Symmetric Approach for Integrable Nonlinear Eolution Equations in 1-Space and 1-Time Dimensions

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§1. Introduction

Since the famous investigation of the KdV equation by GGKM,[1] both existence of infinitely many conservation laws and the hierarchy of nonlinear evolution equations (NLLEs) had been understood as essential for a given NLEE to be integrable. This problem was studied by many authors, but we remark the contribution by Magri,[2,3] who had explained these properties from the view of geometrical point. Starting from symmetries (contravariant quantity), he introduced a potential operator (covariant quantity) and a sympletic operator which maps the covariant to the contravariant. The conseration laws were simply derived and he proposed a "bi-Hamiltonian structure" for integrable systems. Fuchsteiner[4,5] deeply considered symmetries and introduced both concepts of strong symmetries and hereditary symmetries. Fokas[6] had used a Lie-Backlund transformation and also arrived at the hereditary symmetry. Their idear was further developed and connections with the Backlund transformation and with the cannonical structures were made clear.[7,8] The iso-spectral problem is essential for the inverse spectral method (**ISM**) and its relation with the symmetric approach is very interesting. Such relations were first treated by Lax[9,10] for the case of KdV equation and extended to other cases.[11,12]

The motiation of this issue is to detail with the review of the symmetric approach. We specially consider a certain linear integro-differential operator $\mathbf{K}_\pm$ and make clear the role of squared eigenfunctions which is closely related with the 2×2-matrix isospectral problem.[13,14] We inspect that this operator is both strong and hereditary symmetries and make clear the associated cannonical structures. Considering that $\mathbf{K}_\pm$ are also obtained by the compatibility condition of NLLEs, we propose a direct and simple method for developing the hereditary symmetries with a $N\times N$-matrix formula.

§2. Infinitesimal Transformations

We denote the exactly solvable nonlinear equation as

$$\partial_t \mathbf{u} = \mathbf{N}[\mathbf{u}], \tag{2.1}$$

where **u** is on a certain configuration space while the bracket [•] is used to emphasize nonlinaear actions of the operator **N** on **u**. If the infinitesimal transformation,

$$\mathbf{u} \rightarrow \mathbf{u} + \varepsilon \mathbf{Z}[\mathbf{u}], \ (\mid \varepsilon \mid << 1) \tag{2.2}$$

keeps eq. (2.1) invariant, **Z[u]** and its nonlinear mapping **Z[*]** are called symmetries (or contravariants) and symmetric transformation, respectively. By virture of eq.(2.2), the NLEE (2.1) yields a first variational system,

$$\partial_t \mathbf{Z}[\mathbf{u}] = N'_{\mathbf{u}}(\mathbf{Z}[\mathbf{u}]), \tag{2.3}$$

where the italic notation $N'_u(*)$, promissed to represent a linear operator generated from **N[u]**, means the Gateau derivative defined by

$$N'_{\mathbf{u}}(\mathbf{W}) = \lim_{\varepsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbf{N}[\mathbf{u} + \varepsilon \mathbf{w}].$$

Another bracket **(•)** is used to emphahize a linear dependence.Both brackets **[*]** and **(*)** are often omitted for simplicity.

2A) Symmetric Transformations [2,3]

Because $\mathbf{u}(t + \delta t)$ (| δt | <<1) also satisfies eq.(2.1), we obtain

$$Z'_{\mathbf{u}}(\mathbf{N}[\mathbf{u}] = N'_{\mathbf{u}}(\mathbf{Z}[\mathbf{u}]), \tag{2.4}$$

from $\partial_t \mathbf{Z}[\mathbf{u}] = Z'_{\mathbf{u}}(\frac{\partial \mathbf{u}}{\partial t}) = Z'_{\mathbf{u}}(\mathbf{N}[\mathbf{u}])$. There exists a family of symmetries $\mathbf{Z}_i[\mathbf{u}]$ {i=1, 2⋯} , which satisfy eq.(2.4) as far as u is a solution of eq.(2.1). It is not difficult to show **[Z,N]** (**u**) $\{\equiv \mathbf{Z}'_{\mathbf{u}}(\mathbf{N}[\mathbf{u}]) - \mathbf{N}'_{\mathbf{u}}(\mathbf{Z}[\mathbf{u}])\}$ is also symmetry and any symmetries are on Lie algebra (see Appendix-A).[2]

2B) Strong Symmetry [4,5]

Let's take functios $\mathbf{u}(x.t)$ and $\mathbf{Z}(x,t)$ as solutions of eqs.(2.1) and (2.3), respectively. An operator **K[u]** is called strong symmetry, if **K[u]Z[u]** still satisfies eq.(2.3),

$$\partial_t \{\mathbf{K}[\mathbf{u}]\mathbf{Z}[\mathbf{u}]\} = N'_{\mathbf{u}}(\mathbf{K}[\mathbf{u}]\mathbf{Z}[\mathbf{u}]), \tag{2.5}$$

since eq.(2.5) can be iterated, $\mathbf{K}^n[\mathbf{u}]$ and the linear combination $\sum_{n=1}^{M} b_n \mathbf{K}^n[\mathbf{u}]$ are also strong symmetries. On the other hand, we can see

$$\partial_t \{\mathbf{K}[\mathbf{u}] \cdot \mathbf{Z}[\mathbf{u}]\} = \mathbf{K}'_{\mathbf{u}}(\mathbf{N}[\mathbf{u}]) \cdot \mathbf{Z}[\mathbf{u}] + \mathbf{K}[\mathbf{u}] \cdot N'_{\mathbf{u}}(\mathbf{Z}[\mathbf{u}]), \tag{2.6}$$

in which $K'_{\mathbf{u}}(\mathbf{N}) \cdot \mathbf{Z}$ $\{= K'_{\mathbf{u}} \cdot \mathbf{N} \cdot \mathbf{Z}\}$ is regarded as a bilinear functional of $K'_{\mathbf{u}}$ acting on the ordered point $\mathbf{N} \cdot \mathbf{Z}$. Comparing eqs.(2.6) with (2.5), we get the condition for strong symmetry,

$$K'_{\mathbf{u}} \cdot \mathbf{N} \cdot \mathbf{Z} = \{N'_{\mathbf{u}} \cdot \mathbf{K} - \mathbf{K} \cdot N'_{\mathbf{u}}\} \cdot \mathbf{Z} \equiv [N'_{\mathbf{u}}, \mathbf{K}] \cdot \mathbf{Z}, \tag{2.7}$$

where the bracket [,] means usual commutator

$$[N'_{\mathbf{u}}, \mathbf{K}] \equiv N'_{\mathbf{u}} \cdot \mathbf{K} - \mathbf{K} \cdot N'_{\mathbf{u}}.$$

The following three facts are euivalent, **(1)** $\mathbf{K[u]}$ is strong symmetry, **(2)** $\partial_t(\mathbf{K}\cdot\mathbf{Z})=N'_u(\mathbf{K}\cdot\mathbf{Z})$, and **(3)** $\hat{\mathbf{K}}[\mathbf{u},\mathbf{N}]\cdot\mathbf{Z}=0$, where

$$\hat{\mathbf{K}}[\mathbf{u},\mathbf{N}]\equiv K'_u\cdot\mathbf{N}+[\mathbf{K},N'_u].$$

2C) Hereditary Symmetry [4,5]

Fuchsteiner had pointed out that both "strong" and "hereditary" symmetries play a key role for understanding the exactly solvale systems.

For derivation of the Bäcklund transformation, the hereditary symmetry must be studied. As to the hereditary symmetry, we prepare the following lemmas.

[Lemma.1] For a vector $\mathbf{L[u]}\equiv\mathbf{K[u]}\cdot\mathbf{N[u]}$, we obtain

$$\{K'_u\cdot\mathbf{L}-[\mathbf{K},L'_u]\}\cdot\mathbf{w}=[K'_u,\mathbf{K}]\cdot[\mathbf{N},\mathbf{w}]. \tag{2.8}$$

[proof] We list

$$L'_u\cdot\mathbf{w}=K'_u\cdot\mathbf{w}\cdot\mathbf{N}+\mathbf{K}\cdot N'_u\cdot\mathbf{w}, \tag{2.9a}$$

$$[L'_u,\mathbf{K}]\cdot\mathbf{w}=K'_u\cdot\mathbf{K}\cdot\mathbf{w}\cdot\mathbf{N}+\mathbf{K}\cdot N'_u\cdot\mathbf{K}\cdot\mathbf{w}-\mathbf{K}\cdot K'_u\cdot\mathbf{w}\cdot\mathbf{N}-\mathbf{K}^2\cdot N'_u\cdot\mathbf{w}. \tag{2.9b}$$

Eq. (2.7) multiplying **K** from the left direction yields

$$\mathbf{K}^2\cdot N'_u\cdot\mathbf{w}=\mathbf{K}\cdot\{N'_u\cdot\mathbf{K}-K'_u\cdot\mathbf{N}\}\cdot\mathbf{w}, \tag{2.9c}$$

by which we can eliminate the term $\mathbf{K}^2\cdot N'_u\cdot\mathbf{w}$ from eq.(2.9b), then eq.(2.8) is directly obtained.

[QED]

We define a bilinear functional $\mathbf{B}:(\mathbf{u}\cdot\mathbf{v})\rightarrow\mathbf{B}(\mathbf{u}\cdot\mathbf{v})$ and introduce its dual one as $\mathbf{B}^*:(\mathbf{u}\cdot\mathbf{v})\rightarrow\mathbf{B}(\mathbf{v}\cdot\mathbf{u})$, then

$$\mathbf{B}(\mathbf{u}\cdot\mathbf{v})-\mathbf{B}(\mathbf{v}\cdot\mathbf{u})=(\mathbf{B}-\mathbf{B}^*)\cdot(\mathbf{u}\cdot\mathbf{v}). \tag{2.10}$$

If $\mathbf{B}=\mathbf{B}^*$, **B** is called symmetry. The term appearing in the RHS of eq.(2.7) is represented by

$$[K'_u,\mathbf{K}]\cdot[\mathbf{N},\mathbf{w}]=\{[K'_u,\mathbf{K}]-[K'_u,\mathbf{K}]^*\}\cdot\mathbf{N}\cdot\mathbf{w}. \tag{2.11}$$

The operator $\mathbf{K[u]}$ is called hereditary, if $[K'_u,\mathbf{K}]^*=[K'_u,\mathbf{K}]$. These facts are summarized in the following theorem.

[Theor.1]" If the hereditary operator $\mathbf{K[u]}$ satisfies eq.(2.7), $K'_u\cdot\mathbf{N}+[\mathbf{K},N'_u]\doteq0$, we still obtain

$$K'_u\cdot\mathbf{L}-[L'_u,\mathbf{K}]=0. \qquad (\mathbf{L}\equiv\mathbf{K}\cdot\mathbf{N})\text{"} \tag{2.12}$$

As a corollary of Theor.1 we obtain an important fact, that is, if **K** is a hereditary symmetry and a strong symmetry for eq.(2.1), **K** is still a strong symmetry of the equation,

$$\mathbf{u}_t=\{\mathbf{K[u]}\}^n\,\mathbf{N[u]}\quad(n=0,1,2..). \tag{2.13}$$

§3. Potential Operators and Sympletic Structures

Magri[2] had introduced the potential operator **P** {: **u**→**p**} mapping the configuration space to the covariant one **W***, which corresponds to a dual space of contravariant field **W**. Then we can define a bilinear functional as ≪**u**,**p**≫.

