Dressing Method for
The Schrödinger Eigenvalue Problem

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The Riemann-Hilbert problem (**RHP**) plays a key role on the invers scattering transform (**IST**), by which many types of nonlinear evolution equations (**NLEE's**) can be solved. The **IST** of Schrödinger cperator is reviewed in some content, that is, a vector formalism is given. Considerations of the reviews enabled us to derive the **dressing method** (originally developed by Zakharov and Shabat) which is also powerful to solve the **NLEE's**. Connections between both methods are made clear. We find that a Schrödinger problem results in the dressing method and the Gel'fand-Levitan type of integral equations (**GLE**) is also derived with the same spectral function as the one of the **IST**. This fact means to define the scattering data still for the dressing method.

§1. Introduction

The **RHP**[1] plays a key role for developing of the **IST**[2] and various studies had been reported by us. In this section we review the **IST** for the case of the Schrödinger type of eigenvalue problem (**S-EVP**).

The **S-EVP** is given by

$$\mathfrak{L}(\partial_x, q)|u> \equiv \{-\partial_x^2 + q(x,t)\}|u> = \xi^2|u>, \tag{1.1}$$

where the italic letter means a differential operator, $|u>$ is an eigen function of the column vector and a spectral parameter ξ is taken as real (=Re. λ), but will be later extended to a complex λ. Throughout this issue the potential $q(x)$ is assummed to be vanishing rapidely as $x \to \pm\infty$. The general solution $|u>$ of (1.1) is given by two linearly independent solutions $|\varphi_1>$ and $|\varphi_2>$,

$$|u> = -c_2|\varphi_1> + c_1|\varphi_2> \equiv <c|\varphi>, \tag{1.2a}$$

where bra-ket notations are defined by $|\varphi> \equiv (\varphi_1, \varphi_2)^T$, $|c> \equiv (c_1, c_2)^T$ and $<c| \equiv (-c_2, c_1)$, clearly satisfying $<c|c>=0$ and $<c|c'> = -<c'|c>$. Since $q(x)$ is rapidly vanishing, the solution $|u(\xi, x)>$ is generally determined by specifying its far fields as the linear combination of two principal vacuums ($e^{\pm i\xi x}$) as

$$|u(\xi,x)> \to -c_2^{\pm}|1>e^{-i\xi x} + c_1^{\pm}|2>e^{i\xi x}$$
$$= <c^{\pm}(\xi)|J(\xi x)\begin{pmatrix}1\\1\end{pmatrix} \text{ as } x \to \pm\infty, \qquad (1.2\text{b})$$

where $c_{1,2}^{\pm}$ are scattering amplitudes, the vacuum is defined by $J(\xi x)\begin{pmatrix}1\\1\end{pmatrix}\{J(\xi x) \equiv \exp(-i\sigma_3 \xi x)\}$ and superscripts $^{(\pm)}$ are ordered as $x \to \pm\infty$. The scattering matrix $S(\xi)$ is connected with the scattering amplitudes $<c^{\pm}(\xi)|$ as

$$<c^{+}(\xi)| = <c^{-}(\xi)|S(\xi). \qquad (1.3)$$

Correspnding to the vaccume $J(\xi x)\begin{pmatrix}1\\1\end{pmatrix}$, Jost functions $\{=|\varphi^{\pm}(\xi,x)>\}$ are defined by

$$\mathfrak{L}(\partial_x, q)|\varphi^{\pm}(\xi,x)> = \xi^2|\varphi^{\pm}(\xi,x)>, \qquad (1.4\text{a})$$
$$|\varphi^{\pm}(\xi,x)> \to J(\xi x)\begin{pmatrix}1\\1\end{pmatrix} \text{ as } x \to \pm\infty, \qquad (1.4\text{b})$$

where $\mathfrak{L}(\partial_x, q) \equiv -\partial_x^2 + q(x)$, while the scattering matrix $S(\xi\,;t)$ is introduced by

$$|\varphi^{-}(\xi,x)> = S(\xi)|\varphi^{+}(\xi,x)>, \qquad (1.5)$$

Since det $A = <a_1|a_2>$ for a matrix $A = [\boldsymbol{a}_1, \boldsymbol{a}_2]$, the Wronskian of $|\varphi^{\pm}(\xi,x)>$ satisfying (1.4) is given by

$$W_r^{\pm}(\xi) \equiv \det[\boldsymbol{\varphi}^{\pm}, \boldsymbol{\varphi}_x^{\pm}] = <\varphi^{\pm}|\varphi_x^{\pm}> = 2i\xi. \qquad (1.6)$$

Because of (1.5) and its x-derivative, we easily find det $S(\xi) = 1$.

