Direct Potential Proliferation, Connection with the Riccati Equation and Related Tranformation

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1. Introduction

It is interesting to seek the simplest way solving nonlinear evolution equations (NLEE). A contribution by Crum[1] should be emphasized, where a simple way of potential proliferation had given for the one-dimensional Schrödinger operator and the resulting potential is characterized by a parameter. Wadati et al[2] had pointed out that is applicable for the Bäcklund transformation (BT) solving the integrable NLEE. We thought Crum's method well comparable to a version of Riemann-Hilbert transform (RHT)[3~5] except for the contribution from continuous scattering data. Recently various kinds of transformations have been studied for integrable NLEE's. Our hope is to arive at such a situation based on the RHT. Along this thema we specially show a crucial point for deriving the BT in this note.

Discussions are given generally as possible. Based on the RHT, we derive a general and direct potential proliferation by using a projection matrix. The proliferation formula generally solves soliton solutions, but we distinguish this from the BT. according to Alberty et al,[6] we derive the Riccati equation and change the RHT as suitable for description of the associated transformation between solutions of Riccati eqs. The resulting transformation includes a "singular" one and we find it playing a kee role for deriving the BT. For examples the two-dimensional M×M−AKNS class of NLEE's is discussed and we show how to eliminate the Riccati slution from the transformation.

2. Direct Potential Proliferation

We consider matrices $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ with a parameter λ, satisfying

$$[\tilde{\boldsymbol{\Phi}}(\lambda;\cdot)]^{\mathrm{T}}\boldsymbol{\Phi}(\lambda;\cdot) = \boldsymbol{\Xi}(\lambda), \tag{2.1}$$

where orders $(=M)$ of matrices and the number of dimensions are arbitrary, and $\boldsymbol{\Xi}(\lambda)$ is independent on dimensional variables. We assume another pair of matrices $\boldsymbol{\Phi}^1$ and $\tilde{\boldsymbol{\Phi}}^1$,

$$[\tilde{\boldsymbol{\Phi}}^0]^{\mathrm{T}}\boldsymbol{\Phi}^0 = [\tilde{\boldsymbol{\Phi}}^1]^{\mathrm{T}}\boldsymbol{\Phi}^1, \tag{2.2}$$

and define a transformation as

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$$\Phi^1[\Phi^0]^{-1} = [\tilde{\Phi}^{1T}]^{-1}\tilde{\Phi}^{0T} = \chi = [\tilde{\chi}^T]^{-1}, \tag{2.3}$$

$$\Phi^1 = \chi\Phi^0, \qquad \tilde{\Phi}^{1T} = \tilde{\Phi}^{0T}\tilde{\chi}^T, \tag{2.3}$$

Both upper-scripts "0" and "1" are used to distuinguish both original and excited states, respectively.

A 1-form $\Omega(\lambda; \cdot)$ is introduced to give general evolutions of the system,

$$\mathrm{d}\Phi^0 = \Omega^0\Phi^0, \qquad \mathrm{d}\Phi^1 = \Omega^1\Phi^1, \tag{2.5}$$

and $\mathrm{d}\Phi^0$ means the exterior derivative of $\Phi^0\ (\lambda; \cdot)$. The exteriour derivative of eq. (2.5) defines a flat connection, ie., the integrable condition

$$\mathrm{d}\Omega^0 - \Omega^0 \wedge \Omega^0 = 0, \qquad \mathrm{d}\Omega^1 - \Omega^1 \wedge \Omega^1 = 0. \tag{2.6}$$

Because $\mathrm{d}\Phi^1 = \mathrm{d}(\chi\Phi^0) = (\mathrm{d}\chi + \chi\Omega^0)\Phi^0 = \Omega^1\chi\Phi^0$, we obtain

$$\Omega^1 = \mathrm{d}\chi \cdot \chi^{-1} + \chi\Omega^0\chi^{-1}. \tag{2.7}$$

Since $\mathrm{d}(\tilde{\Phi}^T\Phi) = 0$, adjoint relations are similarly given by

$$\mathrm{d}\tilde{\Phi}^{0T} = -\tilde{\Phi}^{0T}\Omega^0, \qquad \mathrm{d}\tilde{\Phi}^{1T} = -\tilde{\Phi}^{1T}\Omega^1. \tag{2.8}$$

It is necessary to determine both matrices χ and $\tilde{\chi}$, then we assume these as

$$\chi(\lambda; \cdot) = 1 - \alpha(\lambda)P(\cdot), \qquad \tilde{\chi}^T(\lambda; \cdot) = 1 - \tilde{\alpha}(\lambda)\mathrm{P}(\cdot), \tag{2.9}$$

where $P(=P^2)$ a projection matrix, while both scalars α and $\tilde{\alpha}$ constants. Because $\tilde{\chi}^T(\lambda)\chi(\lambda) = 1$, we must set

$$\alpha(\lambda) + \tilde{\alpha}(\lambda) = \alpha(\lambda)\tilde{\alpha}(\lambda). \tag{2.10}$$