3A) Conserved Covariants and Integrals

It is basic to define a functional **F[u]**, which is independent of the path as to the one-parameter solution $\mathbf{u}=\mathbf{u}(t)$. That is, the circular integral

$$\mathbf{F[u]}=\mathbf{F[u_0]}+\int_{t_0}^{t} \ll \mathbf{u}_\tau(\tau),\mathbf{P[u}(\tau)\mathbf{]}\gg d\tau \tag{3.1}$$

must be defined uniquely only by specifying both end points (t_o,t). In such a case, **P** is called a potential operator (or gradient) of **F** as to the specified bilinear form ≪·,·≫. We note that the gradient **P[u]** is covariant against that the symmetry **Z[u]** is contravariant. We denote the variations of **F** and **u** as $\delta\mathbf{F}$ $\{\equiv \mathbf{F[u}(t+\delta t)\mathbf{]}-\mathbf{F[u}(t)\mathbf{]}\}$ and $\delta\mathbf{u}$ $\{\equiv \mathbf{u}_t\delta t\}$ as to the time shift δt. Then eq.(3.1) gives

$$\delta\mathbf{F[u]}=\ll\delta\mathbf{u},\mathbf{P[u]}\gg \text{ or } \frac{d}{dt}\mathbf{F[u]}=\ll\mathbf{N[u]},\mathbf{P[u]}\gg, \tag{3.2}$$

which defines some correspondence between **F[u]** and **P[u]**. If ≪**N[u]**,**P[u]**≫=0, **F[u]** is called a conserved functional (or integral) while **P[u]** the conserved covariant (gradient of potentia or integrating operator.). As in the case of symmetry generator, however, we shall only consider the integrating operator for which d**F**/dt {=≪**N[u]**,**P[u]**≫}=0 is identically verified.

If **P** is a potential operator, the variation δ**F[u]** given by eq.(3.1) must vanish along a inifinitesimal closed path **C**. Because the path **C** can be replaced with a parallelogram, it is possible to rewrite eq.(3.2) as

$$\ll\delta\mathbf{u},\ P'_{\mathbf{u}}\,\delta\mathbf{v}\gg=\ll\delta\mathbf{v},\ P_{\mathbf{u}}\,\delta\mathbf{u}\gg, \tag{3.3}$$

where $\delta\mathbf{u}$, $\delta\mathbf{v}\in\mathbf{W}$ and "δ" is used to denote the quantity independent on **u**.[2, 3] As well as symmetries we find a direct relation

$$[N'_{\mathbf{u}}]^{\dagger}\,\mathbf{P}+P'_{\mathbf{u}}\,\mathbf{N}=0 \tag{3.3a}$$

between the covariant **P** and symmetry **N** (use eq. (3.3) for the relation obtained from Gateau derivative of ≪**N,P**≫=0). If a gradient **P** satisfies above relations, the potential must be a conserved quantity.

[Lemma 2]" Let **F** and **P** as a conserved quantity and its gradient of eq.(2.1), then **F** is also the conserved quantity of the hierarhy $\mathbf{u}_t=\mathbf{K}^n\mathbf{N}$, iff $\mathbf{P}_n$ $\{\equiv[\mathbf{K}^{\dagger}]^n\mathbf{P}\}$ is the gradient of $\mathbf{u}_t=\mathbf{K}^n\mathbf{N}$, **[proof]** omitted.

3B) Sympletic Operator

In geomety, metric tensor is introduced to interchange the role of contravariant and covariant tensors. In this case, we must take a certain type of linear sympletic operators ($\mathbf{L}_u$: $\delta\mathbf{P}\rightarrow\delta\mathbf{w}$,

$\delta\mathbf{P}\in\mathbf{W}^*$, $\delta\mathbf{w}\in\mathbf{W}$), constrained by the following skew symmetry and Jacobi's identity,[3)]

$$\langle\langle \mathbf{L}_u(\delta\mathbf{P}),\delta\mathbf{Q}\rangle\rangle+\langle\langle \mathbf{L}_u(\delta\mathbf{Q}),\delta\mathbf{P}\rangle\rangle=0, \tag{3.4a}$$

$$\langle\langle L'_u(\delta\mathbf{P};\mathbf{L}_u\delta\mathbf{Q}),\delta\mathbf{R}\rangle\rangle+\langle\langle L'_u(\delta\mathbf{Q};\mathbf{L}_u\delta\mathbf{R}),\delta\mathbf{P}\rangle\rangle+\langle\langle L'_u(\delta\mathbf{R};\mathbf{L}_u\delta\mathbf{P}),\delta\mathbf{Q}\rangle\rangle=0, \tag{3.4b}$$

where $\{\delta\mathbf{P},\ \delta\mathbf{Q},\ \delta\mathbf{R}\}$ on $\mathbf{W}^*$ are independent on $\mathbf{u}$, $\mathbf{L}_u\delta\mathbf{P}\equiv\mathbf{L}_u(\delta\mathbf{P})$ and $L'_u(\delta\mathbf{P};\ \delta\mathbf{Q})=\frac{\mathbf{d}}{\mathbf{d}\varepsilon}\mathbf{L}_{u+\varepsilon\delta Q}(\delta\mathbf{P})$. As already noted, there is a family of symmetires, hence a family of covariants ia generated by the action of sympletic operator. The symmetry $\mathbf{N}[\mathbf{u}]$ in eq.(2.1) is called a Hamiltonian, if $\mathbf{N}[\mathbf{u}]$ $\{=\mathbf{L}_u\mathbf{P}[\mathbf{u}]\}$ can be related by the sympletic operator satisfying eqs.(3.4) {see **lemma.B4** in appendix-B} .

Let's show that a commutator $[\mathbf{Z}_j,\mathbf{Z}_k][\mathbf{u}]$ $\{\equiv Z'_{j,u}\cdot\mathbf{Z}_k-Z'_{k,u}\cdot\mathbf{Z}_j\}$ is also a Hamiltonian, if $\mathbf{Z}_j$ is a Hamiltonian. For this symmetry $\mathbf{Z}_j$ we can define a potential opearator $\mathbf{P}_j$ by $\mathbf{Z}_j=\mathbf{L}_u\mathbf{P}_j$ and a functional $\mathbf{F}_j[\mathbf{u}]$ by eq.(3.2). We must show that a potential operator $\mathbf{P}_{jk}$ $\{=[\mathbf{L}_u]^{-1}\mathbf{Z}_{jk}\}$ corresponding to the symmetry $\mathbf{Z}_{jk}=[\mathbf{Z}_j,\ \mathbf{Z}_k][\mathbf{u}]$ satisfies the relation,

$$\delta\mathbf{F}_{jk}[\mathbf{u}]=\langle\langle\delta\mathbf{u},\ \mathbf{P}_{jk}[\mathbf{u}]\rangle\rangle. \tag{3.5}$$

This answer is given by the following theorem proved in Appendix-B.

[Theor.2]"The commutator $[\mathbf{Z}_j,\ \mathbf{Z}_k]$ can be factorized by the sympletic operator $\mathbf{L}_u$,

$$[\mathbf{Z}_j,\ \mathbf{Z}_k]=\mathbf{L}_u(P'_{j,u}(\mathbf{Z}_k)+[Z'_{k,u}]^\dagger(\mathbf{P}_j)), \tag{3.6}$$

where $[Z'_{k,u}]^\dagger$ is an adjoint of $Z'_{k,u}$."

Because of eq.(3.6), we see $\mathbf{P}_{jk}=P'_{j,u}(\mathbf{Z}_k)+[Z'_{k,u}]^\dagger(\mathbf{P}_j)$ and the functional $\mathbf{F}_{jk}$ which satisfies eq.(3.5) is given by

$$\mathbf{F}_{jk}[\mathbf{u}]=\langle\langle\mathbf{Z}_k[\mathbf{u}],\ \mathbf{P}_j[\mathbf{u}]\rangle\rangle. \tag{3.7}$$

It is not difficult to derive eq.(3.5) forom the variation of dq.(3.7),

$$\begin{aligned}\delta\mathbf{F}_{jk}[\mathbf{u}]&=\langle\langle\delta\mathbf{Z}_k[\mathbf{u}],\mathbf{P}_j[\mathbf{u}]\rangle\rangle+\langle\langle\mathbf{Z}_k[\mathbf{u}],\delta\mathbf{P}_j[\mathbf{u}]\rangle\rangle\\ &=\langle\langle Z'_{k,u}(\delta\mathbf{u}),\mathbf{P}_j\rangle\rangle+\langle\langle\mathbf{Z}_k,P'_{j,u}(\delta\mathbf{u})\rangle\rangle\\ &=\langle\langle\delta\mathbf{u},\ [Z'_{k,u}]^\dagger\mathbf{P}_j+P'_{j,u}(\mathbf{Z}_k)\rangle\rangle.\end{aligned}$$

3C) Iso-Spectral Problems

The inverse spectral transform(**IST**) is a powerful tool for solving NLEEs and the isospectral eigenvalue problem plays the central role. It is a natural question to connect the IST with the symmetric approach. A direct connection of the strong symmetry **K** to the iso-spectral problem is closely related with the time-independence of the integaral $\mathbf{F}[\mathbf{u}]$.

[Lemma 3a] "Let's μ arbitrary scalar and $\mathbf{Z}[\mathbf{u}]$ as a solution of linearized equation (2.3), then the strong symmetry **K** satisfies the eigenvalue problem,

$$\mathbf{K}\mathbf{Z}=\mu\mathbf{Z}." \tag{3.8}$$

[proof] Since $\mathbf{K}[\mathbf{u}]\mathbf{Z}[\mathbf{u}]$ still satisfies eq.(2.3), we see

$$\frac{\partial}{\partial t}\{(\mathbf{K}[\mathbf{u}]-\mu)\mathbf{Z}[\mathbf{u}]\}=N'_u(\mathbf{K}[\mathbf{u}]-\mu)\mathbf{Z}[\mathbf{u}])-\mu_t\mathbf{Z}[\mathbf{u}].$$

If $\mathbf{KZ}=\mu\mathbf{Z}$ at $t=t_o$, μ_t must be zero, because eq. (2. 3) has a null solution and $\mathbf{Z}[\mathbf{u}(\mathbf{t}_o)]$ is not zero genarally. **[QED]**

[Lemma 3b]' Let consider the eigenvalue problem $\mathbf{KZ}=\mu\mathbf{Z}$, where $\mathbf{K}$ is a strong symmetry, $\mathbf{Z}$ its eigenfunction and μ a spectral parameter. If the problem is iso-spectral ($d\mu/dt=0$), $\mathbf{u}_t=\mathbf{N}$ and $\Psi'_u\mathbf{N}+[N'_u]^\dagger\Psi=0$, we can see

$$\{K'_u\mathbf{N}+[(N'_u)^\dagger,\mathbf{K}]\}\mathbf{Z}=0." \tag{3.8}$$

[proof] The Gateau differential of $\mathbf{KZ}$ is easily reduced to eq. (3. 8), that is, if we take $\mathbf{w}=\mathbf{u}_t=\mathbf{N}$,

$$\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}(\mathbf{KZ})(\mathbf{u}+\varepsilon\mathbf{w})=K'_u\mathbf{N}\Psi+\mathbf{K}Z'_u\mathbf{N}=\mu Z'_u\mathbf{N}.$$

Since $Z'_u\mathbf{N}=-[N'_u]^\dagger\mathbf{Z}$, we can see

$$0=\{K'_u\mathbf{N}-\mathbf{K}[N'_u]^\dagger\}\mathbf{Z}+\mu[N'_u]^\dagger\mathbf{Z}=0.$$ **[QED]**

Is it possible to obtain the stong symmetry from the knowledge of inverse decoppling scheme $\Phi_x=D(\lambda,u)\Phi$? It is not difficult to find the gradient of λ (written as $\mathbf{G}_\lambda$) and the problem is to get such an eigenvalue equation as $\mathbf{KZ}=\mu\mathbf{Z}$. For this problem the following Lemma is useful.

