Generalized AKNS Class of the Nonlinear Evolution Equations and Its Trace Formula and Dynamical Structures

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The generalized theory belonging to the AKNS class of nonlinear evolution equations is reviewed and some topics relating with dynamical natures are discussed rigorously. The general solvable class with a closed formula is given from directly solving its integrable conditions and from analysis of squared eigenstates. Conservbation laws are derived by using both trace method and squared eigenvalue problem. We naturally define a cannonical equation of course equivalent to the generalized equation and the corresponding Poisson bracket. Each constant of motions are prooved to commute each other, then we show an existense of infini tesimal cannonical transformation which allows the system an infinite dimensional abelian symmetry corresponding to the "half" Kac-Moody Lie algebra. This representation directly connects to the infinite conservations of integrable nonlinear systems because of using a cannonical frame.

§ 1. Introduction

The inverse scattering transform (IST)[1] is powerful not only for solving the initial value problem of nonlinear evolution equations (NLEE's) but also for the analysis of that dynamical tstructures. The interpretation of the IST as a cannonical transformation was first given by Zakharov and Faddeeve[2] for the KdV equation, where the sympletic form was used to prove the cannonical nature. The algebraic 2×2-class of NLEE's (say "AKNS -class"[1]), on the other hand, was also treated by several authors, Zakharov-Manakov,[3] Flaschka-Newell,[4] Kodama[5] and Dodd-Bullough[6] etc, where the Poisson bracket was also used.

Since several years ago we have been interested in symmetries, appearing in integrable systems, specially relating with a new mathematical concept "Kac-Moody Lie algebras".[7] The "half" of a Kac-Moody algebra is its subalgebra,

$$[M_a^{(n)},\ M_b^{(m)}] = C_{abc}\ M_c^{(n+m)} \quad \text{for n, m}=0,1,2\cdots\infty. \tag{1.1}$$

That is, this subalgebra is $G\times C[t,t^{-1}]$, which is associated with a finite-parameter simple Lie group G. A representation of this generators$(n\geqq0)$ is $M_a^{(n)}=T^a\times t^n$, where T^a is a generator of G and t is a variable. For example the group SU(2) has three generators $T^a=\sigma_a/2i\ (a=1,2,3)$ and $[T^a,\ T^b]=\varepsilon_{abc}T^C$,

$$M_3^{(n)}=\frac{1}{2i}\sigma_3\times t^n=\frac{1}{2i}\begin{bmatrix} t^n, & 0 \\ 0, & -t^n\end{bmatrix}\quad, \text{ etc.} \tag{1.2}$$

where ε_{abc} $(=C_{abc})$ a complete antisymmetric tensor. Of course the realization of $M_a^{(n)}$ should be different as what a problem we consider.

According to Eichenherr[8] and others,[9] where they based on the Riemann-Hilbert problem,[10] we had considered the symmetric transformations of the N×N-class of NLEE's[11] relating with the Kac-Moody algebras. The matrix algebra through these gave a representation of the Kac-Moody algebras, but we did not find the existence of conservation laws.

In this paper we summarize a rigorous treatment for the generalized AKNS class of NLEE's. In §2, the algebraic class of AKNS solvable equations is determined from the integrable[4] condition, while also obtained by using squared eigenfunctions in §3. The trace method[4] is introduced in §4, which gives a relation between diagonal entries of scattering matrix S and a differential operator of the AKNS eigenvalue problem. The conservation laws are derived in §5, where we use the trace formula and eigenvalue equations of squared eigenfunctions. In §6, we derive a cannonical equation equivalent to the generalized NLEE and define a Poisson bracket naturally. It can be shown that constants of motions commute each other. By this fact we can find an infinitesimal cannonical transformation which allows an infinite dimensional Lie algebra. This is a realization of the Kac-Moody Lie algebras and it surely relates with the infinite conservation laws.

§2. AKNS Equation and Integrable Condition[1]

The AKNS equation is given by

$$u_x = D(\lambda; x,\ t)u, \quad u_t = F(\lambda; x,\ t)u, \tag{2. 1}$$

where D and F "are traiceless 2x2-matices. Specially the matrix D is taken as

$$D(\lambda : x,\ t) = -i\lambda\,\sigma_3 + Q(x,\ t)\ , \tag{2. 2}$$

which consists of a spectral parameter λ, σ_3 one of Pauli spin matrices $\{\sigma_j\,; j=1,2,3\}$ and an off-diagonal potential $Q(x,\ t)$,

$$Q(x,\ t) = \begin{bmatrix} 0, & q(x,\ t) \\ r(x,\ t), & 0 \end{bmatrix} \quad . \tag{2. 3}$$

It is basic to define the Jost functions $\Phi^{\pm}$ and scattering matrix S as

$$\Phi_X^{\pm} = D(\lambda, x)\Phi^{\pm}, \quad \Phi^{\pm}(\lambda, x) \to e^{-i\lambda\sigma_3 x} \quad \text{for } x \to \pm\infty, \tag{2. 4a}$$

$$\Phi^{-}(\lambda, x) = \Phi^{+}(\lambda, x)S(\lambda)\ , \tag{2. 4b}$$

where t is omitted for simplicity. We note det. $\Phi^{\pm} = 1$ and $[\Phi^{\pm}]^{-1} = [\Phi^{\pm}]^{\dagger}$, where "†" means adjoint. The analytical propertiy of vector Jost components $\phi_j^{\pm}$ and diagonal entries s_{jj} of S-matrix is well-known, that is , functions $\{\phi_1^{-}(\lambda, x), \phi_2^{+}(\lambda, x), s_{11}(\lambda)\}$ are analytic on the upper λ-plane, while $\{\phi_1^{+}(\lambda, x), \phi_2^{-}(\lambda, x), s_{22}(\lambda)\}$ on the lower plane.

For eq. (2 . 1) we must provide the integrable condition,

$$D_t - F_X + [D, F] = 0\quad , \tag{2. 5}$$

obtained from cross-differentiation of eq. (2 . 1). If $F(\lambda; x, t)$ is taken as entire as to λ, we possibly find the coefficients of expansions determined recursively. To make clear this procedure, we introduce some conventional notations,

$$F=\begin{bmatrix} A, & B \\ C, & -A \end{bmatrix}, \ |w>=\begin{bmatrix} r \\ q \end{bmatrix}, \ |h>=\begin{bmatrix} C \\ B \end{bmatrix} . \tag{2.6}$$

and a bra vector $<h| \equiv (-h_2, h_1)^T$ adjoint to the ket $|h>=(h_1, h_2)$.
Then the integrable condition (2. 5) is reduced to

$$A_x=<w|h> \quad , \tag{2.7a}$$
$$|h_x>-2i\lambda\sigma_3|h>=|w_t>+2A\sigma_3|w> . \tag{2.7b}$$

We expand the vector $|h>$ and scalar A as to λ,

$$A=\sum_{n=0}^{N} \lambda^n A^{(n)}, \quad |h>=\sum_{n=0}^{N} \lambda^n |h^{(n)}>. \tag{2.8}$$

Substituting these into eqs. (2 . 7), we obtain

$$A_x^{(n)} =<w|h^{(n)}> \quad , \quad (0\leqq n\leqq N) \tag{2.9}$$
$$\sigma_3|h^{(n)}>=0 \quad , \tag{2.10a}$$
$$|h_x^{(n)}>-2i\sigma_3|h^{(n-1)}>=2A^{(n)}\sigma_3|w>, \quad (1\leqq n\leqq N) \tag{2.10b}$$
$$|h_x^{(0)}>=|w_t>+2A^{(0)}\sigma_3|w> \quad . \tag{2.10c}$$

These can be regarded as the differetial-differece equation for unkowns $A^{(n)}$ and $|h^{(n)}>$.
For solving this we define the following intergral-differential operator,

$$\Lambda_-=\frac{i}{2}\{\sigma_3\partial_X-2W_-[x,\mathrm{d}y]\} =\sigma_3\sigma_1[\widetilde{\Lambda}]\sigma_1\sigma_3 \quad , \tag{2.11}$$

where

$$W_\pm[x,\mathrm{d}y] =\sigma_3\sigma_1|w(x)>\int_{\pm\infty}^{X}\mathrm{d}y<w(y)|\sigma_1\sigma_3 \quad . \tag{2.12}$$

