}, \varepsilon)$ for all $\alpha \in \mathbb{N}_0^n$. Then we have

$$\sum_{i=1}^{n} (-1)^{i-1} \varphi_{1 \cdots \widehat{i} \cdots n} = 0.$$

Therefore we have $\overline{\partial} \varphi = (\overline{\partial} \varphi_{1 \cdots \widehat{i} \cdots n}) \in Z^{n-2}(\mathfrak{U}, \varepsilon^{(0,1)})$. Since

$$H^{n-2}(\mathfrak{U}, \varepsilon^{(0,1)}) \cong H^{n-2}(D^n \setminus \{0\}, \varepsilon^{(0,1)}) = 0,$$

there exists $(\varphi_{1 \cdots \widehat{i} \cdots \cdots \widehat{j} \cdots n}) \in C^{n-3}(\mathfrak{U}, \varepsilon^{(0,1)})$ such that

$$\overline{\partial} \varphi = \delta((\varphi_{1 \cdots \widehat{i} \cdots \cdots \widehat{j} \cdots n})) = \left(\sum \pm \varphi_{1 \cdots \widehat{i} \cdots \cdots \widehat{j} \cdots n} \right).$$

Repeating this procedure, we obtain $(\varphi_j) \in C^0(\mathfrak{U}, \varepsilon^{(0,n-2)})$ such that

$$\overline{\partial} \varphi_i = \overline{\partial} \varphi_j \text{ for any } i,\ j \text{ with } U_i \cap U_j \neq \emptyset.$$

Therefore we can define a C^∞ $(0, n-1)$-form Ψ on $D^n \setminus \{0\}$ by

$$\Psi := \overline{\partial} \varphi_i \quad \text{on } U_i.$$

It is clear that $\overline{\partial} \Psi = 0$. This form Ψ is the Dolbeault representative corresponding to $z^{-\alpha-1}$.

We set

$$P_\varepsilon := \{|z_i| \leqq \varepsilon; i = 1, \ldots, n\}$$

for sufficiently small $\varepsilon > 0$. Then we have

$$\int_{|z_1|=\varepsilon, \cdots, |z_n|=\varepsilon} z^{-\alpha-1} \sum c_\beta z^\beta dz_1 \cdots dz_n = (2\pi\sqrt{-1})^n c_\alpha$$

for any convergent series $f = \sum_{\beta \in \mathbb{N}_0^n} c_\beta z^\beta$ in a neighbourhood of 0 in which P_ε is contained. Namely $z^{-\alpha-1}$ are generators characterized by the above property.

Using Stokes's theorem and repeating correspondence between Čech cohomology classes and $\overline{\partial}$-cohomology classes, we obtain

$$\begin{aligned}
&\int_{|z_1|=\varepsilon, \cdots, |z_n|=\varepsilon} z^{-\alpha-1} f dz_1 \wedge \cdots \wedge dz_n \\
&= \sum_{i=1}^{n} \int_{|z_1|=\varepsilon, \cdots, |z_n|=\varepsilon} (-1)^{i-1} \varphi_{1 \cdots \widehat{i} \cdots n} f dz_1 \wedge \cdots \wedge dz_n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_{|z_1|=\varepsilon, \cdots, |z_i| \leqq \varepsilon, \cdots, |z_n|=\varepsilon} \overline{\partial} \varphi_{1 \cdots \widehat{i} \cdots n} f dz_1 \wedge \cdots \wedge dz_n \\
&= \cdots \\
&= \pm \sum \int_{|z_i|=\varepsilon, |z_j| \leqq \varepsilon \ (j \neq i)} \Psi f dz_1 \wedge \cdots \wedge dz_n \\
&= \pm \int_{\partial P_\varepsilon} \Psi f dz_1 \wedge \cdots \wedge dz_n,
\end{aligned}$$

for

$$z^{-\alpha-1} = \sum_{i=1}^{n}(-1)^{i-1}\varphi_{1\cdots\widehat{i}\cdots n}.$$

Hence we have

$$\pm\int_{\partial P_\varepsilon}\Psi f dz_1\wedge\cdots\wedge dz_n = (2\pi\sqrt{-1})^n c_\alpha.$$

Therefore, it holds that for any holomorphic function f on a neighbourhood of $\overline{B}$

$$\int_{\partial B}\Psi f dz_1\wedge\cdots\wedge dz_n = \pm(2\pi\sqrt{-1})^n(\alpha!)^{-1}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0),$$

where B is an open ball with center 0 and sufficiently small radius. For $K_{\alpha+\mathbf{1}} = \omega\wedge\psi_{\alpha+\mathbf{1}}$ it holds that

$$\int_{\partial B} fK_{\alpha+\mathbf{1}} = \frac{(2\pi\sqrt{-1})^n}{(n-1)!}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)$$

by Proposition 1.2 in the previous section. Therefore, the representative in the Dolbeault classes corresponding to $z^{-\alpha-1}$ is $(n-1)!\psi_{\alpha+\mathbf{1}}$ (up to sign).

b) From (a) we see that the series

$$(n-1)!\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha\psi_{\alpha+\mathbf{1}}$$

formally corresponds to

$$(z_1\cdots z_n)^{-1}\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^{-\alpha} \quad\text{with } \lim_{|\alpha|\to\infty}\sqrt[|\alpha|]{|c_\alpha|} = 0.$$

Then it suffices to show that the above series converges absolutely on $\mathbb{C}^n\setminus\{0\}$. Take δ with $0<\delta<1$. Let $\sum_{i=1}^n |z_i|^2 > \delta$. Then we have $|z_i| > \frac{\delta}{n}$ for some i. Therefore we have

$$\sum_{j=1}^{n}|z_j|^{2\alpha_j+2} \geqq |z_i|^{2\alpha_i+2} > \left(\frac{\delta}{n}\right)^{2\alpha_i+2} > \left(\frac{\delta}{n}\right)^{2|\alpha|+2}.$$

Consequently the absolute value of the coefficient of $d\overline{z_1}\wedge\cdots\wedge\widehat{d\overline{z_j}}\wedge\cdots\wedge d\overline{z_n}$ in $\psi_{\alpha+\mathbf{1}}$ is estimated by

$$\begin{aligned}&\prod_{k\neq j}\alpha_k|z_j||z_1|^{\alpha_1}\cdot|z_n|^{\alpha_n}\left(\left(\frac{\delta}{n}\right)^{2|\alpha|+2}\right)^{-n}\\ &=\prod_{k\neq j}\alpha_k|z_j|\frac{1}{\left(\frac{\delta}{n}\right)^{2n}}\left(\frac{|z_1|}{\left(\frac{\delta}{n}\right)^{2n}}\right)^{\alpha_1}\cdots\left(\frac{|z_n|}{\left(\frac{\delta}{n}\right)^{2n}}\right)^{\alpha_n}.\end{aligned}$$

Thus we see that the absolute value of the coefficient of $d\overline{z_1} \wedge \cdots \wedge \widehat{d\overline{z_j}} \wedge \cdots \wedge d\overline{z_n}$ in $\sum c_\alpha \psi_{\alpha+\mathbf{1}}$ is estimated by

$$|z_j| \frac{1}{\left(\frac{\delta}{n}\right)^{2n}} \sum |c_\alpha| w^\alpha, \;\; w_i := \frac{|z_i|}{\left(\frac{\delta}{n}\right)^{2n}}.$$

Hence the series converges uniformly on compact subsets in $\mathbb{C}^n \setminus \{0\}$. $\square$

Chapter 2

Zappa's Results

2.1 Cohomology groups of a punctured torus

Let $T^n = \mathbb{C}^n/\Gamma$ be a complex torus of n-dimension, where Γ is a lattice of $\mathbb{C}^n$. Let $\pi : \mathbb{C}^n \to T^n$ be the canonical projection. We can take a neighbourhood V of 0 in T^n such that

$$\pi^{-1}(V) = \bigsqcup_{\gamma\in\Gamma} U_\gamma \text{ (disjoint union)},$$

$$\pi|_{U_\gamma} : U_\gamma \to V$$

is a biholomorphic mapping, where U_γ is a polydisc with center γ. Applying Mayer-Vietoris' theorem to $T^n = V \cup (T^n \setminus \{0\})$, we obtain a cohomology exact sequence

$$\begin{aligned} 0 &\to H^0(T^n, \mathcal{O}) \to H^0(V, \mathcal{O}) \oplus H^0(T^n \setminus \{0\}, \mathcal{O}) \to \cdots \to H^k(T^n, \mathcal{O}) \to \\ H^k(T^n, \mathcal{O}) &\oplus H^k(T^n \setminus \{0\}, \mathcal{O}) \to \quad H^k(V \setminus \{0\}, \mathcal{O}) \to \cdots \to H^{n-2}(V \setminus \{0\}, \mathcal{O}) \\ &\to H^{n-1}(T^n, \mathcal{O}) \to H^{n-1}(V, \mathcal{O}) \oplus H^{n-1}(T^n \setminus \{0\}, \mathcal{O}) \to H^{n-1}(V \setminus \{0\}, \mathcal{O}) \\ &\to H^n(T^n, \mathcal{O}) \to H^n(V, \mathcal{O}) \oplus H^n(T^n \setminus \{0\}, \mathcal{O}) \to H^n(V \setminus \{0\}, \mathcal{O}) \to 0. \end{aligned}$$

By Lemma1.1 a) in Chapter 1 and

$$H^i(V, \mathcal{O}) = 0 \text{ if } i \geqq 1,$$

we have the following exact sequence

$$\begin{aligned} 0 \to H^{n-1}(T^n, \mathcal{O}) \to H^{n-1}(T^n \setminus \{0\}, \mathcal{O}) \to \\ H^{n-1}(V \setminus \{0\}, \mathcal{O}) \xrightarrow{\delta} H^n(T^n, \mathcal{O}) \to 0. \end{aligned}$$

Therefore we obtain

$$H^{n-1}(T^n \setminus \{0\}, \mathcal{O}) \cong H^{n-1}(T^n, \mathcal{O}) \oplus \mathrm{Ker}\delta.$$

$H^{n-1}(T^n, \mathcal{O})$ is an n-dimensional vector space generated by $(0, n-1)$-forms with constant coefficients. $H^{n-1}(U_\gamma \setminus \{\gamma\})$ is isomorphic to $H^{n-1}(V \setminus \{0\}, \mathcal{O})$. It is a Fréchet space generated by Čech cohomolgy classes expressed by $(z-\gamma)^{-\alpha-\mathbf{1}} \in \Gamma(\cap_{i=1}^n U_i, \mathcal{O})$, $\alpha \in \mathbb{N}_0^n$ (Lemma 1.1). A $\overline{\partial}$-closed $(0, n-1)$-form $(n-1)!\psi_{\alpha+\mathbf{1}}(z,\gamma)$ corresponds to the cohomology class of $(z-\gamma)^{-\alpha-\mathbf{1}}$ by the Dolbeault isomorphism, where

$$\psi_{\alpha+\mathbf{1}}(z,\gamma) = \frac{\sum_{k=1}^n (-1)^k (\overline{z_k} - \overline{\gamma_k})^{\alpha_k+1} \bigwedge_{\substack{j=1 \\ j\neq k}}^n d((\overline{z_j} - \overline{\gamma_j})^{\alpha_j+1})}{\left(\sum_{j=1}^n |z_j - \gamma_j|^{2\alpha_j}\right)^n}$$

(Lemma 1.3). For a sufficiently small $\varepsilon > 0$, we set an open ball $B_\varepsilon(\gamma)$ with radius ε and center γ. By the generalized Martinelli formula, we have

$$\int_{S_\varepsilon(\gamma)} f\psi_{\alpha+\mathbf{1}}(z,\gamma) \wedge dz = \frac{(2\pi\sqrt{-1})^n}{(n-1)!} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(\gamma) \tag{2.1}$$

for any holomorphic function f on a neighbourhood of $\overline{B_\varepsilon(\gamma)}$. Here we note that

$$\psi_{\alpha+\mathbf{1}} \wedge dz = (-1)^{n(n-1)} dz \wedge \psi_{\alpha+\mathbf{1}} = dz \wedge \psi_{\alpha+\mathbf{1}}.$$

If we set $\alpha = (0, \ldots, 0)$ in (2.1), then we have

$$\int_{S_\varepsilon(\gamma)} f\psi_{\mathbf{1}}(z,\gamma) \wedge dz = \frac{(2\pi\sqrt{-1})^n}{(n-1)!} f(\gamma).$$

From the above equality it follows that

$$\frac{\partial^{|\alpha|}}{\partial\gamma^\alpha} \int_{S_\varepsilon(\gamma)} f\psi_{\mathbf{1}}(z,\gamma) \wedge dz = \frac{(2\pi\sqrt{-1})^n}{(n-1)!} \frac{\partial^{|\alpha|} f}{\partial\gamma^\alpha}(\gamma). \tag{2.2}$$

And we see that Kerδ is generated by $\psi_{\alpha+\mathbf{1}}$ ($|\alpha| \geqq 1$). Let

$$\phi_\alpha(z,\gamma) := \frac{(-1)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial z^\alpha} \psi_{\mathbf{1}}(z,\gamma).$$

Since

$$\frac{\partial^{|\alpha|}}{\partial\gamma^\alpha} \psi_{\mathbf{1}}(z,\gamma) = (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial z^\alpha} \psi_{\mathbf{1}}(z,\gamma),$$

it follows from (2.1) and (2.2) that

$$\int_{S_\varepsilon(\gamma)} f(\psi_{\alpha+\mathbf{1}}(z,\gamma) - \phi_\alpha(z,\gamma)) \wedge dz = 0 \tag{2.3}$$

for any holomorphic function f on a neighbourhood of $\overline{B_\varepsilon(\gamma)}$.