[Lemma 4]' Let assume that the strong symmetry of $\mathbf{u}_t=\mathbf{N}$ satisfies an iso-spectral problem,

$$\mathbf{K}^\dagger\mathbf{G}_\lambda=\mu(\lambda)\mathbf{G}_\lambda, \tag{3.9}$$

were $\mathbf{G}_\lambda$ is a gradient of the conserved quantity of $\mathbf{u}_t=\mathbf{N}$. Then the conserved quantity related with gradient $\mathbf{G}_\lambda$ is an eigenvalue of $\Phi_x=D(\lambda,\mathbf{u})\Phi$, $\mathbf{K}$ is a stong symmetry and λ conserved quantity,"

[Proof] We see $\langle\langle\mathbf{K}^n\mathbf{N},\mathbf{G}_\lambda\rangle\rangle=\mu^n\langle\langle\mathbf{N},\mathbf{G}_\lambda\rangle\rangle=0$, while the Lemma.3 teaches us that $\mathbf{K}$ is a strong symmetry of $\mathbf{u}_t=\mathbf{N}$. **[QED]**

Above Lemmas say that the strong symmetry acts on symmetries while the adjoint on conserved covariants.

§4. Bäcklund Transformations[7,8]

In addition to eq. (2. 1), we prepare another equation as

$$\partial_t\mathbf{v}=\mathbf{G}[\mathbf{v}]. \tag{4.1}$$

The Bäcklund transformation between both solutions of eqs. (2. 1) and (4. 1) is defined by the constrain,

$$\mathbf{B}[\mathbf{u}(x,t),\mathbf{v}(x,t)]\equiv 0 \text{ for all } (x,t), \tag{4.2}$$

In the following, we must often use such two-variables functionals as $\mathbf{B}[\mathbf{u},\mathbf{v}]$ and related directional derivatives. For example, the partial derivatives (equivalent to the variation) is similar to the one of single-variable functionals,

$$B'_u(\delta u) \equiv B'_u[u,v](\delta u) = \frac{\partial}{\partial \varepsilon} B[u+\varepsilon(\delta u),v],$$

while the total (exteriour) derivative of eq.(4.2) must be

$$d\_B[u,v] \equiv B'_u(\delta u) + B'_v(\delta v) = 0.$$

It is possible to to take the inverse of linear operator B'_v, which enables us to introduce a linear mapping **T:**$\delta u \rightarrow \delta v$,

$$T[u,v] \equiv A'_v \cdot B'_u \qquad (A'_v \equiv [B'_v]^{-1}). \tag{4.3}$$

It is necessary to calculate a derivative of a functionals **F[u,v]** as to **v** under eq.(4.2). We note that the derivative is given by the variation. If eq.(4.2) is solved as **u** = **b[v]**, the "constarained" derivative is given by

$$\begin{aligned} d_v F[b[v],v](\delta v) &= F[b[v+\delta v],v+\delta v] - F[u,v] \\ &\simeq F_v[b[v],v]\ \{H[b[v],v](b'_v[v](\delta v)) + (\delta v)\} \\ &\equiv F'_v\ \{H \cdot b'_v + 1\}\ (\delta v), \end{aligned} \tag{4.4}$$

where $H \equiv [F'_v[b[v],v]^{-1} F'_u[b[v],v]$. If $F[u,v] = B[u,v]$, eq.(4.4) is surely reduced to $d_v B[u.v](\delta v) = B'_v\{T \cdot b'_v + 1\}(\delta v) = 0$.

A main thema of this section is to examine the relation between both strong symmetries **K[u]** and **M[u]** of eq.(2.1) and (4.1), respectively. The following in this section and the **Appendix-C** is devoted to show the theorem and related lemmas, respectively.

[Theor.3] "The strong symmetry **M[v]** of eq.(4.1) can be reconstructd by **K[u]** of eq.(2.1) as

$$M[v] = A'_v B'_u K[u][A'_u B'_v]^{-1} \equiv TK[u]T^{-1}." \tag{4.5}$$

The following Lemmma is necessary to differentiate the operator **T**. (The proof is shown in **Appendix-C.**)

[Lemma.1] "If **B[u,v]** satisfies eq.(4.2), the derivative of **T[u,v]** is given by

$$d_v T(\delta v)(\delta u) = d_v T(T(\delta u)) \cdot T^{-1}(\delta v)." \tag{4.6}$$

For relating eq.(4.1) with eq.(2.1), the next lemma is used.

[Lemma.2] "Between eqs.(2.1) and (4.1) we can see

$$d_v K(\delta v) = -K'_u(T^{-1}(\delta u)), \tag{4.7a}$$

$$d_v N(\delta v) = -N'_u(T^{-1}(\delta v)), \tag{4.7b}$$

$$G = -T \cdot N." \tag{4.7c}$$

[Lemma.3] "The product of arbitrary functionals **K[u,v]** and **L[u,v]** with the constrain is differentated as

$$d_v\{K \cdot L\}(\delta v) = d_v K(\delta v) \cdot L + K \cdot d_v L(\delta v), \tag{4.8a}$$

$$d_v\{K \cdot L \cdot M\}(\delta v) = d_v K(\delta v) \cdot L \cdot M + K \cdot d_v L(\delta v) \cdot M + K \cdot L \cdot d_v M(\delta v). \tag{4.8b}$$

The derivative of an inverse T^{-1} is also given by

$$\mathbf{d_v T^{-1}(\delta v)} = -\mathbf{T^{-1}} \cdot \mathbf{d_v T(\delta v)} \cdot \mathbf{T^{-1}}.\text{"} \qquad (4.8c)$$

[Lemma.4] "If we set $\mathbf{M[v]} \equiv \mathbf{T} \cdot \mathbf{K[u]} \cdot \mathbf{T^{-1}}$, the following relation is obtained,

$$\mathbf{d_v M(G)(\delta v)} = -\mathbf{d_v T(M\delta v)} \cdot \mathbf{N} + \mathbf{M} \cdot \mathbf{d_v T(\delta v)} \cdot \mathbf{N} + \mathbf{T} \cdot M'_u \cdot \mathbf{N} \cdot \mathbf{T^{-1}(\delta v)}.\text{"} \qquad (4.9)$$

Now we can prove the **Theor.3.** As shown in **2B**), **K** is a recursion operator and $\hat{\mathbf{K}}[\mathbf{u},\mathbf{N}]$ $\{\equiv K'_u + [\mathbf{K},N'_u] = 0\}$. Our problem is to derive $M'_v\,(\mathbf{G}) = [G'_v, \mathbf{M}]$. For this purpose we consider a quantity $\mathbf{T} \cdot K'_u(\mathbf{N}) \cdot \mathbf{T^{-1}(\delta v)}$, which can be represented by $\hat{\mathbf{K}} = 0$ and eq.(4.5) as

$$\begin{aligned}\mathbf{T} \cdot K'_u\,(\mathbf{N}) \cdot \mathbf{T^{-1}(\delta v)} &= \mathbf{T} \cdot N'_u \cdot \mathbf{K} \cdot \mathbf{T^{-1}(\delta v)} - \mathbf{T} \cdot \mathbf{K} \cdot N'_u \cdot \mathbf{T^{-1}(\delta v)} \\ &= \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}} \cdot \mathbf{M(\delta v)} - \mathbf{M} \cdot \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}(\delta v)}.\end{aligned}$$

Substituting this into eq.(4.5), we obtain

$$\begin{aligned}\mathbf{d_v\, M(G)(\delta v)} = &-\mathbf{d_v\, T(M \cdot \delta v)} \cdot \mathbf{N} + \mathbf{M} \cdot \mathbf{d_v\, T(\delta v)} \cdot \mathbf{N} \\ &+ \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}} \cdot \mathbf{M(\delta v)} - \mathbf{M} \cdot \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}(\delta v)}.\end{aligned} \qquad (4.10)$$

Eq.(4.8) gives us the followings,

$$\begin{aligned}&\mathbf{d_v\, \{T \cdot N\}\ M(\delta v)} = \mathbf{d_v T(M(\delta v))} \cdot \mathbf{N} + \mathbf{T} \cdot \mathbf{d_v N(M(\delta v))}, \\ &\mathbf{d_v\, \{T \cdot N\}\ \delta v)} = \mathbf{d_v T(\delta v)} \cdot \mathbf{N} + \mathbf{T} \cdot \mathbf{d_v N(\delta v)}.\end{aligned}$$

By means of these and $\mathbf{G} = -\mathbf{T} \cdot \mathbf{N}$, $G'_v = \mathbf{d_v\, G}$, the commutator $[G'_u, \mathbf{M}]$ is reduced to

$$\begin{aligned}&[G'_v, \mathbf{M}](\delta \mathbf{v}) = G'_v \cdot \mathbf{M(\delta v)} - \mathbf{M} \cdot G'_v\,(\delta \mathbf{v}) = \mathbf{d}_v\mathbf{G} \cdot \mathbf{M(\delta v)} - \mathbf{M} \cdot \mathbf{d_v G(\delta v)} \\ &= -\mathbf{d_V\{T \cdot N\}} \cdot \mathbf{M(\delta v)} + \mathbf{M} \cdot \mathbf{d_V\{T \cdot N\}\ (\delta v)} \\ &= -\mathbf{d_V T(M(\delta v))} \cdot \mathbf{N} - \mathbf{T} \cdot \mathbf{d_V N(M(\delta v))} + \mathbf{M} \cdot \{\mathbf{d_V T(\delta v)} \cdot \mathbf{N} + \mathbf{T} \cdot \mathbf{d_V N(\delta v)}\}\ .\end{aligned} \qquad (4.11)$$

From eqs. (4.10), (4.11) and $\mathbf{d_V N(\delta v)} = -N'_u \cdot \mathbf{T^{-1}(\delta v)}$, we get

$$\begin{aligned}&\mathbf{d_V M(G)(\delta v)} - [\mathbf{G'_V}, \mathbf{M}](\delta \mathbf{v}) \\ &\qquad = \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}} \cdot \mathbf{M(\delta v)} - \mathbf{M} \cdot \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}(\delta v)} \\ &\qquad - \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}(M(\delta v))} + \mathbf{M} \cdot \mathbf{T} \cdot N'_u \cdot \mathbf{T^{-1}(\delta v)} = 0.\end{aligned}$$

That is, $M'_V(\mathbf{G}) = [\mathbf{G'_V}, \mathbf{M}]$, and the proof is completed.

By Theor.3 we can say that the Bäcklund transformation characterized by **T** is only related with the strong symmetries. Then all the equations of a hierarchy generated by **K** possess the same Bäcklund transformation. It remains to determine the Bäcklund transformation for given NLEEs, but this process is not easy. Some known examples, however, tell us that **B(u,v)** plays the role of Miura type of transformations (i.e., the Riccati type of equations) and makes it possible to classify the Bäcklund transformation.[7]

§5. Applications to the AKNS-class of NLEE

In this section, we take a case of 2x2-AKNS class of NLEE,[12] were the inverse decouppling scheme is given by

$$\Phi_X = D[\lambda,Q]\Phi,\ \Phi_t = F[\lambda,Q,\partial_X Q,\partial_X^2 Q,\cdots]\Phi, \qquad (5.1)$$

where $D[\lambda,Q] \equiv -i\lambda\sigma_3 + Q$ "σ_3 is one of Pauli matrices), λ is a spectral parameter, Q $\{=Q(x,t)\}$ is a potential matrix, but F must be determined from the compatibility condition $D_t - F_x + [D, F] = 0$. For the **IST,** the Jost (matrix) functions $\Phi^{\pm}$ $\{=[\varphi_1^{\pm},\varphi_2^{\pm}]\}$ satisfying suitable boundary conditions play the role of auxially variables and linearize the problem.