After that we get

$$\sigma_3|h^{(N)}>=0 \ ,$$
$$|h^{(n-1)}>=\widetilde{\Lambda}|h^{(n)}>+ia_n|w> \qquad (1\leqq n\leqq N), \tag{2.13}$$

where a_n is an integral constant for eq.(2 . 9). The last one of eq. (2.10 c) should represent the solvable nonlinear equation,

$$|w_t>=2i\sigma_3|h^{(-1)}> \ ,$$

where $|h^{(-1)}>$ can be obtained from generalization of the recursion relation (2.13),

$$|h^{(-1)}>=i\Omega(\widetilde{\Lambda})|w> \ ,$$
$$\Omega(z)=a_Nz^N+a_{N-1}z^{N-1}+ \qquad +a_1z+a_0 \quad . \tag{2.14}$$

The solvable class of nonlinear equation can be given by

$$\{\partial_t-2\Omega(\Lambda_-)\sigma_3\}\sigma_1|w>=0 \quad . \tag{2.15}$$

Corresponding to A and $|h>$, the followings are similarly obtained,

$$A=\Omega(\lambda)+i\sum_{k=0}^{N}\lambda^{k-1}\sum_{j=0}^{N-k}a_{N-j}\int_{-\infty}^{X}<w|\tilde{\Lambda}^{N-k-j}|w>\,\mathrm{d}y \quad , \tag{2. 16}$$

$$|h>=i\sum_{k=1}^{N}\lambda^{k-1}\sum_{j=0}^{N-k}a_{N-j}\,\tilde{\Lambda}^{N-k-j}|w> \, . \tag{2. 17}$$

We specially list the case of $N=3$,[1)]

$$A=\Omega(\lambda)-\frac{i}{4}a_3(qr_x-q_xr)+\frac{1}{2}(a_2+\lambda a_3)qr \quad , \tag{2.18 a}$$

$$B=i\,a_3(-\frac{1}{4}q_{xx}+\frac{1}{2}q^2r+\frac{i}{2}\lambda q_x+\lambda^2q)+i\,a_2(\frac{i}{2}q_x+\lambda q)+i\,a_1q \quad , \tag{2.18 b}$$

$$C=i\,a_3(-\frac{1}{4}r_{xx}+\frac{1}{2}qr^2-\frac{i}{2}\lambda r_x+\lambda^2r)+i\,a_2(-\frac{i}{2}r_x+\lambda r)+i\,a_1r \quad . \tag{2.18 c}$$

$$q_t+\frac{i}{4}a_3(q_{xxx}-6qrq_x)+\frac{1}{2}a_2(q_{xx}-2q^2r)-i\,a_1q_x-2a_0q=0 \quad , \tag{2.19 a}$$

$$r_t+\frac{i}{4}a_3(r_{xxx}-6rqr_x)-\frac{1}{2}a_2(r_{xx}-2r^2q)-i\,a_1r_x+2a_0r=0 \quad . \tag{2.19 b}$$

The well-known integrable equations are found as

(1) $a_0=a_1=a_2=0$, $a_3=-4i$.

(1 a) $r=-1$: KdV equation,

$$q_t+6qq_x+q_{xxx}=0 \quad . \tag{2. 20}$$

(1 b) $r=mq(m=\pm1)$: M-KdV equation,

$$q_t-6mq^2q_x+q_{xxx}=0 \quad . \tag{2. 21}$$

(2) $a_0=a_1=a_3=0$, $a_2=-2i$ and $r=mq^*$ $(m=\pm1)$; NLS equation,

$$i\,q_t+q_{xxx}-2m|q|^2q=0 \quad . \tag{2. 22}$$

Specially for eq. (2.22) with independent potential q and r, the matrix F is given by

$$F=\begin{pmatrix} -2i\lambda^2-iqr, & 2\lambda q+iq_x \\ 2\lambda r-ir_x, & 2i\lambda^2+iqr \end{pmatrix} \quad . \tag{2. 23}$$

§ 3. Squared Eigenstates and Solvable System

The AKNS solvable system can be reformulated by the squared eigenfunctions. For this purpose we define

$$\begin{aligned}\Phi^{(j,k)}\equiv|\phi_j><\phi_k|&=\begin{pmatrix} -\phi_{1j}\phi_{2k} \ , & \phi_{1j}\phi_{1k} \\ -\phi_{2j}\phi_{2k} \ , & \phi_{2j}\phi_{1k}\end{pmatrix}\\ &=\Phi_D^{(j,k)}+\Phi_O^{(j,k)},\end{aligned} \tag{3. 1}$$

where Φ_D and Φ_O are diagonal and off-diagonal, respectively. We easily find

$$\Phi_x^{(j,k)}=[D(\lambda,x),\ \Phi^{(j,k)}]. \tag{3. 2}$$

From substitution of eq. (3. 1) into eq. (3. 2), we obtain

$$\Phi_{D,\,x}=[Q,\ \Phi_O] \quad , \tag{3. 3a}$$

$$\Phi_{O,\,x}=-i\lambda[\sigma_3,\ \Phi_O]+[Q,\ \Phi_D] \quad . \tag{3. 3b}$$

We define both scalar and vector types of squared functions,

$$\Phi_S \equiv \Phi_{11} - \Phi_{22} = \langle \phi_j | \sigma_3 | \phi_k \rangle \quad ,$$

$$| \Phi_v^{(j,k)} \rangle \equiv | \phi_j \times \phi_k \rangle = \begin{bmatrix} \phi_{1j} & \phi_{1k} \\ \phi_{2j} & \phi_{2k} \end{bmatrix} \quad ,$$

then eqs. (3. 3) can be reduced to

$$\Phi_{S,X} = 2 \langle w | \sigma_3 \sigma_1 | \Phi_v \rangle \quad , \tag{3. 4a}$$

$$(\partial_X + 2i\lambda\sigma_3) | \Phi_v \rangle = -\Phi_S \sigma_1 | w \rangle \quad , \tag{3. 4b}$$

where $(q\Phi_{21} - r\Phi_{12}) = \langle w | \sigma_3\sigma_1 | \Phi_v \rangle$ is used.

There exist various squared eigenstates, but it is sufficient to deal with three types of squared functions,

$$\{\Phi_S^{+P}, | \Phi_v^{+P} \rangle\} = \{\langle \phi_2^+ | \sigma_3 | \phi_2^+ \rangle, \ | \phi_2^+ \times \phi_2^+ \rangle\}, \tag{3. 5a}$$

$$\{\Phi_S^{-P}, | \Phi_v^{-P} \rangle\} = \{\langle \phi_1^- | \sigma_3 | \phi_1^- \rangle, \ | \phi_1^- \times \phi_1^- \rangle\}, \tag{3. 5b}$$

$$\{\Phi_S^{0P}, | \Phi_v^{0P} \rangle\} = \{\langle \phi_1^- | \sigma_3 | \phi_2^+ \rangle, \ | \phi_1^+ \times \phi_2^+ \rangle\}, \tag{3. 5c}$$

all of which are analytic on the upper λ-plane.