Proposition 2.1 (Proposizione 1 in [9]). *The form $\psi_{\alpha+\mathbf{1}}(z,\gamma) - \phi_\alpha(z,\gamma)$ is $\overline{\partial}$-exact.*

Proof. First we prove that $\psi_{\alpha+\mathbf{1}} - \phi_\alpha$ is $\overline{\partial}$-closed. From (2.3) we have

$$\begin{aligned} 0 &= \int_{S_\varepsilon(\gamma)} f(\psi_{\alpha+\mathbf{1}} - \phi_\alpha) \wedge dz \\ &= \int_{B_\varepsilon(\gamma)} d[f(\psi_{\alpha+\mathbf{1}} - \phi_\alpha) \wedge dz] \\ &= \int_{B_\varepsilon(\gamma)} f\overline{\partial}(\psi_{\alpha+\mathbf{1}} - \phi_\alpha) \wedge dz \end{aligned}$$

for any holomorphic function f on a neighbourhood of $\overline{B_\varepsilon(\gamma)}$. Since f is arbitrary, it must be that $\overline{\partial}(\psi_{\alpha+\mathbf{1}} - \phi_\alpha) = 0$ on $B_\varepsilon(\gamma)$. Furthermore $\psi_{\alpha+\mathbf{1}} - \phi_\alpha$ is $\overline{\partial}$-closed on $\mathbb{C}^n \setminus \{\gamma\}$, because it is real analytic.

Then $\psi_{\alpha+\mathbf{1}} - \phi_\alpha$ defines a $\overline{\partial}$-cohomology class in $H^{n-1}(U_\gamma \setminus \{\gamma\}, \mathcal{O})$. Therefore there exists a $(0, n-2)$-form σ such that

$$\psi_{\alpha+\mathbf{1}} - \phi_\alpha = (n-1)! \sum_{\beta \in \mathbb{N}_0^n} c_\beta \psi_{\beta+\mathbf{1}} + \overline{\partial}\sigma,$$

where $\lim_{|\beta|\to\infty} \sqrt[|\beta|]{|c_\beta|} = 0$. From (2.3) we have

$$\begin{aligned} 0 &= \int_{S_\varepsilon(\gamma)} f\left((n-1)! \sum c_\beta \psi_{\beta+\mathbf{1}} + \overline{\partial}\sigma\right) \wedge dz \\ &= \sum c_\beta (n-1)! \int_{S_\varepsilon(\gamma)} f\psi_{\beta+\mathbf{1}} \wedge dz \\ &= \sum (2\pi\sqrt{-1})^n \frac{c_\beta}{\beta!} \frac{\partial^{|\beta|} f}{\partial z^\beta}(\gamma) \end{aligned}$$

for any holomorphic function f on a neighbourhood of $\overline{B_\varepsilon(\gamma)}$. Since f is arbitrary, we have $c_\beta = 0$ for all $\beta \in \mathbb{N}_0^n$. Thus we obtain

$$\psi_{\alpha+\mathbf{1}} - \phi_\alpha = \overline{\partial}\sigma.$$

□

By a straight calculation we obtain the following explicit representation of ϕ_α

(2.4) $\phi_\alpha(z,\gamma) =$

$$\frac{(|\alpha+\mathbf{1}|-1)!}{(n-1)!(\alpha+\mathbf{1})!} \frac{\sum_{k=1}^n (-1)^k (\alpha_k+1)(\overline{z_k}-\overline{\gamma_k})^{\alpha_k+1} \bigwedge_{j\neq k} d((\overline{z_j}-\overline{\gamma_j})^{\alpha_j+1})}{\left(\sum_{j=1}^n |z_j-\gamma_j|^2\right)^{|\alpha+\mathbf{1}|}}.$$

Proposition 2.2. *When* $\lim_{|\alpha|\to\infty} \sqrt[|\alpha|]{|c_\alpha|} = 0$, *the series* $\sum c_\alpha \psi_{\alpha+\mathbf{1}}$ *and* $\sum c_\alpha \phi_\alpha$ *converge uniformly on any compact set of* $\mathbb{C}^n \setminus \{\gamma\}$, *and give the same class in* $H^{n-1}(\mathbb{C}^n \setminus \{\gamma\}, \mathcal{O})$.

Proof. By Proposition 2.1, it is clear that if $\sum c_\alpha \phi_\alpha$ is convergent, then the above two series give the same class. Threrfore we prove that $\sum c_\alpha \phi_\alpha$ converges.

Take a sufficiently small $\rho_1 > 0$ and a sufficiently large number ρ_2. We estimate the absolute value of any coefficient of $\phi_\alpha(z,\gamma)$ on $\rho_1 \leqq ||z-\gamma|| \leqq \rho_2$. Since

$$\begin{aligned}&(-1)^k(\alpha_k+1)(\overline{z_k}-\overline{\gamma_k})^{\alpha_k+1}\bigwedge_{j\neq k} d((\overline{z_j}-\overline{\gamma_j})^{\alpha_k+1})\\ &=(-1)^k\prod_{\ell=1}^{n}(\alpha_\ell+1)(\overline{z_k}-\overline{\gamma_k})\prod_{j=1}^{n}(\overline{z_j}-\overline{\gamma_j})^{\alpha_j}\bigwedge_{j\neq k} d\overline{z_j},\end{aligned}$$

the absolute value of the coefficient of this $(0,n-1)$-form is estimated by

$$\prod_{\ell=1}^{n}(\alpha_\ell+1)\prod_{j=1}^{n}|z_j-\gamma_j|^{\alpha_j}|z_k-\gamma_k| \leqq |\alpha+\mathbf{1}|^n\rho_2^{|\alpha+\mathbf{1}|}.$$

Therfore the absolute value of the coefficient of $\bigwedge_{j\neq k} d\overline{z_j}$ in ϕ_α is estimated by

$$\frac{(|\alpha+\mathbf{1}|-1)!}{(n-1)!(\alpha+\mathbf{1})!}|\alpha+\mathbf{1}|^n\rho_1^{-2|\alpha+\mathbf{1}|}\rho_2^{|\alpha+\mathbf{1}|}.$$

Hence we see that the absolute value of the coefficient of $\bigwedge_{j\neq k} d\overline{z_j}$ in$\sum c_\alpha\phi_\alpha$ is estimated by

$$\sum |c_\alpha|\frac{(|\alpha+\mathbf{1}|-1)!}{(n-1)!(\alpha+\mathbf{1})!}|\alpha+\mathbf{1}|^n\rho_1^{-2|\alpha+\mathbf{1}|}\rho_2^{|\alpha+\mathbf{1}|}.$$

Since

$$\lim_{|\alpha|\to\infty}\left(|c_\alpha|\frac{(|\alpha+\mathbf{1}|-1)!}{(n-1)!(\alpha+\mathbf{1})!}|\alpha+\mathbf{1}|^n\rho_1^{-2|\alpha+\mathbf{1}|}\rho_2^{|\alpha+\mathbf{1}|}\right)^{\frac{1}{|\alpha|}} = \lim_{|\alpha|\to\infty}\sqrt[|\alpha|]{|c_\alpha|}=0,$$

we obtain the conclusion. □

2.2 Γ-invariant forms

We first prove the following lemma which is the key lemma showing the convergence of series. We state it in a general form in order to use it in the next chapter.

Lemma 2.3. *Let Γ be a discrete subgroup of $\mathbb{C}^n$ with* rank $\Gamma = n+m$ $(1 \leqq m \leqq n)$. *Then the series $\sum_{\gamma\in\Gamma\setminus\{0\}}\|\gamma\|^{-\lambda}$ converges for $\lambda > n+m$.*

Proof. Take generators $\gamma_1,\dots,\gamma_{n+m}$ of Γ. For any $i\in\mathbb{N}_0$ we set

$$\Gamma_i := \{a_1\gamma_1+\cdots+a_{n+m}\gamma_{n+m}; a_1,\dots,a_{n+m}\in\mathbb{Z}, |a_j|\leqq i\ (j=1,\dots,n+m)\}.$$

The number of elements of $\Gamma_i \setminus \Gamma_{i-1}$ is given by

$$
\begin{aligned}
\#(\Gamma_i \setminus \Gamma_{i-1}) &= (2i+1)^{n+m} - (2i-1)^{n+m} \\
&= 2((2i+1)^{n+m-1} + (2i+1)^{n+m-2}(2i-1) + \cdots + (2i-1)^{n+m-1}).
\end{aligned}
$$

Since

$$
(2i+1)^{n+m-j}(2i-1)^{j-1} \leqq 2^{2(n+m-1)} i^{n+m-1}
$$

for $j = 1, \ldots, n+m$, we have

$$
\#(\Gamma_i \setminus \Gamma_{i-1}) \leqq A_{n,m} i^{n+m-1},
$$

where $A_{n,m} = 2(n+m)2^{2(n+m-1)}$. Let k be the distance of the boundary of the parallelotope given by $\Gamma_1 \setminus \{0\}$ from the origin. Then we have $\|\gamma\| \geqq ki$ for all $\gamma \in \Gamma_i \setminus \Gamma_{i-1}$.

Therefore we obtain

$$
\begin{aligned}
\sum_{\gamma \in \Gamma \setminus \{0\}} \|\gamma\|^{-\lambda} &= \sum_{i=1}^{\infty} \sum_{\gamma \in \Gamma_i \setminus \Gamma_{i-1}} \|\gamma\|^{-\lambda} \\
&\leqq \sum_{i=1}^{\infty} \sum_{\gamma \in \Gamma_i \setminus \Gamma_{i-1}} (ki)^{-\lambda} \\
&\leqq \frac{A_{n,m}}{k^{\lambda}} \sum_{i=1}^{\infty} \frac{1}{i^{\lambda-(n+m-1)}}.
\end{aligned}
$$

If $\lambda > n+m$, then $\lambda - (n+m-1) > 1$. Then the series $\sum_{i=1}^{\infty} \frac{1}{i^{\lambda-(n+m-1)}}$ converges. □

For the later use we give the following proposition in a more general form (cf. Proposizione 2 in [9]).

Proposition 2.4. *Let Γ be a discrete subgroup of $\mathbb{C}^n$ with* $\operatorname{rank} \Gamma = n+m$ $(1 \leqq m \leqq n)$. *If a multiindex $\alpha \in \mathbb{N}_0^n$ satisfies $|\alpha| > -n+m+1$, then the series*

$$
\varepsilon_{\Gamma}^{\alpha}(z) := \sum_{\gamma \in \Gamma} \varphi_{\alpha}(z, \gamma) \tag{2.5}
$$

converges uniformly on compact subsets of $\mathbb{C}^n \setminus \Gamma$. Therefore it is a $\overline{\partial}$-closed Γ-invariant $(0, n-1)$-form on $\mathbb{C}^n \setminus \Gamma$.

Proof. Take any $\rho > 0$. Suppose that $\|z\| \leqq \rho$ and $z \notin \Gamma$. Since

$$
\begin{aligned}
&(-1)^k(\alpha_k+1)(\overline{z_k} - \overline{\gamma_k})^{\alpha_k+1} \bigwedge_{j \neq k} d((\overline{z_j} - \overline{\gamma_j})^{\alpha_j+1}) \\
&= (-1)^k \prod_{\ell=1}^{n} (\alpha_\ell + 1)(\overline{z_k} - \overline{\gamma_k}) \prod_{\ell=1}^{n} (\overline{z_\ell} - \overline{\gamma_\ell})^{\alpha_\ell} (d\overline{z})_k,
\end{aligned}
$$

we see that the absolute value of the coefficient of $\bigwedge_{j\neq k} d\overline{z}$ in $\phi_\alpha(z,\gamma)$ for fixed k is estimated from above by

$$\frac{(|\alpha+\mathbf{1}|-1)!|\alpha+\mathbf{1}|^n}{(n-1)!(\alpha+\mathbf{1})!}\frac{\prod_{\ell=1}^n |z_\ell-\gamma_\ell|^{\alpha_\ell}|z_k-\gamma_k|}{\|z-\gamma\|^{-2|\alpha+\mathbf{1}|}}.$$

For all $\gamma\in\Gamma$ except a finite number of elements, we have $||\gamma||>2\rho$. Since $\|z-\gamma\|>\frac{1}{2}\|\gamma\|$ for such a γ, the above estimate is further bounded by

$$\frac{(|\alpha+\mathbf{1}|-1)!|\alpha+\mathbf{1}|^n}{(n-1)!(\alpha+\mathbf{1})!}2^{|\alpha|+2n-1}\|\gamma\|^{-(|\alpha|+2n-1)}.$$

It holds that $|\alpha|+2n-1>n+m$ if $|\alpha|>-n+m+1$. Then the series $\sum_{\gamma\in\Gamma\setminus\{0\}}\|\gamma\|^{-(|\alpha|+2n-1)}$ converges by Lemma 2.3. This shows that the series (2.5) converges uniformly on compact subsets of $\mathbb{C}^n\setminus\Gamma$. It is obvious that $\varepsilon_\Gamma^\alpha$ is $\overline{\partial}$-closed and Γ-invariant. $\square$

The next proposition is also a general form of the result which is stated in [9] (cf. Proposizione 3 in [9]).

Proposition 2.5. *If* $\lim_{|\alpha|\to\infty}\sqrt[|\alpha|]{|c_\alpha|}=0$, *then the series*

$$\sum_{|\alpha|\geqq\varepsilon_0} c_\alpha\varepsilon_\Gamma^\alpha \tag{2.6}$$

uniformly converges on compact subsets in $\mathbb{C}^n\setminus\Gamma$, *and is a* $\overline{\partial}$*-closed* Γ*-invariant* $(0,n-1)$*-form on* $\mathbb{C}^n\setminus\Gamma$, *where*

$$\varepsilon_0=\begin{cases}2 & \text{if } m=n,\\ 1 & \text{if } m=n-1,\\ 0 & \text{if } m<n-1.\end{cases}$$

Proof. Take any $\rho>0$. As in the proof of the previous proposition, the absolute value of the coefficient of $\bigwedge_{j\neq k} d\overline{z_j}$ in the serie (2.6) is estimated by

$$\begin{aligned}&\sum_{|\alpha|\geqq\varepsilon_0}|c_\alpha|d_\alpha 2^{|\alpha|+2n-1}\left(\sum_{\gamma\in\Gamma\setminus\{0\}}\|\gamma\|^{-(|\alpha|+2n-1)}\right)\\ &\leqq\sum_{|\alpha|\geqq\varepsilon_0}|c_\alpha|d_\alpha\left(\frac{A_{n,m}}{k^{|\alpha|+2n-1}}\sum_{i=1}^\infty\frac{1}{i^{|\alpha|+n-m}}\right)\\ &\leqq\sum_{|\alpha|\geqq\varepsilon_0}|c_\alpha|d_\alpha A_{n,m}\left(\frac{2}{k}\right)^{|\alpha|+2n-1}\sum_{i=1}^\infty\frac{1}{i^2}\end{aligned}$$

on $\|z\|\leqq\rho$. Here we set

$$d_\alpha=\frac{(|\alpha+\mathbf{1}|-1)!|\alpha+\mathbf{1}|^n}{(n-1)!(\alpha+\mathbf{1})!},$$

and k and $A_{n,m}$ are constants in the proof of Lemma 2.3. Since we have

$$\lim_{|\alpha|\to\infty}\left(|c_\alpha|d_\alpha A_{n,m}\left(\frac{2}{k}\right)^{|\alpha|+2n-1}\sum_{i=1}^{\infty}\frac{1}{i^2}\right)^{\frac{1}{|\alpha|}}=\frac{2}{k}\lim_{|\alpha|\to\infty}\sqrt[|\alpha|]{|c_\alpha|}=0,$$

the proof is complete. □

We use the above proposition in the case of rank $\Gamma = 2n$. The series $\varepsilon_\Gamma^\alpha$ and (2.6) are Γ-invariant and generate cohomology classes of $H^{n-1}(T^n \setminus \{0\}, \mathcal{O})$. But these do not generate all of kerδ. In fact, classes of $|\alpha| = 1$ are omitted. To construst a class of $|\alpha| = 1$, we have to change it.