The squared eigenfunctions $\Phi_V^{\pm}$ can be introduced by the Jost functions, as $|\Phi_V^+> \equiv |\varphi_2^+ \times \varphi_2^+>$, $|\Phi_V^-> \equiv |\varphi_1^- \times \varphi_1^->$, and satisfy the following eigenvalue problem,[12)]

$$K_\pm(x,t;dy)\sigma_1 | \Phi_V^\pm(\lambda;x,t)> = \lambda\sigma_1 | \Phi_V^\pm(\lambda;x,t)>, \qquad (5.2a)$$

where $| \bullet >$ is a ket corresponding to the usual column vector but the bra is promised to be $<a| = (-a_2, a_1)$, and $K_\pm$ is an integral operator defined as

$$\mathbf{K}_\pm(x,t;y) = -\frac{i}{2}\ \{\sigma_3\partial_X + 2\sigma_3 | u(x,t)> \mathbf{I}_\pm < u(y,t) | \sigma_3\}\ , \qquad (5.2b)$$

$$\mathbf{I}_\pm \equiv \int_{\pm\infty}^{x} dy$$

The ket $|\mathbf{u}>$ $\{= | r,q>\}$ consists of the elements of Q $\{\equiv |1> r <2| + |2> q <1|\}$ while the bra is $<u| = (-q,r)$.

It is an important character of the **IST** that the tempolary dependence of problem is reduced to the one of scattering data which is always solved easily. Considering these facts, the solvable class of **NLEE** can be represented on the number vector space as

$$\partial_t |\mathbf{u}> = |\mathbf{N}[\mathbf{u}]> \equiv -2\sum_{k=1}^{M} \mathbf{a}_K\ \{\mathbf{K}_\pm[\mathbf{u}]\}^{k}\sigma_3 |\mathbf{u}>. \qquad (5.3)$$

This relation must be compared with eq.(2.13), that is, we expect the differential-integral operators defined in eq.(5.2b) are the hereditary symmetry (including strong symmetry). The following of this section is devoted to discuss this fact.

We find that it is possible to impose

$$(\mathbf{K}_-)^n\sigma_3 |\mathbf{u}> = (\mathbf{K}_+)^n\sigma_3 |u> \ (n=0,1,.). \qquad (5.4)$$

Hence, as far as $\mathbf{K}_\pm$ acting on $\sigma_3 |u>$, we can say that $\mathbf{K}_- \equiv \mathbf{K}_+$.

For a rapidely vanishing vector $\mathbf{w}(x,t)$ (as $x \to \pm\infty$), we can define adjoint operators $(\mathbf{K}_\pm^\dagger)$by

$$\mathbf{I}_0 < \mathbf{w} | \sigma_1\mathbf{K}_\pm[\mathbf{u}]\sigma_1 | \Phi_V^\pm> = -\mathbf{I}_0 < \Phi_V^\pm | \sigma_1\mathbf{K}_\pm^\dagger[\mathbf{u}]\sigma_1 | \mathbf{w}>, \qquad (5.5)$$

where

$$\mathbf{I}_0 = \int_{-\infty}^{\infty} dx.$$

By means of eq.(5.4) the following symmetry holds

$$\mathbf{K}_+[\mathbf{u}] = \mathbf{K}_-^\dagger[\mathbf{u}]. \qquad (5.6)$$

Both relations (5.5) and (5.6) result in

$$\mathbf{I}_0 < \mathbf{u} | \sigma_3\ \{\mathbf{K}_\pm[\mathbf{u}]\}^n\sigma_3 | \mathbf{u}> = 0. \qquad (5.7)$$

5A) Strong Symmetry

In eqs. (5. 3) – (5. 7), the variable **u** is treated as a column vector on the configuration space, but this is not basic for our discussions because the column vector may be exchanged with the raw vector. It is only necessary for us to keep mined where the symmetry is defined on. In the following, we assume the symmetries as raw (ket) vector in the bra-ket product, inspite of relations which is often written by the column formula. According to the facts in 2**A**), the recursive nature enables us only to consider the lowest **NLEE** of eq. (5. 8a) for the proof of strong symmetry of **K[u]**,

$$\partial_t \mid \mathbf{u}> = -2\sigma_3 \mid \mathbf{u}>. \tag{5.8}$$

We note that eq. (5. 8) is itself variational system. Let's take $\mid \delta\mathbf{u}>$ as the variational solution, then we can directly show that $\mathbf{K[u]} \mid \delta\mathbf{u}>$ still satisfies eq. (5. 8). This means that **K[u]** is a strong symmetry of eq. (5. 3).

5B) Hereditary Symmetry

The problem is only to show

$$[\mathbf{K}, K'_{\mathbf{u}}] \mid \mathbf{w}> \mid \mathbf{v}> - [\mathbf{K}, K'_{\mathbf{u}}] \mid \mathbf{v}> \mid \mathbf{w}> 0,$$

where $[\mathbf{K}, K'_{\mathbf{u}}] \mid \mathbf{w}> \mid \mathbf{v}> = \mathbf{K[u]}K'_{\mathbf{u}}(\mathbf{w}) \mid \mathbf{r}> - K'_{\mathbf{u}}(\mathbf{K[u]} \mid \mathbf{w}>) \mid \mathbf{v}>$ while $[\mathbf{K}, K'_{\mathbf{u}}]^* \mid \mathbf{w}> \mid \mathbf{v}> = [\mathbf{K}, K'_{\mathbf{u}}] \mid \mathbf{v}> \mid \mathbf{w}>$. We simply denote $\mathbf{K[u]} \simeq \sigma_3\partial_x + 2\sigma_3 \mid \mathbf{u}> \mathrm{I} < \mathbf{u} \mid \sigma_3$, then its derivative is given by $\mathbf{K}'_{\mathbf{u}}(\mathbf{w}) = 2\sigma_3 \{ \mid \mathbf{w}> \mathrm{I} < \mathbf{u} \mid + \mid \mathbf{u}> \mathrm{I} < \mathbf{w} \mid \} \sigma_3$ while $K'_{\mathbf{u}}(\mathbf{w}) \mid \mathbf{v}> = 2\sigma_3 \mid \mathbf{w}> \mathrm{I} < \mathbf{u} \mid \sigma_3 \mid \mathbf{v}> + (\mathrm{sym})$. The term (sym) $\{ = 2\sigma_3 \mid \mathbf{u}> \mathrm{I} < \mathbf{w} \mid \sigma_3 \mid \mathbf{v}> \}$ can be eliminated because it cancels with the same term appering in $[K'_{\mathbf{u}}, \mathbf{K}]^*$. Then we can see

$$\begin{aligned} &\mathbf{K[u]}K'_{\mathbf{u}}(\mathbf{w}) \mid \mathbf{v}> \\ &\simeq 2 \mid \mathbf{w}_x> \mathrm{I} < \mathbf{u} \mid \mathbf{v}> + 2 \mid \mathbf{w}> \mathrm{I} < \mathbf{u} \mid \sigma_3\mathbf{v}> + 4\sigma_3 \mid \mathbf{u}> \mathrm{I} < \mathbf{u} \mid \mathbf{w}> \mathrm{I} < \mathbf{u} \mid \sigma_3\mathbf{v}>, \end{aligned}$$

and

$$\begin{aligned} [\mathbf{K}, K'_{\mathbf{u}}] \mid \mathbf{w}> \mid \mathbf{v}> &\simeq 2 \mid \mathbf{w}> < \mathbf{u} \mid \sigma_3\mathbf{v}> - 2\sigma_3 \mid \mathbf{u}> \mathrm{I} < \mathbf{v} \mid \mathbf{w}_y> \\ &+ \mathbf{4}\sigma_3 \mid \mathbf{u}> \{ \mathrm{I} < \mathbf{u} \mid \mathbf{w}> \mathrm{I} < \mathbf{u} \mid \sigma_3\mathbf{v}> + \mathrm{I} < \mathbf{u} \mid \mathbf{v}> \mathrm{I} < \mathbf{u} \mid \sigma_3\mathbf{w}> \} \end{aligned}$$

After all, we can obtain

$$\begin{aligned} &\{ [\mathbf{K}, K'_{\mathbf{u}}] - [\mathbf{K}, K'_{\mathbf{u}}]^* \} \mid \mathbf{w}> \mid \mathbf{v}> \\ &= \mathbf{2} \mid \mathbf{w}> < \mathbf{u} \mid \sigma_3\mathbf{v}> - \mathbf{2} \mid \mathbf{v}> < \mathbf{u} \mid \sigma_3\mathbf{w}> + \mathbf{2}\sigma_3 \mid \mathbf{u}> < \mathbf{w} \mid \mathbf{v}> = \mathbf{0}. \end{aligned}$$

5C) Cannonical Structure [11, 12, 13]

It is necessary to give the bilinear functional $\ll \cdot, \cdot \gg$. explicitely. For the strong symmetry operator **K[u]** and the lowest symmetry $\sigma_3 \mid \mathbf{u}>$, we define a class of symmetries by $< \mathbf{Z}_0 \mid$,

$$\mathbf{Z}_0[\mathbf{u}] \equiv \sum_{j=0}^{M} a_j \{\mathbf{K[u]}\}^j \sigma_3 \mid \mathbf{u}>. \tag{5.9}$$

Eq. (5. 7) gives the next theorem.

[Theor. 4] "The adjoint of $|\mathbf{Z}_0[\mathbf{u}]>$ in eq. (5. 9) is a conservd covariant as to the bilinear functional,

$$\langle\langle \delta\mathbf{z}, \delta\mathbf{p}\rangle\rangle = \mathbf{I}_0 < \delta\mathbf{z} \mid \delta\mathbf{p} > = -\mathbf{I}_0 < \delta\mathbf{p} \mid \delta\mathbf{z} >, \qquad (5.\ 10)$$

where $\delta\mathbf{p}$ is covariant while $\delta\mathbf{z}$ contravariant."

[proof] Since $<\delta\mathbf{u} \mid \delta\mathbf{v}> = -<\delta\mathbf{v} \mid \delta\mathbf{u}>$, the variantion of eq. (5 • 9) with only a_1 is easily given by

$$|Z'_{\mathbf{u}}(\delta\mathbf{u})> = |\delta \mathrm{u}_x> + 2\sigma_3 \mid \mathbf{u}>\mathrm{I}\ \{<\delta\mathbf{u} \mid \mathbf{u}> + \mathbf{u} \mid \delta\mathbf{u}>\} = |\delta\mathbf{u}_x>.$$

It becomes clear that $\mathbf{Z}$ is a potential operator, because eq. (3. 3) is satisfied as

$$\langle\langle \delta\mathbf{u}, Z'_{0,\mathbf{u}}\delta\mathbf{v}\rangle\rangle - \langle\langle \delta\mathbf{v}, Z'_{0,\mathbf{u}}\delta\mathbf{u}\rangle\rangle = \mathbf{I}_0\ \{\delta\mathbf{u} \mid \delta\mathbf{v}_x> - <\delta\mathbf{v} \mid \delta\mathbf{u}_x>\} = 0.$$

On the other hand, from eq. (5. 4) we easily find $\langle\langle \mathbf{N}[\mathbf{u}], \mathbf{Z}[\mathbf{u}]\rangle\rangle = 0$, then $\mathbf{Z}_o$ is a conserved covariant. **[QED]**

We note that the linear conbination of kets, $\sum_{k=0}^{N} a_k(\mathbf{K}[\mathbf{u}])^k\sigma_3 \mid \mathbf{u}>$, are also covariant.