§2. Inverse Spectral Transform

We derive the **IST** by means of the **RHP** and the **GLE** by which the solutions of **NLEE's** can be solved. For developing the **IST** it is always necessary to make clear analyticities of Jost functions and diagonal entries of the scattering matrix. This can be accomplished by rewriting (1.4) to its equivalent integral equations,

$$|\varphi^{\pm}(\xi,x)> = J(\xi x)\begin{pmatrix}1\\1\end{pmatrix} + \int_{\pm\infty}^{x} \frac{\sin\xi\,(x-y)}{\xi} q(y)|\varphi^{\pm}(\xi,y)>\mathrm{d}y \qquad (2.1\text{a})$$

The type of function $|\varphi>$ is often exchanged by $|\psi>\{\equiv J(-\xi x)|\varphi>\}$, then instead of (2.1a) we get

$$|\psi_1^{\pm}(\xi,x)> = \begin{pmatrix}1\\1\end{pmatrix} + \int_{\pm\infty}^{x} \frac{e^{2i\xi(x-y)}-1}{2i\xi} q(y)|\psi_1^{\pm}(\xi,y)>\mathrm{d}y, \text{ etc..} \qquad (2.1\text{b})$$

By virture of Neumann series expansions, we obtain the theorem.
[Theorem. 1] "functions $\{|\psi_1^{-}(\lambda,x)>, |\psi_2^{+}(\lambda,x)>, s_{11}(\lambda)\}$ are analytic on the upper λ-plane, while the lower plane allows $\{|\psi_1^{+}(\lambda,x)>, |\psi_2^{-}(\lambda,x)>, s_{22}(\lambda)\}$."
[Theorem. 2] On each domain of analiticities, both sets of functions, $\{|\psi_1^{-}>, |\psi_2^{+}>, s_{11}\}$ and $\{|\psi_1^{+}, |\psi_2^{-}>, s_{22}\}$ behave as

$$|\psi^{\pm}>\to\begin{pmatrix}1\\1\end{pmatrix},\ |\psi^{\pm}_x>\to 0,\ s_{11}\to 1 \tag{2.2}$$

around at $|\lambda|=\infty$.

To make clear the analytic region, we define such new types of vectors as $|\psi^P>=|\psi_1^-,\ \psi_2^+>$ and $|\psi^N>=|\psi_1^+,\ \psi_2^->$, where superscripts "P, N" represent that Im. λ is positive and negative, respectively. While, in far regions of x, these approach to the triangular states listed in

$$|\psi^P(\lambda,x)>\to D_U^P(\lambda)J(-\lambda x)S_U^P(\lambda)J(\lambda x)\begin{pmatrix}1\\1\end{pmatrix} \text{ as } x\to\ \infty,$$
$$\to D_L^P(\lambda)J(-\lambda x)S_L^P(\lambda)J(\lambda x)\begin{pmatrix}1\\1\end{pmatrix} \text{ as } x\to-\infty, \tag{2.3 a}$$
$$|\psi^N(\lambda,x)>\to D_L^N(\lambda)J(-\lambda x)S_L^N(\lambda)J(\lambda x)\begin{pmatrix}1\\1\end{pmatrix} \text{ as } x\to\ \infty,$$
$$\to D_U^N(\lambda)J(-\lambda x)S_U^N(\lambda)J(\lambda x)\begin{pmatrix}1\\1\end{pmatrix} \text{ as } x\to-\infty, \tag{2.3 b}$$

Matrices $D_{L,U}^{P,N}$ are diagonal,

$$D_N^P(\lambda)=\begin{bmatrix}s_{11}, & 0\\ 0, & 1\end{bmatrix}=\{D_L^P(\lambda)\}^{\dagger},\ D_L^N(\lambda)=\begin{bmatrix}1, & 0\\ 0, & s_{22}\end{bmatrix}=\{D_U^N(\lambda)\}^{\dagger},$$

while $S_{L,U}^{P,N}$ are *strongly* triangular (its diagonal entries are unity),

$$S_U^P(\lambda)=\begin{bmatrix}1, & \rho_+^P\\ 0, & 1\end{bmatrix},\ S_L^P(\lambda)=\begin{bmatrix}1 & , 0\\ -\rho_-^P, & 1\end{bmatrix},$$
$$S_L^N(\lambda)=\begin{bmatrix}1, & 0\\ \rho_+^N, & 1\end{bmatrix},\ S_U^N(\lambda)=\begin{bmatrix}1, & -\rho_-^N\\ 0, & 1\end{bmatrix}, \tag{2.4 a}$$

and

$$\rho_+^P=\frac{s_{12}}{s_{11}},\ \rho_+^N=\frac{s_{21}}{s_{22}},\ \rho_-^P=\frac{s_{21}}{s_{11}},\ \rho_-^N=\frac{s_{12}}{s_{22}}. \tag{2.4 b}$$

Functions $\psi^{P,N}$ are further modified by diagonal matrices,

$$|\psi_+^P(\lambda,x)>=[D_U^P(\lambda)]^{-1}|\psi^P(\lambda,x)>,\ |\psi_-^P(\lambda,x)>=[D_L^P(\lambda)]^{-1}|\psi^P(\lambda,x)>,$$
$$|\psi_+^N(\lambda,x)>=[D_L^N(\lambda)]^{-1}|\psi^N(\lambda,x)>,\ |\psi_-^N(\lambda,x)>=[D_U^N(\lambda)]^{-1}|\psi^N(\lambda,x)>. \tag{2.5}$$

