Paileve Transcendents and Its Linearizations

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§1. Introduction

Painleve and his co-workers had studied what kind of ordinary differential equations (**ODE's**) belonging to the second-order class does not admit any movable singular point (**MCP**) in its solutions.[1] They found six so-called irreducible Painleve transcendents, which can be integrated in terms of elliptic functions,

$$P_1: \frac{d^2f}{dx^2}=6f^2+x, \tag{1.1a}$$

$$P_2: \frac{d^2f}{dx^2}=2f^3+xf+\alpha, \tag{1.1b}$$

$$P_3: \frac{d^2f}{dx^2}=\frac{1}{f}\left(\frac{df}{dx}\right)^2-\frac{1}{x}\frac{df}{dx}+\frac{1}{x}\left(\alpha f^2+\beta\right), \tag{1.1c}$$

$$P_4: \frac{d^2f}{dx^2}=\frac{1}{2f}\left(\frac{df}{dx}\right)^2-\frac{3}{2}f^3+4xf^2+2(x^2-\alpha)f+\frac{\beta}{f}, \tag{1.1d}$$

$$P_5: \frac{d^2f}{dx^2}=\left(\frac{1}{2f}+\frac{1}{f-1}\right)\left(\frac{df}{dx}\right)^2-\frac{1}{x}\frac{df}{dx}$$
$$+\left(\frac{f-1}{x}\right)^2\left(\alpha f+\frac{\beta}{f}\right)+\gamma\frac{f}{x}+\delta\frac{f(f+1)}{f-1}, \tag{1.1e}$$

$$P_6: \frac{d^2f}{dx^2}=\frac{1}{2}\left(\frac{1}{f}+\frac{1}{f-1}+\frac{1}{f-x}\right)\left(\frac{df}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{f-x}\right)\frac{df}{dx}$$
$$+\frac{f(f-1)(f-x)}{x^2(x-1)^2}\left\{\alpha+\beta\frac{x}{f^2}+\gamma\frac{x-1}{(f-1)^2}+\delta\frac{x(x-1)}{(f-x)^2}\right\}, \tag{1.1f}$$

where $\alpha, \beta, \gamma, \delta$ are constant. On the other hand, the group thoretical analysis for differential equations was advocated by S. C. Lie during 19-th century,[2] then many contributions had been published as to the symmetric transformations and similarity solutions.[3] Despite the considerable effort to these earlier studies, few advances for solutions of the Painleve equations were made until recently.

More than ten years ago Ablowitz et al[4,5] found that exactly integrable nonlinear partial differential equations (**PDE's**) allows similarity solutions and are closely related with the **Painleve** transcendents.

Flaschka and Newell[6] considered the problem deeply and obtained a principal method for solving the 4-th Painleve type of ordinary nonlinear equations (**P-ODE**) in a global sence.

In this issue we consider the connection between linearizations of **PDE's** and **P-ODE's** simply. The principal idear depends on the similarity solutions of **PDE's**.

§2. Similarity Transformations

The group theoretical analysis for differential equations had contributed to the symmetric transformatiohs and similarity of solutions.[3)] To review this, we consider

$$N(x, t, q, q_x, q_t, q_{xt}, \cdots)=0. \qquad (2.1)$$

The basic idear is to consider the invariance of tangential equations under one (or several) parameter ($=\varepsilon$), where the transformation group acts on variables (x, t, q) and generates (x', t', q),

$$x'=f(x, t, q; \varepsilon),\ t'=g(x, t, q; \varepsilon),\ q'=h(x, t, q; \varepsilon), \qquad (2.2)$$

where the case $\varepsilon=0$ is set to be identity,

$$x=f(x, t, q; 0),\ t=g(x, t, q; 0),\ q=h(x, t, q; 0).$$

Denoting a solution of (2.1) as $q=\phi(x, t)$, we replace these variables with q', x' and t' then obtain

$$N(x', t', q', q'_{x'}, q'_{t'}, q'_{x't'}, \cdots)=0. \qquad (2.3)$$

since ε is a parameter. That is, (2.3) also allows $q'=\phi(x', t')$,

$$\phi(f(x, t, \phi; \varepsilon),\ g(x, t, \phi; \varepsilon))=h(x, t, \phi(x, t); \varepsilon). \qquad (2.4)$$

We say this as the invariant condition, which enables us to find such infinitesimal transformations as

$$x'=x+\varepsilon\xi(x, t, q),\ t'=t+\varepsilon\tau(x, t, q),\ q'=q+\varepsilon\eta(x, t, q). \qquad (2.5)$$

The problem is reduced to find three functions $\xi(x, t, q)$, $\tau(x, t, q)$ and $\eta(x, t, q)$.