Caused from boundary conditions of Jost functions, the asymptotic behaviour of scalar functions are made clear,

$$\Phi_S^{\pm P}(\lambda, x) \to 0 \qquad \text{as} \qquad x \to \pm\infty \quad , \tag{3. 6a}$$

$$\Phi_S^{0P}(\lambda, x) \to -s_{11} \qquad \text{as} \qquad x \to \pm\infty \quad . \tag{3. 6b}$$

From eqs. (3. 4) and (3. 5) the condition (3. 6a) yields

$$\Phi_S^{\pm P}(\lambda, x) = 2\int_{\pm\infty}^{X} \langle w(y) | \sigma_3\sigma_1 | \Phi_v^{\pm P}(\lambda, x) \rangle dy \quad , \tag{3. 7a}$$

$$(\partial_X + 2i\lambda\sigma_3) | \Phi_v^{\pm P}(\lambda, x) \rangle = -2\sigma_1 | w(x) \rangle \int_{\pm\infty}^{X} \langle w(y) | \sigma_3\sigma_1 | \Phi_v^{\pm P}(\lambda, y) \rangle dy. \tag{3. 7b}$$

While the condition (3. 6b) similarly results in

$$\Phi_S^{0P}(\lambda, x) = 2\int_{\pm\infty}^{X} \langle w(y) | \sigma_3\sigma_1 | \Phi_v^{0P}(\lambda, y) \rangle dy - s_{11}(\lambda) \quad . , \tag{3. 8a}$$

$$\begin{aligned}(\partial_X + 2i\lambda\sigma_3) | \Phi_v^{0P}(\lambda, x) \rangle \\ = -2\sigma_1 | w(x) \rangle \int_{\pm\infty}^{X} \langle w(y) | \sigma_3\sigma_1 | \Phi_v^{0P}(\lambda, y) \rangle dy + \sigma_1 | w(x) \rangle s_{11}(\lambda) \ .\end{aligned} \tag{3. 8b}$$

Both eqs. (3. 7b) and (3. 8b) may be regarded as eigenvalue problems,

$$\Lambda^{\pm}(x) | \Phi_v^{\pm P}(\lambda, x) \rangle = \lambda | \Phi_v^{\pm P}(\lambda, x) \rangle \quad , \tag{3. 9a}$$

$$\Lambda^{\pm}(x) | \Phi_v^{0P}(\lambda, x) \rangle = \lambda | \Phi_v^{0P}(\lambda, x) \rangle + (i/2)\sigma_3\sigma_1 | w(x) \rangle s_{11}(\lambda), \tag{3. 9b}$$

where $\Lambda^{\pm}$ are λ-independent integro-differential operators already defined in eq. (2.12),

$$\Lambda^{\pm}(x) = (i/2)\{\sigma_3\partial_X - 2\sigma_3\sigma_1 | w(x) \rangle \int_{\pm\infty}^{X} dy \langle w(y) | \sigma_1\sigma_3 \ \} \quad . \tag{3. 10}$$

The variation of S-matrix caused from potentials is given by

$$\delta S(\lambda) = \int_{-\infty}^{\infty} [\Phi^+(\lambda, y)]^{-1} \delta Q(y) \cdot \Phi^-(\lambda, y) dy \quad . \tag{3. 11}$$

Regarding this variation depending on t, we obtain

$$S^{-1}S_t=\int_{-\infty}^{\infty}[\Phi^-]^{-1}\ Q_t\ \Phi^-\mathrm{d}y \quad . \tag{3. 12}$$

We introduce a matrix $H(\lambda)$, which is independent on $t-x$ and still commutes with σ_3, then

$$\frac{\partial}{\partial x}\{[\Phi^-]^{-1}H\Phi^-\}=[\Phi^-]^{-1}[H,\ Q]\Phi^- \quad . \tag{3. 13}$$

Considering the boundary condition in eq. (2. 4 a) and $[H,\sigma_3]=0$, we integrate eq.(3.13) and obtain

$$S^{-1}[H,S]=\int_{-\infty}^{\infty}[\Phi^-]^{-1}[H,Q]\Phi^-\mathrm{d}x \quad . \tag{3. 14}$$

If we impose the following t-dependence,

$$S_t=[H,S] \quad , \tag{3. 15}$$

both eqs. (3.12) and (3.14) are reduced to [12]

$$\int_{-\infty}^{\infty}[\Phi^-]^{-1}\{Q_t-[H,Q]\}\Phi^-\mathrm{d}y=0 \quad . \tag{3. 16}$$

If we set $H(\lambda)=h(\lambda)\sigma_3$,

$$Q_t-[H,Q]=\begin{bmatrix} 0, & q_t-2h(\lambda)q \\ r_t+2h(\lambda)r, & 0 \end{bmatrix} \quad ,$$

$$\begin{aligned} <u|\{Q_t-[H,Q]\}|v> &= <u\times v|\sigma_1|w_t+2h\sigma_3 w> \\ &= <w|\sigma_1\overleftarrow{\partial_t}-2h\sigma_3\sigma_1|u\times v> \quad . \end{aligned}$$

We further note

$$[\Phi^-]^{\dagger}\ Q_t\ \Phi^-=\begin{bmatrix} -<\phi_2^+\times\phi_1^-|\sigma_1|w_t>, & -<\phi_2^+\times\phi_2^+|\sigma_1|w_t> \\ <\phi_1^-\times\phi_1^-|\sigma_1|w_t>, & <\phi_1^-\times\phi_2^+|\sigma_1|w_t> \end{bmatrix} \quad .$$

After all eq. (3.16) can be reduced to

$$\int_{-\infty}^{\infty}<\phi_1^-\times\phi_1^-|\ \sigma_1\ \{\partial_t+2h(\lambda)\sigma_3\}|w>\mathrm{d}x=0 \quad , \tag{3.17 a}$$

$$\int_{-\infty}^{\infty}<\phi_2^+\times\phi_2^+|\ \sigma_1\ \{\partial_t+2h(\lambda)\sigma_3\}|w>\mathrm{d}x=0 \quad . \tag{3.17 b}$$

As shown by AKNS, [1] squared vectors spun a vector space and above relations means a vector $\sigma_1\{\partial_t+2h(\lambda)\sigma_3\}|w>$ to be zero. But this is not true since in eqs. (3.17) $h=h(\lambda)$ exists. We must eliminate this λ-dependence. Considering $(\sigma_1\sigma_3)^{\dagger}=\sigma_3\sigma_1=-\sigma_1\sigma_3$ and $<\Phi_v|A|w>=-<w|A^{\dagger}|\Phi_v>$, we use eq. (3. 9 a) then find

$$\begin{aligned} <\phi_1^-\times\phi_1^-|h(\lambda)\sigma_1\sigma_3|w> &= <w|\sigma_1\sigma_3 h(\lambda)|\phi_1^-\times\phi_1^-> \\ &= <w|\sigma_1\sigma_3 h(\Lambda^-)|\phi_1^-\times\phi_1^-> \quad . \end{aligned}$$

According to eq. (A. 4) shown in Appendix A, relating parts of eqs. (3.17) are written as

$$\int_{-\infty}^{\infty}<w|\sigma_1\sigma_3 h(\Lambda^{\pm})|\Phi_v^{\pm P}>\mathrm{d}x=-\int_{-\infty}^{\infty}<\Phi_v^{\pm P}|h([\Lambda^{\pm}]^{\dagger})\sigma_3\sigma_1|w>\mathrm{d}x \quad .$$

Hence we finally obtain

$$\{\partial_t-2h([\Lambda^{\pm}]^{\dagger})\sigma_3\}\sigma_1|w>=0 \quad . \tag{3. 19}$$

This is completely equivalent to eq. (2.15) and represents the nonlinear equation which can be solved from analysis of AKNS eigenvalue problem.