From (2.2) its asymptotic behaviours are given by

$$|\psi_\pm^{P,N}(\lambda,x)>=\begin{pmatrix}1\\1\end{pmatrix}+O(1/\lambda) \tag{2.6}$$

on each domains of analyticicities. If piecewice analytic functions

$$|\Psi^{\pm}(\lambda)>=\begin{cases}|\psi_\pm^P(\lambda,x)> & (\text{Im. } \lambda>0),\\ |\psi_\pm^N(\lambda,x)> & (\text{Im. } \lambda<0),\end{cases} \tag{2.7}$$

are introduced, we can define the following **RHP's**,

$$|\Psi^{\pm}(\xi+i0,x)>-|\Psi^{\pm}(\xi-i0,x)>=\Omega^{\pm}(\xi\,;x,x)|\psi^{\pm}(\xi,x)>, \tag{2.8}$$

where

$$\Omega^{\pm}(\xi\,;x,y)=\begin{bmatrix}0, & \rho_\pm(\xi)e^{i\xi(x+y)}\\ -\rho_\pm(\xi)e^{-i\xi(x+y)}, & 0\end{bmatrix}=\Omega^{\pm}(\xi\,;x+y) \tag{2.9}$$

and both reflectional coefficients ρ_+^P {or ρ_-^N} and ρ_+^N {or ρ_-^P} must be located in (1.2) and (2.1), respectively. Following to Plemelii's formula, we can get

$$|\Psi^{\pm}(\lambda,x)>=\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\lambda}\cdot\Omega^{\pm}(\xi\,;x,x)|\psi^{\pm}(\xi,x)>. \qquad (2.10)$$

It is possible to introduce the vectors independent on ξ,

$$|K_{\pm}^{P}(x,y)>=\frac{1}{2\pi}\int_{-\infty}^{\infty}J[\xi(x-y)]\{|\Psi^{\pm}(\xi+i0,x)>-\begin{pmatrix}1\\1\end{pmatrix}\}d\xi, \qquad (2.11a)$$

$$|K_{\pm}^{N}(x,y)>=\frac{1}{2\pi}\int_{-\infty}^{\infty}J[\xi(x-y)]\{|\Psi^{\pm}(\xi-i0,x)>-\begin{pmatrix}1\\1\end{pmatrix}\}d\xi; \qquad (2.11b)$$

which are called "kernels" and independent on λ. As shown in Appendix-A, substitution of (2.10) into (2.11) and deformations of paths results in

$$|K_{\pm}^{P}(x,y)>=\begin{bmatrix}\theta(x-y),\ 0\\ 0\ ,\theta(y-x)\end{bmatrix}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Omega^{\pm}(\xi\,;y,x)|\psi^{\pm}(\xi,x)>d\xi, \qquad (2.12a)$$

$$|K_{\pm}^{N}(x,y)>=-\begin{bmatrix}\theta(y-x),\ 0\\ 0\ ,\theta(x-y)\end{bmatrix}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Omega^{\pm}(\xi\,;y,x)|\psi^{\pm}(\xi,x)>d\xi. \qquad (2.12b)$$

These kernels are on suppoprts of half line,

$$<1|K_{\pm}^{P}(x,y) \text{ and } <2|K_{\pm}^{N}(x,y) \sim \theta(x-y),$$
$$<1|K_{\pm}^{N}(x,y) \text{ and } <2|K_{\pm}^{P}(x,y) \sim \theta(y-x). \qquad (2.12)$$

For convenience of later use, we define new type of kernel vectors as $|K^{\pm}(x,y)> \simeq \theta[\pm(y-x)]$, where

$$|K^{+}>=[<1|K_{+}^{N},\ <2|K_{+}^{P}]^{T},\ |K^{-}>=[<1|K_{-}^{P},\ <2|K_{-}^{N}]^{T}. \qquad (2.13)$$

In a sence of (2.11) we see correspondence between kernel and Jost functions. Actually the corresponding Jost functions with (2.13) are given by

$$|\psi^{+}>\equiv[<1|\psi_{+}^{N},\ <2|\psi_{+}^{P}]^{T},\ |\psi^{-}>\equiv[<1|\psi_{-}^{P},\ <2|\psi_{-}^{N}]^{T}, \qquad (2.14)$$

For these $|K^{\pm}(x,y)>$, we can arrange (2.11) to

$$|K^{\pm}(x,y)>=\frac{1}{2\pi}\int_{-\infty}^{\infty}J[\xi(x-y)]\{|\psi^{\pm}(\xi,x)>-\begin{pmatrix}1\\1\end{pmatrix}\}d\xi. \qquad (2.15a)$$

with its inverted formula,

$$|\psi^{\pm}(\lambda,x)>=\begin{pmatrix}1\\1\end{pmatrix}-(\pm)\int_{\pm\infty}^{x}J[\lambda(y-x)]|K^{\pm}(x,y)>dy. \qquad (2.15b)$$

While eqs. (2.12) are reduced to

$$|K^{\pm}(x,y)>=-(\pm)\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma_3\Omega^{\pm}(\xi\,;y,x)|\,\psi^{\pm}(\xi,x)>d\xi. \qquad (2.16)$$

From (2.15) and (2.16) we can easily eliminate Jost functions,

$$|K^{\pm}(x,y)>=-(\pm)\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma_3\Omega^{\pm}(\xi\,;y,x)d\xi\cdot\begin{pmatrix}1\\1\end{pmatrix}$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma_3\Omega^{\pm}(\xi\,;y,x)d\xi\int_{\pm\infty}^{x}J[\xi(z-x)]|K^{\pm}(x,z)>dz.$$