§3. Similarities of NLPDE's

In this section we consider the typical type of **NLPDE's**,

$$\textbf{KdV}\ :\ q_t-6qq_x+q_{xxx}=0, \qquad (3.1)$$
$$\textbf{mKdV}:\ q_t-6q^2q_x+q_{xxx}=0. \qquad (3.2)$$

2A) Korteweg-de Vries Equation

We first remark that the **K-dV** equation (3.1) is invariant under transformations of independent variables $\tilde{t}=\alpha t+\beta x$, $\tilde{x}=\nu t+\delta x$, and dependent variable as

$$\tilde{q}(x,t) = \varkappa \cdot q(\tilde{x}, \tilde{t}), \tag{3.3}$$

where $\varkappa$ is a constant. Actually the **similarity condition** $q(x,t) = \tilde{q}(x,t)$ holds under conditions $\alpha = \delta^3$, $\varkappa = \delta^2$ and $\beta = \gamma = 0$, by which the invariance is given by $q(x,t) = \delta^2 q(\delta x, \delta^3 t)$, caused from that the **K-dV** equation at least allows us 1-parameter solution. Since $\delta^2 q(\delta x, \delta^3 t) \to \{1 + \delta(2 + x\partial_x + 3t\partial_t)\} q(x,t)$ as $\delta \to 1 + \delta$, the invariance is reduced to

$$x q_x + 3t q_t = -2q. \tag{3.4}$$

As shown in **Appendix-A**, characteristic equations, $\dfrac{dx}{x} = \dfrac{dt}{3t} = -\dfrac{dq}{2q}$, are important for solving the general solution. By means of two independent solutions $q = c_1 x^{-2}$ and $q = c_2 t^{-2/3}$, the general solution can be given by $F(qx^2, qt^{2/3}) = 0$. Since $F(*, *)$ is an arbitrary function, we can obtain $c_1 = qx^2 = f(c_2) = f(qt^{2/3})$ or equivalently

$$q = x^{-2} f(x^{-2} t^{2/3}), \quad q = t^{-2/3} f(x^2 t^{-2/3}) \text{ etc.}, \tag{3.5}$$

which reduces the **KdV** to an **ODE**. If we take $q = t^{-2/3} f(xt^{-1/3})$ as an example, the **KdV** eq. is reduced to

$$f''' - 6f'f - \frac{1}{3} z f' - \frac{2}{3} f = 0, \tag{3.6}$$

where $f \equiv f(z)$, $f' = df/dz$ and $z = xt^{-1/3}$.

2B) Modified KdV Equation

For the **m-KdV** eq. (3.2), we also obtain

$$q(x,t) = \tilde{q}(x,t) \equiv \delta q(\delta x, \delta^3 t). \tag{3.7}$$

The self-similar solution is easily derived,

$$q(x,t) = (3t)^{-1/3} f(x(3t)^{-1/3}). \tag{3.8}$$

From eqs. (3.2) and (3.8), the second **P-ODE** is deived,

$$f'' = 2f^3 + zf + \nu, \tag{3.9}$$

where ν is a constant. We are sure that similarities reduce both **KdV** and **mKdV** to **P-ODE's**. This fact holds for many kinds of **PDE** which can be solved by so-called exact method, inverse spectral transform (**IST**), Backlund Transform, and so on. While the **Painlve's type** of equations had been studicd by many authors and its mathematical properties were made clear in various points. We specially refer to the connection between **Painleve transendents** and **IST**, which was found by Ablowitz and his co-workers.[5] The **IST** decouples the **PDE** into a set of linear problems, one of them is an eigenvalue problem. From this aspect it is natural to expect such a decoupling scheme for Painleve transcendents.