Let δ_k^i be kronecker's delta. We set $\delta^i = (\delta_1^i, \dots, \delta_n^i) \in \mathbb{N}_0^n$. We write $\phi_{\delta^i} = \phi^i$. Consider the series

$$\wp_\Gamma^i(z) := \phi^i(z,0) + \sum_{\gamma\in\Gamma\setminus\{0\}}\left(\phi^i(z,\gamma) - G^i(\gamma)\right), \tag{2.7}$$

where

$$G^i(\gamma) = \frac{n\sum_{k=1}^n(-1)^k\overline{\gamma_i\gamma_k}\bigwedge_{j\neq k}d\overline{z_j}}{\|\gamma\|^{2n+2}}.$$

Considering $\overline{\gamma_i\gamma_k} = ((\overline{\gamma_i} - \overline{z_i}) + \overline{z_i})((\overline{\gamma_k} - \overline{z_k}) + \overline{z_k})$, we obtain

$$\begin{aligned}
&\wp_\Gamma^i(z) = \phi^i(z,0)\\
&+\sum_{\gamma\in\Gamma\setminus\{0\}}\left(\frac{n(\|\gamma\|^{2n+2}-\|z-\gamma\|^{2n+2})}{(\|\gamma\|\|z-\gamma\|)^{2n+2}}\sum_{k=1}^n(-1)^k(\overline{z_i}-\overline{\gamma_i})(\overline{z_k}-\overline{\gamma_k})\bigwedge_{j\neq k}d\overline{z_j}\right)\\
&+\frac{n}{\|\gamma\|^{2n+2}}\sum_{k=1}^n(-1)^k\left((\overline{z_i}-\overline{\gamma_i})\overline{z_k}+(\overline{z_k}-\overline{\gamma_k})\overline{z_i}-\overline{z_iz_k})\bigwedge_{j\neq k}d\overline{z_j}\right)
\end{aligned}$$

by a simple calculation. We have

$$\left|\|\gamma\|^{2n+2} - \|z-\gamma\|^{2n+2}\right| \leqq c\|z\|\|\gamma\|^{2n+1}$$

with a suitable constant c for $\|\gamma\| - \|z-\gamma\| \leqq \|z\|$. Therefore the absolute value of any coefficient of the series (2.7) is estimated by

$$d\sum_{\gamma\in\Gamma\setminus\{0\}}\rho\|\gamma\|^{-(2n+1)}$$

on $\|z\| \leqq \rho$ for any $\rho > 0$, where d is a suitable constant. Then $\wp_\Gamma^i$ converges uniformly on any compact set of $\mathbb{C}^n \setminus \Gamma$. We can prove that $\wp_\Gamma^i$ is Γ-invariant in an analogous way to the proof for Weierstrass' $\wp$-function.

Restrictions of $\wp_\Gamma^i$ and $\varepsilon_\Gamma^\alpha$ constructed as above to $U_\gamma \setminus \{\gamma\}$ generate ker δ. By a simple calculation we obtain the following formulas for these forms

$$\begin{aligned}
\frac{\partial}{\partial z_k}\varepsilon_\Gamma^\alpha &= -(\alpha_k+1)\varepsilon_\Gamma^{\alpha+\delta^k} \quad \text{for } |\alpha| \geqq 2,\\
\frac{\partial}{\partial z_k}\wp_\Gamma^i &= -(\delta_k^i+1)\varepsilon_\Gamma^{\delta^i+\delta^k}.
\end{aligned}$$

2.3 Mittag-Leffler type theorem for the classes of Dolbeault

For the sake of simplicity, we represent

$$(dz)_k = \bigwedge_{j \neq k} dz_j, \ (d\overline{z})_k = \bigwedge_{j \neq k} d\overline{z_j}$$

from now on. We write the Martinelli kernel as follows

$$\psi(z,\zeta) := \psi_{\mathbf{1}}(z,\zeta) = \frac{\sum_{k=1}^{n} (-1)^k (\overline{z_k} - \overline{\zeta_k})(d\overline{z})_k}{\left(\sum_{j=1}^{n} |z_j - \zeta_j|^2\right)^n}.$$

The Taylor expantion of $\psi(z,\zeta)$ is formally given by

$$\begin{aligned}\psi(z,\zeta) &= \psi(0,\zeta) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} \overline{z_j} \frac{\partial}{\partial \overline{z_j}} \right)^k \psi(0,\zeta) \\ &=: \sum_{k=0}^{\infty} \psi_k,\end{aligned}$$

where

$$\left(\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} \overline{z_j} \frac{\partial}{\partial \overline{z_j}} \right)^k \psi(0,\zeta) = \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{\substack{|\alpha|=\ell \\ |\beta|=k-\ell}} \frac{\partial^k \psi}{\partial z^\alpha \partial \overline{z}^\beta}(0,\zeta) z^\alpha \overline{z}^\beta.$$

Let $|\psi_k|$ be the maximum value among the absolute values of coefficients of ψ_k.

Lemma 2.6 (Lemma 1 in [10])**.** *We have*

$$|\psi_k| \leqq 8^k \binom{n+k-1}{n-1} \|z\|^k \|\zeta\|^{-2n-k+1} \leqq \frac{2^{n-1}}{\|\zeta\|^{2n-1}} \left(\frac{16\|z\|}{\|\zeta\|} \right)^k.$$

Proof. We have

$$|\psi_k| \leqq \frac{2^k}{k!} \|z\|^k \max_{|\alpha|+|\beta|=k} \left| \frac{\partial^k \psi}{\partial z^\alpha \partial \overline{z}^\beta}(0,\zeta) \right|.$$

We note

$$\left| \frac{\partial^k \psi}{\partial z^\alpha \partial \overline{z}^\beta}(0,\zeta) \right| = \left| \frac{\partial^k \psi}{\partial z^\alpha \partial \overline{z}^\beta}(\zeta,0) \right|.$$

It follows that

$$\frac{\partial^{|\alpha|} \psi}{\partial z^\alpha}(z,0) = \sum_{\ell=1}^{n} (-1)^{|\alpha|+\ell-1} \frac{(|\alpha|+n-1)!}{(n-1)!} \overline{z}^{\alpha+\delta^\ell} \|z\|^{-2(n+|\alpha|)} (d\overline{z})_k.$$

We have the following estimation

$$\begin{aligned}
&\left|\frac{\partial^{|\beta|}}{\partial\overline{z}^{\beta}}\left(\overline{z}^{\alpha+\delta^{\ell}}\|z\|^{-2(n+|\alpha|)}\right)\right| \\
&=\left|\sum_{\eta+\nu=\beta}\frac{\beta!}{\eta!}\frac{\partial^{|\eta|}}{\partial\overline{z}^{\eta}}\left(\overline{z}^{\alpha+\delta^{\ell}}\right)\frac{\partial^{|\nu|}}{\partial\overline{z}^{\nu}}\left(\|z\|^{-2(n+|\alpha|)}\right)\right| \\
&\leqq 2^{|\beta|}\max_{\eta+\nu=\beta}\left|\frac{\partial^{|\eta|}}{\partial\overline{z}^{\eta}}\left(\overline{z}^{\alpha+\delta^{\ell}}\right)\frac{\partial^{|\nu|}}{\partial\overline{z}^{\nu}}\left(\|z\|^{-2(n+|\alpha|)}\right)\right|.
\end{aligned}$$

Furthermore we have

$$\left|\frac{\partial^{|\eta|}}{\partial\overline{z}^{\eta}}\left(\overline{z}^{\alpha+\delta^{\ell}}\right)\right| \leqq \prod_{\eta_i\neq 0}(\alpha_i+\delta_i^{\ell})\cdots(\alpha_i+\delta_i^{\ell}-\eta_i+1)\|z\|^{|\alpha-\eta+\delta^{\ell}|},$$

$$\left|\frac{\partial^{|\nu|}}{\partial\overline{z}^{\nu}}\left(\|z\|^{-2(n+|\alpha|)}\right)\right| \leqq (n+|\alpha|)\cdots(n+|\alpha|+|\nu|-1)\|z\|^{-2n-2|\alpha|-|\nu|}.$$

Since

$$\begin{aligned}
\prod_{\eta_i\neq 0}(\alpha_i+\delta_i^{\ell})\cdots(\alpha_i+\delta_i^{\ell}-\eta_i+1) &= \prod_{\eta_i\neq 0}\frac{(\alpha_i+\delta_i^{\ell})!}{(\alpha_i-\eta_i)!} \\
&\leqq \prod_{i=1}^{n}\sum_{\eta_i=0}^{\alpha_i}\frac{(\alpha_i+\delta_i^{\ell})!}{(\alpha_i-\eta_i)!} \\
&= \prod_{i=1}^{n}2^{\alpha_i+\delta_i^{\ell}} \\
&= 2^{|\alpha|+1},
\end{aligned}$$

we obtain

$$\left|\frac{\partial^{|\eta|}}{\partial\overline{z}^{\eta}}\left(\overline{z}^{\alpha+\delta^{\ell}}\right)\frac{\partial^{|\nu|}}{\partial\overline{z}^{\nu}}\left(\|z\|^{-2(n+|\alpha|)}\right)\right| \leqq 2^{|\alpha|+1}\frac{(n+|\alpha|+|\beta|-1)!}{(n+|\alpha|-1)!}\|z\|^{-2n-k+1}.$$

Hence we have

$$\begin{aligned}
\left|\frac{\partial^k\psi}{\partial z^{\alpha}\partial\overline{z}^{\beta}}(0,\zeta)\right| &= \left|\frac{\partial^k\psi}{\partial z^{\alpha}\partial\overline{z}^{\beta}}(\zeta,0)\right| \\
&\leqq \frac{(|\alpha|+n-1)!}{(n-1)!}2^{|\beta|}2^{|\alpha|+1}\frac{(n+|\alpha|+|\beta|-1)!}{(n+|\alpha|-1)!}\|\zeta\|^{-2n-k+1} \\
&= 2^k 2\frac{(n+k-1)!}{(n-1)!}\|\zeta\|^{-2n-k+1}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
|\psi_k| &\leqq \frac{2^k}{k!}\|z\|^k 2^k 2\frac{(n+k-1)!}{(n-1)!}\|\zeta\|^{-2n-k+1} \\
&\leqq 8^k\begin{pmatrix} n+k-1 \\ n-1 \end{pmatrix}\|z\|^k\|\zeta\|^{-2n-k+1}.
\end{aligned}$$

$\square$

We can represent ψ_k as follows:

$$\begin{aligned}\psi_k &= \frac{1}{k!}\sum_{\ell=0}^{k}\begin{pmatrix} k \\ \ell \end{pmatrix}\sum_{\substack{|\alpha|=\ell \\ |\beta|=k-\ell}}\frac{\partial^k\psi}{\partial z^\alpha\partial\overline{z}^\beta}(0,\zeta)z^\alpha\overline{z}^\beta \\ &= \sum_{|\alpha|+p=k}\sum_{|\beta|=p}\frac{1}{k!}\begin{pmatrix} k \\ k-p \end{pmatrix}\frac{\partial^k\psi}{\partial z^\alpha\partial\overline{z}^\beta}(0,\zeta)z^\alpha\overline{z}^\beta \\ &=: \sum_{|\alpha|+p=k}\psi_{\alpha,p}.\end{aligned}$$

Coefficients of $\psi_{\alpha,p}$ are homogeneous polynomials of degree p in $\overline{z_1},\dots,\overline{z_n}$, and we have the representation

$$\psi = \sum_{\alpha}\sum_{p=0}^{\infty}\psi_{\alpha,p}.$$

Similarly coefficients of $\overline{\partial}\psi_{\alpha,p}$ are homogeneous polynomials of degree $p-1$ in $\overline{z_1},\dots,\overline{z_n}$. Since

$$\overline{\partial}\psi = \sum_{\alpha}\sum_{p=0}^{\infty}\overline{\partial}\psi_{\alpha,p}$$

and ψ is $\overline{\partial}$-closed, it must be that $\overline{\partial}\psi_{\alpha,p} = 0$. Hence we have $\overline{\partial}\psi_k = 0$. From the above fact and Lemma 2.6, we obtain the following lemma.

Lemma 2.7 (Lemma 2 in [10]). *The Martinelli kernel $\psi(z,\zeta)$ ($\zeta \neq 0$) is expanded into Taylor series $\sum_{k=0}^{\infty}\psi_k$ of $\overline{\partial}$-closed forms which has norm convergence in a neighbourhood of the origin. The series is uniformly convergent on any compact set in the open ball with radius $\frac{\|\zeta\|}{16}$.*

Definition 2.8 ([10]). *Let ω be a $\overline{\partial}$-closed $(0,n-1)$-form on $\mathbb{C}\setminus\{\xi\}$. If there exists a form*

$$\sum_{|\beta|\leqq k}c_\beta\psi_\beta(z,\xi) \text{ with } c_\beta \neq 0 \text{ for some } \beta \text{ of } |\beta| = k$$

such that $\omega - \sum_{|\beta|\leqq k}c_\beta\psi_\beta(z,\xi)$ is $\overline{\partial}$-exact, then ξ is called a singularity of polar type of order k of ω.