We can find a simplest solution of eq. (2.10) as

$$\alpha(\lambda) = \frac{\lambda_0 - \tilde{\lambda}_0}{\lambda - \tilde{\lambda}_0}, \qquad \tilde{\alpha}(\lambda) = \frac{\tilde{\lambda}_0 - \lambda_0}{\lambda - \lambda_0}. \tag{2.11}$$

The projection matrix can be chosen as

$$P = \frac{|\phi_0\rangle\langle\tilde{\phi}_0|}{\langle\tilde{\phi}_0|\phi_0\rangle}, \qquad P^2 = P, \tag{2.12}$$

where $|\phi_0\rangle = \Phi^0(\lambda_0)|c\rangle$, $\langle\tilde{\phi}_0| = \langle\tilde{c}|\tilde{\Phi}^0(\tilde{\lambda}_0)$, c and $\tilde{c}$ are arbitrary constants.

It is not difficult to get the exterior derivateve of eq. (2.12),
the exrivative of eq. (2.12),

$$\mathrm{d}P = (1-P)\Omega^0(\lambda_0)P - P\Omega^0(\tilde{\lambda}_0)(1-P). \tag{2.13}$$

From eqs. (2.7), (2.9), (2.10) and (2.13) we can obtain the proliferation of λ-dependent potential $\Omega(\lambda; \cdot)$,

$$\begin{aligned} \Delta\Omega &\equiv \Omega^1 - \Omega^0 \\ &= \tilde{\alpha}(\lambda)(1-P)\{\Omega^0(\lambda_0) - \Omega^0(\lambda)\}P \\ &\quad + \alpha(\lambda)P\{\Omega^0(\tilde{\lambda}_0) - \Omega^0(\lambda)\}(1-P). \end{aligned} \tag{2.14}$$

3. Riccati Equation and Connection to Projection Matrix

We take a quantity $y_k{}^j = y^j/y^k$, where $|y>\{=|y^1, y^2,...>\}$ is a vector satisfying eq. (2.5), then its derivative is given by[6)]

$$dy_{kj} = \sum_q \Omega_q{}^j y_k{}^q - \sum_q y_k{}^j \Omega_q{}^k y_k{}^q. \tag{3.1}$$

This can be regarded as the Riccati equation, and the conservation laws can be derived actually. For this purpose we define

$$\omega_j = \sum_q \Omega_q{}^j y_j{}^q. \tag{3.3}$$

Here ω_j is a closed form, that is,

$$d\omega_j = \sum_{p,q} (\Omega_p{}^j \wedge \Omega_q{}^j) y^p{}_j y_j{}^q = 0, \tag{3.3}$$

because $(\Omega_p{}^j \wedge \Omega_q{}^j)$ is anti-symmetric while $y_j{}^p y_j{}^q$ symmetric as to (p, q). If we expand $d\omega_j(\lambda)$ as to λ, infinite conservation laws are obtained.[6)]

From eq. (3.1) we can define a vector type of Riccati equation,

$$|dy_j> = (\Omega - \omega_j)|y_j>, \qquad \omega_j = <j|\Omega|y_j>, \tag{3.4}$$

where $|y_j> = |y>/y^j$. Here we must note

$$d\omega_j = <j|\{d\Omega - \Omega\wedge\Omega\}|y_j> + \omega_j\wedge\omega_j = 0,$$

that is, ω_j is a closed 1-form and just equal to the one already introduced in eq. (3.2). Including adjoint relations, we list

$$|d\phi_j> = \{\Omega(\lambda) - \omega_j\}|\phi_j>, \qquad \omega_j = <j|\Omega(\lambda)|\phi_j>, \tag{3.5a}$$

$$<d\tilde{\phi}_k| = -<\tilde{\phi}_k|\{\Omega(\lambda) - \tilde{\omega}_k\}, \qquad \tilde{\omega}_k = <\tilde{\phi}_k|\Omega(\lambda)|_k>, \tag{3.5b}$$

where the upper script "0" meaning the ground state is omitted and

$$|\phi_j(\lambda)> = |\phi>/\phi^j, \qquad <\tilde{\phi}_k(\lambda)| = <\tilde{\phi}|/\phi^k, \tag{3.6}$$

We remark a fact $<j|\phi_j> = 1$, then $0 = <j|d\phi_j>$ also results in $\omega_j = <j|\Omega(\lambda)|\phi_j>$.