§ 4. Green function and Trace Formula

The AKNS eigenvalue problem can be written as

$$L\Phi^{\pm}\equiv i\sigma_3(\partial_X - Q)\Phi^{\pm}=\lambda\Phi^{\pm} \quad . \tag{4. 1}$$

For eq. (4. 1) we can define the Green function as

$$(L-\lambda)G(\lambda;x,y)=\delta(x-y). \tag{4. 2}$$

It is well-known that the Green function corresponds to the resolvent kernel or inverse operator of $(L-\lambda)$. It is not difficult to write down the Green function,[6)]

$$G(\lambda;x,y)=\begin{cases} G_P(\lambda;x,y) & (\mathrm{Im.}\lambda>0) \quad , \\ G_N(\lambda;x,y) & (\mathrm{Im.}\lambda<0) \quad , \end{cases} \tag{4. 3}$$

where $G_P(\lambda)$ and $G_N(\lambda)$ are analytic on the upper and lower λ-plane, respectively,

$$G_P(\lambda;x,xy)=\begin{cases} -i|\psi_2^+(\lambda,x)>\dfrac{e^{i\lambda(x-y)}}{s_{11}(\lambda)}<\psi_1^-(\lambda,y)|\sigma_3 & (y<x), \\ -i|\psi_1^-(\lambda,x)>\dfrac{e^{i\lambda(y-x)}}{s_{11}(\lambda)}<\psi_2^+(\lambda,y)|\sigma_3 & (y>x), \end{cases} \tag{4. 4a}$$

$$G_N(\lambda;x,y)=\begin{cases} i|\psi_1^+(\lambda,x)>\dfrac{e^{i\lambda(y-x)}}{s_{22}(\lambda)}<\psi_2^-(\lambda,y)|\sigma_3 & (y<x), \\ i|\psi_2^-(\lambda,x)>\dfrac{e^{i\lambda(x-y)}}{s_{22}(\lambda)}<\psi_1^+(\lambda,y)|\sigma_3 & (y>x), \end{cases} \tag{4. 4b}$$

and $\psi_1=\phi_1\exp(i\lambda x)$, $\psi_2=\phi_2\exp(-i\lambda x)$.

If we denote a trivial potential as $Q^0(\equiv 0)$ and the corresponding operator L^0, the trace formula $R(\lambda)$ and its kernel $g(x,y)$ are defined by[4)]

$$R(\lambda)=\mathrm{Tr}.D'(\lambda)=\int_{-\infty}^{\infty}\mathrm{Tr}.g(x,x)\,\mathrm{d}x \quad , \tag{4. 5}$$

where

$$D'(\lambda)=(L-\lambda)^{-1}-(L^0-\lambda)^{-1} \quad , \tag{4. 6}$$

$$g(x,y)=G(\lambda;x,y)-G^0(\lambda;x,y) \quad . \tag{4. 7}$$

From eqs. (4. 4a) we can obtain

$$G_P(\lambda;x,x)=-i|\phi_2^+(\lambda,x)>\frac{1}{s_{11}(\lambda)}<\phi_1^-(\lambda,x)|\sigma_3 \quad ,$$

$$G_P^0(\lambda;x,x)=i\begin{bmatrix} 0, & 0 \\ 0, & 1 \end{bmatrix} \quad ,$$

and note the relation

$$\mathrm{Tr}.\ \{|\phi_2^+><\phi_1^-|\sigma_3\}=<\phi_1^-|\sigma_3|\phi_2^+>\equiv\Phi_S^{0P} \quad .$$

By these relations the function $R(\lambda)$ is given by

$$R(\lambda)=-i\int_{-\infty}^{\infty}\left\{\frac{\Phi_S^{0P}(\lambda,x)}{s_{11}(\lambda)}+1\right\}\mathrm{d}x \quad . \tag{4. 8}$$

We specially take the following components of AKNS equation,

$$< \phi_{1,x}^{-} | = - < \phi_{1}^{-} | (-i\lambda\sigma_3 + Q) \quad , \tag{4. 9a}$$
$$| \dot{\phi}_{2,x}^{+} > = (-i\lambda\sigma_3 + Q) | \dot{\phi}_2^{+} > - i\sigma_3 | \phi_2^{+} >, \tag{4. 9b}$$

where $\dot{\phi} = d\phi / d\lambda$. Adding above relations, we get

$$\frac{\partial}{\partial x} < \phi_1^{-} | \dot{\phi}_2^{+} > = -i\,\Phi_S^{0P} \quad . \tag{4. 10}$$

We want to integrate eq. (4.10), but there remains a trouble, that is, the integral diverges. To remove this, we first take a sufficiently larege but finite region $(-a, a)$. After that we make a limit of $a \to \infty$ and impose the boundary condition, then obtain

$$< \phi_1^{-} | \dot{\phi}_2^{+} > \Big|_{x=-a}^{x=a} = -\dot{s}_{11} + 2ias_{11} \quad .$$

After all we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log s_{11}(\lambda) = i\int_{-\infty}^{\infty} \{ \Phi_S^{0P}/s_{11} + 1 \} \mathrm{d}x \quad . \tag{4. 11}$$

Comparing both eqs. (4. 8) and (4.11), we obtain

$$R(\lambda) = -\frac{\mathrm{d}}{\mathrm{d}\lambda} \log s_{11}(\lambda) \quad , \tag{4. 12}$$

which is the well-known relation $R = -\Delta'/\Delta$, that is, $\Delta\,(=s_{11})$ is the Fredholm determinant. From eqs. (45) and (46) we also get

$$-\frac{\mathrm{d}}{\mathrm{d}\lambda} \log s_{11}(\lambda) = \mathrm{Tr.}\{(L-\lambda)^{-1} - (L^0 - \lambda)^{-1}\}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{Tr.}\{\log(L-\lambda) - \log(L^0 - \lambda)\} \quad ,$$

and

$$\log s_{11}(\lambda) = -\mathrm{Tr.}\{\log(L-\lambda) - \log(L^0-\lambda)\} \quad . \tag{4. 13}$$

§ 5. Conservation Laws

We substituste eq. (3. 8a) into eq. (4.11),

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log s_{11} = 2i \int_{-\infty}^{\infty} \mathrm{d}x \int_{\pm\infty}^{x} < w(y) | \sigma_3 \sigma_1 | \Omega^P(\lambda, x) > \mathrm{d}y \quad , \tag{5. 1}$$

where

$$| \Omega^P(\lambda, x) > = | \Phi_v^{0P}(\lambda, x) / s_{11}(\lambda) > \quad .$$

As well-known the conservation laws are derived from $s_{11}(\lambda)$ expanding into $1/\lambda$-series. Instead for the derivation of eq. (5. 1) we use eq. (3. 8b), that is,

$$\Omega^P(\lambda) = (i/2)(\Lambda^{\pm}(\cdot) - \lambda)^{-1} \sigma_3 \sigma_1 | w(\cdot) > \quad . \tag{5. 2}$$

The inverse operator in eq. (5. 2) can be expanded into[13)]

$$(\Lambda^{\pm}-\lambda)^{-1}=-\frac{1}{\lambda}\cdot\frac{1}{1-(\Lambda^{\pm}/\lambda)}=-\sum_{n=0}^{\infty}\frac{(\Lambda^{\pm})^{n}}{\lambda^{n+1}} ,$$

by which the relation (5. 1) is reduced to

$$\frac{d}{d\lambda}\log s_{11}=-\int_{-\infty}^{\infty}dx\int_{\pm\infty}^{x}<w(y)|\sigma_3\sigma_1(\Lambda^{\pm}-\lambda)^{-1}\ \sigma_3\sigma_1|w(\cdot)>dy$$

$$=\sum_{n=1}^{\infty}\frac{1}{\lambda^{n+1}}\int_{-\infty}^{\infty}c_n^{\pm P}\ (x,t)\ dx ,$$

where conserved density $c_n^{\pm P}$ {vanishing for $n=0$} is given by

$$c_n^{\pm P}\ (x,t)=\int_{\pm\infty}^{x}<w(y)|\sigma_1\sigma_3(\Lambda^{\pm})^{n}\ \sigma_3\sigma_1|w(\cdot)>dy\ . \tag{5. 4}$$

Because of eq. (3.19) this surely results in a polinomial of potentials and its derivatives. If we set the expansion as

$$\log s_{11}(\lambda)=\sum_{n=1}^{\infty}\lambda^{-n}\ C_n^P , \tag{5. 5}$$

the conserved quantity c_n^P (constant of motion) shoud result in an integral related with the density. ,

$$C_n^P=-\frac{1}{n}\int_{-\infty}^{\infty}c_n^P\ (x,t)\ d\ dx\ . \tag{5. 6}$$

We examine eq. (5. 1) in a different manner. Because of the relation $\Phi^{-}=\Phi^{+}S$ the cross type of suared eigenstates are related to

$$|\Phi_v^{0P}>=s_{11}|\phi_1^{+}\times\phi_2^{+}>+s_{21}|\Phi_v^{+P}> ,$$
$$|\Phi_v^{0N}>=s_{12}|\Phi_v^{+N}>+s_{22}|\phi_1^{+}\times\phi_2^{+}>\ .$$