This is just the **GLE**,

$$|K^{\pm}(x,y)>\pm F^{\pm}(x,y)\binom{1}{1}=\int_{\pm\infty}^{x}F^{\pm}(z,y)|K^{\pm}(x,z)>\mathrm{d}z, \tag{2.17}$$

where $F^{\pm}(x,y)$ are off-diagonal spectral functions,

$$F^{\pm}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma_3\Omega^{\pm}(\xi;y,x)\mathrm{d}\xi, \tag{2.18}$$

The fact that (2.15b) must satisfy the **S-EVP** enables the kernels to satisfy

$$\frac{\partial^2 K^{\pm}(x,y)}{\partial^2 x}-\frac{\partial^2 K^{\pm}(x,y)}{\partial^2 y}=q(x)K^{\pm}(x,y), \tag{2.19a}$$

$$q(x)=2\frac{\mathrm{d}}{\mathrm{d}x}K^{\pm}(x,x). \tag{2.19b}$$

as shown in **Appendix B**. These define Cauchy problem, and the uniquness of solution is already known. Specially (2.19b) recovers the potential once the **GLE** is solved as to kernels.

§3. Integral Operators for The IST

We consider to develop the dressing method for the Shrödinger operator. The scalar case was already treated by Zakharov and Schabat,[3] but in our case their formula must be modified to the vector formula. We remark that the modification is not trivial. It is a key to give a mapping from the vacuum to nontrivial states. Considering (2.15b), we define

$$[\boldsymbol{\Phi}_0\boldsymbol{F}](x)=\boldsymbol{\Phi}_0(\xi,y)\boldsymbol{F}(y;x)=\int_{-\infty}^{\infty}\boldsymbol{\Phi}_0(\xi,y)F(y;x)\mathrm{d}y, \tag{3.1a}$$

$$[\boldsymbol{\Phi}_0|\boldsymbol{W}^{\pm}>](x)=\boldsymbol{\Phi}_0(\xi,y)\binom{1}{1}-(\pm)\int_{\pm\infty}^{x}\boldsymbol{\Phi}_0(\xi,y)\,|K^{\pm}(y;x)>\mathrm{d}y, \tag{3.1b}$$

wher $\boldsymbol{\Phi}_0$ is a vacuum (matrix) and $\boldsymbol{F}$ is a Fredholm type of matrix operators while $|\boldsymbol{K}^{\pm}>$ $\{|\boldsymbol{W}^{\pm}>\equiv\mathbf{1}\binom{1}{1}+|\boldsymbol{K}^{\pm}>\}$ are Volterra type one. We introduce the following relation,

$$|\boldsymbol{W}^{-}(\xi,x)>=\{\mathbf{1}+\boldsymbol{F}(y,x)\}|\boldsymbol{W}^{+}(\xi,x)>, \tag{3.2}$$

which can be reduced to the **GLE** as shown in the following.

For convenience we exchange both kernels $K^{\pm}(x,y)$

$$K^{\pm}(y;x)=K^{\pm}(y;x)\theta[\pm(y-x)], \tag{3.3}$$

by which the Volterra integrals are represented by

$$[\boldsymbol{\Phi}_0|\boldsymbol{K}^{\pm}>](x)\equiv\int_{-\infty}^{\infty}\boldsymbol{\Phi}_0(\xi,y)|K^{\pm}(y;x)>\theta[\pm(y-x)]\mathrm{d}y. \tag{3.4}$$

On the other hand (3.2) is reduced to $|\boldsymbol{K}^{-}>=\boldsymbol{F}\binom{1}{1}+|\boldsymbol{K}^{+}>+\boldsymbol{F}|\boldsymbol{K}^{+}>$. To this relation, we operate the matrix function $\boldsymbol{\Phi}_0$ from the left side, then for $z>x$ we obtain such a **GLE** as

$$F(z;x)\binom{1}{1}+|K^{+}(z;x)>+\int_{x}^{\infty}F(z;y)|K^{+}(y;x)>\mathrm{d}y=0. \tag{3.5}$$

This is exactly same with (2.17), but we can not find any imformations for the spectrum function $F(z;x)$. While the spectral function F^{+} in (2.18) include $\Omega^{+}(\xi;x,x)$ which consists of reflectional coefficients (scatteringn data). It is natural to connects the spectral function in (3.5) with the one of the **IST**.

We remember the treatments of §2 and arrange them by Volterra operators, which generates $\psi^{\pm}(\xi,x)$ from the vacuum state $J(\xi x)$,

$$\psi^{\pm}(x)=J[\xi(y-x)]\{\mathbf{1}\binom{1}{1}+|\boldsymbol{K}^{\pm}(y;x)>\}$$

$$=\binom{1}{1}-(\pm)\int_{\pm\infty}^{x}J[\xi(x-y)]K^{\pm}(y;x)\mathrm{d}y. \qquad (3.6)$$

Let's consider the following vectors,

$$\psi_0^{-}=[<1|\psi_{+}^{P},\ <2|\psi_{+}^{N}]^{T}=\begin{bmatrix} s_{11}, & 0 \\ 0, & s_{22}\end{bmatrix}^{-1}\psi^{-},$$

$$\psi_0^{+}=[<1|\psi_{-}^{N},\ <2|\psi_{-}^{P}]^{T}=\begin{bmatrix} s_{22}, & 0 \\ 0, & s_{11}\end{bmatrix}^{-1}\psi^{+}. \qquad (3.7)$$