Since operators $\overline{\partial}$ and $\frac{\partial}{\partial z_j}$ are commutative, the form

$$\begin{aligned}\phi_\alpha(z,\xi) &= \frac{(-1)^{|\alpha|}}{\alpha!}\frac{\partial^{|\alpha|}\psi}{\partial z^\alpha}(z,\xi) \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{|\alpha|}}{\alpha!}\frac{\partial^{|\alpha|}\psi}{\partial z^\alpha}(z,\xi)\end{aligned}$$

is a sum of $\overline{\partial}$-closed forms. This series converges uniformly on $\|z\| \leqq \frac{\|\xi\|}{32}$ (Lemma 2.7).

The following theorem is a generalization of the theorem of Mittag-Leffler.

Theorem 2.9 (Teorema 1 in [10])**.** *Let $\Xi = \{\xi_k; k \in \mathbb{N}\}$ be a discrete set in $\mathbb{C}^n$. We suppose that for every ξ_k a form*

$$G_k(z) = \sum_{|\alpha| \leqq m_k} a_{k,\alpha}\phi_\alpha(z, \xi_k)$$

with singularity ξ_k is given. Then there exists a $\overline{\partial}$-closed $(0, n-1)$-form $F(z)$ on $\mathbb{C}^n \setminus \Xi$ satisfying the following:
For any ξ_k, there exist an open neighbourhood U_k of ξ_k and a $(0, n-2)$-form $H_k(z)$ in U_k such that

$$F(z) - G_k(z) = \overline{\partial} H_k(z).$$

Proof. Renumbering if it is necessary, we may assume $\|\xi_{k+1}\| \geqq \|\xi_k\|$. By Lemma 2.7, we can take a $\overline{\partial}$-closed form $P_k(z)$ which consists of a partial sum in Taylor expansion of $G_k(z)$ such that

$$|G_k(z) - P_k(z)| \leqq \left(\frac{1}{2}\right)^k \text{ for } \|z\| \leqq \frac{1}{32}\|\xi_k\|.$$

The series $\sum_{k=1}^\infty (G_k(z) - P_k(z))$ is uniformly convergent on any compact subset of $\mathbb{C}^n \setminus \Xi$ and $\overline{\partial}$-closed. We set

$$F(z) := \sum_{k=1}^\infty (G_k(z) - P_k(z)).$$

We show that the form $F(z)$ satisfies the required condition.

For any ξ_k, we take a Stein neighbourhood U_k of ξ_k such that $U_k \cap \Xi = \{\xi_k\}$. $\sum_{k' \neq k}(G_{k'}(z) - P_{k'}(z))$ is a $\overline{\partial}$-closed form on U_k. The sum is $\overline{\partial}$-exact for U_k is Stein. Furthermore, $P_k(z)$ is $\overline{\partial}$-exact for it is a partial sum. Hence,

$$F(z) - G_k(z) = -P_k(z) + \sum_{k' \neq k}(G_{k'}(z) - P_{k'}(z))$$

is $\overline{\partial}$-exact on U_k. □

As in Theorem 2.9, let $\Xi = \{\xi_k; k \in \mathbb{N}\}$ be a discrete set in $\mathbb{C}^n$ with $\|\xi_{k+1}\| \geqq \|\xi_k\|$. Moreover, without loss of generality, we may assume $\|\xi_k\| \geqq 2$ $(k \in \mathbb{N})$. By Lemma 2.6, we have

$$|\psi_k(z, \xi_\ell)| < \left(\frac{1}{2}\right)^k \text{ for } \|z\| \leqq \frac{\|\xi_\ell\|}{32}.$$

If we set

$$P_\ell^h(z) := \psi(0, \xi_\ell) + \sum_{k=1}^h \psi_k(z, \xi_\ell),$$

then coefficients of $P_\ell^h(z)$ are polynomials of order h in z_j and $\overline{z_j}$. By the above estimate, we see that

$$\sum_{k=1}^{\infty} \left(\psi(z,\xi_k) - P_k^k(z)\right)$$

converges uniformly on any compact subset of $\mathbb{C}^n \setminus \Xi$.

Definition 2.10 ([10])**.** *We define the exponent of convergence of Ξ by*

$$\mathcal{C} := \inf\{\gamma \in \mathbb{N}; \sum_{k=1}^{\infty} \frac{1}{\|\xi_k\|^\gamma} < +\infty\}.$$

In the case that $\sum_{k=1}^{\infty} \frac{1}{\|\xi_k\|^\gamma} = +\infty$ for any $\gamma \in \mathbb{N}$, we set $\mathcal{C} := \infty$.

Lemma 2.11 ([10])**.** *If a discrete set $\Xi = \{\xi_k; k \in \mathbb{N}\}$ in $\mathbb{C}^n$ have the exponent of convergence $\mathcal{C} \in \mathbb{N}$, then*

$$\sum_{k=1}^{\infty} \left(\psi(z,\xi_k) - P_k^{\mathcal{C}-2n}(z)\right)$$

converges uniformly on any compact subset of $\mathbb{C}^n \setminus \Xi$.

Proof. Since $\|\xi_k\| \to \infty$ $(k \to \infty)$, for any $R > 0$ there exists a natural number $N(R)$ such that

$$\|\xi_k\| > 32R \quad \text{for any } k \geqq N(R).$$

Since we have the estimate

$$\begin{aligned}
&\left| \sum_{k=N(R)}^{\infty} \left(\psi(z,\xi_k) - P_k^{\mathcal{C}-2n}(z)\right) \right| \\
&= \left| \sum_{k=N(R)}^{\infty} \sum_{\ell=\mathcal{C}-2n+1} \psi_\ell(z,\xi_k) \right| \\
&\leqq \sum_{k=N(R)}^{\infty} \sum_{\ell=\mathcal{C}-2n+1}^{\infty} \frac{2^{n-1}}{\|\xi_k\|^{2n-1}} \left(\frac{16\|z\|}{\|\xi_k\|}\right)^\ell \\
&\leqq \sum_{k=N(R)}^{\infty} \sum_{\ell=\mathcal{C}-2n+1}^{\infty} \left(\frac{16R}{\|\xi_k\|}\right)^\ell \frac{2^{n-1}}{\|\xi_k\|^{2n-1}} \\
&= \sum_{k=N(R)}^{\infty} \frac{2^{n-1}(16R)^{\mathcal{C}-2n+1}}{\|\xi_k\|^{\mathcal{C}}} \left(\sum_{\ell=\mathcal{C}-2n+1}^{\infty} \left(\frac{16R}{\|\xi_k\|}\right)^{\ell+2n-1+\mathcal{C}} \right) \\
&\leqq 2^{n-1}(16R)^{\mathcal{C}-2n+1} \sum_{k=N(R)}^{\infty} \frac{1}{\|\xi_k\|^{\mathcal{C}}} \sum_{\ell=\mathcal{C}-2n+1}^{\infty} \left(\frac{1}{2}\right)^{\ell+2n-1+\mathcal{C}} < +\infty
\end{aligned}$$

on $\|z\| \leqq R$, the proof is complete. $\square$

Theorem 2.12 (Teorema 2 in [10]). *Let $\Xi = \{\xi_k; k \in \mathbb{N}\}$ be a discrete set in $\mathbb{C}^n$ with exponent of convergence $\mathcal{C}$. Then a $\overline{\partial}$-close form ω on $\mathbb{C}^n \setminus \Xi$ with singularity of polar type of minimum order (= 1) at each point of Ξ is given by*

$$\omega = \sum_{k=1}^{\infty} \left(\psi(z, \xi_k) - P_k^i(z)\right),$$

where we set

$$\begin{cases} i = k & \text{if } \mathcal{C} = \infty \\ i = \mathcal{C} - 2n & \text{if } 2n \leqq \mathcal{C} < \infty \\ P_k^i = 0 & \text{if } \mathcal{C} < 2n. \end{cases}$$

And $\frac{(n-1)!}{(2\pi\sqrt{-1})^n}\omega \wedge dz$ has the residue 1 at each point of Ξ.

Proof. (I) In the case of $\mathcal{C} = \infty$. We have already proved that $\sum_{k=1}^{\infty}(\psi(z, \xi_k) - P_k^i(z))$ converges. As in the proof of Theorem 2.9, we see that

$$\omega - \psi(z, \xi_k) = -P_k^k(z) + \sum_{\ell \neq k}(\psi(z, \xi_k) - P_\ell^\ell(z))$$

is $\overline{\partial}$-exact on a neighbourhood of ξ_k for any ξ_k ($k \in \mathbb{N}$). Therefore ξ_k is a singularity of polar type of order 1 for ω.

(II) In the cace of $2n \leqq \mathcal{C} < \infty$. By Lemma 2.11 $\sum_{k=1}^{\infty}(\psi(z, \xi_k) - P_k^{\mathcal{C}-2n}(z))$ is convergent. The same argument as in (I) shows that ξ_k is a singularity of polar type of order 1 for ω.

(III) In the case of $\mathcal{C} < 2n$. It is enough to prove that $\sum_{k=1}^{\infty} \psi(z, \xi_k)$ converges uniformly on any compact set in $\mathbb{C} \setminus \Xi$.

For any $R > 0$, there exists a natural number $N(R)$ such that

$$\|\xi_k\| > 32R \ \text{ for any } k \geqq N(R).$$

Then, as in the proof of Lemma 2.7, we obtain

$$\left| \sum_{k=N(R)}^{\infty} \psi(z, \xi_k) \right| \leqq 2^n \sum_{k=1}^{\infty} \frac{1}{\|\xi_k\|^{2n-1}} < +\infty$$

for $\|\xi_k\| \geqq R$.

(IV) Since ω has the minimum order 1 at each ξ_k, we obtain

$$\begin{aligned} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \int_{S_\varepsilon(\xi_k)} \omega \wedge dz &= \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \int_{S_\varepsilon(\xi_k)} \psi(z, \xi_k) \wedge dz \\ &= \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \frac{(2\pi\sqrt{-1})^n}{(n-1)!} = 1 \end{aligned}$$

by Proposition 1.2. □

2.4 Generalization of the Legendre relation

In this section, we construct a $\overline{\partial}$-closed $(0, n-1)$-form corresponding to Weierstrass' ζ-function. And we give a formula which generalizes the Legendre relation.

If Γ is a lattice of maximal rank in $\mathbb{C}^n$, then its exponent of convergence is $2n+1$ (see Lemma 2.3). Therefore, we can define the following form

$$\begin{aligned}\zeta_\Gamma(z) &:= \psi(z,0) + \sum_{\gamma\in\Gamma\setminus\{0\}} (\psi(z,\gamma) - P^1_\gamma(z)) \\ &= \psi(z,0) + \sum_{\gamma\in\Gamma\setminus\{0\}} \left(\psi(z,\gamma) - \psi(0,\gamma) - \sum_{i=1}^n \left(\frac{\partial\psi}{\partial z_i}(0,\gamma) z_i + \frac{\partial\psi}{\partial \overline{z_i}}(0,\gamma)\overline{z_i}\right)\right),\end{aligned}$$

where

$$P^1_\gamma(z) = \psi(0,\gamma) + \psi_1(z,\gamma).$$

We have the representation of $P^1_\Gamma(z)$ defined by (2.7) as follows:

$$P^1_\Gamma(z) = -\frac{\partial\psi}{\partial z_i}(z,0) + \sum_{\gamma\in\Gamma\setminus\{0\}} \left(\frac{\partial\psi}{\partial z_i}(z,\gamma) - \frac{\partial\psi}{\partial z_i}(0,\gamma)\right).$$

We note that $\zeta_\Gamma(z)$ has the following properties

$$\begin{aligned}\zeta_\Gamma(z) &= -\zeta_\Gamma(-z), \\ \frac{\partial}{\partial z_i}\zeta_\Gamma(z) &= -\wp^i_\Gamma(z).\end{aligned}$$

Fix any $\gamma \in \Gamma$. Noting that $\psi(z+\gamma, \gamma') = \psi(z, \gamma'-\gamma)$ for all $\gamma' \in \Gamma$, we obtain

$$\begin{aligned}&\zeta_\Gamma(z+\gamma,0) - \zeta_\Gamma(z) \\ &= \psi(z,-\gamma) - \psi(z,0) + \sum_{\gamma'\in\Gamma\setminus\{0\}} (\psi(z,\gamma'-\gamma) - \psi(z,\gamma')) \\ &\qquad - \sum_{\gamma'\in\Gamma\setminus\{0\}} \sum_{i=1}^n \left(\frac{\partial\psi}{\partial z_i}(0,\gamma')\gamma_i - \frac{\partial\psi}{\partial \overline{z_i}}(0,\gamma')\overline{\gamma_i}\right) \\ &= -\sum_{\gamma'\in\Gamma\setminus\{0\}} \sum_{i=1}^n \left(\frac{\partial\psi}{\partial z_i}(0,\gamma')\gamma_i - \frac{\partial\psi}{\partial \overline{z_i}}(0,\gamma')\overline{\gamma_i}\right) \\ &=: \eta_\gamma.\end{aligned}$$

Let $\gamma_1, \ldots, \gamma_{2n}$ be generators of Γ. We write $\gamma_j = (\gamma_{1j}, \ldots, \gamma_{nj}) \in \mathbb{C}^n$ $(j = 1, \ldots, 2n)$. We set

$$G := (\gamma_1, \ldots, \gamma_{2n}) = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1,2n} \\ \vdots & & \vdots \\ \gamma_{n1} & \cdots & \gamma_{n,2n}\end{pmatrix}.$$

Let $C_G(0)$ be the fundamental parallelotope with center at the origin. Using faces

$$F_k^{\pm} := \left\{ z \in \mathbb{C}^n ; z = \pm\frac{1}{2}\gamma_k + \sum_{\substack{j=1 \\ j\neq k}}^{2n} \lambda_j\gamma_j,\ |\lambda_j| \leqq \frac{1}{2} \right\},\ k = 1, \ldots, 2n,$$

we can represent the boundaty $\partial C_G(0)$ of $C_G(0)$ as

$$\partial C_G(0) = \sum_{k=1}^{2n}(-1)^{k-1}F_k^+ + \sum_{k=1}^{2n}(-1)^k F_k^-.$$