Then we get

$$\frac{1}{s_{11}}|\Phi_v^{0P}>-\frac{1}{s_{22}}|\Phi_v^{0N}>=\rho^{P}|\Phi_v^{+P}>-\rho^{N}|\Phi_v^{+N}> , \tag{5. 7}$$

where $\rho^{P}=s_{21}/s_{11}$ and $\rho^{N}=s_{12}/s_{22}$. From Plemelj's formula, we obtain

$$\Omega^{P}(\lambda)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\lambda}\{\rho^{P}|\Phi_v^{+P}>-\rho^{N}|\Phi_v^{+N}>\}\quad(\mathrm{Im.}\,\lambda>0). \tag{5. 8}$$

Substituting this into eq. (5. 1), we get

$$\frac{d}{d\lambda}\log s_{11}(\lambda)=-R(\lambda)$$

$$=-\frac{1}{\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\lambda}\{\rho^{P}<w|\sigma_3\sigma_1|\Phi_V^{+P}>-\rho^{N}<w|\sigma_3\sigma_1|\Phi_V^{+N}>\}(\xi,y)dy$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\lambda}\{\rho^{P}\Phi_s^{+P}-\rho^{N}\Phi_s^{+N}\}(\xi,x) , \tag{5. 9}$$

where eq. (3. 7a) is used. We further note

$$\frac{<\phi_2^{+}|\sigma_3|\phi_1^{-}>}{s_{11}}-\frac{<\phi_1^{+}|\sigma_3|\phi_2^{-}>}{s_{22}}=\rho^{P}\Phi_s^{+P}-\rho^{N}\Phi_s^{+N} ,$$

from which the following relation is derived.

$$\frac{\Phi_S^{0P}(\lambda,x)}{s_{11}(\lambda)}+1=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\lambda}\{\rho^{P}\Phi_s^{+P}-\rho^{N}\Phi_s^{+N}\}\ .\quad(\mathrm{Im.}\,\lambda>0) \tag{5. 10}$$

If we substitute eq. (5.10) into eq.(5. 9), eq. (4.11) is again obtained . Expanding eq.(5. 9) as to λ^{-1} and comparig it with eq. (5. 3), we can obtain

$$c_n^P = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^n \{\rho^P \Phi_s^{+P} - \rho^N \Phi_s^{+N}\} \, d\xi \quad . \qquad (5.\ 11)$$

This represents the density by the scattering data.

As shown by Flaschka, we can show another type of conservation laws. From eq. (3.14) and $[\Phi^+]^\dagger = S[\Phi^-]^\dagger$ we obtbain

$$\delta s_{11} = -s_{11} \int_{-\infty}^{\infty} <\phi_2^-| \delta Q | \phi_1^-> dx + s_{12} \int_{-\infty}^{\infty} <\phi_1^-| \delta Q | \phi_1^-> dx \quad . \qquad (5.\ 12)$$

Because of $s_{11}\phi_2^- = \phi_2^+ + s_{12}\phi_1^-$, the term of the first integral is written as

$$-s_{11} <\phi_2^-| \delta Q | \phi_1^-> = -<\phi_2^+| \delta Q | \phi_1^-> - s_{12} <\phi_1^-| \delta Q | \phi_1^-> .$$

Hence eq. (5.12) is reduced to

$$\delta (\log s_{11}) = -\frac{1}{s_{11}} \int_{-\infty}^{\infty} <\phi_2^+| \delta Q | \phi_1^-> dy \quad . \qquad (5.\ 13)$$

This can be again expanded into $1/\lambda$-seriese as eq. (5. 3),

$$\delta (\log s_{11}) = -\frac{1}{s_{11}} \int_{-\infty}^{\infty} <\delta w | \sigma_1 | \Phi_v^{0P} > dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \delta C_n^P \quad , \qquad (5.\ 14)$$

which gives the variation δC_n^P as

$$\delta C_n^P = (i/2) \int_{-\infty}^{\infty} <\delta w | \sigma_1 [\Lambda^\pm]^n \sigma_3 \sigma_1 | w > dx \quad . \qquad (5.\ 15)$$

The trace formula is again useful for derivation of eq. (5.15). By means of eq. (4.13) we can also evaluate the variation of log (s_{11}),

$$\delta (\log s_{11}) \cong \mathrm{Tr.} \{(L-\lambda)^{-1} \delta L\} \qquad (5.\ 16)$$

Because $L = i\sigma_3(\partial_X - Q)$ and

$$\mathrm{Tr.} \{(L-\lambda)^{-1} \delta L\} = -(1/s_{11}) \int_{-\infty}^{\infty} \mathrm{Tr.} \{|\phi_2^+><\phi_1^-| \delta Q\} dx ,$$

$$\mathrm{Tr.} (|\phi_2^+><\phi_1^-| \delta Q) = <\delta w | \sigma_1 | \phi_1^- \times \phi_2^+> \quad ,$$

the relation (5.15) is reduced to

$$\delta (\log s_{11}) = -\int_{-\infty}^{\infty} <\delta w | \sigma_1 | \Phi_v^{0P} / s_{11} > dx \quad . \qquad (5.\ 17)$$

This is just eq. (5.14).

§ 6 . Hamiltonian Structures

In this secstion we consider functional derivatives and make clear the dynamical structure of problems. First we give the folowing Proposition.

Prop. 1 : "For the scalar variations,

$$\delta a = \int_{-\infty}^{\infty} <\delta u(x) | \sigma_1 v(x) > dx, \quad \delta b = \int_{-\infty}^{\infty} <u(x) | \delta B(x) | v(x) > dx \quad ,$$

its functional derivatives are given by

$$\frac{\delta a}{\delta | u>} = \sigma_3 | v(x)>, \quad \frac{\delta b}{\delta B}(x) = [| v(x)> < u(x)|]^T \quad ,$$

respectively. The alginment of RHS in the former sholud be same as the ket $| u>$, while the one in the later as the matrix B of LHS."

By this Prop. 1 we can take the varliation of eq. (5.17),

$$\frac{\delta}{\delta | w>} \log s_{11} = -\sigma_3 | \Omega^P > \quad , \tag{6. 1}$$

which can be expanded into λ^{-1}-series via eqs. (5. 2) and (5. 5). That is, we obtain

$$-2i\sigma_3 \frac{\delta}{\delta | w>} C^P_{n+1} = [\Lambda^{\pm}]^n \sigma_3 \sigma_1 | w> \quad . \tag{6. 2}$$

This is also derived from eq. (5.15) and suggests a clsose connection with the generalized NLEE formula (3.19). Considering $[\Lambda^+]^\dagger = \Lambda^-$ and $[\Lambda^\dagger]^\dagger = \Lambda$ shown in eq. (A. 8), we can regard $[\Lambda^{\pm}]^\dagger \cong \Lambda^{\pm}$.

Then both eqs. (3.19) and (6. 2) result in

$$\sigma_1 \partial_t | w> = -4i\sigma_3 \sum_{n=0}^{N} a_n \frac{\delta}{\delta | w>} C^P_{n+1} \quad , \tag{6. 3}$$

by which the Hamiltonian H^P $\{=H^P(x,t)\}$ may be introduced,

$$H^P = -4i \sum_{n=0}^{N} a_n C^P_{n+1} \quad . \tag{6. 4}$$

We remark that eq. (6. 4) can be derived from the variational formula (5.17) connected with the analysis of trace formula. In the following, however, we show another way giving the same result. The (1−1) entry of eq. (3.11)

$$\delta s_{11}(\xi) = -\int_{-\infty}^{\infty} <\phi_2^+(\xi, x)| \delta Q(x)| \phi_1^-(\xi, x)> dx \tag{6. 5}$$

defines a functional derivative, and from Prop. 1 we obtain

$$\left(\frac{\delta s_{11}}{\delta Q}(x)\right)^T = -\Big(| \phi_1^-(x)> < \phi_2^+(x)|\Big)_{off} = [\Xi^P(x)]_{off}, \tag{6. 6}$$

and dividing both sides by s_{11}

$$\left(\frac{\delta}{\delta Q} \log s_{11}\right)^T = \left(\frac{\Xi^P(\lambda, x)}{s_{11}(\lambda)}\right)_{off}, \tag{6. 7}$$

where

$$\Xi^P = -| \phi_1^-> < \phi_2^+| \quad , \quad \Xi^N = | \phi_2^-> < \phi_1^+| \quad .$$

We must note that eq. (6. 1) corresponds to a vector relation of eq. (6. 7), because off-diagonal parts of Ξ^P are directly connected with Φ_v^{0P}. On the other words, the trace formula not only gives the conservation laws but also contributs on the Hamiltonian structure.