Because the metric operator is not unique, we may assume two kinds of metric operators $\mathbf{L}_\mathbf{u}$ and $\mathbf{M}_\mathbf{u}$ and set them as

$$\mathbf{Z}_{j+1}[\mathbf{u}] = \mathbf{L}_\mathbf{u}\ \mathbf{P}^{(j+1)}[\mathbf{u}] = \mathbf{M}_\mathbf{u}\ \mathbf{P}^{(j)}[\mathbf{u}], \qquad (5.\ 11)$$

where $\mathbf{Z}_j\ \{=\mathbf{K}^j\mathbf{Z}_o\}$ is a symmetry while $\mathbf{P}^{(j)}$ covariant. To determine $\mathbf{L}_u$, we factorize the symmetry $\mathbf{N}[\mathbf{u}]\ \{\equiv \mathbf{Z}[\mathbf{u}]\}$ as

$$\mathbf{N}[\mathbf{u}(x)] = -2\mathbf{P}_o[\mathbf{u}]. \qquad (5.\ 12)$$

where $\mathbf{P}_o[\mathbf{u}] = \sum_{k=1}^{M} a_k\ \{\mathbf{K}[\mathbf{u}]\}^k\sigma_3 \mid \mathbf{u}>$. Then $\mathbf{L}_\mathbf{u} = -2$ and $\mathbf{M}_\mathbf{u} = -2\mathbf{K}$. To confirm $\mathbf{L}_\mathbf{u}$ and $\mathbf{M}_\mathbf{u}$ as sympletic, we must substitute eq. (4. 5) into eq. (4. 1). The case of $\mathbf{L}_\mathbf{u}$ is trivial, while another case of $\mathbf{M}_\mathbf{u}$ is not easy (see Appendix-D). Eq. (5. 11) recursively determines the covariant $\mathbf{P}_j$ and also symmetries $\mathbf{Z}_j$. This fact can be interpreted as that symmetries have two kinds of decomposition by the gradient of conseved quantities (bi-Hamiltonian structure).

By means of eqs. (3. 2) and (5.10), we can see

$$\delta\mathbf{F}_j[\mathbf{u}] = \mathbf{I}_0 < \delta\mathbf{u} \mid \mathbf{P}_j[\mathbf{u}]>, \qquad (5.\ 13)$$

where $\mathbf{P}_j$ is regarded as the gradient of $\mathbf{F}_j$ and the usual sence of functional derivatives gives us

$$\frac{\delta}{|\delta\mathbf{u}>}\mathbf{F}_j = -\sigma_3\sigma_1 \mid \mathbf{P}_j>, \left(\frac{\delta}{|\delta\mathbf{u}>} \equiv \frac{\delta}{\delta\mathbf{r}} \mid 1> + \frac{\delta}{\delta\mathbf{q}} \mid 2>\right). \qquad (5.\ 14)$$

Regarding the variations in eq. (5. 13) caused from time-shift $(t \to t+\triangle t)$, we see that the functional $\mathbf{F}_j$ must be integral by eqs. (5. 7) and (5. 9). Further we compare the RHS in eq. (5. 3) with eq. (5.14) and use $\mathbf{P}_o[\mathbf{u}] = \sum_{k=1}^{M} \mathrm{a}_k\ \{\mathbf{K}[\mathbf{u}]\}^k\sigma_3 \mid \mathbf{u}>$, then the NLLE is reduced to the canonical form,

$$\partial_t \mid \mathbf{u}> = 2\sigma_1\sigma_3\frac{\delta}{|\delta\mathbf{u}>}\mathbf{F}_0, \qquad (5.\ 15)$$

where $\mathbf{F}_0\ \{\equiv \mathbf{H}\}$ is the Hamiltonian.

We can apply eq. (5. 13) to the relation (3. 7) then obtain

$$\mathbf{F}_{jk}[\mathbf{u}]=\mathrm{I}_0\ \{\frac{\delta}{<\delta\mathbf{u}\ |}\mathbf{F}_j[\mathbf{u}]\cdot\sigma_1\sigma_3\cdot\mathbf{L}_u\sigma_3\sigma_1\frac{\delta}{|\ \delta\mathbf{u}>}\mathbf{F}_k[\mathbf{u}]\}\ .$$

Considering $\mathbf{L}_u=-2$, we represent this as

$$\mathbf{F}_{jk}[\mathbf{u}]=\ \{\{\mathbf{F}_j[\mathbf{u}],\ \mathbf{F}_k[\mathbf{u}]\}\}\ .$$

The bracket $\{\{*,*\}\}$ corresponding to the Posson's bracket defined by

$$\{\{\mathbf{F}_j,\ \mathbf{F}_k\}\}\ \equiv-2\mathrm{I}_0\ \{\frac{\delta}{<\delta\mathbf{u}\ |}\mathbf{F}_j[\mathbf{u}]\cdot\frac{\delta}{|\ \delta\mathbf{u}>}\mathbf{F}_k[\mathbf{u}]\} \qquad (5.\ 16)$$

which is untisymmetry and still constitutes a Lie algebra.

§6. Concludings and Remarks

In this note we detailed the symmetric approach resulting in the sympletic structures of the soliton equations in $1+1$ dimensions, in which we studied such important concepts as strong and hereditary symmetries and connections with Bäcklund tranformation, bi-Hamiltonian and sympletic structures. As far as in $1+1$ dimensions it can be done successfully to develope the symmetric approach, but it is rather difficult for physicists.

We found that differential integral operators $\mathbf{K}_\pm$ acting on squared eigenfunctions are the hereditary symmetry. This fact enables us to propose a simpler way for deriving the hereditary symmetry, because it is well-known that $\mathbf{K}_\pm$ are also derived from the compatibility condition of the inverse scattering scheme of NLLEs. We simply show the treatment for the NxN-matrix order problem with the following inverse scattering scheme,

$$\Phi_{\mathrm{X}}=\ \{\lambda\mathrm{A}+\mathbf{U}(\mathrm{x},\ \mathrm{t})\}\Phi,\qquad \Phi_t=\mathbf{F}[\lambda,\ \mathbf{U}]\Phi, \qquad (6.\ 1)$$

where A is a diagonal constant, λ is a spectral parameter, $\mathbf{U}(x,\ t)$ is an off-diagonal matrix. We specially take $\mathbf{F}=\sum_{\mathbf{k}=0}^{\mathbf{M}}\lambda^k\mathbf{F}^{(k)}$ (M is called a rank), then the compatibility can be reduced to $\mathbf{F}^{(\mathrm{M})}_{off}=0$ and

$$\mathbf{F}^{(n)}_{dia,\mathrm{x}}+[\mathbf{F}^{(n)}_{off},\ \mathbf{U}]=\mathbf{0}, \qquad (6.\ 2a)$$

$$[\mathbf{A},\ \mathbf{F}^{(n-1)}_{off}]=\mathbf{F}^{(n)}_{off,\mathrm{x}}+[\mathbf{F}^{(n)}_{dia},\ \mathbf{U}]+[\mathbf{F}^{(n)}_{off},\ \mathbf{U}]_{off} \qquad (6.\ 2b)$$

for $n=1,\ 2,\ ..,\ M$ and

$$\mathbf{U}_t=\mathbf{F}^{(o)}_{off,\mathrm{x}}+[\mathbf{F}^{(o)}_{dia},\ \mathbf{U}]+[\mathbf{F}^{(o)}_{off},\ \mathbf{U}]_{off}, \qquad (6.\ 2c)$$

where suffixes "dia" and "off" mean to take the diagonal and off-diagonal parts of matrices, respectively. Each off-diagonal matrix $\mathrm{F}^{(n)}_{off}$ can be determined recursively by eq. (6. 2b) extending the range of n to $\{0,\ 1,\ ...,\ M\}$. That is, eq. (6. 2c) is reduced to the NLLE with rank M as

$$\frac{\partial\mathbf{U}}{\partial\mathbf{t}}=\mathbf{N}[\mathbf{U}]\equiv[\mathbf{A},\ \mathbf{F}^{(-1)}_{off}]. \qquad (6.\ 3)$$

Let consider a different NLLE with rank $\widetilde{M}(=M-1)$, where related quantities are distuinguished by a superscript "$\sim$". Now we set a conjecture

$$\widetilde{\mathbf{F}}^{(-1)}_{off} = \mathbf{F}^{(o)}_{off}, \tag{6. 4}$$

by which the NLLEE with $\widetilde{\mathbf{N}}$ $(\equiv \widetilde{\mathbf{N}}_{jk}])$ is transformed to $\mathbf{N}$ as $\mathbf{K}:\widetilde{\mathbf{N}} \rightarrow \mathbf{N}$,

$$\begin{aligned} N_{jk} &= \frac{1}{A_j - A_k} \frac{\partial}{\partial x} \widetilde{N}_{jk} + \sum_{\ell} \left(\frac{\widetilde{N}_{j\ell} U_{\ell k}}{A_j - A_l} \right) \\ &+ \sum_{\ell} U_{jk} \int^{x} \left(\frac{\widetilde{N}_{j\ell} U_{\ell j} + U_{j\ell} \widetilde{N}_{\ell j}}{A_\ell - A_j} + \frac{\widetilde{N}_{k\ell} U_{\ell k} + U_{k\ell} \widetilde{N}_{\ell k}}{A_k - A_\ell} \right) dy. \end{aligned} \tag{6. 5}$$

This mapping **K** can be expected as the hereditary symmetry. Although the analytical proof is not completed , the computer calculations show that in some content.

We must note that such an approach results in difficulty for the case of multi-dimensions and it is still open. Under this situation, however, some treatments have been reported still by means of symmetric approach.[15)]

The strong symmetry may be applicable to solve such a special problem as perturbation problems appearing in equtions with nonintegrable terms. In such a case we need the complete basis of solutions of the linearized equation. This problem was solved by using squared eigenfunctions, but the elimination of squared eigenfunction is rather tedious. The strong symmetry may give the basis of solutions, which expands the nonintegrable functions.