The projection matrix in eq. (2.12) can be replaced with

$$P = \frac{|\phi_j(\lambda_0)><\tilde{\phi}_k(\tilde{\lambda}_0)|}{<\tilde{\phi}_k(\tilde{\lambda}_0)|\phi_j(\lambda_0)>}, \tag{3.7}$$

which of course gives the same derivative as in eq. (2.13). The transformation in eq. (2.4) may be translated for the Riccati equattion and it is our interst. Then it is necessary to construct $<\tilde{\phi}_k{}^1|$ and $|\phi_j{}^1>$,

$$|\phi_j{}^1> = (1 - aP)\rho_{1j}{}^0|\phi_j>, \qquad \rho_{1j}{}^0 = \phi^j/\phi^{1j}. \tag{3.8}$$

which must satisfy $|d\phi_j{}^1> = (\Omega^1 - \omega_j{}^1)|\phi_j{}^1>$. It is not dfficult to show this directly, by using $d\rho_{1j}{}^0 = \rho_{1j}{}^0(\omega_j - \omega_j{}^1)$, etc..

The adjoint case is obtained similarly,

$$<\tilde{\phi}_k^{\,1}| = <\tilde{\phi}_k|(1-\tilde{\alpha}P)\tilde{\rho}_{1k}^{\,0}, \tag{3.9}$$

where $\tilde{\rho}_{1k}^{\,0} = \tilde{\phi}_k/\tilde{\phi}^{1k}$, $d\tilde{\rho}_{1k}^{\,0}(\tilde{\omega}_k^{\,1}-\tilde{\omega}_k)$ and $\tilde{\omega}_k = <\tilde{\phi}_k|\Omega'|k>$. We can again show $<\tilde{\phi}_k^{\,1}|$ satisfying $<d\tilde{\phi}_k^{\,1}| = -<\tilde{\phi}_k^{\,1}|(\Omega^1-\tilde{\omega}_k^{\,1})$. We remark that the unknown scalar factor $\rho_{1j}^{\,0}$ in eq. (3.8) does not give any difficuty to construct the next-order projection matrix P^1. This makes it possible to reconstruct the solutions recursively.

Under j, k fixed, it is better to represent the solution of Riccati eq. as

$$|\phi_j> = \sum_p |p>\Phi_j^{\,p}, \qquad <\tilde{\phi}_k| = \sum_q <q|\tilde{\Phi}_q^{\,k}, \tag{3.10}$$

where $\Phi_j^{\,j}(\lambda) = \tilde{\Phi}_k^{\,k}(\lambda) = 1$. From eqs. (3.5) and (3.10) the λ-dependent Riccati eq. can be obtained,

$$d\Phi_j^{\,p} = V_j^{\,p} - V_j^{\,j}\Phi_j^{\,p}, \qquad -d\tilde{\Phi}_p^{\,j} = \tilde{V}_p^{\,j} - \tilde{V}_j^{\,j}\tilde{\Phi}_p^{\,j}, \tag{3.11}$$

where

$$V_j^{\,p}(\lambda) = \sum_k \Phi_j^{\,k}(\lambda)\Omega_k^{\,p}(\lambda), \qquad \tilde{V}_q^{\,j}(\lambda) = \sum_k \tilde{\Phi}_k^{\,j}(\lambda)\Omega_q^{\,k}(\lambda). \tag{3.12}$$

Both eqs. (3.11) and (3.12) self-consistently satisfy $\Phi_q^{\,p}\Phi_j^{\,q} = \Phi_j^{\,p}$.

4. Transformation of Riccati Solutions

We can directly proliferate the solutions of Riccati equation. If we take the j-th component of eq. (3.8),

$$\phi^{1j}(\lambda) = \phi^j(\lambda) - \alpha(\lambda)\frac{<\tilde{\phi}_{0j}|\phi_j(\lambda)>}{<\tilde{\phi}_{0j}|\phi_{0j}>}, \tag{3.8}$$

where $|\phi_{0j}> = |\phi_j(\lambda_0)>$ and $<\tilde{\phi}_{0k}| = <\tilde{\phi}_k(\tilde{\lambda}_0)|$, the following is obtained,

$$|\phi_j^{\,1}(\lambda)> = \frac{|\phi_j(\lambda)><\tilde{\phi}_{0j}|\phi_{0j}> - \alpha(\lambda)|\phi_{0j}><\tilde{\phi}_{0j}|\phi_j(\lambda)>}{<\tilde{\phi}_{0j}|\phi_{0j}> - \alpha(\lambda)<\tilde{\phi}_{0j}|\phi_j(\lambda)>}, \tag{4.1a}$$

$$<\tilde{\phi}_k^{\,1}(\lambda)| = \frac{<\tilde{\phi}_{0k}|\phi_{0k}><\tilde{\phi}_k(\lambda)| - \tilde{\alpha}(\lambda)<\tilde{\phi}_k(\lambda)|\phi_{0k}><\tilde{\phi}_{0k}|}{<\tilde{\phi}_{0k}|\phi_{0k}> - \tilde{\alpha}(\lambda)<\tilde{\phi}_k(\lambda)|\phi_{0k}>}, \tag{4.1b}$$