§4. Linearization and Compatibility Conditions

We consider a typical set linear equations, consisting of the 2×2-matrx order equations,

$$\varphi_x = D(\lambda; q)\varphi, \quad (4.1a) \qquad \varphi_t = F(\lambda; q)\varphi, \quad (4.1b)$$

where λ is a spectral parameter and both matrices $\boldsymbol{D}$ and $\boldsymbol{F}$ satisfy the integrable condition as to λ,

$$\boldsymbol{D}_t - \boldsymbol{F}_x + [\boldsymbol{D}, \boldsymbol{F}] = 0. \qquad (4.2)$$

The coefficient matrix $\boldsymbol{D}(\lambda; q)$ of (4.1a) is specified, while $\boldsymbol{F}(\lambda; q)$ is determined by the integrability (4.2). For the case of **mKdV** eq., both matrices $\boldsymbol{D}$ and $\boldsymbol{F}$ are given by

$$\boldsymbol{D} = -i\lambda\sigma_3 + Q, \quad \boldsymbol{F} = \sigma_3 f(\lambda; q) + F_0(\lambda; q), \qquad (4.3)$$

where σ_3 and σ_1 are Pauli spin matrices, $Q(x, t) = q(x, t)\cdot\sigma_1$, F_0 is chosen as $F_0 = \begin{bmatrix} 0, & g \\ h, & 0 \end{bmatrix}$,

$$f \equiv f(\lambda, q) = -4i\lambda^3 - 2miq^2\lambda, \qquad (4.4a)$$
$$g \equiv g(\lambda; q) = 4q\lambda^2 + 2iq_x\lambda - q_{xx} + 2mq^3, \qquad (4.4b)$$
$$h \equiv h(\lambda; q) = 4mq\lambda^2 - 2imq_x\lambda - mq_{xx} + 2q^3 (m = \pm 1). \qquad (4.4c)$$

The **mKdV** eq. is actually obtained as

$$q_t - 6mq^2 q_x + q_{xxx} = 0 (m = \pm 1), \qquad (4.5)$$

where the potential $q = q(x, t)$ is determined under a initial condition $(q = q_0(x))$. Because both matrices $\boldsymbol{D} = \boldsymbol{D}(\lambda; q)$ and $\boldsymbol{F} = \boldsymbol{F}(\lambda; q)$ depend on (x, t), the eigenfunction of (4.1) can be denoted as

$$\varphi \equiv \varphi(\lambda; x, t; q) \equiv \varphi(\lambda; x, t), \qquad (4.6)$$

Now we define the following transformations,

$$\tilde{x} = \delta x, \quad \tilde{t} = \delta^3 t, \quad \tilde{q} = q/\delta, \quad \tilde{\lambda} = \frac{\lambda}{\delta}, \qquad (4.7)$$

by which the following lemmas are deduced.

[**Lemma. 1**] By means of (4.7) the linear set (4.1) can be transformed to

$$\varphi_{\tilde{x}} = \boldsymbol{D}(\tilde{\lambda}; \tilde{q})\varphi, \quad (4.8a) \qquad \varphi_{\tilde{t}} = \boldsymbol{F}(\tilde{\lambda}; \tilde{q})\varphi. \quad (4.8b)$$

and this solution is represented by

$$\varphi = \varphi(\tilde{\lambda}; \tilde{x}, \tilde{t}; \tilde{q}). \qquad (4.9)$$

(**proof**) Because of (4.7) the differential operators $(\partial/\partial x, \partial/\partial t)$ are transformed to $\{\delta(\partial/\partial\tilde{x}), \delta^3(\partial/\partial\tilde{t})\}$. By this facts we can see

$$\boldsymbol{D}(\lambda; q) \to \delta\cdot\boldsymbol{D}(\tilde{\lambda}; \tilde{q}), \quad \boldsymbol{F}(\lambda; q) \to \delta^3 \boldsymbol{F}(\tilde{\lambda}; \tilde{q}). \qquad (4.10)$$

The components of $\boldsymbol{F}$ are also transformed as

$$f(\lambda;q)\to\delta^3 f(\tilde{\lambda};\tilde{q}),\ g(\lambda;q)\to\delta^3 g(\tilde{\lambda};\tilde{q}),\ h(\lambda;q)\to\delta^3 h(\tilde{\lambda};q).$$

The coupled set (4.1) are transformed to $\varphi_{\tilde{x}}=\boldsymbol{D}(\tilde{\lambda};\tilde{q})\varphi$, $\varphi_{\tilde{t}}=\boldsymbol{F}(\tilde{\lambda};\tilde{q})\varphi$, which are invariant with (4.2) and the solution is given by $\varphi=\varphi(\tilde{\lambda};\tilde{x},\tilde{t};\tilde{q})$. **[QED]**