References

1) C.S.Gardner, J.M.Greene, M.D.Kruskal and R.M.Miura :" Korteweg-de Vries equation and generalization. 6, method for exact solutions", Commun. Pure Appl. Math., **27**(1975) 97-133

2) F.Magri :"An operator approach to Polisson brackets", Anals of Physics **99**, (1976) 196-228

3) F.Magri :"A simple model of the integrable Hamiltonian equation", J.Math.Phys. **19**(5), (1978) 1156-1162

4) B.Fuchsteiner :"pure soliton solutions of some nonlinear partial differential equations", Commun. Math.Phy., **55** (1977) 187-194

5) B. Fuchsteiner :"application of hereditary symmetries to nonlinear evolution equations", Nonlinear Anal. Theory & Appl. **3** (1979) 849-862

6) A.S.Fokas :"a symmetry approach to exactly solvable evolution equations", J.Math.Phys., **21** (1980) 1318-1325

7) B.Fuchsteiner and A.S.Fokas :"sympletic structures, their Bäcklund transformations and hereditary symmetries", Physica **4D**, (1981) 47-66

8) A.S.Fokas and B.Fuchsteiner :"Bäcklund Transformations for Hereditary Symeties", Nonlinear Theor. Method & Appl., **3** (1981) 423-432

9) P.D.Lax :"integral of nonlinear equations of evolution and solitary waves", Commun. Pure Apll. Math., **21** (1968) 467-490

10) P.D.Lax :"periodic solutions of the KdV equation", Commun. Pure Appl. Math., **118** (1975) 144-188

11) A.S.Fokas and R.L.Anderson :"on the use of isospectral eigenvvalue problems for obtaining hereditary symmetries for Hamiltonian systems", J.Math.Phys., **23** (1982) 1066-1175

12) M.Boiti, F.Pempinelli and G.Z.Tu :"Canonical Structures of Soliton Equations via Isospectral Eigenvalu Problems", Nuovo Cimento **79B** (1984) 231-266

13) T.Kawata :"generalized AKNS class of the nonlinear evolution equations and its trace formula and dynamical structures", Bultain Letter, Faculty of Toyam University 1987
14) J.M.Alberty, T.Koikawa and R.Sasaki : Physica **5D** (1982) 43
15) P.M.Santini and A.S.Fokas :"Recursion Operators and Bi-Hamiltonian Structures in Multidmensions.I", Commun.Math.Phys., **115** (1988) 375-419

Appendix A : Symmetry Algebras

Before considerling the Gateau derivative of a commutator,

$$Z_k[u] \equiv [Z_i, Z_j][u] = Z'_{i,u}(Z_j[u]) - Z'_{j,u}(Z_i[u]), \qquad \text{(A. 1)}$$

we refer to the notation of the second Gateau derivative,

$$Z''_u(v,w) \equiv \frac{\partial^2}{\partial\varepsilon\partial\nu} Z[u+\varepsilon v+\nu w] = Z''_u(w, v), \qquad \text{(A. 2)}$$

Since $v \neq w$ and

$$\frac{\partial^2}{\partial\varepsilon\partial\nu} Z[u+\varepsilon v+\nu w] = \frac{\partial}{\partial\varepsilon} Z'_u[u+\varepsilon v](w) = \frac{\partial}{\partial\varepsilon} Z'_u[u+\varepsilon w](v), \qquad \text{(A. 3)}$$

we find $Z'_u[u+\varepsilon v] \neq Z'_u[u+\varepsilon w]$. To emphacize this fact, we use the notation, $Z'_{u+\varepsilon v}(w) \equiv Z'_u[u+\varepsilon v](w)$. From eqs. (A. 2) and (A. 3) we obtain $Z''_u(v, w) \simeq \frac{1}{\varepsilon}\{Z'_{u+\varepsilon v} - Z'_u\}(w)$, that is,

$$Z'_{u+\varepsilon v}(w) = Z'_u(w) + \varepsilon Z''_u(v, w). \qquad \text{(A. 4)}$$

The Gateau derivative of eq. (A. 1) is given by

$$\begin{aligned} Z'_{k,u}(w) &= Z'_{i,u}(Z'_{j,u}(w)) + Z''_{i,u}(Z_j[u], w) \\ &\quad - Z'_{j,u}(Z'_{i,u}(w)) - Z''_{j,u}(Z_i[u], w), \end{aligned}$$

by which we can calculate

$$\begin{aligned} &[Z_k, [Z_i, Z_j]][u] \\ &= Z'_{k,u}(Z'_{i,u}(Z_j[u])) - Z'_{k,u}(Z'_{j,u}(Z_i[u])) - Z'_{i,u}(Z'_{j,u}(Z_k[u])) \\ &\quad - Z''_{i,u}(Z_j[u], Z_k[u]) + Z'_{j,u}(Z'_{i,u}(Z_k[u])) + Z''_{j,u}(Z_i[u], Z_k[u]). \end{aligned}$$

If we assume the symmetry $Z''_{k,u}(\delta u, \delta v) = Z''_{k,u}(\delta v, \delta u)$, the cyclic summation $\sum_{i,j,k}[Z_k, [Z_i, Z_j]][u]$ results in zero.

Appendix-B The proof of Theor. 2

Before proving the theorem we list some necessary lemmas.

[Lemma. B1] If L_u is skew symmetric and $P[u]$ is conserved covariant, we obtain

$$\langle\langle L_u(P'_u(L_u Q)), P\rangle\rangle = \langle\langle L_u(P'_u(L_u(P))), Q\rangle\rangle \qquad \text{(B. 1)}$$

[proof] Because of eqs. (3. 3) and (3. 4a), we find

$$\begin{aligned} &\langle\langle L_u(P'_u(L_u Q)), P\rangle\rangle = -\langle\langle L_u(P), P'_u(L_u Q)\rangle\rangle \\ &= -\langle\langle L_u(Q), P'_u(L_u(P))\rangle\rangle = \langle\langle L_u(P'_u(L_u((P))), Q\rangle\rangle. \end{aligned}$$ **[QED]**

[Lemma. B2] If **w** $\{\equiv \delta \mathbf{u}\}$ is some variation of $\mathbf{u}(x, t)$, we can obtain

$$\langle\langle L'_u(\delta\mathbf{Q};\mathbf{w}), \delta\mathbf{P}\rangle\rangle + \langle\langle L'_u(\delta\mathbf{P};\mathbf{w}), \delta\mathbf{Q}\rangle\rangle = 0, \qquad \text{(B. 2)}$$

where $\delta\mathbf{P}$ and $\delta\mathbf{Q}$ are $\mathbf{u}$-independent.

[proof] We consider the functional $\mathbf{I[u]} = \langle\langle \mathbf{L}_u(\delta\mathbf{Q}), \delta\mathbf{Q}\rangle\rangle$, then its variation is directly evaluated as

$$\delta\mathbf{I[u]} = \langle\langle L'_u(\delta\mathbf{Q};\delta\mathbf{u}), \delta\mathbf{P}\rangle\rangle,$$

while eq. (3. 3) gives another formula as

$$\delta\mathbf{I[u]} = -\langle\langle L'_u(\delta\mathbf{P};\delta\mathbf{u}), \delta\mathbf{Q}\rangle\rangle.$$

[QED]

[Lemma. B3] The Gateau derivative of symmetrien **Z[u]** generated from the covariants **P[u]** $\{\mathbf{Z} = \mathbf{L}_u\mathbf{P}\}$ is given by

$$Z'_u(\mathbf{w}) = L'(\mathbf{P[u]};\mathbf{w}) + \mathbf{L}_u(P'_u(\mathbf{w})). \qquad \text{(B. 3)}$$

[proof] From eq.(A. 4)we find

$$\frac{\partial}{\partial\varepsilon}\{\mathbf{L}_{u+\varepsilon v}(\mathbf{P[u]})\} = L'_u(\mathbf{P[u]}) + o(\varepsilon),$$

by which $Z'_u(\mathbf{w})$ is reduced to

$$\begin{aligned} Z'_u(\mathbf{w}) &= \frac{\partial}{\partial\varepsilon}\mathbf{L}_{u+\varepsilon v}\cdot\mathbf{P[u]} + \frac{\partial}{\partial\varepsilon}\mathbf{L}_{u+\varepsilon v}\cdot(\mathbf{P[u+\varepsilon w]}) - \mathbf{P[u]}) \\ &= \frac{\partial}{\partial\varepsilon}\mathbf{L}_{u+\varepsilon v}\cdot\mathbf{P[u]} + \mathbf{L}_{u+\varepsilon v}(P'_u(\mathbf{w}) + \varepsilon\cdot\frac{\partial}{\partial\varepsilon}\mathbf{L}_{u+\varepsilon v}(P'_u(\mathbf{w})) \\ &\simeq L'_u(\mathbf{P[u]};\mathbf{w}) + \mathbf{L}_u(P'_u(\mathbf{w})). \end{aligned}$$

[QED]

[Lemma. B4] For $\mathbf{Z} = \mathbf{L}_u\mathbf{P}$, the following relation is obtained,

$$\langle\langle Z'_u(\mathbf{L}_u\delta\mathbf{Q}), \delta\mathbf{P}\rangle\rangle - \langle\langle Z'_u(\mathbf{L}_u\delta\mathbf{P}), \delta\mathbf{Q}\rangle\rangle = \langle\langle L'_u(\delta\mathbf{Q};\mathbf{Z[u]}), \delta\mathbf{P}\rangle\rangle. \qquad \text{(B. 4)}$$

[proof] Let's substitute eq. (B. 3) into the LHS of eq. (B. 4)

$$\langle\langle Z'_u(\mathbf{L}_u\delta\mathbf{Q}), \delta\mathbf{P}\rangle\rangle = \langle\langle L'_u(\mathbf{P[u]};\mathbf{L}_u\delta\mathbf{Q}) + \mathbf{L}_u(P'_u(\mathbf{L}_u\delta\mathbf{Q})), \delta\mathbf{P}\rangle\rangle \quad \text{etc..}$$

Because of eqs. (B. 1), (B. 2), and further (3. 4b), we can obtain

$$\begin{aligned} &\langle\langle Z'_u(\mathbf{L}_u\delta\mathbf{Q}), \delta\mathbf{P}\rangle\rangle - \langle\langle Z'_u(\mathbf{L}_u\delta\mathbf{P}), \delta\mathbf{Q}\rangle\rangle \\ &\quad = \langle\langle L'_u(\mathbf{P[u]};\mathbf{L}_u\delta\mathbf{Q}), \delta\mathbf{P}\rangle\rangle - \langle\langle L'_u(\mathbf{P[u]};\mathbf{L}_u\delta\mathbf{P}), \delta\mathbf{Q}\rangle\rangle \\ &\quad = -\langle\langle L'_u(\delta\mathbf{P};\mathbf{L}_u\delta\mathbf{Q}), \mathbf{P}\rangle\rangle - \langle\langle L'_u(\mathbf{P[u]};\mathbf{L}_u\delta\mathbf{P}), \delta\mathbf{Q}\rangle\rangle \\ &\quad = \langle\langle L'_u(\delta\mathbf{Q};\mathbf{L}_u\mathbf{P}), \delta\mathbf{P}\rangle\rangle. \end{aligned}$$

[QED]

[proof of Theor.2] By virture of **Lemmas** listed above and definition of the commutator, we see

$$\begin{aligned} &\langle\langle [\mathbf{Z}_j, \mathbf{Z}_k][\mathbf{u}], \delta\mathbf{P}\rangle\rangle = \langle\langle Z'_{j,u}(\mathbf{Z}_k), \delta\mathbf{P}\rangle\rangle - \langle\langle Z'_{k,u}(\mathbf{Z}_j), \delta\mathbf{P}\rangle\rangle \\ &= \langle\langle \mathbf{L}_u(P'_{j,u}(\mathbf{Z}_k)), \delta\mathbf{P}\rangle\rangle + \langle\langle L'_u(\mathbf{P}_j;\mathbf{Z}_k), \delta\mathbf{P}\rangle\rangle - \langle\langle Z'_{k,u}(\mathbf{Z}_j), \delta\mathbf{P}\rangle\rangle. \end{aligned}$$

Since $\mathbf{Z}_k = \mathbf{L}_u\mathbf{P}_k$ and from **Lemma. B4,** we get

$$\ll Z'_{k,u}(\mathbf{Z}_j),\ \delta\mathbf{P}\gg = \ll Z'_{k,u}(\mathbf{L_u P}_j),\ \delta\mathbf{P}\gg$$
$$= \ll Z'_{k,u}(\mathbf{L_u}\delta\mathbf{P}),\ \mathbf{P}_j\gg + \ll L'_u(\mathbf{P}_j;\ \mathbf{L_u P}_k),\ \delta\mathbf{P}\gg.$$