We note that both vectors $[<1|\psi_{-}^{N},\ <2|\psi_{-}^{P}]^{T}$ and $[<1|\psi_{+}^{P},\ <2|\psi_{+}^{N}]^{T}$ are different from the one used in (2.14). Corresponding to $\psi_0^{\pm}$, new kernels

$$K_0^{+}=[<1|K_{-}^{N},\ <2|K_{-}^{P}],\ K_0^{-}=[<1|K_{+}^{P},\ <2|K_{+}^{N}], \qquad (3.8)$$

are introduced on supports $\theta(y-x)$ and $\theta(x-y)$, respectively. Now reconstruction formulae obtained from (2.11),

$$|\psi_{+}^{P}(\lambda,x)>=\binom{1}{1}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi-\lambda}\Omega^{+}(\xi;x,x)|\psi^{+}(\xi,x)>, \qquad (3.9a)$$

$$|\psi_{+}^{N}(\lambda,x)>=\binom{1}{1}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi-\lambda}\Omega^{+}(\xi;x,x)|\psi^{+}(\xi,x)>, \qquad (3.9b)$$

are represented at $\lambda=\xi$ as

$$|\psi_{+}^{P}(\xi,x)>=\binom{1}{1}+\frac{1}{2}\Omega^{+}(\xi;x,x)|\psi^{+}(\xi,x)>$$

$$+\frac{1}{2\pi i}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi'-\xi}\Omega^{+}(\xi';x,x)|\psi^{+}(\xi',x)>, \qquad (3.10a)$$

$$|\psi_{+}^{N}(\xi,x)>=\binom{1}{1}-\frac{1}{2}\Omega^{+}(\xi;x,x)|\psi^{+}(\xi,x)>$$

$$+\frac{1}{2\pi i}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi'-\xi}\Omega^{+}(\xi';x,x)|\psi^{+}(\xi',x)>, \qquad (3.10b)$$

where P.V. means the Cauchy principal value. It can be also summarized as

$$|\psi_0^{-}(\xi,x)>=\binom{1}{1}+\frac{1}{2}\sigma_3\Omega^{+}(\xi;x,x)|\psi^{+}(\xi,x)>$$

$$+\frac{1}{2\pi i}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi'-\xi}\sigma_3\Omega^{+}(\xi';x,x)|\psi^{+}(\xi',x)>. \qquad (3.11)$$

On the contrary to these we can see

$$|\psi^{+}(\lambda,x)>=\binom{1}{1}-\frac{1}{2}\sigma_3\Omega^{+}(\xi;x,x)|\psi^{+}(\xi,x)>$$

$$+\frac{1}{2\pi i}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi'}{\xi'-\xi}\sigma_3\Omega^{+}(\xi';x,x)|\psi^{+}(\xi',x)>. \qquad (3.12)$$

Eliminating the principal value integral from (3.11) and (3.12), we can obtain

$$|\psi_0^-(\xi,x)\rangle=\{E+\sigma_3\Omega^+(\xi;x,x)\}|\psi^+(\xi,x)\rangle. \tag{3.13}$$

Since $|\psi_0^-(\xi,x)\rangle$ also satisfies the **S-EVP**, it has also a kernel and similarly to (3.6) we define it as

$$|\psi_0^-(\xi,x)\rangle=J[\xi(y-x)]\{\mathbf{1}\begin{pmatrix}1\\1\end{pmatrix}+|\boldsymbol{K}_0^-(y;x)\rangle\}. \tag{3.14}$$

Both formulae (3.6) and (3.14) reduce (3.13) to

$$J(\xi y)\{\mathbf{1}\begin{pmatrix}1\\1\end{pmatrix}+|\boldsymbol{K}_0^-(y;x)\rangle\}$$

$$=\{E+\sigma_3\Omega^+(\xi;0,0)\}J(\xi y)\{\mathbf{1}\begin{pmatrix}1\\1\end{pmatrix}+|\boldsymbol{K}^+(y;x)\rangle\}, \tag{3.15}$$

which can be represented by

$$|\boldsymbol{W}_0^-(y;x)\rangle=\{\mathbf{1}+\boldsymbol{F}(y;z)\}|\boldsymbol{W}^+(z;x)\rangle. \tag{3.16}$$

The relation (3.10) really determin $\boldsymbol{F}$ as follows,

$$\sigma_3\Omega^+(\xi;0,0)J(\xi y)=J(\xi z)\boldsymbol{F}(z;y). \tag{3.17}$$

The kernel F of operator $\boldsymbol{F}$ satisfies

$$\int_{-\infty}^{\infty}J[\xi(z-x)]F(z;y)\mathrm{d}z=\sigma_3\Omega^+(\xi;x,y). \tag{3.18}$$

The RHS is a Fourier integral and we can invert it as

$$F(x;y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma_3\Omega^+(\xi;x,y)\mathrm{d}\xi. \tag{3.19}$$

This relation is basically derived from two kinds of Jost functions $|\psi_0^-(\lambda,x)\rangle$, $|\psi^+(\lambda,x)\rangle$ and from operator relation (3.16). It is easily seen that another type of relations can exist.