[**Lemma. 2**] We denote the solution of the **m-KdV** equation (4.6) as $q=q(x,t)$. Under (4.7) the solution $\tilde{q}$ satisfies the same **m-KdV** equation with independent variables $\tilde{x}$ and $\tilde{t}$, then we obtain

$$\tilde{q}=q(\tilde{x},\tilde{t})/\delta. \tag{4.11}$$

(**proof**) The **m-KdV** equation (4.6) can be transformed to

$$q_t-6mq^2q_x+q_{xxx}=\delta^4\{\tilde{q}_{\tilde{t}}-6m\tilde{q}^2\tilde{q}_{\tilde{x}}+\tilde{q}_{\tilde{x}\tilde{x}\tilde{x}}\}=0.$$

Both **m-KdV** equations as to q and $\tilde{q}$ are invariant under $(x,t)\to(\tilde{x},\tilde{t})$. Hence $\tilde{q}=q(\tilde{x},\tilde{t})/\delta$ is obtained. **[QED]**

The self-similar solution q_s in (3.9) shows us

$$\begin{aligned}q_s(x,t)&\equiv(3t)^{-1/3}f(x(3t)^{-1/3})\\&=\delta(3t\cdot\delta^3)^{-1/3}f((x\cdot\delta)(3t\cdot\delta^3)^{-1/3})\\&=\delta(3\tilde{t})^{-1/3}f(\tilde{x}(3\tilde{t})^{-1/3})=\delta\cdot q_s(\tilde{x},\tilde{t}),\end{aligned} \tag{4.12a}$$

then from (4.11) we find

$$\tilde{q}_s=q_s(x,t)/\delta=q_s(\tilde{x},\tilde{t}) \tag{4.12b}$$

and $\tilde{q}_s\equiv(3\tilde{t})^{-1/3}f(\tilde{x}(3\tilde{t})^{-1/3})$. We note that the potential of $\varphi\left(\lambda;x,\frac{1}{3}\right)$is $q_s\left(x,\frac{1}{3}\right)=f(x)$.

[**Theor.**] The potential is assummed to be self-similar. Then from (4.12b) both solutions of linear sets (4.1) and (4.9) must be related with

$$\varphi(\lambda;x,t)=\varphi(\tilde{\lambda};\tilde{x},\tilde{t}). \tag{4.13}$$

[**proof**] Because of self-similarity (4.12b), the set of (4.9) is written by $\varphi_{\tilde{x}}=\boldsymbol{D}(\tilde{\lambda};q_s(\tilde{x},\tilde{t}))\varphi$, $\varphi_{\tilde{t}}=\boldsymbol{F}(\tilde{\lambda};q_s(\tilde{x},\tilde{t}))\varphi$, with a solution $\varphi(\tilde{\lambda};\tilde{x},\tilde{t})$, because it are invariant with (4.1). **[QED]**

On the $\tilde{x}$-$\tilde{t}$ space, we may set $\tilde{t}=\frac{1}{3}$ and define

$$\psi(\tilde{\lambda},\tilde{x})=\varphi\left(\tilde{\lambda};\tilde{x},\frac{1}{3}\right), \tag{4.14}$$

then $\varphi(\lambda;x,t)=\psi(\tilde{\lambda},\tilde{x})$. In this case the parameter δ must be taken as $\delta=(3t)^{-1/3}$ and we denote

$$\lambda'=\lambda(3t)^{1/3},\ x'=x(3t)^{-1/3}. \tag{4.15}$$

Since ψ must satisfy (4.1), we obtain

$$\begin{aligned}\varphi_x&=(3t)^{-1/3}\psi_{x'}=\boldsymbol{D}(\lambda,q_s)\psi,\\\varphi_t&=(3t)^{-1}\{-x'\psi_{x'}+\lambda'\psi_{\lambda'}\}=\boldsymbol{F}(\lambda,q_s)\psi,\end{aligned}$$

where we used differential operators given by

$$\frac{\partial}{\partial t}=(3t)^{-1}\left\{-x'\frac{\partial}{\partial x'}+\lambda'\frac{\partial}{\partial \lambda'}\right\},\ \frac{\partial}{\partial x}=(3t)^{-1/3}\frac{\partial}{\partial \tilde{x}}.$$