By eqs. (B. 6) and (B. 7) we obtain

$$\ll[\mathbf{Z}_j,\ \mathbf{Z}_k][\mathbf{u}],\ \delta\mathbf{P}\gg$$
$$= \ll \mathbf{L_u}(P'_{j,u}(\mathbf{Z}_k)),\ \delta\mathbf{P}\gg - \ll Z'_{k,u}(\mathbf{L_u}\,\delta\mathbf{P}),\ \mathbf{P}_j\gg$$
$$+ \ll L'_u(\mathbf{P}_j;\mathbf{Z}_k),\ \delta\mathbf{P}\gg - \ll L'_u(\mathbf{P}_j;\mathbf{L_u\,P}_k),\ \delta\mathbf{P}\gg$$
$$= \ll \mathbf{L_u}(P'_{j,u}(\mathbf{Z}_k)),\ \delta\mathbf{P}\gg - \ll Z'_{k,u}(\mathbf{L_u}\,\delta\mathbf{P}),\ \mathbf{P}_j\gg.$$

The second term of RHS can be arranged to

$$-\ll Z'_{k,u}(\mathbf{L_u}\,\delta\mathbf{P}),\ \mathbf{P}_j\gg = -\ll \mathbf{L_u}\,\delta\mathbf{P},\ [Z'_{k,u}]^{\dagger}\,\mathbf{P}_j>>$$
$$= \ll \mathbf{L_u}([Z'_{k,u}]^{\dagger}\,\mathbf{P}_j),\ \delta\mathbf{P}\gg.$$

Substituting this into eq. (B. 8), we get

$$\ll[\mathbf{Z}_j,\ \mathbf{Z}_k][\mathbf{u}],\ \delta\mathbf{P}\gg = \ll \mathbf{L_u}(P'_{j,u}(\mathbf{Z}_k)),\ \delta\mathbf{P}\gg + \ll \mathbf{L_u}(Z'_{k,u}]^{\dagger}\,\mathrm{P}_j),\ \delta\mathrm{P}\gg$$
$$= \ll \mathbf{L_u}\{P'_{j,u}(\mathbf{Z}_k) + [Z'_{k,u}]^{\dagger}(\mathbf{P}_j)\},\ \delta\mathbf{P}\gg.$$

[QED]

Appendix-C Proofs of Lemmas in §. 4

The second directional derivative also appears

$$\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\mu}\mathbf{B}[\mathbf{u}+\varepsilon\delta\mathbf{u},\mathbf{v}+\mu\delta\mathbf{v}] = \frac{\partial}{\partial\varepsilon}B'_v[\mathbf{u}+\varepsilon\delta\mathbf{u},\ \mathbf{v}](\delta\mathbf{v}) = B''_{v,u}(\delta\mathbf{v})(\delta\mathbf{u})$$
$$\equiv \frac{\partial}{\partial\mu}B'_u[\mathbf{u},\ \mathbf{v}+\mu\delta\mathbf{v}](\delta\mathbf{u}) = B''_{u,v}(\delta\mathbf{u})(\delta\mathbf{v}),$$

so we note

$$B'_u[\mathbf{u}+\varepsilon\delta\mathbf{u},\ \mathbf{v}] \simeq B'_u[\mathbf{u},\mathbf{v}] + \varepsilon B''_{u,u}[\mathbf{u},\ \mathbf{v}](\delta\mathbf{u}),$$
$$B'_u[\mathbf{u},\ \mathbf{v}+\varepsilon\delta\mathbf{v}] \simeq B'_u[\mathbf{u},\ \mathbf{v}] + \varepsilon B''_{u,v}[\mathbf{u},\ \mathbf{v}](\delta\mathbf{v}). \qquad (C.\ 1)$$

The product, $\mathbf{B} = \mathbf{G}\cdot\mathbf{H}$, is treated by usual manner,

$$B'_u[\mathbf{u},\ \mathbf{v}](\delta\mathbf{w}) = \frac{\partial}{\partial\varepsilon}\mathbf{G}[\mathbf{u}+\varepsilon\delta\mathbf{w},\ \mathbf{v}]\mathbf{H}[\mathbf{u}+\varepsilon\delta\mathbf{w},\ \mathbf{v}]$$
$$= \frac{\partial}{\partial\varepsilon}\mathbf{G}[\mathbf{u}+\varepsilon\delta\mathbf{w},\ \mathbf{v}]\cdot\mathbf{H}[\mathbf{u}+\varepsilon\delta\mathbf{w},\ \mathbf{v}] + \mathbf{G}[\mathbf{u}+\varepsilon\delta\mathbf{w},\ \mathbf{v}]\cdot\frac{\partial}{\partial\varepsilon}\mathbf{H}[\mathbf{u}+\varepsilon\delta\mathbf{w},\ \mathbf{v}]$$
$$= \mathbf{G_u}[\mathbf{u},\ \mathbf{v}](\delta\mathbf{w})\cdot\mathbf{H}[\mathbf{u},\ \mathbf{v}] + \mathbf{G}[\mathbf{u},\ \mathbf{v}]\cdot\mathbf{H_u}[\mathbf{u},\ \mathbf{v}](\delta\mathbf{w}).$$

C1) Proof of Lemma. 1

It is easy to bive the partial derivatives of **B**, but the derivative of inverse is remained. Since $B'_v\cdot A'_v = 1$, the variation is reduced to

$$0 = B'_v[\mathbf{u},\ \mathbf{v}+\delta\mathbf{v}]\cdot A'_v[\mathbf{u},\ \mathbf{v}+\delta\mathbf{v}] - B'_v[\mathbf{u},\ \mathbf{v}]\cdot A'_v[\mathbf{u},\ \mathbf{v}]$$
$$= \{B'_v + B''_{v,v}(\delta\mathbf{v})\}\cdot\{A'_v + A''_{v,v}(\delta\mathbf{v})\} - B'_v\cdot A'_v$$
$$\simeq B''_{v,v}(\delta\mathbf{v})\cdot A'_v + B'_v\cdot A''_{v,v}(\delta\mathbf{v}).$$

That is, we get

$$A''_{v,v}(\delta v) = -A'_v \cdot A''_{v,v}(\delta v) \cdot A'_v,\ A''_{v,v}(\delta u) = -A'_v \cdot B''_{v,v}(\delta u) \cdot A'_v. \quad \text{(C. 1)}$$

By these relations we can calculate the derivative of $T = A'_v \cdot B'_u$,

$$\begin{aligned} \mathbf{T}_v[u, v](\delta v) &= A'_v[u, v+\delta v] \cdot B'_u[u, v+\delta v] - A'_v[u, v] \cdot B'_u[u, v] \\ &= A''_{v,v}(\delta v) \cdot B'_u + A'_v \cdot A''_{u,v}(\delta v) \\ &= -A'_v \cdot B''_{v,v}(\delta v) \cdot \mathbf{T} + A'_v \cdot B''_{u,v}(\delta v), \quad \text{(C. 2a)} \\ \mathbf{T}_u[u, v](\delta u) &= -\mathbf{A}'_v \cdot B''_{v,u}(\delta u) \cdot \mathbf{T} + A'_v \cdot B''_{u,u}(\delta u) \quad \text{(C. 2b)} \end{aligned}$$

The constrained dervative is obtained by imposing $\delta v = -\mathbf{T}(\delta u)$,

$$\begin{aligned} d_v\mathbf{T}[u, v](\delta v) &= \mathbf{T}_u(\mathbf{T}^{-1}(\delta u)) + \mathbf{T}_v(\delta v) \\ &= -A'_v \cdot B''_{v,v}(\delta v) \cdot \mathbf{T} + A'_v \cdot B''_{u,v}(\delta v) \\ &\quad + A'_v \cdot B''_{v,u}(\mathbf{T}^{-1}(\delta u)) \cdot \mathbf{T} - A'_v \cdot B''_{u,u}(\mathbf{T}^{-1}(\delta u)), \end{aligned}$$

then we get

$$\begin{aligned} d_v\mathbf{T}(\delta v)(\delta u) &= -A'_v \cdot B''_{v,v}(\delta v) \cdot (\delta v) + A'_v \cdot B''_{u,v}(\delta v)(\delta u) \\ &\quad + A'_v \cdot B''_{v,u}(\delta v) \cdot \mathbf{T}(\delta u) - A'_v \cdot B''_{u,u}(\delta v)(\delta u). \quad \text{(C. 3)} \end{aligned}$$

On the other hand, from eqs. (C. 2) we get

$$\begin{aligned} \mathbf{T}_v(\mathbf{T} \cdot \delta u) &= -A'_v \cdot B''_{v,v}(\mathbf{T} \cdot \delta u) \cdot \mathbf{T} + A'_v \cdot B''_{u,v}(\mathbf{T} \cdot \delta u), \\ \mathbf{T}_u(\delta u) &= -A'_v \cdot B''_{v,u}(\delta u) \cdot \mathbf{T} + A'_v \cdot B''_{u,u}(\delta u), \quad \text{(C. 4)} \end{aligned}$$

from which the constrained derivative is obtained,

$$\begin{aligned} d_v\mathbf{T}(\delta v)(\delta u) = &\{-A'_v \cdot B''_{v,v}(\delta v) \cdot \mathbf{T} + A'_v \cdot B''_{u,v}(\delta v) \\ &+ A'_v \cdot B''_{v,u}(\mathbf{T}^{-1}(\delta v)) \cdot \mathbf{T} - A'_v \cdot B''_{u,u}(\mathbf{T}^{-1}(\delta v))\}\ (\delta u). \quad \text{(C. 5)} \end{aligned}$$

By using eqs. (C. 4) and (C. 5) we can obtain

$$\begin{aligned} d_v\mathbf{T}(\mathbf{T}(\delta u))\mathbf{T}^{-1}(\delta v) &= \mathbf{T}_v(\mathbf{T} \cdot \delta u) \cdot \mathbf{T}^{-1}(\delta v) - \mathbf{T}_u(\mathbf{T} \cdot \delta u)(\delta u) \\ &= -A'_v \cdot B''_{v,v}(\mathbf{T} \cdot \delta u)(\delta v) + A'_v \cdot B''_{u,v}(\mathbf{T} \cdot \delta u)\mathbf{T}^{-1}(\delta v) \\ &\quad + A'_v \cdot B''_{v,u}(\delta v) \cdot (\delta v) - A'_v \cdot B''_{u,u}(\delta v)(\delta u). \quad \text{(C. 6)} \end{aligned}$$

Comparing both eqs. (C. 3) and (C. 6), we proof eq. (4. 6). **[EQD]**

C2) Proof of Lemma. 2

We estimeate the variation of strong symmetry $\mathbf{K}[u]$ and nonlinear operator $\mathbf{N}[u]$,

$$d_v\mathbf{K}(\delta v) \simeq K'_u[u[v]](b'_v[v](\delta v)) \equiv K'_v \cdot \mathbf{T}^{-1}(\delta v),$$

$$d_v\,\mathbf{N}(\delta v) \simeq -N'_u \cdot \mathbf{T}^{-1}(\delta v).$$

On the other hand, we see

$$\frac{d}{dt}\mathbf{B}[u, v] \simeq \frac{1}{\Delta t}\{\mathbf{B}[u(t) + \triangle t \cdot u_t, v(t) + \triangle t \cdot v_t] - \mathbf{B}[u(t), v(t)]\}$$

$$\simeq B'_u[\mathbf{u(t), v(t)}](\mathbf{u}_t)+B'_v[\mathbf{u(t), v(t)}](\mathbf{v}_t)$$
$$=B'_u\cdot\mathbf{N[u]}+B'_v\cdot\mathbf{G[v]}=0,$$

from which $\mathbf{G}=-\mathbf{T}\cdot\mathbf{N}$ is obtained. **[QED]**

C3) Proof of Lemma. 3

From definition of constrained derivative, we can see

$$\mathbf{d_v\{K\cdot L\}\ (\delta v)}=\delta\{\mathbf{K[u[v], v]\cdot L[u[v], v]}\}$$
$$=\delta\mathbf{K[u[v], v]\cdot L[u[v], v]+K[u[v]], v]\cdot}\ \delta\mathbf{L[u[v], v]}$$
$$=\mathbf{d_v\,K(\delta v)\cdot L+K\cdot d_v\,L(\delta v).}$$