Another case of Im. $\lambda < 0$ is similarly treated. The squared eigenstates are given by $\Phi_s^{0N} = \mathrm{Tr.}\ (\sigma_3 | \phi_2^-> < \phi_1^+|) = < \phi_2^-| \sigma_3 | \phi_1^+>$ and $| \Phi_v^{0N}> = | \phi_2^- \times \phi_1^+>$, which satisfy

$$\Phi_{s,x} = 2 < w| \sigma_3 \sigma_1 | \Phi_v>, \quad (\partial_x + 2i\lambda\sigma_3)| \Phi_v> = -\Phi_s \sigma_1 | w> \ . \tag{6. 8}$$

Because of boundary condition, $\Phi_s^{0N}(\lambda, x) \to -s_{22}(\lambda)$ for $x \to \pm\infty$, we get the functional form as eq. (3. 9b),

$$\Lambda^{\pm}(x,dy)|\Omega^N(\lambda)>=\lambda|\Omega^N(\lambda)>+\frac{i}{2}\sigma_3\sigma_1|w> \quad , \tag{6. 9}$$

where $|\Omega^N>=|\Phi_v^{0N}/s_{22}>$. This is same as eq. (5. 2), while eq. (6 . 1) must be replaced with

$$\frac{\delta}{\delta|w>}\log s_{22}=\sigma_3|\Omega^N> \quad . \tag{6. 10}$$

Again taking the expansion

$$\log(s_{22})=\sum_{n=1}^{\infty}\lambda^{-n}C_n^N \quad , \tag{6. 11}$$

we can reduce eq. (3.19) to

$$\sigma_1\partial_t|w>=+4i\sigma_3\sum_{n=0}^{N}a_n\frac{\delta}{\delta|w>}C_{n+1}^N \quad . \tag{6. 12}$$

Both relations (6. 3) and (6.12) can be reduced to

$$\sigma_1\partial_t|w>=\sigma_3\frac{\delta H^P}{\delta|w>}=-\sigma_3\frac{\delta H^N}{\sigma|w>} \quad , \tag{6. 13}$$

where the Hamiltonian is given by

$$H(x,t)\equiv H^P=-H^N=-4i\sum_{n=0}^{N}a_n C_{n+1}^P \quad . \tag{6. 14}$$

The components of eqs. (6.13) shows the cannonical form,

$$\partial_t r=-\frac{\delta H}{\delta q} \quad , \quad \partial_t q=\frac{\delta H}{\delta r} \quad . \tag{6. 15}$$

We define the Poisson's bracket as follows

$$\{A,B\}\equiv-\int_{-\infty}^{\infty}\frac{\delta A}{\delta<w|}\cdot\frac{\delta B}{\delta|w>}\,dx$$

$$=\int_{-\infty}^{\infty}(\frac{\delta A}{\delta q}\cdot\frac{\delta B}{\delta r}-\frac{\delta A}{\delta r}\cdot\frac{\delta B}{\delta q})\,dx=-\{B,A\} \quad . \tag{6. 16}$$

which of course satisfies the Jacobi identity,

$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0 \quad . \tag{6. 17}$$

Since $\delta q/\delta r=0$, $\delta q/\delta q=\delta(x-y)$ etc., eqs. (6.15) are reduced to

$$\partial_t r=\{r,H\} \quad , \quad \partial_t q=\{q,H\} \quad . \tag{6. 18}$$

The quantity $C_n^{P,N}$ were of course conserved because $s_{jj}(\lambda)$ is independent of t via eq. (3.15). However, it is still possible to show this by using Hamiltonian structures. We consider the bracket for both C_m^P and C_n^P,

$$\{C_m^P, C_n^P\}=(\frac{i}{2})^2\int_{-\infty}^{\infty}<[\Lambda^{\pm}]^{m-1}\sigma_3\sigma_1 w|[\Lambda^{\pm}]^n\sigma_3\sigma_1 w>dx. \quad .$$

From eqs. (A.1), (A.11) and (A.12) we obtain

$$\{C_m^P, C_n^P\}=(\frac{i}{2})^2\int_{-\infty}^{\infty}<w|\sigma_1\sigma_3[\Lambda^{\pm}]^{m+n-2}\sigma_3\sigma_1|w>dx=0. \tag{6. 19}$$

Since each C_n^P commutes with the Hamiltonian, we find $C_{n,t}^P=0$.

Now we can discuss the cannonical transformation for our class and the basic points are shown in Appendix B. First we comment on the action-angle variables developed by Zakharov and his co-workers originally. We can list eqs. (6. 1), (6.10) and

$$\frac{\delta(\log s_{12})}{\delta|w>} = -\sigma_3|\frac{\phi_2^+\times\phi_2^-}{s_{12}}>, \quad \frac{\delta(\log s_{21})}{\delta|w>} = \sigma_3|\frac{\phi_1^+\times\phi_1^-}{s_{21}}> . \tag{6.20}$$

On the real axis $\xi=\mathrm{Re.}\ \lambda$, it is not difficult to calculate various brackets as $\{\log s_{ij}, \log s_{mn}\}$. For example we show

$$\{\log s_{11},\ \log s_{21}\}_{(\xi)} = \frac{1}{2i(\xi-\xi')} - \frac{\pi}{2}\delta(\xi-\xi') \quad . \tag{6.21}$$

To eliminate the first term in R.H.S., we must take complex conjugate quantities. The action-angle variables are defined by

$$P(\xi)=2\cdot\log|s_{11}(\xi)| \ , \quad Q(\xi)=(2/\pi)\arg s_{21} \ , \text{ etc.}$$

Details for this had been reported by Kodama.[5]

As a new topic, we refer to the infinitesimal cannonical transformation,[6,14]

$$I.C.T:\ |w>\to|W>=|w>+\varepsilon\sigma_1\sigma_3\frac{\delta}{\delta|w>}C_n^P \quad , \tag{6.22}$$

where $|W>=|R,Q>$ and $0<\varepsilon\ll1$. Denoting the infinitesimal term as $|\Delta r,\Delta q>$, one of the Poisson brackets is given by

$$\{Q,R\}=\{q,r\}+\{\Delta q,r\}+\{q,\Delta r\}+O(\varepsilon^2)$$

$$=\delta(x-x') + \varepsilon(\frac{\delta}{\delta q}\cdot\frac{\delta}{\delta r}-\frac{\delta}{\delta r}\cdot\frac{\delta}{\delta q})\ C_n^P+O(\varepsilon^2)\cong\delta(x-x'). \tag{6.23}$$

From discussions in Appendix B, the transformation (6.22) is surely cannonical and both Hamiltonians should be related as $H[w]=H'[W]$, where H' means the transformed one. We substitute eq. (6.22) into this invariance,

$$H'[W]=H[W-\Delta w]=H[W]-\delta H[w,\Delta w]+O(\varepsilon^2) \quad . \tag{6.24}$$

From eqs. (B. 1) and (6.19) we obtain a symmetry, $H'[W]\cong H[W]$. This fact means that the Hamiltonian system has an infinite abelian group of symmetry transformations.