In these relations both factors $\alpha(\lambda)$ and $\tilde{\alpha}(\lambda)$ diverge at $\lambda = \tilde{\lambda}_0$ and λ_0, respectively. Hence we obtain

$$|\phi_j^{\,1}(\tilde{\lambda}_0)> = |\phi_{0j}>, \qquad <\tilde{\phi}_k^{\,1}(\lambda_0)| = <\tilde{\phi}_{0k}|. \tag{4.2}$$

From substitution of eq. (4.2) into eq. (3.11), the λ-independent Riccati eqs. are obtained as

$$d\Phi_0{}^p{}_j = V_0{}^p{}_j - V_{0j}{}^j\Phi_{0j}{}^p = V_{0j}^{\,1p} - V_{0j}^{\,1j}\Phi_{0j}{}^p, \tag{4.3a}$$

$$-d\tilde{\Phi}_{0p}{}^j = \tilde{V}_{0p}{}^j - \tilde{V}_{0j}{}^j\tilde{\Phi}_{0p}{}^j = \tilde{V}_{0p}^{\,1j} - \tilde{V}_{0j}^{\,1j}\tilde{\Phi}_{0p}{}^j, \tag{4.3b}$$

where $\Phi_0 = \Phi(\lambda_0)$, $V_0 = V(\lambda_0)$ etc., and from these we can get

$$W_{0j}{}^p = W_{0j}{}^j\Phi_{0j}{}^p, \qquad \tilde{W}_{0p}{}^j = \tilde{W}_{0j}{}^j\tilde{\Phi}_{0p}{}^j \qquad (p\neq j), \tag{4.4}$$

where

$$W_{0j}{}^{q} = \sum_k \Lambda_{0k}{}^{q}\Phi_{0j}{}^{k}, \qquad \tilde{W}_{0q}{}^{j} = \sum_k \tilde{\Lambda}_{0q}{}^{k}\tilde{\Phi}_{0k}{}^{j}, \tag{4.5a}$$

$$\Lambda_0 \equiv \Omega^1(\tilde{\lambda}_0) - \Omega(\lambda_0), \qquad \tilde{\Lambda}_0 \equiv \Omega^1(\lambda_0) - \Omega(\tilde{\lambda}_0). \tag{4.5b}$$

Because of

$$\omega_j{}^1(\tilde{\lambda}_0) - \omega_j(\lambda_0) = \langle j|\Lambda_0|\phi_{0j}\rangle, \qquad \tilde{\omega}_j{}^1(\lambda_0) - \tilde{\omega}_j(\tilde{\lambda}_0) = \langle\tilde{\phi}_{0j}|\tilde{\Lambda}_0|j\rangle,$$

relations (4.3) can be written as the eigenvalue equations,

$$\Lambda_0|\phi_{0j}\rangle = |\phi_{0j}\rangle\langle j|\Lambda_0|\phi_{0j}\rangle, \qquad \langle\tilde{\phi}_{0k}|\tilde{\Lambda}_0 = \langle\tilde{\phi}_{0k}|\tilde{\Lambda}_0|k\rangle\langle\tilde{\phi}_{0k}|. \tag{4.6}$$

This relatio makes it pssible to determine both vectors $|\phi_{0j}\rangle$ and $\langle\tilde{\phi}_{0j}|$ from matrices Λ_0 and $\tilde{\Lambda}_0$, but this process is not so clear. Let's consider the first of eq. (4.6),

$$\sum_q \Lambda_{0q}{}^{p}\Phi_{0j}{}^{q} = \sum_q \Lambda_{0q}{}^{j}\Phi_{0j}{}^{q}\Phi_{0j}{}^{p}.$$

If we multiply $\Phi_{0p}{}^{j}$ on the both sides, it can be reduced to

$$\sum_q \Lambda_{0q}{}^{p}\Phi_{0p}{}^{q} = \sum_q \Lambda_{0q}{}^{j}\Phi_{0j}{}^{q}. \qquad (p \neq j) \tag{4.7}$$

As shown later by this relation we can solve $\Phi_{0j}{}^{p}$ basically.