By only replacement, we can easily extend eq. (C. 7) to

$$\mathbf{d_v\{K\cdot L\cdot M\}\ (\delta v)=d_v\{K\cdot L\}\ (\delta v)\cdot M+K\cdot L\cdot d_v\,M(\delta v)}$$
$$\mathbf{=d_v\,K(\delta v)\cdot L\cdot M+K\cdot d_v\,L(\delta v)\cdot M+K\cdot L\cdot d_v\,M(\delta v).}$$

By using eq. (4. 8a), the constrahned derivative of $\mathbf{T}^{-1}$ is also obtained,

$$\mathbf{d_v\{T\cdot T^{-1}\}\ (\delta v)=d_v\,T(\delta v)\cdot T^{-1}+T\cdot d_v\,T^{-1}(\delta v)}=0. \qquad \textbf{[QED]}$$

C4) Proof of Lemma. 4

By means of eqs. (4. 8b) and (4. 6), we obtain

$$\mathbf{d_v\{T\cdot K\cdot T^{-1}\}\ (\delta v)}$$
$$\mathbf{=d_v\,T(\delta v)\cdot K\cdot T^{-1}+T\cdot d_v\,K(\delta v)\cdot T^{-1}-T\cdot K\cdot T^{-1}\cdot d_v\,T(\delta v)\cdot T^{-1},}$$

and

$$\mathbf{d_v\,K(G)(\delta v)}$$
$$\mathbf{=d_v\,T(G)\cdot KT^{-1}(\delta v)+T\cdot d_v\,K(G)\cdot T^{-1}(\delta v)-M\cdot d_v\,T(G)\cdot T^{-1}(\delta v),} \qquad \text{(C. 7)}$$

respectively. By means of eqs. (C. 2), $\mathbf{G}=-\mathbf{T}\cdot\mathbf{K}$ and $\mathbf{d_v\,K(G)}=K'_u\cdot\mathbf{N}$, each term of RHS can be reduced to

$$\mathbf{d_v\,T(G)\cdot K\cdot T^{-1}(\delta v)=-d_v\,T(M\cdot\delta v)\cdot N,}$$
$$\mathbf{T\cdot d_v\,K(G)\cdot T^{-1}(\delta v)=T\cdot}\,K'_u\,\mathbf{\cdot N\cdot T^{-1}(\delta v),}$$
$$\mathbf{M\cdot d_v\,T(G)\cdot T^{-1}(\delta v)=-M\cdot d_v\,T(\delta v)\cdot N.} \qquad \text{(C. 8)}$$

From both eqs. (C. 7) and (C. 8), we obtain eq. (4. 9). **[QED]**

Appendix-D

It is necessary to proof that the operator K in eq. (5. 2b) is sympletic in the sence of eqs. (3. 4).

Skew-symmetry: By means of eq. (5. 9) we see

$$\ll\mathbf{a, K[u]b}\gg\simeq I_0<\mathbf{a}\mid\sigma_3\mathbf{b}_x>+2I_0<\mathbf{a}\mid\sigma_3\mathbf{u}>I_\pm<\mathbf{u}\mid\sigma_3\mathbf{b}>,$$
$$\ll\mathbf{b, K[u]a}\gg\simeq I_0<\mathbf{b}\mid\sigma_3\mathbf{a}_x>+2I_0<\mathbf{b}\mid\sigma_3\mathbf{u}>I_\pm<\mathbf{u}\mid\sigma_3\mathbf{a}>.$$

Throughout preceeding discussions we had used the integrator $I_\pm$, but it can be replaced with

$$I=2I_\pm=\left\{\int_{-\infty}^{x}+\int_{+\infty}^{x}\right\}dy. \qquad \text{(D. 1)}$$

Because of integral by part, we see

$$I_0<\mathbf{a}\mid\sigma_3\mathbf{b}_X>+I_0<\mathbf{b}\mid\sigma_3\mathbf{a}_X>=I_0\left\{\frac{\partial}{\partial x}<\mathbf{a}\mid\sigma_3\mathbf{b}>\right\}=0,$$

and obtain

$$I_\pm f I_0 g=-I_0 g I_\pm f. \qquad \text{(D. 2)}$$

After all we obtain

$$\begin{aligned}&\langle\!\langle \mathbf{a}, K[\mathbf{u}]\mathbf{b}\rangle\!\rangle+\langle\!\langle \mathbf{b}, K[\mathbf{u}]\mathbf{a}\rangle\!\rangle\\ &\simeq 2I_0<\mathbf{a}\mid\sigma_3\mathbf{u}>I_\pm<\mathbf{u}\mid\sigma_3\mathbf{b}>+2I_0<\mathbf{b}\mid\sigma_3\mathbf{u}>I_\pm<\mathbf{u}\mid\sigma_3\mathbf{a}>\\ &=I_0<\mathbf{u}\mid\sigma_3\mathbf{a}>I<\mathbf{u}\mid\sigma_3\mathbf{b}>+2I_0<\mathbf{u}\mid\sigma_3\mathbf{b}>I<\mathbf{u}\mid\sigma_3\mathbf{a}>=0.\end{aligned}$$

[QED]

Jacobi Equation :

To check the Jacobi relation, let's substitute $\mathbf{v}=K[\mathbf{u}]\mid\mathbf{q}>$ into

$$\begin{aligned}M'_{\mathbf{u}}(\mathbf{p},\mathbf{v})&=\frac{d}{d\varepsilon}M_{u+\varepsilon v}(\mathbf{p})=\frac{d}{d\varepsilon}K[\mathbf{u}+\varepsilon\mathbf{v}]\mid\mathbf{p}>\\ &=\sigma_3\mid\mathbf{v}>I<\mathbf{u}\mid\sigma_3\mathbf{p}>+\sigma_3\mid\mathbf{u}>I<\mathbf{v}\mid\sigma_3\mathbf{p}>,\end{aligned}$$

then obtain

$$\begin{aligned}M'_{\mathbf{u}}(\mathbf{p};K[\mathbf{u}]\mid\mathbf{q}>)&\simeq\mid\mathbf{q}_X>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{p}>+\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{p}>\\ &+\sigma_3\mid\mathbf{u}>I<\mathbf{p}\mid\mathbf{q}_X>+\sigma_3\mid\mathbf{u}>I<\mathbf{p}\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>. \qquad \text{(D. 3)}\end{aligned}$$

According to eq. (3. 4b), the broblem is reduced to the estimation of the following term,

$$\begin{aligned}&\sum_{p,q,r}\langle\!\langle K'_{\mathbf{u}}(\mathbf{p};K[\mathbf{u}]\mid\mathbf{q}>),\mathbf{r}\rangle\!\rangle\\ &\simeq\sum_{p,q,r}\{I_0<\mathbf{r}\mid\mathbf{q}_X>I<\mathbf{u}\mid\sigma_3\mathbf{p}>+I_0<\mathbf{r}\mid\sigma_3\mathbf{u}>I<\mathbf{p}\mid\mathbf{q}_X>\}\\ &+\sum_{p,q,r}\{I_0<\mathbf{r}\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{p}>+I_0<\mathbf{r}\mid\sigma_3\mathbf{u}>I<\mathbf{p}\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>. \qquad \text{(D. 4)}\end{aligned}$$

The contribution of the first term in the RHS is given by

$$\begin{aligned}&\sum_{p,q,r}\{I_0<\mathbf{r}\mid\mathbf{q}_X>I<\mathbf{u}\mid\sigma_3\mathbf{p}>+I_0<\mathbf{r}\mid\sigma_3\mathbf{u}>I<\mathbf{p}\mid\mathbf{q}_X>\}\\ &=I_0<\mathbf{p}\mid\mathbf{q}>_x I<\mathbf{u}\mid\sigma_3\mathbf{r}>+I_0<\mathbf{q}\mid\mathbf{r}>_x I<\mathbf{u}\mid\sigma_3\mathbf{p}>+I_0<\mathbf{r}\mid\mathbf{p}>_x I<\mathbf{u}\mid\sigma_3\mathbf{q}>\\ &=-I_0<\mathbf{u}\mid\sigma_3\{\mid\mathbf{p}><\mathbf{q}\mid\mathbf{r}>+\mid\mathbf{q}><\mathbf{r}\mid\mathbf{p}>+\mid\mathbf{r}><\mathbf{p}\mid\mathbf{q}>\}=0, \qquad \text{(D. 5)}\end{aligned}$$

because of relations,

$$\begin{aligned}&I_0<\mathbf{p}\mid\mathbf{q}>_x I<\mathbf{u}\mid\sigma_3\mathbf{r}>=<\mathbf{p}\mid\mathbf{q}>I<\mathbf{u}\mid\sigma_3\mathbf{r}>\Big|_{x=-\infty}^{x=\infty}-I_0<\mathbf{p}\mid\mathbf{q}><\mathbf{u}\mid\sigma_3\mathbf{r}>,\\ &\mid\mathbf{p}><\mathbf{q}\mid\mathbf{r}>+\mid\mathbf{q}><\mathbf{r}\mid\mathbf{p}>+\mid\mathbf{r}><\mathbf{p}\mid\mathbf{q}>=0.\end{aligned}$$

We next consider a term included in the second term in eq. (D. 4),

$$\begin{aligned}&I_0<\mathbf{r}\mid\sigma_3\mathbf{u}>I<\mathbf{p}\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>=-I_0<\mathbf{u}\mid\sigma_3\mathbf{r}>I<\mathbf{u}\mid\mathbf{p}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>\\ &=-\{I<\mathbf{u}\mid\sigma_3\mathbf{r}>I<\mathbf{u}\mid\mathbf{p}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>\}_{x=-\infty}^{x=+\infty}+I_0<\mathbf{u}\mid\mathbf{p}>\{I<\mathbf{u}\mid\sigma_3\mathbf{r}>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{q}>\}\\ &=I_0<\mathbf{u}\mid\mathbf{p}>\{I<\mathbf{u}\mid\sigma_3\mathbf{r}>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{q}>\}.\end{aligned}$$

Then second term is reduced to

$$\begin{aligned}&\sum_{p,q,r}\{I_0<\mathbf{r}\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{p}>+I_0<\mathbf{r}\mid\sigma_3\mathbf{u}>I<\mathbf{p}\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>\}\\ &=\sum_{p,q,r}I_0<\mathbf{r}\mid\mathbf{u}>I<\mathbf{u}\mid\sigma_3\mathbf{q}>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{p}>\\ &-\sum_{p,q,r}I_0<\mathbf{u}\mid\mathbf{q}>\{I<\mathbf{u}\mid\sigma_3\mathbf{r}>\cdot I<\mathbf{u}\mid\sigma_3\mathbf{p}>\}=0.\end{aligned}$$

[QED]