§ 7 . Concludings and Discussions

The generalized AKNS class of NLEE's were given with a closed formula still containing integral differential operators $\Lambda^\pm$, from both the integrable condition (2. 5) and the given S-matrix relation (3.15). Constants of motions ($=C_n$) are derived by using the trace formula and we also saw that λ^{-1}−expansions of $(\Lambda^\pm-\lambda)|\Omega^{P,N}>=(i/2)\sigma_3\sigma_1|w>$ give the conservation laws. We obtained such a canonical system $\sigma_1|w_t>=\sigma_3(\delta H/|\delta w>)$ equivalent to the generalized formula, from which the Poisson bracket was defined naturally. Considering the property of $\Delta^\pm$, we found that $C_n^{P,N}$ commutes each other as to the Poisson bracket. This fact enables us to find an infinitesimal cannonical transformation which gives the system an infinite dimensional abelian symmetry already mentioned in § 1. This can be regarded not only as the Lie-Backlund transformation but also as the Kac-Moody Lie algebra of the system.

References

(1) M.J. Ablowitz, D.J. Kaup, A.C. Newell and H.Segur: "The inverse Scattering Trans-

form-Fourier Analysis for Nonlinear Problems", Stud. Appl. Math. 53(1974)249−314
(2) V.E. Zakharov and L. D. Faddeev: "Korteweg-de Vries equation: a completely integrable Hamiltonian system". Func. Anal. Appl., 5 (1971) 280−287
(3) V. E. Zakharov and S. V. Manakov : "on the completely integrability of a nonlinear Schröd-inger equation", Theor. Math. Phys., 19(1974) 332−343
(4) H. Flaschka and A. C. Newell : "integrable systems of nonlinear evoulution equations", Lecture Notes in Physics in 38: "Dynamical Systems and Applications", Springer 1974, p. p. 355 −440
(5) Y. Kodama : "complete integrability of nonlinear evolution equations", Prog. Theor. Phys., Vol. 59 (1975) 669 −686
(6) R. K. Dodd and R. K. Bullough : "The generalized Marchenko equation and the cannon-ical structure of the AKNS-ZS inverse method", Physica Scripta, Vol. 20(1979)514 −530
(7) L. Dolan : "Why Kac-Moody Subalgebras are interesting in Physics". Lectures in Appl. Math. Vol. 21 (1985)307 −324, Providence
(8) H. Eichenherr : "symmetry algebras of the Heisenberg model and the nonlinear Schöd-inger equation". Phys. Lett., 115B (1982)385 −388
(9) L. L. Chaw and T. Koikawa : "understanding of the symmetric space σ-models through the soliton connection ", private commnication
(10) T. Kawata : "2 × 2-matrix Riemann-Hibert transform and its connction to the contin-uous scattering data", J. Phys. Soc. Jpn., 53 (1984) 2879−2884
(11) T. Kawata : "some transformation property and its algebraic structure relating with the soliton equation", Bultain of Faculty of Engingeering in Toyama Univ., 37 (1986)49−59
(12) A. C. Newell : "the general structure of integrable evolution equation", Proc. R. Soc. Lond. A, 365 (1979) 283 −311
(13) V. S. Gerdjikov and E. Kh Kritov : Bulg. J. Phys. 7(1980)119 −133
(14) J. M. Alberty, T. Koikawa and R. Sasaki : "canonical structure of soliton equations. I", Phisica 5 D(1982) 43−65

Appendix A: Adjoint Operator

We define an adjoint operator $\Lambda^{\dagger}$ for Λ given by eq. (2.12a),

$$\int_{-\infty}^{\infty} < p_0(x) | \Lambda^{\pm}(\cdot) | \Phi_v^{\pm P}(x) > dx = -\int_{-\infty}^{\infty} < \Phi_v^{\pm P}(x) | [\Lambda^{\pm}(\cdot)]^{\dagger} | p_0(x) > dx \quad , \qquad (A.1)$$

where $q_0(x)$ is a rapidly decreasig function. Once the adjoint $\Lambda^{\dagger}$ is determined, it is possiblo to generalize eq. (A.1). For tlhis purpose we introduce

$$p_n(x) = \{[\Lambda^{\pm}(\cdot)]^{\dagger}\}^n | p_0(x) > \quad (n=0,1.. \quad) \quad , \qquad (A.2)$$

which also vanishes rapidly. For example

$$\int_{-\infty}^{\infty} < p_0 | [\Lambda^{\pm}]^2 | \Phi_v^{\pm P} > dx = -\int_{-\infty}^{\infty} < \Phi_v^{\pm P} | \lambda [\Lambda^{\pm}]^{\dagger} | p_0 > dx$$

$$= -\int_{-\infty}^{\infty} < \Phi_v^{\pm P} | \lambda | p_1 > dx = \int_{-\infty}^{\infty} < p_1 | \Lambda^{\pm} | \Phi_v^{\pm P} > dx = -\int_{-\infty}^{\infty} < \Phi_v^{\pm P} | [\Lambda^{\pm}]^{\dagger} | p_1 > dx .$$

Then we obtain

$$\int_{-\infty}^{\infty} < p_0 | [\Lambda^{\pm}]^2 | \Phi_v^{\pm P} > \mathrm{d}x = -\int_{-\infty}^{\infty} < \Phi_v^{\pm P} | \{ \Lambda^{\pm}]^{\dagger} \}^2 \; | p_0 > \mathrm{d}x. \tag{A.3}$$

Repeating this, we get

$$h^{\dagger}([\Lambda^{\pm}]) = h([\Lambda^{\pm}]^{\dagger}) \quad . \tag{A.4}$$

It is not so easy to give the explicite formula of $\Lambda^{\dagger}$, that is, we prepare the formula exchaging integrations,

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{x}^{\infty} f(x, y) \mathrm{d}y = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} f(y, x) \; \mathrm{d}y \quad . \tag{A.5}$$

By this formula both integral operators in eq. (2.12b) are relating with each adjoint as

$$W_{-}^{\dagger}[x, \mathrm{d}y] = W_{+}[x, \mathrm{d}y] \; , \quad W_{+}^{\dagger}[x, \mathrm{d}y] = W_{-}[x, \mathrm{d}y] \quad . \tag{A.6}$$

It is reasonable to define the operators $\Lambda^{\pm}$ as

$$\Lambda^{\pm}(x) = (i/2)(\sigma_3 \partial_X - 2W_{\pm}[x, \mathrm{d}y]) \quad . \tag{A.7}$$

For example we show the first one of eq. (A.6). Taking a rapidely vanishing function $p(x)$, we calculate

$$\int_{-\infty}^{\infty} < p(x) | W_{+}[x, \mathrm{d}y] | \Phi_v^{-P}(\lambda, x) > \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} < p(x) | \sigma_3 \sigma_1 | w(x) > \int_{+\infty}^{X} \mathrm{d}y < w(y) | \sigma_1 \sigma_3 | \Phi_v^{-P}(\lambda, y) > \mathrm{d}x$$

$$= -\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{X} \mathrm{d}y < p(y) | \sigma_3 \sigma_1 | w(y) > < w(x) | \sigma_1 \sigma_3 | \Phi_v^{-P}(x) >$$

$$= -\int_{-\infty}^{\infty} \mathrm{d}x < \Phi_v^{-P}(x) | \sigma_3 \sigma_1 | w(x) > \int_{-\infty}^{x} \mathrm{d}y < w(y) | \sigma_1 \sigma_3 | p(y) >$$

$$= -\int_{-\infty}^{\infty} < \Phi_v^{-P}(x) | W_{-}[x, \mathrm{d}y] | p(y) > \mathrm{d}x \quad ,$$

which is just the first one of eq. (A.6).