Because of $\boldsymbol{D}(\lambda;q)=(3t)^{-1/3}\boldsymbol{D}(\lambda';q')$ and $\boldsymbol{F}(\lambda;q)=(3t)^{-1}\boldsymbol{F}(\lambda';q')$, we obtain

$$\psi_{x'}=\boldsymbol{D}(\lambda',q_s)\psi,\ \ \psi_{\lambda'}=\boldsymbol{R}(\lambda';q_s)\psi, \tag{4.16a}$$

where

$$\boldsymbol{R}(\lambda,x;q)\equiv\{x\cdot\boldsymbol{D}(\lambda,q_s)+\boldsymbol{F}(\lambda;q_s)\}/\lambda. \tag{4.16b}$$

We change $\{(3t)^{-1/3}x,(3t)^{1/3}\lambda\}$ to (x,λ) for briefness. Since $\tilde{q}_s=f(x)$, the matrix $\boldsymbol{R}$ can be given by

$$\boldsymbol{R}=\begin{bmatrix}-i(4\lambda^2+x+2mf^2), & 4\lambda f+2if'\\ 4mf\lambda-2imf', & i(4\lambda^2+x+2mf^2)\end{bmatrix}-(f''-2mq^3-xf)\begin{bmatrix}0, & 1\\ m, & 0\end{bmatrix}.$$

If $f(x)$ satisfy

$$f''=2mf^3+xf+\nu\ (\nu:\text{const.}), \tag{4.17}$$

we can obtain the following linear set,

$$\partial_x\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}-i\lambda, & f\\ f, & i\lambda\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}, \tag{4.18a}$$

$$\partial_\lambda\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}-i(4\lambda^2+x+2mf^2), & 4\lambda f+2if'+\nu\\ 4mf\lambda-2imf'+m\nu, & i(4\lambda^2+x+2mf^2)\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}. \tag{4.18b}$$

We note that the integrability of formula (4.18) again yields (4.17) which is known as the second Painleve's equation.

Similar treatments can be also performed for the **KdV** equation (3.1). Under transformations of variables,

$$\tilde{x}=\delta\cdot x,\ \ \tilde{t}=\delta^3 t,\ \ \tilde{q}=q\cdot\delta^{-2}, \tag{4.19}$$

the **KdV** (3.1) is invariant and the solution $\tilde{q}$ is given by $\tilde{q}=q(\tilde{x},\tilde{t})$ or $q(x,t)=\delta^2 q(\tilde{x},\tilde{t})$. This is a self-similar condition and we take the self-similar solution q_s as

$$q_s=q_s(x,t)\equiv t^{-2/3}f(x^2t^{-2/3}), \tag{4.20}$$

which really shows $q_s(x,t)=\delta^2\tilde{t}^{-2/3}f(\tilde{x}^2\tilde{t}^{-2/3})=\delta^2 q_s(\tilde{x},\tilde{t})$. The inverse scheme of (4.1) is well-known as

$$\varphi_{xx}=(q_s-\lambda)\varphi,\ \ \varphi_t=(4\lambda+2q_s)\varphi_x-q_{s,x}\varphi, \tag{4.25}$$

where λ is a spectral parameter. Adding to (4.19), we define $\tilde{\lambda}=\lambda\cdot\delta^{-2}$. Then (4.25) is again invariant for such a transformation of variables. We see $\varphi(\lambda;x,t)=\varphi(\tilde{\lambda};\tilde{x},\tilde{t})$ and choose $\tilde{t}=1$ (corresponding to $\delta=t^{-1/3}$). By this setting we can define a function $\varphi_0(\tilde{\lambda},\tilde{x})=\varphi(\tilde{\lambda};\tilde{x},\tilde{t}=1)$ and obtain

$$\varphi(\lambda;x,t)=\varphi_0(\lambda t^{2/3},\ xt^{-2/3}). \tag{4.26}$$

by which the following linear set of φ_0 is obtained,

$$\varphi_{0,xx}=\{q_s(x,t)-\lambda\}\varphi_0,$$
$$\varphi_{0,\lambda}\ =\{6+\frac{1}{\lambda}(\frac{x}{2}+3q_s(x,t)\}\varphi_{0,x}-\frac{3}{2\lambda}q_{s,x}\varphi_0, \tag{4.27}$$

where $\lambda t^{2/3}$ and $xt^{-2/3}$ are denoted by λ and x for simplicity. The integrability of (4.27) is directly reduced to (3.7).