We set

$$\zeta_\Gamma(z) = \sum_{i=1}^{n} \zeta_i(z)(d\overline{z})_i,$$

$$\eta_{\gamma_j} = \sum_{i=1}^{n} \eta_{ij}(d\overline{z})_i, \quad j = 1, \ldots, 2n.$$

By the generalized Martinelli formula, we have

$$\frac{(2\pi\sqrt{-1})^n}{(n-1)!} = \int_{\partial C_G(0)} \zeta_\Gamma(z) \wedge dz.$$

We represent the detarminant of the matrix which omits the i-th row and the k-th column from $\begin{pmatrix} \overline{G} \\ G \end{pmatrix}$ as $\left| \begin{matrix} \overline{G} \\ G \end{matrix} \right|_{i,k}$. Take real variables $(t_1, \ldots, t_{2n})$ such that

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = G \begin{pmatrix} t_1 \\ \vdots \\ t_{2n} \end{pmatrix}.$$

Then the above formula is

$$\begin{aligned}
\frac{(2\pi\sqrt{-1})^n}{(n-1)!} &= \sum_{k=1}^{2n}(-1)^{k-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{i=1}^{n} \zeta_i(z(t))\Big|_{F_k^+} \left| \begin{matrix} \overline{G} \\ G \end{matrix} \right|_{i,k} dt_1 \cdots \widehat{dt_k} \cdots dt_{2n} \\
&\quad + \sum_{k=1}^{2n}(-1)^{k} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{i=1}^{n} \zeta_i(z(t))\Big|_{F_k^-} \left| \begin{matrix} \overline{G} \\ G \end{matrix} \right|_{i,k} dt_1 \cdots \widehat{dt_k} \cdots dt_{2n} \\
&= \sum_{k=1}^{2n}(-1)^{k-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{i=1}^{n} \left(\zeta_i(z(t))\Big|_{F_k^+} - \zeta_i(z(t))\Big|_{F_k^-} \right) \\
&\qquad\qquad \times \left| \begin{matrix} \overline{G} \\ G \end{matrix} \right|_{i,k} dt_1 \cdots \widehat{dt_k} \cdots dt_{2n} \\
&= \sum_{k=1}^{2n}(-1)^{k-1} \sum_{i=1}^{n} \eta_{ik} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \begin{matrix} \overline{G} \\ G \end{matrix} \right|_{i,k} dt_1 \cdots \widehat{dt_k} \cdots dt_{2n}.
\end{aligned}$$

Then we get a generalization of the Legendre relation

$$\frac{(2\pi\sqrt{-1})^n}{(n-1)!} = \sum_{i=1}^{n}\sum_{k=1}^{2n}(-1)^{k-1}\eta_{ik}\left|\begin{array}{c}\overline{G}\\ G\end{array}\right|_{i,k}. \tag{2.8}$$

2.5 Generalization of Weierstrass' $\wp$-function

In this section, we assume that an n-dimentional complex torus $M = \mathbb{C}^n/\Gamma$ is an abelian variety. For the sake of simplicity, we write $\wp^i(z) = \wp^i_\Gamma(z)$ omitting Γ. Let $\pi : \mathbb{C}^n \to M$ be the canonical projection. Consider $\wp^i(z - \widetilde{p})$ for any $\widetilde{p} \in \pi^{-1}(p)$. We can consider $\wp^i(z)$ as a form on M, because it is Γ-invariant. We write $\wp^i(z - p)$ when $\wp^i(z - \widetilde{p})$ is considered on M. We treat other Γ-invariant forms and functions in the same manner. Let B be a sufficiently small open ball containing p. It holds by (2.1) that

$$\frac{(2\pi\sqrt{-1})^n}{(n-1)!}\frac{\partial f}{\partial z_i}(p) = \int_{\partial B} f(z)\wp^i(z-p)\wedge dz \tag{2.9}$$

for any holomorphic function f on a neighbourhood of $\overline{B}$.

Definition 2.13 ([11])**.** *We define a $\overline{\partial}$-closed $(n-1, n-1)$-form $\wp^{ij}(z)$ on $M \setminus \{0\}$ by*

$$\wp^{ij}(z) := \frac{(n-1)!}{(2\pi\sqrt{-1})^{n-1}}(-1)^{j-1}\wp^i(z)\wedge(dz_j). \tag{2.10}$$

Remark . *Dolbeault classes of $\wp^{ij}$ and its derivatives generate $H^{n-1}(M \setminus \{p\}, \Omega^{n-1})$ as a Frechét space.*

There exists a theta function θ ($\not\equiv 0$) for M is an abelian variety. We can take a positive C^∞ function h such that $\omega = h|\theta|^2$ is Γ-invariant. Let Θ be the divisor on M defined by $\theta = 0$.

Proposition 2.14 (Proposizione 1 in [11])**.** *Let $p \notin \Theta$, and take $\widetilde{p} \in \pi^{-1}(p)$. Let Q be a fundamental parallelotope such that $\widetilde{p}$ is not contained in the interior Q° of Q. Then we have*

$$\int_\Theta \wp^{ij}(z-p) = -\frac{\partial^2}{\partial z_i \partial z_j}\log\theta(\widetilde{p}) + \frac{\sqrt{-1}}{2\pi}\int_{\partial Q}\partial \log h \wedge \wp^{ij}(z-\widetilde{p}). \tag{2.11}$$

Proof. Take an open ball B with center p and sufficiently small rudius such that $\pi^{-1}(B) \cap \partial Q = \emptyset$. Let T be a tubular neighbourhood of Θ with piecewise differentiable boundary such that $\overline{T} \cap \overline{B} = \emptyset$. We can take a C^∞ function ρ with $\rho \geqq 0$ on M such that

$$\rho(q) = \begin{cases} 1 & \text{if } q \in \overline{T} \\ 0 & \text{if } q \in \overline{B}. \end{cases}$$

We have

$$\begin{aligned}(2.12)\qquad \int_{\Theta} \wp^{ij} &= \int_{\Theta} \rho \wp^{ij} \\ &= \left\langle -\frac{\sqrt{-1}}{\pi} \overline{\partial} \partial \log |\theta|, \rho \wp^{ij} \right\rangle \\ &= -\frac{\sqrt{-1}}{2\pi} \left(\int_M \overline{\partial} \partial \log \omega \wedge \rho \wp^{ij} - \int_M \overline{\partial} \partial \log h \wedge \rho \wp^{ij} \right) \\ &= -\frac{\sqrt{-1}}{2\pi} \left(\int_M \partial \log \omega \wedge \overline{\partial} \rho \wedge \wp^{ij} - \int_M \overline{\partial} \partial \log h \wedge \rho \wp^{ij} \right)\end{aligned}$$

by the Poincaré-Lelong equation. Since $\overline{\partial}\partial \log |\theta|^2 = 0$ on $M \setminus \Theta$, we obtain

$$\begin{aligned}(2.13)\qquad & \int_M \partial \log \omega \wedge \overline{\partial} \rho \wedge \wp^{ij} \\ &= \int_{M \setminus (B \cup T)} \partial \log \omega \wedge \overline{\partial} \rho \wedge \wp^{ij} \\ &= \int_{M \setminus (B \cup T)} \overline{\partial} \partial \log \omega \wedge \rho \wp^{ij} - \overline{\partial} (\partial \log \omega \wedge \rho \wp^{ij}) \\ &= \int_{M \setminus (B \cup T)} \overline{\partial} \partial \log h \wedge \rho \wp^{ij} + \int_{\partial B} \log \omega \wedge \rho \wp^{ij} + \int_{\partial T} \log \omega \wedge \wp^{ij} \\ &= \int_{M \setminus (B \cup T)} \overline{\partial} \partial \log h \wedge \rho \wp^{ij} + \int_{\partial T} \log \omega \wedge \wp^{ij}.\end{aligned}$$

On the other hand, we have

$$(2.14)\qquad \int_M \overline{\partial} \partial \log h \wedge \rho \wp^{ij} = \int_T \overline{\partial} \partial \log h \wedge \wp^{ij} + \int_{M \setminus (B \cup T)} \overline{\partial} \partial \log h \wedge \rho \wp^{ij}.$$

From (2.12), (2.13) and (2.14) it follows that

$$(2.15)\qquad \int_{\Theta} \wp^{ij} = \frac{\sqrt{-1}}{2\pi} \left(-\int_{\partial T} \partial \log \omega \wedge \wp^{ij} + \int_T \overline{\partial} \partial \log h \wedge \wp^{ij} \right).$$

By Stokes' theorem, we have

$$\begin{aligned}(2.16)\qquad \int_M \overline{\partial} \partial \log \omega \wedge \wp^{ij} &= \int_M d(\partial \log \omega \wedge \wp^{ij}) \\ &= -\int_{\partial T} \partial \log \omega \wedge \wp^{ij} - \int_{\partial B} \partial \log \omega \wedge \wp^{ij}.\end{aligned}$$

Moreover, since $\overline{\partial}\partial \log h = \overline{\partial}\partial \log \omega$ on $M \setminus T$, we have

$$\begin{aligned}-\int_{\partial T} \partial \log \omega \wedge \wp^{ij} &= \int_{M \setminus (B \cup T)} \overline{\partial} \partial \log \omega \wedge \wp^{ij} + \int_{\partial B} \partial \log \omega \wedge \wp^{ij} \\ &= \int_{M \setminus (B \cup T)} \overline{\partial} \partial \log h \wedge \wp^{ij} + \int_{\partial B} \partial \log \omega \wedge \wp^{ij}.\end{aligned}$$

Therefore, we obtain

$$\int_\Theta \wp^{ij} = \frac{\sqrt{-1}}{2\pi}\left(\int_{M\setminus B} \overline{\partial}\partial \log h \wedge \wp^{ij} + \int_{\partial B} \partial \log \omega \wedge \wp^{ij}\right) \tag{2.17}$$

by (2.15). Setting $B' := \pi^{-1}(B) \cap Q$, we have

$$\begin{aligned}\int_{M\setminus B} \overline{\partial}\partial \log h \wedge \wp^{ij} &= \int_{Q\setminus B'} d(\partial \log h \wedge \wp^{ij}) \\ &= \int_{\partial Q} \partial \log h \wedge \wp^{ij} - \int_{\partial B'} \partial \log h \wedge \wp^{ij}.\end{aligned} \tag{2.18}$$

On the other hand, since $\partial \log |\theta|^2 = \partial \log \theta$, we obtain

$$\begin{aligned}\int_{\partial B} \partial \log \omega \wedge \wp^{ij} &= \int_{\partial B'} (\partial \log h + \partial \log |\theta|^2) \wedge \wp^{ij} \\ &= \int_{\partial B'} (\partial \log h + \partial \log \theta) \wedge \wp^{ij}.\end{aligned} \tag{2.19}$$

By the definition of $\wp^{ij}$ and (2.9), we have

$$\begin{aligned}\int_{\partial B'} \partial \log \theta \wedge \wp^{ij} &= \frac{(n-1)!}{(2\pi\sqrt{-1})^{n-1}} \int_{\partial B'} \frac{\partial}{\partial z_j} \log \theta \; \wp^{ij}(z - \widetilde{p}) \wedge dz \\ &= 2\pi\sqrt{-1} \frac{\partial^2}{\partial z_i \partial z_j} \log \theta(\widetilde{p}).\end{aligned}$$

We finally obtain

$$\int_\Theta \wp^{ij} = -\frac{\partial^2}{\partial z_i \partial z_j} \log \theta(\widetilde{p}) + \frac{\sqrt{-1}}{2\pi} \int_{\partial Q} \partial \log h \wedge \wp^{ij}.$$

□

Lemma 2.15 (Proposizione 2 in [11])**.** *The term* $\frac{\sqrt{-1}}{2\pi}\int_{\partial Q} \partial \log h \wedge \wp^{ij}(z - \widetilde{p})$ *in (2.11) is a constant independent of* p.

Proof. By the definition of $\wp^{ij}$, we have

$$\begin{aligned}&\frac{\sqrt{-1}}{2\pi} \int_{\partial Q} \partial \log h \wedge \wp^{ij}(z - \widetilde{p}) \\ &= \frac{\sqrt{-1}}{2\pi} \frac{(n-1)!}{(2\pi\sqrt{-1})^{n-1}} (-1)^{j-1} \int_{\partial Q} \partial \log h \wedge \wp^{ij}(z - \widetilde{p}) \wedge (dz)_j \\ &= \frac{(n-1)!}{(2\pi\sqrt{-1})^{n-1}} (-1)^n \int_{\partial Q} \frac{\partial}{\partial z_j} \log h \; \wp^i(z - \widetilde{p}) \wedge dz.\end{aligned}$$

Since h is the exponent of a polynomial of degree 2 in z_i and $\overline{z_i}$, $\frac{\partial}{\partial z_j}\log h$ is represented as the following linear polynomial

$$\sum_{r=1}^{n}(a_{jr}z_r + b_{jr}\overline{z_r}) + c_j.$$

It holds that

$$\begin{aligned}(2.20)\qquad &\int_{F_\ell^+\cup F_\ell^-}\frac{\partial}{\partial z_j}\log h\ \wp^i(z-\widetilde{p})\wedge dz\\ &=\sum_{r=1}^{n}(a_{jr}\gamma_{r\ell}+b_{jr}\overline{\gamma_{r\ell}})\int_{F_\ell^-}\wp^i(z-\widetilde{p})\wedge dz,\end{aligned}$$

because $\wp^i\wedge dz$ is Γ-invariant. Therefore, it is sufficient to show that the integral

$$\int_{F_\ell^-}\wp^i(z-\widetilde{p})\wedge dz$$

is independent of p.

Let $H_\ell := \pi(F_\ell^-)$ be the real $(2n-1)$-dimensional hypersurface of M determined by F_ℓ^-. We have

$$\int_{F_\ell^-}\wp^i(z-\widetilde{p})\wedge dz = \int_{H_\ell}\wp^i(z-p)\wedge dz = \int_{H_\ell-p}\wp^i(z)\wedge dz.$$

We take a different point $q\in M$ which does not lie on H_ℓ. We denote by D the domain in M surrounded by $H_\ell - p$ and $H_\ell - q$. Then we have

$$\int_{H_\ell-p}\wp^i(z)\wedge dz - \int_{H_\ell-q}\wp^i(z)\wedge dz = \int_D d(\wp^i(z)\wedge dz) = 0.$$

□

Combining Propositon 2.14 with Lemma 2.15, we obtain the following theorem which shows that $\wp^{ij}$ is a generalization of Weierstrass' $\wp$-function.