Since $[\sigma_3 \partial_X]^{\dagger} = \sigma_3 \partial_X$, the adjoints are given by

$$[\Lambda^{+}(x)]^{\dagger} = (i/2)(\sigma_3 \partial_X - 2W_{+}[x, \mathrm{d}y]) = \Lambda^{-} \quad , \tag{A.8a}$$

$$[\Lambda^{-}(x)]^{\dagger} = (i/2)(\sigma_3 \partial_X - 2W_{-}[x, \mathrm{d}y]) = \Lambda^{+} \quad . \tag{A.8b}$$

Both relations (A.8) consistently satisfy

$$[\{[\Lambda^{-}]^{\dagger}\}^{\dagger} = (i/2)(\sigma_3 \partial_x - 2W_{+}[x, \mathrm{d}y])^{\dagger}$$

$$= (i/2)(\sigma_3 \partial_x - 2W_{-}[x, \mathrm{d}y]) = \Lambda^{-} \tag{A.9}$$

Some specific properties are found in these operators. Relating these we list directly a few of the first terms of $[\Lambda^{-}]^n \sigma_3 \sigma_1 | w >$,

$$[\Lambda^{-}] \sigma_3 \sigma_1 | w > = \frac{i}{2} \; \sigma_1 | w_X > \quad ,$$

$$[\Lambda^{-}]^2 \sigma_3 \sigma_1 | w > = (\frac{i}{2})^2 \sigma_3 \sigma_1 \{ | w_{XX} > + | w > < w | \sigma_3 w > \} \quad ,$$

$$[\Lambda^-]^3\sigma_3\sigma_1|w> = (\frac{i}{2})^3\sigma_1\{|w_{xxx}> + 2\sigma_3|w><w|w_x> + [|w><w|\sigma_3 w>]_x\} \quad , \quad (A.10)$$

It also becomes

$$[\Lambda^-]^n\sigma_3\sigma_1|w> = [\Lambda^+]^n\sigma_3\sigma_1|w> \quad . \qquad (A.11)$$

From eqs. (A.1), (A.8) and (A.9) we can see

$$\int_{-\infty}^{\infty} <w|\sigma_1\sigma_3[\Lambda^-]^n\sigma_3\sigma_1|w>dx = -\int_{-\infty}^{\infty} <w|\sigma_1\sigma_3([\Lambda^-]^n)^\dagger\sigma_3\sigma_1|w>dx$$
$$= -\int_{-\infty}^{\infty} <w|\sigma_1\sigma_3[\Lambda^+]^n\sigma_3\sigma_1|w>dx \quad .$$

That is, from eq. (A.11) we obtain

$$\int_{-\infty}^{\infty} <w|\sigma_1\sigma_3[\Lambda^\pm]^n\sigma_3\sigma_1|w>dx = 0 . \qquad (A.12)$$

Appendix B. Cannonical Transformation

We start from the cannonical equations, $\sigma_1|w_t> = \sigma_3(\delta H/|\delta w>)$. The variation of an arbitrary functional $J[w]$ is given by

$$\delta J[w, \delta w] = \int_{-\infty}^{\infty} <\delta w|\delta\sigma_1\sigma_3 \frac{\delta J}{\delta|w>} dx \quad , \qquad (B.1)$$

and all of variations are regarded as

$$\delta F = F_t\Delta t \ , \quad \delta q = q_t\Delta t \ , \quad \delta r = r_t\Delta t \ .$$

Then eq. (B.1) is written as

$$J_t = \int_{-\infty}^{\infty} <w_t|\sigma_1\sigma_3 \frac{\delta J}{\delta|w>} dx$$
$$= -\int_{-\infty}^{\infty} \frac{\delta J}{<\delta w|}\cdot\frac{\delta H}{|\delta w>} dx \equiv \{J, H\}_{(x)} \quad . \qquad (B.2)$$

This reduce the cannonical sytem to

$$\{q, r\}_{(x)} = \delta(x-x') \ , \ \{q, q\}_{(x)} = \{r, r\}_{(x)} = 0 \ , \qquad (B.3)$$

The cannonical equation can be reduced to

$$q_t = \{q, H\}_{(x)}, \qquad r_t = \{r, H\}_{(x)} \quad . \qquad (B.4)$$

We next consider another cannonical system, $\sigma_1|W_t> = \sigma_3(\delta H'/|\delta W>)$, and assume that w is a functional of W, $w = w[W]$.
The variation in eq. (B.1) is repretented by

$$\delta J = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty} \{(\frac{\delta J}{\delta q}\cdot\frac{\delta q}{\delta Q} + \frac{\delta J}{\delta r}\cdot\frac{\delta r}{\delta Q})\delta Q + (\frac{\delta J}{\delta q}\cdot\frac{\delta q}{\delta R} + \frac{\delta J}{\delta r}\cdot\frac{\delta r}{\delta R})\delta R\}\ d\xi \quad .$$

Again we reduce this to the form as eq. (B.2),

$$J_t = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty} \{(\frac{\delta J}{\delta q}\cdot\frac{\delta q}{\delta Q} + \frac{\delta J}{\delta r}\cdot\frac{\delta r}{\delta Q})\frac{\delta H}{\delta R} - (\frac{\delta J}{\delta q}\cdot\frac{\delta q}{\delta R} + \frac{\delta J}{\delta r}\cdot\frac{\delta r}{\delta R})\frac{\delta H}{\delta Q}\}\ d\xi$$
$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \{\frac{\delta J}{\delta q}(\frac{\delta q}{\delta Q}\cdot\frac{\delta H}{\delta R} - \frac{\delta q}{\delta R}\cdot\frac{\delta H}{\delta Q}) + \frac{\delta J}{\delta r}(\frac{\delta r}{\delta Q}\cdot\frac{\delta H}{\delta R} - \frac{\delta r}{\delta R}\cdot\frac{\delta H}{\delta Q})\}\ d\xi \ .$$

Another Poisson bracket is naturally introduced,

$$\{J, H\}_{(\xi)} = -\int_{-\infty}^{\infty} \frac{\delta J}{<\delta W|} \cdot \frac{\delta H}{|\delta W>} \mathrm{d}\xi \quad , \tag{B.5}$$

by which above relation is reduced to

$$J_t = \int_{-\infty}^{\infty} \left(\frac{\delta J}{\delta q} \cdot \{q, H\}_{(\xi)} + \frac{\delta J}{\delta r} \cdot \{r, H\}_{(\xi)}\right) \mathrm{d}x \quad . \tag{B.6}$$

Instead for eq. (B. 4) we obtain the one in the ξ −space,

$$q_t = \{q, H\}_{(\xi)} \quad , \quad r_t = \{r, H\}_{(\xi)} \quad . \tag{B.7}$$

From eq. (B.4) we get

$$\{q, H\}_{(x)} = \{q, H\}_{(\xi)} \quad , \quad \{r, H\}_{(x)} = \{r, H\}_{(\xi)} \quad . \tag{B.8}$$

Of course we obtain

$$Q_t = \{Q, H\}_{(\xi)} \quad , \quad R_t = \{R, H\}_{(\xi)} \quad , \tag{B.9}$$

$$\{Q, R\}_{(\xi)} = \delta(\xi - \xi'), \quad \{Q, Q\}_{(\xi)} = \{R, R\}_{(\xi)} = 0 \quad . \tag{B.10}$$

If we introduce an arbitrary functional $K[Q, R]$ defined in the ξ −space, above considerations are repeated similarly. The results are exactly symmetric,

$$Q_t = \{Q, H\}_{(x)} \quad , \quad R_t = \{R, H\}_{(x)} \quad . \tag{B.11}$$

Both relations (B.9) and (B.11) result in

$$\{Q, H\}_{(\xi)} = \{Q, H\}_{(x)}, \quad \{R, H\}_{(\xi)} = \{R, H\}_{(x)} \quad . \tag{B.12}$$

Relations (B.3) and (B.8) as to (q, r) exactly correspond to both eqs. (B.10) and (B.12). For these set of equations we can choose the Hamiltonian as arbitrary functionals of (q, r) and (Q, R), then it can be writen as

$$\{q, r\}_{(x)} = \{q, r\}_{(\xi)} \ , \ \{q, q\}_{(x)} = \{q, q\}_{(\xi)} \ ,$$
$$\{r, r\}_{(x)} = \{r, r\}_{(\xi)} \ , \tag{B.13}$$

$$\{Q, R\}_{(x)} = \{Q, R\}_{(\xi)}, \ \{Q, Q\}_{(x)} = \{Q, Q\}_{(\xi)},$$
$$\{R, R\}_{(x)} = \{R, R\}_{(\xi)}, \tag{B.14}$$

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