§5. K-P Hierarchy

The K-P equation is interesting since its has (2+1) dimensions. It may be seen difficult and its corresponding Painleve formula was not shown yet. According to Sato,[7] we introduce an scalar psed-differential operator $\mathfrak{L}$,

$$\mathfrak{L}(\partial)=\partial+u_2(x)\partial^{-1}+u_3(x)\partial^{-2}+u_4(x)\partial^{-3}+\cdots, \tag{5.1}$$

where $u_n(n=2,3,\cdots)$ are functions depending on x and also on infinitely many variables $t=(t_0,t_1,t_2,\cdots)$. The operator $\mathfrak{L}^n(\partial)$ has differential parts, which is represented by $\mathfrak{B}_n=[\mathfrak{L}^n(\partial)]_+$. After some calculations, we can get

$$\mathfrak{B}_1=\partial,$$
$$\mathfrak{B}_2=\partial^2+2u_2,$$
$$\mathfrak{B}_3=\partial^3+3u_2\partial+3u_3+3\frac{\partial u_2}{\partial x_1},$$
$$\mathfrak{B}_4=\partial^4+4u_2\partial^2+\left(4u_3+6\frac{\partial u_2}{\partial x_1}\right)\partial+4u_4+6\frac{\partial u_3}{\partial x_1}$$
$$+4\frac{\partial^2 u_2}{\partial x_1^2}+6(u_2)^2,\ \cdots\cdots. \tag{5.2}$$

[**Theorem**] If eigen functions of $\mathfrak{L}$ are introduced by

$$\mathfrak{L}(\partial)\varphi(\lambda,x)=\lambda\varphi(\lambda,x),\ \frac{\partial\varphi}{\partial t_n}=\mathfrak{B}_n(\partial)\varphi, \tag{5.3}$$

for $n=1,2,\cdots$, instead for (5.3) we obtain

$$\frac{\partial\mathfrak{L}}{\partial t_n}=[\mathfrak{B}_n,\mathfrak{L}], \tag{5.4a}$$

$$\frac{\partial\mathfrak{B}_n}{\partial t_m}-\frac{\partial\mathfrak{B}_m}{\partial t_n}=[\mathfrak{B}_n,\mathfrak{B}_m]. \tag{5.4b}$$

(**proof**) (5.4a) is easily derived by taking derivatives of $\mathfrak{L}\varphi\equiv\lambda\varphi$. The derivative $\partial^2\mathfrak{L}/\partial t_m\partial t_n$, of (5.4a), is arranged to

$$\frac{\partial}{\partial t_m}[\mathfrak{B}_n,\mathfrak{L}]=[\mathfrak{B}_{n,t_m},\mathfrak{L}\}+[\mathfrak{B}_n,[\mathfrak{B}_m,\mathfrak{L}]],$$

hence the compatibility results in

$$\frac{\partial}{\partial t_m}[\mathfrak{B}_n,\mathfrak{L}]-\frac{\partial}{\partial t_n}[\mathfrak{B}_m,\mathfrak{L}]$$
$$=[\mathfrak{B}_{n,t_m}-\mathfrak{B}_{m,t_n},\mathfrak{L}]+[\mathfrak{B}_n,[\mathfrak{B}_m,\mathfrak{L}]]-[\mathfrak{B}_m,[\mathfrak{B}_n,\mathfrak{L}]]$$
$$=[\mathfrak{B}_{n,t_m}-\mathfrak{B}_{m,t_n}+[\mathfrak{B}_m,\mathfrak{B}_n],\mathfrak{L}]=0,$$

where we used Jacobi's relation,

$$[\mathfrak{B}_n, [\mathfrak{B}_m, \mathfrak{L}]] - [\mathfrak{B}_m, [\mathfrak{B}_n, \mathfrak{L}]] = [\mathfrak{L}, [\mathfrak{B}_n, \mathfrak{B}_m]].$$ **[QED]**