Theorem 2.16 (Teorema in [11])**.** *Let Θ be the above divisor. Then we have*

$$(2.21)\qquad \int_\Theta \wp^{ij}(z-p) = -\frac{\partial^2}{\partial z_i\partial z_j}\log\theta(\widetilde{p}) + c_{ij},$$

where c_{ij} is a constant independent of p.

Chapter 3

Generalization to Quasi-Abelian Varieties

3.1 Toroidal groups and quasi-abelian varieties

Connected complex Lie groups on which holomorphic functions are only constants are called toroidal groups. It is known that a toroidal group is commutative. Then it is considered as a quotient $\mathbb{C}^n/\Gamma$ of $\mathbb{C}^n$ by a discrete subgroup Γ with rank $\Gamma = n+m$ $(1 \leqq m \leqq n)$. When rank $\Gamma = 2n$, it is a complex torus. In this section, we treat the case of non-compact toroidal groups.

Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group with rank $\Gamma = n+m$ $(1 \leqq m \leqq n-1)$. Take generators $\gamma_1 = (\gamma_{11}, \ldots, \gamma_{1n}), \ldots, \gamma_{n+m} = (\gamma_{n+m,1}, \ldots, \gamma_{n+m,n})$ of Γ. The matrix

$$P = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{n+m,1} \\ \vdots & & \vdots \\ \gamma_{1n} & \cdots & \gamma_{n+m,n} \end{pmatrix}$$

given by these generators is called a period matrix of X. By a suitable change of variables and generators, we can write P as

$$P = \begin{pmatrix} 0 & I_m & T \\ I_{n-m} & R_1 & R_2 \end{pmatrix}, \tag{3.1}$$

where I_k is the unit matrix of degree k, the matrix $(I_m\ T)$ is a period matrix of an m-dimensional complex torus, and $(R_1\ R_2)$ is a real matrix. We say that coordinates in the expression (3.1) are toroidal coordinates. The condition $H^0(X, \mathcal{O}) = \mathbb{C}$ is written in terms of $(R_1\ R_2)$ (see [2] for details). We write the toroidal coordinates as

$$z = (z', z'') = (z_1, \ldots, z_m, z_{m+1}, \ldots, z_n).$$

In the following we use these coordinates. Let $\mathbb{R}^{n+m}_{\Gamma}$ be the real linear subspace of $\mathbb{C}^n$ spanned by Γ. Then

$$\mathbb{C}^m_\Gamma := \mathbb{R}^{n+m}_\Gamma \cap \sqrt{-1}\mathbb{R}^{n+m}_\Gamma$$

is the maximal complex linear subspace contained in $\mathbb{R}^{n+m}_\Gamma$, which is of complex dimension m. A toroidal group X is called a quasi-abelian variety if there exists a hermition form $\mathcal{H}$ on $\mathbb{C}^n$ such that
(a) $\mathcal{H}$ is positive definite on $\mathbb{C}^m_\Gamma$,
(b) the imaginary part $\mathcal{A} := \mathfrak{Im}\mathcal{H}$ is $\mathbb{Z}$-valued on $\Gamma \times \Gamma$.
A hermitian form $\mathcal{H}$ satisfying the above conditions (a) and (b) is said to be an ample Riemann form for X. We set $\mathcal{A}_\Gamma := \mathcal{A}|_{\mathbb{R}^{n+m}_\Gamma \times \mathbb{R}^{n+m}_\Gamma}$ for an ample Riemann form $\mathcal{H}$. Since $\mathcal{A}_\Gamma$ is an alternating form, we have

$$\operatorname{rank} \mathcal{A}_\Gamma = 2(m+k),\ 0 \leqq k \leqq \frac{1}{2}(n-m).$$

In this case we say that $\mathcal{H}$ is of kind k. If a quasi-abelian variety X has an ample Riemann form of kind k, then it also has an ample Riemann form of kind k' for any k' with $k \leqq k' \leqq \frac{1}{2}(n-m)$. Then the kind of a quasi-abelian variety was defined in [3] as follows.

Definition 3.1. *The kind of a quasi-abelian variety X is the smallest integer k with $0 \leqq k \leqq \frac{1}{2}(n-m)$ such that there exists an ample Riemann form of kind k for X.*

Let $X = \mathbb{C}^n/\Gamma$ be a quasi-abelian variety of kind 0. Then the matrix $(I_m\ T)$ in (3.1) is a period matrix of an m-dimensional abelian variety A. The projection $z \mapsto z'$ induces a $(\mathbb{C}^*)^{n-m}$-bundle $\sigma : X \to A$ over A. Replacing fibres $(\mathbb{C}^*)^{n-m}$ with $(\mathbb{P}^1)^{n-m}$, we obtain the associated $(\mathbb{P}^1)^{n-m}$-bundle $\overline{\sigma} : \overline{X} \to A$ over A. We say that $\overline{X}$ is the standard compactification of a quasi-abelian variety X of kind 0.

3.2 Cohomology groups and the Dolbeault isomorphism

Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group with rank $\Gamma = n+m$ $(1 \leqq m \leqq n-1)$. It has the canonical projection $\pi : \mathbb{C}^n \to X$. Let 0 be the unit element of X. We can take a neighbourhood V of 0 in X such that

$$\pi^{-1}(V) = \bigsqcup_{\gamma \in \Gamma} U_\gamma \text{ (disjoint union)},$$
$$\pi|_{U_\gamma} : U_\gamma \to V$$

is a biholomorphic mapping, where U_γ is a polydisc with center γ. In the same way as in Section 2.1 in Chapter 2, we obtain the following cohomology exact sequence

$$
\begin{aligned}
&0 \to H^0(X,\mathcal{O}) \to H^0(V,\mathcal{O}) \oplus H^0(X \setminus \{0\},\mathcal{O}) \to \cdots \to H^k(X,\mathcal{O}) \to \\
&H^k(V,\mathcal{O}) \oplus H^k(X \setminus \{0\},\mathcal{O}) \to H^k(V \setminus \{0\},\mathcal{O}) \to \cdots \to H^{n-2}(V \setminus \{0\},\mathcal{O}) \\
&\to H^{n-1}(X,\mathcal{O}) \to H^{n-1}(V,\mathcal{O}) \oplus H^{n-1}(X \setminus \{0\},\mathcal{O}) \to H^{n-1}(V \setminus \{0\},\mathcal{O}) \\
&\to H^n(X,\mathcal{O}) \to H^n(V,\mathcal{O}) \oplus H^n(X \setminus \{0\},\mathcal{O}) \to H^n(V \setminus \{0\},\mathcal{O}) \to 0.
\end{aligned}
$$

When rank $\Gamma = n + m$, the toroidal group $X = \mathbb{C}^n/\Gamma$ is strongly $(m+1)$-complete. Then we have

$$H^i(X,\mathcal{O}) = 0 \quad for \quad i \geqq m+1.$$

Since $n \geqq 2$, we have

$$H^0(X \setminus \{0\},\mathcal{O}) = H^0(X,\mathcal{O}) = \mathbb{C}.$$

Furthermore we have

$$H^i(V \setminus \{0\},\mathcal{O}) = 0 \quad for \quad i \neq 0, n-1$$

(Lemma 1.1 in Chapter 1). Substituting these results to the above exact sequence, we obtain the following proposition.

Proposition 3.2 (Proposition 1 in [1])**.** *For a toroidal group $X = \mathbb{C}^n/\Gamma$ we have*

$$H^i(X \setminus \{0\},\mathcal{O}) \cong H^i(X,\mathcal{O}) \quad for \quad i \neq n-1,$$
$$H^{n-1}(X \setminus \{0\},\mathcal{O}) \cong H^{n-1}(X,\mathcal{O}) \oplus H^{n-1}(V \setminus \{0\},\mathcal{O}).$$

Since cohomology groups $H^i(X,\mathcal{O})$ of a toroidal group X is comletely determined by [7] and [8], the above proposition shows that if $H^{n-1}(V \setminus \{0\},\mathcal{O})$ is decided, then we can understand cohomology groups of a punctured toroidal group $X \setminus \{0\}$. We note that

$$H^{n-1}(V \setminus \{0\},\mathcal{O}) \cong H^{n-1}(U_\gamma \setminus \{\gamma\},\mathcal{O})$$

for any $\gamma \in \Gamma$. Especially, when $1 \leqq m < n-1$, we have $H^{n-1}(X,\mathcal{O}) = 0$ and

$$H^{n-1}(X \setminus \{0\},\mathcal{O}) \cong H^{n-1}(U_\gamma \setminus \{\gamma\},\mathcal{O}).$$

When $m = n-1$, $H^{n-1}(X \setminus \{0\},\mathcal{O})$ is generated by $H^{n-1}(U_\gamma \setminus \{\gamma\},\mathcal{O})$ and $H^{n-1}(X,\mathcal{O})$. The $(n-1)$-th cohomology group $H^{n-1}(X,\mathcal{O})$ can be of n-dimension or non-Hausdorff.

3.3 Definition of $\wp^{ij}$

The $(n-1)$-th cohomology group $H^{n-1}(U_\gamma \setminus \{\gamma\},\mathcal{O})$ is generated by $\{\phi_\alpha(z,\gamma); \alpha \in \mathbb{N}_0^n\}$ as shown in Chapter 2. Now we treat the case that $1 \leqq m \leqq n-1$. Then we have $-1 + m + 1 \leqq 0$. Therefore,

$$\varepsilon_\Gamma^\alpha(z) := \sum_{\gamma \in \Gamma} \phi_\alpha(z,\gamma)$$

converges for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqq 1$ by Proposition 2.4. Then, we can define $\wp_\Gamma^i(z)$ without a modification as in (2.7). Hence, we set

$$\wp_\Gamma^i(z) := \varepsilon_\Gamma^{\delta^i}(z).$$

By the definition of $\varepsilon_\Gamma^\alpha$ the following equality is obvious

$$\frac{\partial}{\partial z_k}\varepsilon_\Gamma^\alpha = -(\alpha_k + 1)\varepsilon_\Gamma^{\alpha+\delta^k},$$

especially

$$\frac{\partial}{\partial z_k}\wp_\Gamma^i(z) = -(\delta_k^i + 1)\varepsilon_\Gamma^{\delta^i+\delta^k}.$$

For the sake of simplicity we write $\wp^i(z) = \wp_\Gamma^i(z)$ omitting Γ. It is obvious that $\wp^i(z)$ has the property (2.9) in Section 2.5 in Chapter 2. We give the following definition of $\wp^{ij}$ as the case of a complex torus.

Definition 3.3. *We define a $\overline{\partial}$-closed $(n-1, n-1)$-form $\wp^{ij}(z)$ on a punctured toroidal group $X \setminus \{0\}$ by*

$$\wp^{ij}(z) := \frac{(n-1)!}{(2\pi\sqrt{-1})^{n-1}}(-1)^{j-1}\wp^i(z) \wedge (dz)_j. \tag{3.2}$$

While $\wp^i(z)$ is a $\overline{\partial}$-closed $(0, n-1)$-form on $\mathbb{C}^n$ with singularities Γ, it is Γ-invariant. Then we can consider it as a $\overline{\partial}$-closed $(0, n-1)$-form on X with a singularity 0. Let $p \in X$ and $\tilde{p} \in \pi^{-1}(p)$. If we write $\wp^i(z-p)$, then it is considered as a form on X. And if we write $\wp^i(z-\tilde{p})$, then it is a form on $\mathbb{C}^n$. We treat $\wp^{ij}$ and other Γ-invariant forms and functions in the same manner.

3.4 Positive divisors

Let $X = \mathbb{C}^n/\Gamma$ be a quasi-abelian variety of kind 0. We take generators $\gamma_1, \ldots, \gamma_{n+m}$ of Γ such that the period matrix $P = (\gamma_1, \ldots, \gamma_{n+m})$ is of the form in (3.1). For any $j = 1, \ldots, n-m$, we set

$$v_j := (\delta_1^{m+j}, \ldots, \delta_{m+j-1}^{m+j}, \sqrt{-1}\delta_{m+j}^{m+j}, \delta_{m+j+1}^{m+j}, \ldots, \delta_n^{m+j}).$$

Then $\gamma_1, \ldots, \gamma_{n+m}, v_1, \ldots, v_{n-m}$ are a basis of $\mathbb{C}^n$ over $\mathbb{R}$. Any $z \in \mathbb{C}^n$ is represented uniquely by

$$z = \sum_{i=1}^{n+m} s_i\gamma_i + \sum_{j=1}^{n-m} t_j v_j, \quad s_i, t_j \in \mathbb{R}.$$

For a fixed $z^0 \in \mathbb{C}^n$ we define the fundamental parallelotope Q of X with center z^0 by

$$Q := \left\{ z^0 + z; z = \sum_{i=1}^{n+m} s_i \gamma_i + \sum_{j=1}^{n-m} t_j v_j, \ -\frac{1}{2} < s_i < \frac{1}{2} \ (i = 1, \dots, n+m), \right.$$
$$\left. t_j \in \mathbb{R} \ (j = 1, \dots, n-m) \right\}.$$

We denote an $(n-m)$-tuple of positive numbers $R_1, \dots, R_{n-m}$ by $R = (R_1, \dots, R_{n-m})$. Let R and R' be two $(n-m)$-tuples of positive numbers. Then

$$D_{R,R'} := \mathbb{C}^m \times \{(z_{m+1}, \dots, z_n) \in \mathbb{C}^{n-m}; -R'_j < \mathfrak{Im} z_{m+j} < R_j \ (j = 1, \dots, n-m)\}$$

is a subdomain of $\mathbb{C}^n$. Since the period matrix P of X is of the form in (3.1), Γ acts on $D_{R,R'}$ for any R and R'. Then we can define a subdomain $X_{R,R'} := D_{R,R'}/\Gamma$ of X. Let $\overline{X}_{R,R'}$ be the closure of $X_{R,R'}$ in X. We denote

$$Q_{R,R'} := Q \cap D_{R,R'} \quad \text{and} \quad (\partial Q)_{R,R'} := \partial Q \cap D_{R,R'},$$

where ∂Q is the boundary of Q in $\mathbb{C}^n$.