5. Two-Dimensional M×M-AKNS problem and Consevation Laws

In this section we limit the discussion to the case of two dimensios, and give the 1-form Ω explicitely;

$$\Omega(\lambda;\cdot) = \sum_{n=0}^{N} \lambda^n \Omega_n(\cdot) = D(\lambda;x,t)\,dx + F(\lambda;x,t)\,dt, \tag{5.1}$$

and consider the conservation laws which is important to derive the Hamiltonian formalism of problems. The principle is that "$\omega_j(\lambda)$ is a closed form; $d\omega_j = 0$". We denote

$$\omega_j(\lambda) = J_j(\lambda)dx + K_j(\lambda)dt, \tag{5.2}$$

where

$$J_j(\lambda) = \langle j|D(\lambda)|\phi_j(\lambda)\rangle, \qquad K_j(\lambda) = \langle j|F(\lambda)|\phi_j(\lambda)\rangle. \tag{5.3}$$

Taking the exterior derivative of eq. (5.2), we easily obtain

$$\partial_t J_j(\lambda) = \partial_x K_j(\lambda). \tag{5.4}$$

If $K_j(\lambda)$ vanishes rapidly as $x \to \pm\infty$, the λ^{-1}-expansion of $J_j(\lambda)$,

$$J_j(\lambda) = \sum_n \lambda^{-n} J_j{}^{(n)}, \tag{5.5}$$

should give infinite conserved densities under solvable conditions.

As a primitive case for example, we take the M×M−AKNS system defined by

$$D(\lambda;x,t) = i\lambda A + Q(x,t), \tag{5.6}$$

where A is constant and diagonal, while Q is an off-diagonal potential matrix.

We specially take and denote the x-component of eq.(3.5a) as

$$\partial_x|\phi_j> = [D(\lambda) - J_j(\lambda)]\,|\phi_j>, \tag{5.7}$$

and use an expansion for convenience of calculations,

$$|\phi_j(\lambda)> = |j> + \sum_{n=1}^{\infty}\lambda^{-n}\sum_{p}\Phi_{j,n}{}^{k}|k>. \tag{5.8}$$

After tedious calculations we can obtain an explicit formula of Riccati equation,

$$\partial_x\Phi_j{}^k = q_j{}^k + i\lambda(a_k - a_j)\Phi_j{}^k + \sum_m q_m{}^k\Phi_{mj} - \sum_m q_m{}^j\Phi_j{}^m\Phi_j{}^k, \tag{5.9}$$

and expand it as the λ^{-1}-series as

$$\Phi_j{}^k(\lambda) = \sum_{n=1}^{\infty}\lambda^{-n}\Phi_{j,n}{}^k. \qquad (k \neq j) \tag{5.10}$$

Substituting eq.(5.10) into eq.(5.9), we get

$$\lambda^{-0}\ :\ 0 = q_j{}^k + i(a_k - a_j)\Phi_{j,1}{}^k, \tag{5.11a}$$

$$\lambda^{-1}\ :\ \partial_x\Phi_{j,1}{}^k = i(a_k - a_j)\Phi_{j,2}{}^k + \sum_m q_m{}^k\Phi_{j,1}{}^m. \tag{5.11b}$$

It is necessary to take care of the order: $\lambda^{-n}(2 \leqq n)$, from eq.(5.10a) we obtain

$$\partial_x\Phi_{j,n}{}^k = i(a_k - a_j)\Phi_{j,n+1}{}^k + \sum_m q_m{}^k\Phi_{j,n}{}^m - \sum_{p=1,m=1}^{n-1}\sum q_m{}^j\Phi_{j,p}{}^m\Phi_{j,n-p}{}^k. \tag{5.11c}$$

These eqs.(5.17) show that $\Phi_{j,n}{}^k$ can be solved recursively, and from substitution of eqs.(5.3), (5.8) into (5.5), we obtain the infinite conserved densities as

$$J_j{}^{(n)} = \sum_k q_k{}^j\Phi_{j,n}{}^k. \qquad (n=1,2,..) \tag{5.12}$$

6. Derivation of Bäcklund Transformation

As shown in §5 we examine the two-dimensional M×M−AKNS system. The evolution of t is omitted for simlicity.

6A) 2×2−AKNS System: We denote the 2×2-AKNS equation as[4] $|\phi_x> = D(\lambda;Q)\,|\phi>$, where $D(\lambda;Q) = -i\lambda\sigma_3 + Q$, σ_3 one of Pauli's spin matrices. The ket $|\phi>\{=|\phi(\lambda,x)>\}$ means usual column vector, while the bra $<\tilde{\phi}| = (-\tilde{\phi}_2, \tilde{\phi}_1)$ satisfies its adjoint equation, $<\tilde{\phi}_x| = -<\tilde{\phi}|D(\lambda,x)$. The projection matrix is given by

$$P = \frac{|\phi_0><\tilde{\phi}_0|}{<\tilde{\phi}_0|\phi_0>},$$

which still satisfy eq.(2.13) and the potential proliferation (2.15) is reduced to

$$\Delta Q \equiv Q^{(1)} - Q = i(\lambda_0 - \tilde{\lambda}_0)[P, \sigma_3]. \tag{6,2}$$

Elements of Q are denoted as $Q_2{}^1 = q$ and $Q_1{}^2 = r$. If the Riccati eq. is taken as $\Gamma_x = q - r\Gamma^2$