The inverse scheme of the K-P equation

$$(4u_t - u_{xxx} - 3u_x u)_x - 3u_{yy} = 0 \tag{5.5}$$

is given by

$$\frac{\partial \varphi}{\partial y} = \varphi_{xx} + 2u\varphi, \tag{5.5a}$$

$$\frac{\partial \varphi}{\partial t} = \varphi_{xxx} + 3u\varphi_x + 3(v + u_x)\varphi, \tag{5.5b}$$

where $y \equiv t_0$, $t \equiv t_1$ and $u \equiv u_2$, $v \equiv u_3$,

$$2v_x + u_{xx} - u_y = 0. \tag{5.5c}$$

By means of (5.3) the eigenfunction $\varphi = \varphi(\lambda; x, y, t)$ is defined, and we consider the following transformations of variables,

$$\begin{aligned} &\tilde{x} = \beta x, \ \ \tilde{y} = \beta^2 y, \ \ \tilde{t} = \beta^3 t, \\ &\lambda = \beta\tilde{\lambda}, \ \ u = \beta^2 \tilde{u}, \ \ v = \beta^3 \tilde{v}, \end{aligned} \tag{5.6}$$

under which relations (5.5) are invariant. If we further assume

$$\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = u(\tilde{x}, \tilde{y}, \tilde{z}), \ \ \tilde{v}(\tilde{x}, \tilde{y}, \tilde{t}) = v(\tilde{x}, \tilde{y}, \tilde{z}), \tag{5.7}$$

the eigenfunction is still invariant,

$$\varphi(\lambda; x, y, t) = \varphi(\tilde{\lambda}; \tilde{x}, \tilde{y}, \tilde{t}). \tag{5.8}$$

For new variables $\{\tilde{\lambda}, \tilde{x}, \tilde{y}, \tilde{t}\}$ we may take $\tilde{t}$ as const $(=1/3)$, while the parameter β must be set as

$$\beta = (3t)^{-1/3}. \tag{5.9}$$

In this case (5.6) defines

$$\lambda' = \lambda(3t)^{1/3}, \ \ x' = x(3t)^{-1/3}, \ \ y' = \beta^2 y(3t)^{-2/3}, \tag{5.10}$$

while from (5.7) dependent variables satisfy

$$u' = (3t)^{2/3} u, \ \ v' = (3t)^{-1} v. \tag{5.11}$$

The eigen function can be denoted by

$$\varphi(\tilde{\lambda}; \tilde{x}, \tilde{y}, \tilde{t}) = \varphi(\lambda'; x', y', \frac{1}{3}) \equiv \psi(\lambda', x', y'). \tag{5.12}$$

By eqs. (5.10) and (5.12) we may change the derivatives as

$$\frac{\partial}{\partial x} = (3t)^{-1/3} \frac{\partial}{\partial x'}, \ \ \frac{\partial}{\partial y} = (3t)^{-2/3} \frac{\partial}{\partial y'},$$

$$\frac{\partial}{\partial t}=(3t)^{-1}\left(\lambda'\frac{\partial}{\partial x'}-x\frac{\partial}{\partial x'}-2y'\frac{\partial}{\partial y'}\right). \tag{5.13}$$

Then eqs. (5.5) are transformed to

$$\frac{\partial\psi}{\partial y'}=\frac{\partial^2\psi}{\partial x'^2}+2u'\psi,$$

$$\lambda'\frac{\partial\psi}{\partial\lambda'}=\left\{\frac{\partial^3}{\partial x'^3}+2y'\frac{\partial^2}{\partial x'^2}+(3u'+x')\frac{\partial}{\partial x'}+\left(3v'+3\frac{\partial u'}{\partial x'}+4y'u'\right)\right\}\psi,$$

$$2\frac{\partial v'}{\partial x'}+\frac{\partial^2 u'}{\partial x'^2}-\frac{\partial u'}{\partial y'}=0. \tag{5.14}$$

The integrability of (5.14) is reduced to

$$\{u_{xxx}+(12u+4x)u_x+8yu_y+12u\}_x+3u_{yy}=0, \tag{5.15}$$

where we replaced u', v' and x', y' with u, v and x, y.

Now, denoting $u'\left(x', y', \frac{1}{3}\right)=u_s(x', y')$ and from (5.7) and (5.11), we can see

$$u(x, y, t)=(3t)^{-2/3}u_s(x', y'),$$
$$v(x, y, t)=(3t)^{-1}v_s(x', y'). \tag{5.16}$$

After sustitution of (5.16) into the K-P eq., we obtain

$$\{u_{xxx}+(12u+4x)u_x+8yu_y+4u\}_x+3u_{yy}=0, \tag{5.17}$$

which is slightly different from (5.15) but it is trivially removed by adding a shift to x.