§4. Dressing Method for The Schrödinger Eigenvalue Problem

In §3 we found an operator relation (3.16) consisting of both Fredholm and Volterra types of operators. In this section we remark that this is not only important but also is necessary to obtain the integrable conditions. We take a scalar differential operator $\mathfrak{L}(\partial,q)$ which is redefined acting to the left term side as $\varphi\to\varphi\mathfrak{L}$, while the matrix still acts to the right. The Schrödinger equation is represented by

$$|\varphi^\pm(\xi,x)\rangle\mathfrak{L}(\partial,q)=\xi^2|\varphi^\pm(\xi,x)\rangle, \tag{4.1 a}$$

$$J(\xi x)\mathfrak{L}_0(\partial)=\xi^2 J(\xi x), \tag{4.1 a}$$

where $\mathfrak{L}_0$ is simply set as a trivial differential operator ($\equiv-\partial^2$). Discussions of §3 suggests us to redefine Volterra operators,

$$|\varphi^\pm(\xi,x)\rangle=J(\xi y)\boldsymbol{W}^\pm(y;x)\begin{pmatrix}1\\1\end{pmatrix},\ (\boldsymbol{W}^\pm=\mathbf{1}+\boldsymbol{K}^\pm), \tag{4.2}$$

because $|\boldsymbol{W}^\pm\rangle=\boldsymbol{W}^\pm\begin{pmatrix}1\\1\end{pmatrix}$. However, these basically represent scalar relations, then we can take $\boldsymbol{W}^{\pm}$ diagonal. Substituting (4.2) into (4.1a), we obtain

$$J(\xi y)\boldsymbol{W}^{\pm}(y;x)\mathfrak{L}(\partial,q)=J(\xi y)\boldsymbol{W}^{\pm}(y;x)\xi^{2}=J(\xi y)\mathfrak{L}_{0}(\partial)\boldsymbol{W}^{\pm}(y;x).$$

Eliminating $J(\xi y)$ from this, we get

$$\boldsymbol{W}^{\pm}(y;x)\mathfrak{L}(\partial,q)=\mathfrak{L}_{0}(\partial)\boldsymbol{W}^{\pm}(y;x). \tag{4.3}$$

The nontrivial operator $\mathfrak{L}^{(m)}(\partial,q)$ can be generated because the scalar operator $\boldsymbol{W}^{\pm}(\equiv|1>\boldsymbol{W}_{1}^{\pm}<1|+|2>\boldsymbol{W}_{2}^{\pm}<2|)$ can be inverted. We denote this as

$$[\boldsymbol{W}^{\pm}]^{\dagger}\boldsymbol{W}^{\pm}=\boldsymbol{W}^{\pm}[\boldsymbol{W}^{\pm}]^{\dagger}=\mathbf{1}, \tag{4.4}$$

then $[\boldsymbol{K}^{\pm},[\boldsymbol{K}^{\pm}]^{\dagger}]=0$. These facts are generalized to the case of m-th differential operator $\mathfrak{A}_{0}^{(m)}$ (is set as ∂^{m} for simplicity) and we reduces (4.3) to

$$\mathfrak{A}^{(m)}(\partial,q)=[\boldsymbol{W}^{\pm}]^{\dagger}\mathfrak{A}_{0}^{(m)}\boldsymbol{W}^{\pm}, \tag{4.5}$$

which generates a nontrivial operator denoted as

$$\mathfrak{A}^{(m)}(\partial,q)=\partial^{m}+\sum_{k=1}^{m}q_{k}^{(m)}(x)\partial^{m-k}. \tag{4.6}$$

The function $q_{k}(x)$ is called as the potential. All of these potentials can be determined recursively if both orders of $\mathfrak{A}^{(m)}$ and $\mathfrak{A}_{0}^{(m)}$ are equal. On the other hand, from (4.5) we obtain

$$[\mathfrak{A}_{0}^{(m)},\ \boldsymbol{F}^{\pm}]=0, \tag{4.7}$$

where

$$\mathbf{1}+\boldsymbol{F}^{+}=\boldsymbol{W}^{+}[\boldsymbol{W}^{-}]^{\dagger},\ \mathbf{1}+\boldsymbol{F}^{-}=\boldsymbol{W}^{-}[\boldsymbol{W}^{+}]^{\dagger}. \tag{4.8}$$

We assume two trivial differential operators which satisfy

$$[\mathfrak{A}_{0}^{(m)},\ \mathfrak{A}_{0}^{(n)}]=0, \tag{4.9}$$

from which the integrable condition is derived. From the left and right sides we operate $[\boldsymbol{W}^{\pm}]^{\dagger}$ and $\boldsymbol{W}^{\pm}$ to (4.9) then obtain

$$\begin{aligned}0&=[\boldsymbol{W}^{\pm}]^{\dagger}[\mathfrak{A}_{0}^{(m)},\ \mathfrak{A}_{0}^{(n)}]\boldsymbol{W}^{\pm}\\&=[[\boldsymbol{W}^{\pm}]^{\dagger}\mathfrak{A}_{0}^{(m)}\boldsymbol{W}^{\pm},\ [\boldsymbol{W}^{\pm}]^{\dagger}\mathfrak{A}_{0}^{(n)}\boldsymbol{W}^{\pm}].\end{aligned}$$