$-2i\lambda\Gamma$, the projection matrix is given by

$$P=\frac{1}{\Gamma_0-\tilde{\Gamma}_0}\begin{pmatrix}\Gamma_0, & -\Gamma_0\tilde{\Gamma}_0\\ 1 & , -\tilde{\Gamma}_0\end{pmatrix}, \tag{6.3}$$

where $\Gamma_0=\Gamma(\lambda_0;x)$ and $\tilde{\Gamma}_0=\Gamma(\tilde{\lambda}_0;x)$. From eqs. (6.2) and (6.3) we can det

$$q^{(1)}=q+\frac{2i(\lambda_0-\tilde{\lambda}_0)\Gamma_0\tilde{\Gamma}_0}{(\Gamma_0\tilde{\Gamma}_0)}, \tag{6.4a}$$

$$r^{(1)}=r+\frac{2i(\lambda_0-\tilde{\lambda}_0)}{(\Gamma_0-\tilde{\Gamma}_0)}. \tag{6.4b}$$

It is also possible to proliferate the solutions of Riccati equations by the fractional formula (4.1) corresponding to

$$\Gamma^{(1)}(\lambda)=\frac{\Gamma(\lambda)(\Gamma_0-\tilde{\Gamma}_0)-\alpha(\lambda)\Gamma_0(\Gamma(\lambda)-\tilde{\Gamma}_0)}{(\Gamma_0-\tilde{\Gamma}_0)-\alpha(\lambda)(\Gamma(\lambda)-\tilde{\Gamma}_0)}, \tag{6.5a}$$

$$\tilde{\Gamma}^{(1)}(\lambda)=\frac{\tilde{\Gamma}(\lambda)(\tilde{\Gamma}_0-\Gamma_0)-\tilde{\alpha}(\lambda)\tilde{\Gamma}_0(\tilde{\Gamma}(\lambda)-\Gamma_0)}{(\tilde{\Gamma}-\Gamma_0)-\tilde{\alpha}(\lambda)(\tilde{\Gamma}(\lambda)-\Gamma_0)}. \tag{6.5b}$$

Both relations (6.4) and (6.5) give a direct and recursive soliton construction, because the Riccati equation is easily solved for the trivial (null) potential. Furthermore this process is simpler than using eqs. (2.4), (2.9), (6.1) and (6.2). It is not difficult to calculate a composite consisting of two transformations characterized by sets of parameters. $(\lambda_0,\tilde{\lambda}_0)$ and $(\lambda_1, \tilde{\lambda}_1)$. It is interesting to examine the commutivity of two possible composits orderd by $\{(\lambda_0, \tilde{\lambda}_0), (\lambda_1, \tilde{\lambda}_1)\}$ and $\{(\lambda_1, \tilde{\lambda}_1), (\lambda_0, \tilde{\lambda}_0)\}$, repectively. This can be shown as abelian then the process of N-soliton can be done uniquely without introducing any freedom.

We can derive the BT without any meditation (P or Γ). It is basically impossible to eliminate the projection by using only eq. (6.2). Then we provide its differentiation as another independent relation. If eq. (2.13) is rewritten as

$$P_x=-i\lambda_0\sigma_3P+i\tilde{\lambda}_0P\sigma_3+i(\lambda_0-\tilde{\lambda}_0)P\sigma_3P+[Q,P], \tag{6.6}$$

the RHS of eq. (6.2) can be differentiated as

$$[P,\sigma_3]_x=i(\lambda_0+\tilde{\lambda}_0)[P,\sigma_3]\sigma_3+i(\lambda_0-\tilde{\lambda}_0)[P\sigma_3P,\sigma_3]+[[Q,P],\sigma_3]. \tag{6.7}$$

The second and third terms of this relation still consist of the projection. From eq. (6.2), however, the projection included in these terms can be eliminated basically, because eq. (6.2) is also solvable inversely. These two terms are written as

$$P\sigma_3P=\rho_0P, \qquad [[Q,P],\sigma_3]=2\rho_0, \tag{6.8}$$

where $\rho_0=(\Gamma_0+\tilde{\Gamma}_0)/(\Gamma_0-\tilde{\Gamma}_0)$. From substitution of eqs. (6.7) and (6.8) into eq. (6.2) we can get

$$(Q^{(1)}-Q)_x=i(\lambda_0+\tilde{\lambda}_0)(Q^{(1)}-Q)\sigma_3+i(\lambda_0-\tilde{\lambda}_0)\rho_0(Q^{(1)}+Q), \tag{6.9a}$$

$$\rho_0=\pm\sqrt{1-\Delta q\cdot\Delta r/(\lambda_0-\tilde{\lambda}_0)^2}. \tag{6.9b}$$

The meaning of these relations becomes clear, applying to such a special case as the nonlinear Schrödinger equation. We simply take a case $r = -q^*$, and the spectrum point as pure imaginary $\lambda_0 = i\eta_0(\eta_0>0)$, then

$$\Gamma^*_0 = -\tilde{\Gamma}_0^{-1}, \qquad \rho_0 = \frac{|\Gamma_0|^2-1}{|\Gamma_0|^2+1}. \tag{6.10}$$