Let Θ be a positive devisor on X. It determines a holomorphic line bundle $L = [\Theta]$ over X. It is well-known that L is given by a factor of automorphy $\rho(\gamma, z)$. A system of local defining functions $\{\theta_i\}$ of Θ corresponds to an automorphic form θ for $\rho(\gamma, z)$, that is, θ is an entire function on $\mathbb{C}^n$ satisfying

$$\theta(z + \gamma) = \rho(\gamma, z)\theta(z)$$

for all $z \in \mathbb{C}^n$ and $\gamma \in \Gamma$. We note that if X is an abelian variety, then θ is a theta function. A hermitian fibre metric of L gives a positive valued C^∞ function h on $\mathbb{C}^n$ such that

$$\omega = h|\theta|^2 \tag{3.3}$$

is a C^∞ function on $\mathbb{C}^n$ with period Γ. Then ω is considered as a function on X. It follows from this property that

$$\log h(z + \gamma) + \log |\rho(\gamma, z)|^2 = \log h(z) \tag{3.4}$$

for all $z \in \mathbb{C}^n$ and $\gamma \in \Gamma$. For a positive divisor Θ we set

$$\Theta_{R,R'} := \Theta \cap X_{R,R'}.$$

3.5 Formula on a subdomain

Let $X = \mathbb{C}^n/\Gamma$ be a quasi-abelian variety. Suppose that Θ is a generalized theta divisor, that is, it is a positive divisor of which factor of automorphy $\rho(\gamma, z)$ is the exponent of a linear polynomial. We take a point $p \in X$ with $p \notin \Theta$. Let $\tilde{p} \in \pi^{-1}(p)$. We can take a fundamental parallelotope Q of X such as $\tilde{p}$ is an interior point of Q. Let B be a small open ball centered at p such that $\partial Q \cap \pi^{-1}(\overline{B}) = \phi$. We denote by T an open neighbourhood of Θ with piecewise differentiable boundary such that $\overline{T} \cap \overline{B} = \phi$. For any two $(n-m)$-tuples R and R' of positive numbers, we set

$$T_{R,R'} := T \cap X_{R,R'},\ (\partial T)_{R,R'} := \partial T \cap X_{R,R'}$$

and

$$E_{R,R'} := \partial X_{R,R'} \setminus (\partial T_{R,R'} \setminus (\partial T)_{R,R'}),$$

where $\partial X_{R,R'}$ and $\partial T_{R,R'}$ are boundaries of $X_{R,R'}$ and $T_{R,R'}$ in X respectively. We take a C^∞ function ρ on X with $0 \leqq \rho \leqq 1$ such that $\rho = 1$ on $\overline{T}$ and $\rho = 0$ on $\overline{B}$.

Proposition 3.4 (Proposition 3 in [1])**.** *We have*

$$\begin{aligned}\int_{\Theta_{R,R'}} \wp^{ij}(z-p) &= -\frac{\partial^2}{\partial z_i \partial z_j} \log \theta(\tilde{p}) \\ &+ \frac{\sqrt{-1}}{2\pi} \int_{\partial Q_{R,R'}} \partial \log h \wedge \wp^{ij}(z-p) \\ &+ \frac{\sqrt{-1}}{2\pi} \int_{E_{R,R'}} \partial \log \omega \wedge (\rho - 1)\wp^{ij}(z-p),\end{aligned} \tag{3.5}$$

where θ, h and ω are functions as in the previous section.

Proof. We first note that the current on $X_{R,R'}$ determined by $\Theta_{R,R'}$ is extended to a linear functional on the space of C^∞ $(n-1, n-1)$-forms on $\overline{X}_{R,R'}$ since $\overline{X}_{R,R'}$ is compact. Furthermore, the Poincaré-Lelong equation holds for such an extended linear functional. Then we have by (3.3)

$$\begin{aligned}&\int_{\Theta_{R,R'}} \wp^{ij}(z-p) \\ &= \int_{\Theta_{R,R'}} \rho \wp^{ij}(z-p) \\ &= \left\langle -\frac{\sqrt{-1}}{\pi} \overline{\partial}\partial \log |\theta|, \rho\wp^{ij} \right\rangle \\ &= -\frac{\sqrt{-1}}{2\pi} \left(\langle \partial \log \omega, \overline{\partial}\rho \wedge \wp^{ij} \rangle - \int_{X_{R,R'}} \overline{\partial}\partial \log h \wedge \rho\wp^{ij} \right).\end{aligned} \tag{3.6}$$

Since $\overline{\partial}\partial \log |\theta|^2 = 0$ on $X_{R,R'} \setminus T_{R,R'}$, $\rho = 1$ on $\overline{T}$ and $\rho = 0$ on $\overline{B}$, we obtain

$$\begin{aligned}
(3.7)\quad & \langle \partial \log \omega, \overline{\partial}\rho \wedge \wp^{ij} \rangle \\
&= \int_{X_{R,R'}\setminus(B\cup T_{R,R'})} \partial \log \omega \wedge \overline{\partial}\rho \wedge \wp^{ij} \\
&= \int_{X_{R,R'}\setminus(B\cup T_{R,R'})} (\overline{\partial}\partial \log \omega \wedge \rho\wp^{ij} - \overline{\partial}(\partial \log \omega \wedge \rho\wp^{ij})) \\
&= \int_{X_{R,R'}\setminus(B\cup T_{R,R'})} \overline{\partial}\partial \log h \wedge \rho\wp^{ij} - \int_{E_{R,R'}} \partial \log \omega \wedge \rho\wp^{ij} \\
&\qquad + \int_{(\partial T)_{R,R'}} \partial \log \omega \wedge \wp^{ij}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
(3.8)\quad \int_{X_{R,R'}} \overline{\partial}\partial \log h \wedge \rho\wp^{ij} &= \int_{T_{R,R'}} \overline{\partial}\partial \log h \wedge \wp^{ij} \\
&\qquad + \int_{X_{R,R'}\setminus(B\cup T_{R,R'})} \overline{\partial}\partial \log h \wedge \rho\wp^{ij}.
\end{aligned}$$

Then it follows from (3.6), (3.7) and (3.8) that

$$\begin{aligned}
(3.9)\quad & \int_{\Theta_{R,R'}} \wp^{ij}(z-p) \\
&= \frac{\sqrt{-1}}{2\pi}\left(- \int_{(\partial T)_{R,R'}} \partial \log \omega \wedge \wp^{ij} + \int_{T_{R,R'}} \overline{\partial}\partial \log h \wedge \wp^{ij} \right. \\
&\qquad \left. + \int_{E_{R,R'}} \partial \log \omega \wedge \rho\wp^{ij} \right).
\end{aligned}$$

By Stokes' theorem we have

$$\begin{aligned}
& \int_{X_{R,R'}\setminus(B\cup T_{R,R'})} \overline{\partial}\partial \log \omega \wedge \wp^{ij} \\
&\qquad = \int_{E_{R,R'}} \partial \log \omega \wedge \wp^{ij} - \int_{(\partial T)_{R,R'}} \partial \log \omega \wedge \wp^{ij} - \int_{\partial B} \partial \log \omega \wedge \wp^{ij}.
\end{aligned}$$

Moreover, since $\overline{\partial}\partial \log \omega = \overline{\partial}\partial \log h$ on $X_{R,R'} \setminus T_{R,R'}$, we obtain

$$\begin{aligned}
(3.10)\quad & - \int_{(\partial T)_{R,R'}} \partial \log \omega \wedge \wp^{ij} \\
&= \int_{X_{R,R'}\setminus(B\cup T_{R,R'})} \overline{\partial}\partial \log h \wedge \wp^{ij} + \int_{\partial B} \partial \log \omega \wedge \wp^{ij} - \int_{E_{R,R'}} \partial \log \omega \wedge \wp^{ij}.
\end{aligned}$$

Then it follows from (3.9) that

$$\begin{aligned}(3.11)\qquad &\int_{\Theta_{R,R'}} \wp^{ij}(z-p) \\ &\quad = \frac{\sqrt{-1}}{2\pi}\left(\int_{X_{R,R'}\setminus B} \overline{\partial}\partial \log h \wedge \wp^{ij} + \int_{\partial B} \partial \log \omega \wedge \wp^{ij} \right.\\ &\qquad\qquad \left. + \int_{E_{R,R'}} \partial \log \omega \wedge (\rho - 1)\wp^{ij}\right).\end{aligned}$$

We set $B' := \pi^{-1}(B) \cap Q$. By (3.4) it is obvious that $\overline{\partial}\partial \log h$ is Γ-invariant for Θ is a generalized theta divisor. Then we have

$$\begin{aligned}(3.12)\qquad \int_{X_{R,R'}\setminus B} \overline{\partial}\partial \log h \wedge \wp^{ij} &= \int_{Q_{R,R'}\setminus B'} \overline{\partial}\partial \log h \wedge \wp^{ij} \\ &= \int_{\partial Q_{R,R'}} \partial \log h \wedge \wp^{ij} - \int_{\partial B'} \partial \log h \wedge \wp^{ij}.\end{aligned}$$

On the other hand, noting $\partial \log |\theta|^2 = \partial \log \theta$, we obtain

$$(3.13)\qquad \int_{\partial B} \partial \log \omega \wedge \wp^{ij} = \int_{\partial B'} \partial \log \omega \wedge \wp^{ij} = \int_{\partial B'} (\partial \log h + \partial \log \theta) \wedge \wp^{ij}.$$

By the definition (3.2) of $\wp^{ij}$ and its property (cf. (2.10)), we have

$$\begin{aligned}\int_{\partial B'} \partial \log \theta \wedge \wp^{ij}(z-\tilde{p}) &= \frac{(n-1)!}{(2\pi\sqrt{-1})^{n-1}} \int_{\partial B'} \frac{\partial}{\partial z_j} \log \theta \wp^{ij}(z-\tilde{p}) \wedge dz \\ &= 2\pi\sqrt{-1}\frac{\partial^2}{\partial z_i \partial z_j} \log \theta(\tilde{p}).\end{aligned}$$

Then we obtain by (3.13)

$$(3.14)\qquad \int_{\partial B} \partial \log \omega \wedge \wp^{ij} = \int_{\partial B'} \partial \log h \wedge \wp^{ij} + 2\pi\sqrt{-1}\frac{\partial^2}{\partial z_i \partial z_j} \log \theta(\tilde{p}).$$

Thus, substituting (3.12) and (3.14) in (3.11), we finally obtain the desired equality (3.5). □

3.6 Main result

Let $R = (R_1, \ldots, R_{n-m})$ be an $(n-m)$-tuple of positive numbers. When $R_1, \ldots, R_{n-m} \to +\infty$, we simply write $R \to +\infty$. In the previous section we have given the formula (3.5) on a subdomain $X_{R,R'}$. A passage to the limit as $R, R' \to +\infty$ implies the main formula.

We assume throughout this section that X is a quasi-abelian variety of kind 0 with the standard compactification $\overline{X}$ and Θ is a generalized theta divisor.

Proposition 3.5. *If Θ has the holomorphic extension $\overline{\Theta}$ on $\overline{X}$, then $\wp^{ij}(z-p)$ is integrable on Θ and we have*

$$\int_{\Theta} \wp^{ij}(z-p) = \lim_{R,R'\to+\infty} \int_{\Theta_{R,R'}} \wp^{ij}(z-p) \tag{3.15}$$
$$= -\frac{\partial^2}{\partial z_j \partial z_i} \log \theta(\tilde{p}) + \frac{\sqrt{-1}}{2\pi} \int_{\partial Q} \partial \log h \wedge \wp^{ij}(z-\tilde{p}),$$

where

$$\frac{\sqrt{-1}}{2\pi} \int_{\partial Q} \partial \log h \wedge \wp^{ij}(z-\tilde{p}) = \lim_{R,R'\to+\infty} \frac{\sqrt{-1}}{2\pi} \int_{(\partial Q)_{R,R'}} \partial \log h \wedge \wp^{ij}(z-\tilde{p}).$$

Proof. By the assumption, the function ω and the hermitian fibre metric $\{h_i\}$ which gives h are extendable smoothly to $\overline{X}$. Moreover we can take a neighbourhood T of Θ whose closure $\overline{T}$ in $\overline{X}$ is a tubular neighbourhood of $\overline{\Theta}$. Then coefficients of $\partial \log \omega$ are bounded on a neighbourhood of $(\overline{X} \setminus X) \setminus (\overline{T} \setminus T)$.