That is an integrable condition,

$$[\mathfrak{A}^{(m)},\ \mathfrak{A}^{(n)}]=0. \tag{4.10}$$

To derive the **NLEE's**, we must extend $\mathfrak{A}_{0}^{(m)}$ to $\mathfrak{L}_{0}^{(m)}$ consisting of derivatives as to other variables t_{k},

$$\mathfrak{L}_{0}^{(m)}=\sum_{k}\alpha_{k}\frac{\partial}{\partial t_{k}}+\mathfrak{A}_{0}^{(m)}(\partial,q), \tag{4.11a}$$

where α_{k} is constant. Similarly to (4.7) and (4.10) the **NLEE's** can be obtained from

$$[\mathfrak{L}^{(m)},\ \mathfrak{L}^{(n)}]=0. \tag{4.12}$$

For the case of **KdV** equation, $\mathfrak{L}^{(2)}=\mathfrak{A}^{(2)}$, $\mathfrak{L}^{(3)}=\frac{1}{4}\partial_t+\mathfrak{A}^{(3)}$ where

$$\mathfrak{A}^{(2)}(=-\mathfrak{L})=\partial^2-q(x,t),\ \mathfrak{A}^{(3)}=\partial^3-\frac{3}{2}q(x,t)-\frac{3}{4}q_x. \tag{4.11 b}$$

The t-dependent **EVP** $|\varphi_j^{\pm}>\mathfrak{L}=|\varphi_j^{\pm}>\xi^2$ is again introduced by setting $\mathfrak{L}\equiv-\mathfrak{L}^{(2)}$ and $\varphi^{\pm}(\xi,x|t)$ $=J(\xi y)\boldsymbol{W}(y;x|t)\begin{pmatrix}1\\1\end{pmatrix}$. Also from (1.5) the S-matrix is given by

$$J(\xi y)\boldsymbol{W}^{-}(y;x)=S(\xi,t)J(\xi y)\boldsymbol{W}^{+}(y;x). \tag{4.13}$$

by which the well-known relation $S_t=4i\xi^3[\sigma_3,S]$ must be obtained. If the integrable condition (3.11) acts on $|\varphi^{\pm}>$, we see from (4.12) that $H(\xi)|\varphi^{\pm}>$ and $|\varphi^{\pm}>\mathfrak{L}^{(3)}$ are eigenfunctions of $\mathfrak{L}$,

$$|\varphi^{\pm}>\mathfrak{L}^{(3)}\mathfrak{L}=|\varphi^{\pm}>\mathfrak{L}^{(3)}\xi^2. \tag{4.14}$$

We may take $H(\xi)$ as constant and diagonal, and impose

$$|\varphi^{\pm}>\mathfrak{L}^{(3)}+H|\varphi^{\pm}>\rightarrow J(\xi x)\begin{pmatrix}1\\1\end{pmatrix}\ \text{as}\ x\rightarrow\pm\infty.$$

That is, $H(\xi)=E-i\xi^3\sigma_3$ and $|\varphi^{\pm}>\mathfrak{L}^{(3)}=i\xi^3\sigma_3|\varphi^{\pm}>$. Since $|\varphi^{\pm}>=J(\xi y)|\boldsymbol{W}^{\pm}(y;x|t)>$ and $[J,H]=0$, we can find

$$\frac{1}{4}\boldsymbol{W}_t^{\pm}=\partial^3\boldsymbol{W}^{\pm}-\boldsymbol{W}^{\pm}\mathfrak{A}^{(3)}. \tag{4.15}$$

On the other hand. The t-derivative of (4.8) and (4.15) give us

$$\boldsymbol{F}_t^{-}=\boldsymbol{W}_t^{-}[\boldsymbol{W}^{-}]^{\dagger}(\boldsymbol{1}+\boldsymbol{F}^{-})-(\boldsymbol{1}+\boldsymbol{F}^{-})\boldsymbol{W}_t^{+}[\boldsymbol{W}^{-}]^{\dagger},$$
$$\boldsymbol{W}_t^{\pm}[\boldsymbol{W}^{\pm}]^{\dagger}=4\mathfrak{A}_0^{(3)}-4\boldsymbol{W}^{\pm}\mathfrak{A}^3[\boldsymbol{W}^{\pm}]^{\dagger}.$$

Then also from (4.7) we get

$$\boldsymbol{F}_t^{\pm}=4[\mathfrak{A}_0^{(m)},\boldsymbol{F}^{\pm}],\ [\partial^2,\boldsymbol{F}^{\pm}]=0, \tag{4.16}$$

by which the relations are obtained,

$$\frac{\partial F^{\pm}}{\partial t}+4\left(\frac{\partial^3F^{\pm}}{\partial y^3}+\frac{\partial^2F^{\pm}}{\partial x^3}\right)=0,\ \frac{\partial^2F^{\pm}}{\partial y^2}-\frac{\partial^2F^{\pm}}{\partial x^2}=0. \tag{4.17}$$

If the coordinates are transformed to $\mu=x+y$ and $\nu=x-y$,. we find that $F^{\pm}(y;x|t)=F^{\pm}(\mu|t)$ are only nontrivial and obtain

$$F(x+y|t)=\int_{-\infty}^{\infty}J[-\xi(x+y+8\xi^3t)]\widetilde{F}(\xi|0)d\xi, \tag{4.18}$$

where $\widetilde{F}(\xi|0)$ is determined by the initial condition. This gives a same structure as the spectral function in (2.18).