Eqs. (6.4) and (6. 9) are simplified to

$$\Delta q = \frac{4\eta_0\Gamma_0}{1+|\Gamma_0|^2}, \qquad \Delta q_x = -2\eta_0\rho_0(q+q^{(1)}), \tag{6.11a}$$

$$q+q^{(1)} = \frac{2(\Gamma_{0,x}-\Gamma_0^2\Gamma^*_{0x})}{1-|\Gamma_0|^4}. \tag{6.11b}$$

On the other hand we can formally set $\lambda = \tilde{\lambda}_0$ in eq. (6.5), then $\Gamma^{(1)}(\lambda)$ yields a finite value,

$$\Gamma^{(1)}(\tilde{\lambda}_0) = \Gamma_0, \qquad \tilde{\Gamma}^{(1)}(\lambda_0) = \tilde{\Gamma}_0. \tag{6.12}$$

From the view point of RHP, however, this setting is abnormal. This can be said as a "singular" transformation, but plays another key role for deriving the BT (see the former of eq. (6.10)) and the result is exactly same as eqs. (6.9). The intuitional method by konno and Wadati[7] just corresponds to this.

6B) M×M-AKNS System: The case of $M=3$ had been treated by Case and Chiu,[8] only by using the potential proliferation formula,

$$\Delta Q = i(\lambda_0-\tilde{\lambda}_0)[A,P], \tag{6.13}$$

obtained from eq. (2.14), which yet remains the projection. To obtain a complete BT, we must eliminate the projection as shown in the 2×2-case. In this case, however, we note that the variation ΔQ can not be given by P, because the freedom of P is less than the potential's. That is, the potential is under a certain constrain. We must emphasize eq. (6.13) still avairable under such a situation, and its inverse is given by,

$$P_k^{\ j} = \frac{\Delta Q_k^{\ j}}{i(\lambda_0-\tilde{\lambda}_0)(a_j-a_k)} = [\sum_q \tilde{\Phi}_{0q}^{\ k}\Phi_{0j}^{\ q}]^{-1}. \tag{6.14}$$

Because of

$$P^2 = P \qquad \text{or} \qquad P_j^{\ j} = \sum_q P_q^{\ j}P_j^{\ q},$$

the diagonal entry is calculated as

$$P_j^{\ j} = (1/2)\left\{1 \pm \sqrt{1-4\sum_{q(\neq j)} P_q^{\ j}P_j^{\ q}}\right\}. \tag{6.14b}$$

Similaly to eq. (6.7) we consider

$$P_x = [Q,P] - i(\lambda_0-\tilde{\lambda}_0)PAP + i\lambda_0 AP - i\tilde{\lambda}_0 PA,$$
$$[A,P]_x = i\lambda_0 A[A,P] - i\tilde{\lambda}_0[A,P]A - i(\lambda_0-\tilde{\lambda}_0)[A,PAP] + [A,[Q,P]].$$

The potential is differentiated as

$$\Delta Q_x = i(\lambda_0 A\cdot\Delta Q - \tilde{\lambda}_0\Delta Q\cdot A) + i(\lambda_0-\tilde{\lambda}_0)[A,[Q,P]] + (\lambda_0-\tilde{\lambda}_0)^2[A,PAP]. \tag{6.15}$$

For elimination of P, we take the j-k element rf eq. (6.15) and use eq. (6.14),

$$<j|[A,[Q,P]]|k> = \frac{(a_j-a_k)}{i(\lambda_0-\tilde{\lambda}_0)}\sum_q \left\{ \frac{Q_q^{\ j}\Delta Q_k^{\ q}}{a_q-a_k} - \frac{Q_k^{\ q}\Delta Q_q^{\ j}}{a_j-a_q} \right\} - (a_j-a_k)(P_j^j-P_k^k)Q_k^j,$$

$$<j|[A,PAP]|k> = -\frac{a_j-a_k}{(\lambda_0-\tilde{\lambda}_0)^2}\sum_q \frac{\Delta Q_q^{\ j}a_q\Delta Q_k^{\ q}}{(a_j-a_q)(a_q-a_k)} + (a_j-a_k)(a_jP_j^j+a_kP_k^k)P_k^j,$$

where Σ must be taken for $q \neq j, k$. Eq. (6.15) can be reduced to

$$\Delta Q_{k,x}^{\ j} = i(\lambda_0 a_j-\tilde{\lambda}_0 a_k)\Delta Q_k^{\ j} - i(\lambda_0-\tilde{\lambda}_0)\left\{ (a_jQ_k^{\ 1j}-a_kQ_k^{\ j})P_j^j + (a_kQ_k^{\ 1j}-a_jQ_k^{\ j})P_k^k \right\}$$

$$+\sum_q \left\{ \frac{\Delta Q_k^{\ q}}{a_k-a_q}(a_kQ_q^{\ 1j}-a_jQ_q^{\ j}) + \frac{\Delta Q_q^{\ j}}{a_q-a_j}(a_jQ_k^{\ 1q}-a_kQ_k^{\ q}) \right\}. \qquad (6.16)$$