We recall the definition of

$$\wp^i(z) = \sum_{\gamma \in \Gamma} \varphi_i(z, \gamma).$$

Here we write $\varphi_i(z,\gamma) = \phi^{\delta^i}(z,\gamma)$. We have the following explicit representation of $\varphi_i(z,\gamma)$

$$\varphi_i(z,\gamma) = n \frac{(\overline{z_i} - \overline{\gamma_i}) \sum_{k=1}^{n} (-1)^k (\overline{z_k} - \overline{\gamma_k})(d\overline{z})_k}{(\sum_{j=1}^{n} |z_j - \gamma_j|^2)^{n+1}}.$$

Then the absolute value of any coefficient of $\varphi_i(z,\gamma)$ is estimated from above by $n/\|z-\gamma\|^{2n}$. Therefore, considering the definition (3.2) of $\wp^{ij}(z)$, we see that the absolute value of any coefficient of $\wp^{ij}(z)$ is bounded by the following series multiplied by a constant

$$\sum_{\gamma \in \Gamma} \frac{1}{\|z-\gamma\|^{2n}}.$$

We rewrite the last term in the formula (3.5) in Proposition 3.4 as follows:

$$\int_{E_{R,R'}} \partial \log \omega \wedge (\rho - 1) \wp^{ij}(z-p) = \int_{\partial Q_{R,R'} \setminus (\partial Q)_{R,R'}} \partial \log \omega \wedge (\rho - 1) \wp^{ij}(z-\tilde{p}).$$

Since $\Gamma \subset \mathbb{R}_{\Gamma}^{n+m}$ and

$$D_{R,R'} \cong \mathbb{R}_{\Gamma}^{n+m} \times \prod_{j=1}^{n-m} (-R'_j, R_j),$$

there exists $M > 0$ such that

$$\|(z', \mathfrak{Re}\, z'')\| \leqq M$$

for any $z \in \overline{Q}_{R,R'}$ and any $(n-m)$-tuples R, R' of positive numbers, where we write $\mathfrak{Re}z'' = (\mathfrak{Re}z_{m+1}, \ldots, \mathfrak{Re}z_n)$. Let $\tilde{p} = (w', w'')$ be the representation of $\tilde{p}$ in toroidal coordinates. If we set

$$\Gamma_0 := \{\gamma \in \Gamma; \|(w', \mathfrak{Re}w'') + \gamma\| \leqq 2M\},$$

then Γ_0 is a finite set. For any $z \in \mathbb{C}^n$ and $\gamma \in \Gamma$ we have

$$\|z - \gamma\|^2 = \|(z', \mathfrak{Re}z'') - \gamma\|^2 + \|\mathfrak{Im}z''\|^2,$$

where $\mathfrak{Im}z'' = (\mathfrak{Im}z_{m+1}, \ldots, \mathfrak{Im}z_n)$ and $\|\mathfrak{Im}z''\|^2 = \sum_{j=1}^{n-m} |\mathfrak{Im}z_{m+j}|^2$. For any $(n-m)$-tuples R and R' of positive numbers, we set $R_i^0 := \min\{R_i, R_i'\}$ for $i = 1, \ldots, n-m$ and $R^0 := (R_1^0, \ldots, R_{n-m}^0)$. We denote $\|R^0\|^2 := \sum_{j=1}^{n-m} (R_j^0)^2$. We may assume that R and R' are so large that

$$\|\mathfrak{Im}z'' - \mathfrak{Im}w''\| \geqq \frac{1}{2}\|R^0\|$$

for all $z \in \partial Q_{R,R'} \setminus (\partial Q)_{R,R'}$. Then we have

$$\begin{aligned}
\|z - \tilde{p} - \gamma\|^2 &= \|(z', \mathfrak{Re}z'') - (w', \mathfrak{Re}w'') - \gamma\|^2 + \|\mathfrak{Im}z'' - \mathfrak{Im}w''\|^2 \\
&\geqq (\|(w', \mathfrak{Re}w'') + \gamma\| - \|(z', \mathfrak{Re}z'')\|)^2 + \|\mathfrak{Im}z'' - \mathfrak{Im}w''\|^2 \\
&> \frac{1}{4}\|(w', \mathfrak{Re}w'') + \gamma\|^2 + \frac{1}{4}\|R^0\|^2
\end{aligned}$$

for any $\gamma \in \Gamma \setminus \Gamma_0$ and any $z \in \partial Q_{R,R'} \setminus (\partial Q)_{R,R'}$. Therefore the absolute value of any coefficient of $\partial \log \omega \wedge (\rho - 1)\wp^{ij}(z - \tilde{p})$ is estimated from above by

$$C\left(\sum_{\gamma \in \Gamma} \frac{1}{\|z - \tilde{p} - \gamma\|^{2n}} + \sum_{\gamma \in \Gamma \setminus \Gamma_0} \frac{1}{(\|(w', \mathfrak{Re}w'') + \gamma\|^2 + \|R^0\|^2)^n}\right)$$

on $\partial Q_{R,R'} \setminus (\partial Q)_{R,R'}$, where C is a constant independent of R and R'. If $R, R' \to +\infty$, then $R^0 \to +\infty$ and

$$\sum_{\gamma \in \Gamma_0} \frac{1}{\|z - \tilde{p} - \gamma\|^{2n}} \to 0.$$

Since

$$\sum_{\gamma \in \Gamma \setminus \Gamma_0} \frac{1}{(\|(w', \mathfrak{Re}w'') + \gamma\|^2 + \|R^0\|^2)^n} < \sum_{\gamma \in \Gamma \setminus \Gamma_0} \frac{1}{\|(w', \mathfrak{Re}w'') + \gamma\|^{2n}} < +\infty,$$

we have

$$\sum_{\gamma \in \Gamma \setminus \Gamma_0} \frac{1}{(\|(w', \mathfrak{Re}w'') + \gamma\|^2 + \|R^0\|^2)^n} \to 0 \quad \text{as} \quad R, R' \to +\infty.$$

Then we obtain

$$\frac{\sqrt{-1}}{2\pi}\int_{\partial Q_{R,R'}\setminus(\partial Q)_{R,R'}} \partial\log\omega\wedge(\rho-1)\wp^{ij}(z-\tilde{p})\to 0 \quad \text{as} \quad R,R'\to+\infty.$$

Next we rewrite the second term in the right-hand side of (3.5) in Proposition 3.4 as follows

$$\frac{\sqrt{-1}}{2\pi}\int_{\partial Q_{R,R'}} \partial\log h\wedge\wp^{ij}(z-\tilde{p})$$
$$=\frac{\sqrt{-1}}{2\pi}\int_{(\partial Q)_{R,R'}} \partial\log h\wedge\wp^{ij}(z-\tilde{p})+\frac{\sqrt{-1}}{2\pi}\int_{\partial Q_{R,R'}\setminus(\partial Q)_{R,R'}} \partial\log h\wedge\wp^{ij}(z-\tilde{p}).$$

In the same manner as above we can see that

$$\frac{\sqrt{-1}}{2\pi}\int_{\partial Q_{R,R'}\setminus(\partial Q)_{R,R'}} \partial\log h\wedge(\rho-1)\wp^{ij}(z-\tilde{p})\to 0 \quad \text{as} \quad R,R'\to+\infty.$$

By the definition of $\wp^{ij}$ we have

$$\frac{\sqrt{-1}}{2\pi}\int_{(\partial Q)_{R,R'}} \partial\log h\wedge(\rho-1)\wp^{ij}(z-\tilde{p})=$$
$$-\frac{(n-1)!}{(2\pi\sqrt{-1})^n}\int_{(\partial Q)_{R,R'}}\frac{\partial}{\partial z_j}\log h\wedge\wp^{ij}(z-\tilde{p})\wedge dz.$$

Since Θ is a generalized theta divisor, h is the exponent of a polynomial of degree 2 in z_k and $\overline{z_k}$. Then we can write

$$\frac{\partial}{\partial z_j}\log h=\sum_{k=1}^{n}(a_{jk}z_k+b_{jk}\overline{z_k})+c_j. \tag{3.16}$$

Without loss of generality we may assume that the fundamental parallelotope Q is centered at the origin. Then $(\partial Q)_{R,R'}$ consists of the following faces

$$F_\ell^{\pm}(R,R'):=\Bigg\{z\in\mathbb{C}^n; z=\pm\frac{1}{2}\gamma_\ell+\sum_{\substack{i=1\\ i\neq\ell}}^{n+m}s_i\gamma_i+\sum_{j=1}^{n-m}t_jv_j,$$
$$|s_i|\leqq\frac{1}{2}(i=1,\dots,\widehat{\ell},\dots,n+m),-R'_j\leqq t_j\leqq R_j(j=1,\dots,n-m)\Bigg\}$$

for $\ell=1,\dots,n+m$. If we set

$$F_\ell^{\pm}:=\Bigg\{z\in\mathbb{C}^n; z=\pm\frac{1}{2}\gamma_\ell+\sum_{\substack{i=1\\ i\neq\ell}}^{n+m}s_i\gamma_i+\sum_{j=1}^{n-m}t_jv_j,$$
$$|s_i|\leqq\frac{1}{2}(i=1,\dots,\widehat{\ell},\dots,n+m),t_j\in\mathbb{R}(j=1,\dots,n-m)\Bigg\},$$

then ∂Q is the union of all $F_\ell^\pm$. It follows from (3.16) that for any $z \in F_\ell^-(R, R')$ we have

$$\frac{\partial}{\partial z_j} \log h(z + \gamma_\ell) - \frac{\partial}{\partial z_j} \log h(z) = c_{j\ell}.$$

On the other hand, $\wp^i(z - \tilde{p}) \wedge dz$ is the same on two opposite faces $F_\ell^+(R, R')$ and $F_\ell^-(R, R')$ for it is Γ-invariant. Then we obtain

$$\int_{F_\ell^+(R,R') \cup F_\ell^-(R,R')} \frac{\partial}{\partial z_j} \log h \wp^i(z - \tilde{p}) \wedge dz = c_{j\ell} \int_{F_\ell^-(R,R')} \wp^i(z - \tilde{p}) \wedge dz.$$

The absolute value of any coefficient of $\wp^i(z) \wedge dz$ is bounded from above by the series $\sum_{\gamma \in \Gamma} 1/\|z - \gamma\|^{2n}$ multiplied by a constant. From a similar argument as above it follows that $\wp^i(z - \tilde{p}) \wedge dz$ is integrable on $F_\ell^\pm$ and

$$\int_{F_\ell^\pm(R,R')} \wp^i(z - \tilde{p}) \wedge dz \to \int_{F_\ell^\pm} \wp^i(z - \tilde{p}) \wedge dz \quad \text{as} \quad R, R' \to +\infty.$$

Thus we finally obtain

$$\lim_{R,R' \to +\infty} \int_{\Theta_{R,R'}} \wp^{ij}(z - p) = -\frac{\partial^2}{\partial z_i \partial z_j} \log \theta(\tilde{p}) + \frac{\sqrt{-1}}{2\pi} \int_{\partial Q} \partial \log h \wedge \wp^{ij}(z - \tilde{p}).$$

$\square$

Lemma 3.6 (Lemma 2 in [1]). *The term* $\frac{\sqrt{-1}}{2\pi} \int_{\partial Q} \partial \log h \wedge \wp^{ij}(z - \tilde{p})$ *in (3.15) is a constant independent of* p.

Proof. It is sufficient to show that the integral

$$\int_{F_\ell^-} \wp^i(z - \tilde{p}) \wedge dz$$

is independent of p.

Let $H_\ell := \pi(F_\ell^-)$ be the real $(2n-1)$-dimensional hypersurface of X determined by F_ℓ^-. We have

$$\int_{F_\ell^-} \wp^i(z - \tilde{p}) \wedge dz = \int_{H_\ell} \wp^i(z - p) \wedge dz = \int_{H_\ell - p} \wp^i(z) \wedge dz.$$

We take a different point $q \in X$ which does not lie on H_ℓ. We denote by $D_{p,q}$ the subdomain of X surrounded by $H_\ell - p$ and $H_\ell - q$. For any $(n-m)$-tuple R of positive numbers we set $D_{p,q}(R) := D_{p,q} \cap X_{R,R}$, $H_p(R) := (H_\ell - p) \cap X_{R,R}$ and $H_q(R) := (H_\ell - q) \cap X_{R,R}$. Let $E_{p,q}^+(R)$ and $E_{p,q}^-(R)$ be two components of $\partial D_{p,q}(R) \cap \partial X_{R,R}$. Then we have

$$\partial D_{p,q}(R) = H_p(R) \cup H_q(R) \cup E_{p,q}^+(R) \cup E_{p,q}^-(R).$$

By Stokes' theorem we have

$$\int_{H_p(R)} \wp^i(z) \wedge dz - \int_{H_q(R)} \wp^i(z) \wedge dz + \int_{E^+_{p,q}(R)} \wp^i(z) \wedge dz - \int_{E^-_{p,q}(R)} \wp^i(z) \wedge dz$$
$$= \int_{D_{p,q}(R)} d(\wp^i(z) \wedge dz) = 0.$$

From the same argument as in the proof of Proposition 3.5, it follows that

$$\int_{E^{\pm}_{p,q}(R)} \wp^i(z) \wedge dz \to 0 \quad \text{as} \quad R \to +\infty.$$

Hence we obtain

$$\int_{H_\ell - p} \wp^i(z) \wedge dz = \int_{H_\ell - q} \wp^i(z) \wedge dz,$$

which completes the proof. □

Combining Proposition 3.5 with Lemma 3.6, we obtain the following theorem.

Theorem 3.7 (Theorem 1 in [1]). *Let X be a quasi-abelian variety of kind 0 with the standard compactification $\overline{X}$. Let Θ be a positive divisor on X given by a holomorphic function θ on $\mathbb{C}^n$. If Θ is holomorphically extendable to $\overline{X}$, then $\wp^{ij}$ is integrable on Θ and we have*

$$\int_{\Theta} \wp^{ij}(z - p) = -\frac{\partial^2}{\partial z_i \partial z_j} \log \theta(\tilde{p}) + c_{ij}$$

for any $p \in X \setminus \Theta$, where $\tilde{p} \in \pi^{-1}(p)$ and c_{ij} is a constant independent of p.

Bibliography

[1] Y. Abe and A. Kogie, A generalization of Weierstrass' $\wp$-function to quasi-abelian varieties, *Toyama Math. J.*, **35**(2012), 15-33.

[2] Y. Abe and K. Kopfermann, Toroidal Groups, *Lecture Notes in Mathematics*, **1759**, Springer, Berlin, 2001.

[3] Y. Abe and T. Umeno, On quasi-abelian varieties of kind k, *Kyushu J. Math.*, **60** (2006), 305-316.

[4] A. Andreotti and F. Norguet, Problème de Levi convexité holomorphe pour les classes de cohomologie, *Ann. Sc. Norm. Sup. Pisa*, **20** (1966), 197-241.

[5] A. Andreotti and F. Norguet, Cycles of algebraic manifolds and $\partial\overline{\partial}$-cohomology, *Ann. Sc. Norm. Sup. Pisa*, **25** (1971), 69-114.

[6] J. Frenkel, Cohomologie non abélienne et espaces fibrés, *Bull. Soc. Math. France*, **20** (1957), 135-230.

[7] H. Kazama, $\overline{\partial}$ Cohomology of (H, C)-groups, *Publ. RIMS. Kyoto Univ.*, **20** (1984), 297-317.

[8] H. Kazama and T. Umeno, Complex abelian Lie groups with finite-dimensional cohomology groups, *J. Math. Soc. Japan*, **36** (1984), 91-106

[9] P. Zappa, Osservazioni sui nuclei di Bochner-Martinelli, *Att. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, (8) **67** (1979), 21-26

[10] P. Zappa, Sulle classi di Dolbeault di tipo $(0, n-1)$ con singolarità in un insieme discreto, *Att. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, (8) **70** (1981), 87-95

[11] P. Zappa, Su una generalizzazione della $\wp$ di Weierstrass, *Bollettino U. M. I.*, (6) **2-A** (1983), 245-252