§5. Concluding Remarks

The usual dressing method[3)] does not give a correnspondence with the spectral function obtained in the **IST**, which was first studied in a case of the AKNS eigenvalue problem.[4] The case of **S-EVP** starts from definitions of Jost functions, and we find it still possible to choose a different set of Jost functions from the case of **IST**. This selection results in the dressing method, where we obtain the principal factorization formula of Fredholm operator (4.8), the integrable condition (4.12) and other necessary relations (4.7), (4.15), (4.16).

We are interested in the **S-EVP**, which is convenient to investigate the relation of the **IST** with the **Sato** theory,[5] which is useful to introduce the τ-function. It is our next plan how to solve the initial value problem of the Sato theory, in spite of that both **IST** and dressing method solve it.

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Appendix-A Complex Integrals

We consider integrals,

$$\mathrm{I}^{\pm}(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi'-\xi\pm i0}\cdot J(\xi z).$$

Instead of this, we consider its scalar type and obtain

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi'-\xi+i0}\cdot \mathrm{e}^{-i\xi z}=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi-(\xi'+i0)}\cdot \mathrm{e}^{-i\xi z}$$

$$=-\theta(-z)\mathrm{e}^{-i\xi' z},$$

by means of Jordan's theorem. This easily gives us

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi'-\xi+i0}\cdot J(\xi z)=-\begin{bmatrix}\theta(-z), & 0\\ 0, & \theta(z)\end{bmatrix}J(\xi' z). \tag{A.1}$$

We similarly obtain

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi'-\xi-i0}\cdot J(\xi z)=+\begin{bmatrix}\theta(z), & 0\\ 0, & \theta(-z)\end{bmatrix}J(\xi' z). \tag{A.2}$$

Appendix-B Kernel Representation of Jost Functions

We consider the relation in eq. (2.16),

$$\varphi^{\pm}(\xi,x)=J(\xi x)\begin{pmatrix}1\\1\end{pmatrix}-(\pm)\int_{\pm\infty}^{x}J(\xi y)K^{\pm}(x,y)\mathrm{d}y.$$

Its derivative $\varphi^{\pm}_{xx}$ is given by

$$\varphi^{\pm}_{xx}(\xi,x)=-\xi^2 J(\xi x)\begin{pmatrix}1\\1\end{pmatrix}-(\pm)\int_{\pm\infty}^{x} J(\xi y)\frac{\partial^2 K^{\pm}(x,y)}{\partial x^2}\mathrm{d}y$$

$$-(\pm)J(\xi x)\{\frac{\partial K^{\pm}(x,y)}{\partial x}|_{y=x}+\frac{\mathrm{d}}{\mathrm{d}x}K^{\pm}(x,x)-i\xi\sigma_3 K^{\pm}(x,x)\}, \qquad (\mathrm{B}.1)$$

while the term $(q(x)-\xi^2)\varphi^{\pm}$ is

$$(q(x)-\xi^2)\varphi^{\pm}=(q(x)-\xi^2)J(\xi x)\begin{pmatrix}1\\1\end{pmatrix}$$

$$-(\pm)\int_{\pm\infty}^{x} J(\xi y)\{q(x)K^{\pm}(x,y)+\frac{\partial^2 K^{\pm}(x,y)}{\partial y^2}\}\mathrm{d}y$$

$$-(\pm)\{J_x(\xi x)K^{\pm}(x,x)-J(\xi x)\frac{\partial K^{\pm}(x,y)}{\partial y}|_{y=x}\}. \qquad (\mathrm{B}.2)$$

For this derivation we had used $\xi^2 J(\xi y)=-J_{yy}(\xi y)$ and

$$\int_{\pm\infty}^{x} J(\xi y)\xi^2 K^{\pm}(x,y)\mathrm{d}y=-\int_{\pm\infty}^{x} J_{yy}(\xi y)K^{\pm}(x,y)\mathrm{d}y$$

$$=-\int_{\pm\infty}^{x} J(\xi y)\frac{\partial^2 K^{\pm}(x,y)}{\partial y^2}\mathrm{d}y-J_x(\xi x)K^{\pm}(x,x)+J(\xi x)\frac{\partial K^{\pm}(x,y)}{\partial y}.$$

From (B.1) and (B.2) we can obtain

$$(\pm)\int_{\pm\infty}^{x} J(\xi y)\{q(x)K^{\pm}(x,y)+\frac{\partial^2 K^{\pm}(x,y)}{\partial y^2}-\frac{\partial^2 K^{\pm}(x,y)}{\partial x^2}\}\mathrm{d}y$$

$$=q(x)J(\xi x)\begin{pmatrix}1\\1\end{pmatrix}+(\pm)J(\xi x)\{\frac{\partial K^{\pm}(x,y)}{\partial x}+\frac{\partial K^{\pm}(x,y)}{\partial y}+\frac{\mathrm{d}}{\mathrm{d}x}K^{\pm}(x,x)\}, \qquad (\mathrm{B}.3)$$

by which the Cauchy problems are obtained,

$$\frac{\partial^2 K^{\pm}(x,y)}{\partial y^2}-\frac{\partial^2 K^{\pm}(x,y)}{\partial x^2}+q(x)K^{\pm}(x,y)=0, \qquad (\mathrm{B}.4\mathrm{a})$$

$$q(x)\pm 2\frac{\mathrm{d}}{\mathrm{d}x}K^{\pm}(x,x)=0. \qquad (\mathrm{B}.4\mathrm{b})$$