The singular transformation (4.1) is expected to give the same rsult as eq. (6.16), but we omit it here. However, we can show that functions Φ^j_{0k} may be determined from eq. (4.4), which is explicitly given by

$$\Delta Q_j^{\ p} + \sum_q \Delta Q_q^{\ p}\Phi_{0j}^{\ q} - \left\{ i(\lambda_0-\tilde{\lambda}_0)(a_p-a_j) + \sum_q \Delta Q_q^{\ j}\Phi_{0j}^{\ q} \right\}\Phi_{0j}^{\ p} = 0, \quad (6.17a)$$

$$\Delta Q_p^{\ j} + \sum_q \Delta Q_p^{\ q}\tilde{\Phi}_{0j}^{\ q} - \left\{ i(\lambda_0-\tilde{\lambda}_0)(a_p-a_j) + \sum_q \Delta Q_j^{\ q}\tilde{\Phi}_{0q}^{\ j} \right\}\tilde{\Phi}_{0p}^{\ j} = 0. \quad (6.17b)$$

If we introduce

$$\Delta g'_j = \sum_q \Delta Q_q^{\ j}\Phi_{0j}^{\ q}, \qquad (6.18)$$

eq. (6.17a) can be reduced to

$$\Delta g'_j - \Delta g'_k = i(\lambda_0-\tilde{\lambda}_0)\cdot(a_j-a_k). \qquad (6.19)$$

A general solution is easily given by

$$\Delta g'_j = i(\lambda_0-\tilde{\lambda}_0)a_j + \zeta, \qquad (j=1,2..,M) \qquad (6.20)$$

and we remark that an arbitrary function ζ is equal to the value of both sides in eq. (4.7). From eqs. (6.4) and (6.6) we can omit $\Delta g'_j$ and change the notation to the one consisting of components $\phi_0^{\ k}\ (k=1,2,..,M)$,

$$[\zeta - \{ D^1(\tilde{\lambda}_0) - D(\lambda_0) \}]|\phi_0> = 0. \qquad (6.21)$$

This is homogeneous as to $|\phi_0> \simeq |\phi_{0j}>$, that is, equal to eq. (4.6) and $\zeta = <j|\{ D^1(\tilde{\lambda}_0) - D(\lambda_0) \}|\phi_{0j}>$. If we want to get a nontrivial solution, the characteristic equation

$$\det[\zeta - \{ D^1(\tilde{\lambda}_0) - D(\lambda_0) \}] = 0, \qquad (6.22)$$

must be solved but it is not easy to solve when the order is higher. At least we say that the eigenvalue problem can be solved.

7. Concluding Remarks and Discussions

A direct potential proliferation formula had been derived for generally integrable NLEE and translated into the one of Riccati equation. That a "fractional" transformation is obtained. The

RHT of course enables us to describe contributions from continuous scattering data, but we can not get a good sucsess relating the analysis for continuous scattering data[9] with the present case. We must note the form of projection in eq. (2.9) is not general, ie., such a sine-Gordon type of equation can not be treated without some improvements. Furthermore we can employ a more complicated projection matrix,[5] which may effect to the freedom in eq.(2.14).

It was shown that the BT can be derived in two ways. The one directly uses the potential proliferation (6.2), while a singular transformation (4.2) gives the other. Along this line we obtained the BT, that is, eqs.(6.9) or (6.16) is regarded as a nonlinear equation recursively determine potentials $\{Q^1, Q\}$. As shown in eqs.(6.21) and (6.22) it is also possible to obtain a relation between potentials $\{Q^1, Q\}$ and Riccati solutions solutions $\{\Phi_{0k}{}^{j}, \tilde{\Phi}_{0k}{}^{j}\}$, without any their derivatives. For the case of $M = 2$, we obtain

$$\Phi_{01}{}^2 = \{ i(\tilde{\lambda}_0 - \lambda_0) + \sqrt{\Delta q \cdot \Delta r - (\lambda_0 - \tilde{\lambda}_0)^2} \} / \Delta q. \tag{7.1}$$

It is well-known that a psedopotential plays an important role for deriving the BT and infinite dimensional algebra relating with symmetric transformation. We must note $\Phi_{01}{}^2$ regarded as a psedopotential. We can expect to connect our method with the infinite dimensional algebra. It is interesting to research the connection of that algebra with the scattering matrix defined in the RHT or inverse scattering method.

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