§6. Concludings and Remarks

We mentioned the derivation of **P-ODE's** by means of similarities from related **PDE's**, assummed to be in a class of equations solvable by means of the **IST**. Ablowitz et al.[4,5] had found a connection between the **PDE's** and the **P-ODE's**, by the dressing method, developed by Zakharov and Shabat.[8]

In this paper we extend the derivation of the **P-ODE** from the **PDE's** with a couppled set similar to the **IST** formula, where an invariance of eigenfunction is introduced. This was also applied to the K-P equation in (2+1) dimensions. It is important to obtain such couppled sets of **P-ODE**, because Ablowitz had developed the monodromy inverse transform (**MIT**), by which he showed it possible to obtain a global solution of Painleve transcendents.[9]

References

1) E. L. Ince: "ordinary differential equations", (1927), Dover, NY (1956)
2) S. V. Coggeshell and R. A. Axford: "Lie group invariance of radiation hydrodynamics equation and their associated similarities", Phys. Fluids **29(8)** (1986) 2398
3) M. Lakshmanan and P. Kaliappan: "Lie transformations, nonlinear evolution equations, and Painleve forms", J. Math. Phys., **24(4)** (1983) pp. 795-

4) M. J. Ablowitz, A. Ramani and H. Segur: Lett. Nuovo. Cimento **23** (1978) 333
5) M. J. Ablowitz, A. Ramani and H. Segur: "a connection between nonlinear evolution equations and ordinary differential equations", J. Math. Phys., **21** (1980) 715-721
6) H. Flaschka and A. C. Newell: Commun. Math. Phys. **76** (1980) 67
7) M. Sato: RIMS Koukyuroku (Kyoto Univ.) **439** (1981) 30
8) V. Zakharov and A. B. Shabat: Func. Anal. Appl., **8** (1974) 43
9) M. J. Ablowitz: "Painleve Equations and the inverse scattering monodromy transforms", private communications.

Appendix-A Partial Differential Equation and Characteristic Coordinates

We consider the **PDE** as

$$z=F\left(x,\frac{\partial z}{\partial x}\right)+G\left(y,\frac{\partial z}{\partial y}\right). \tag{A.1}$$

The complete solution containing two arbitrary constants is constructed by

$$z\equiv f(x,a)+g(y,b), \tag{A.2}$$

where $f(x,a)$ and $g(y,b)$ are solutions of ordinary differential equations,

$$X=F\left(x,\frac{\partial X}{\partial x}\right),\quad Y=G\left(y,\frac{\partial Y}{\partial y}\right), \tag{A.3}$$

respectively. On the other hand, the general solution must contain arbitrary functions. To get the general solution $z=f(x,y)$, we use the equation

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z), \tag{A.4}$$

which has a form solved as to the differentials of z. We can denote the solution as the 1-parameter family of t,

$$x=\phi(t),\quad y=\psi(t),\quad z=\varkappa(t).$$

Then $z=f(x,y)$ is understood a curve on which $z=\varkappa(t)$ crosses a sylinder $\{(\phi(t),\ \psi(t)|t\in \boldsymbol{R}\}$. This means that a point on the solution surface moves as t. Hence the variations of z : $\mathrm{d}z=f_x\mathrm{d}x+f_y\mathrm{d}y$ is obtained by the total deribative as to t, we get

$$\frac{\mathrm{d}z}{\mathrm{d}t}=f_x\frac{\mathrm{d}x}{\mathrm{d}t}+f_y\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{A.5}$$

Comparing this with eq. (A.4), we obtain

$$R(x,y,z)=\frac{\mathrm{d}z}{\mathrm{d}t},\quad P(x,y,z)=\frac{\mathrm{d}x}{\mathrm{d}t},Q\ (x,y,z)=\frac{\mathrm{d}y}{\mathrm{d}t},$$

or equivalentry

$$\frac{\mathrm{d}x}{P(x,y,z)}=\frac{\mathrm{d}y}{Q(x,y,z)}=\frac{\mathrm{d}z}{R(x,y,z)}, \tag{A.6}$$

which is called as the characteristic equations of (A.4). If we denote two solutions of (A.6) as

$$u(x,y,z)=c_1,\quad v(x,y,z)=c_2, \tag{A.7}$$

the general solution is given by

$$F(u, v)=0, \tag{A.8}$$

where $\boldsymbol{F}$ is an arbitrary function.