Parametric Amplifications in the Nonlinear Transmission Line

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Summary

The parametric amplification in a transmission line with nonlinear capacitors is analysed theoretically using the equations of three wave interactions. Since this line has two modes: high frequency and low frequency modes, there may occur some modecoupling phenomena through the resonant interactions. We consider three waves with wave number k_j and frequency ω_j in resonance with each other, that is, $\omega_1 + \omega_2 = \omega_3$ and $k_1 + k_2 = k_3$, where $0 \leqq \omega_1 \leqq \omega_2 \leqq \omega_3$ and $k_3 \geqq 0$. Such conditions are realized in our network and there exist two states: "forward state" (each group velocity is positive) and "backward state" (one of the group velocities is negative). The coupled equations of three waves has two constant pumps: high frequency(HF) pump and low frequency(LF) pump. Using linear approximations, we examine the possible types of parametric amplification and obtain the power gains depending on the frequency deviation. For only the case of HF pump we get the gain between signals with seme frequency and also get the gain from the low frequency signal to the high frequency signal("up-conversion") for the LF pump. The nonlinear analysis gives the exact relation between input and output signals. For the forward state the gain is absolutely suppressed by the ratio of pumping power to input power, while the gain of backward state has no finite maximum and there may appear an "oscillating state" if the pumping power is comparatively small.

1. Introduction.

The parametric effect is well-known as a mechanism of signal amplifications in a lumped electric circuit with a nonlinear capacitor.[1] This effect depends on the resonant interaction among three signals. We often encounter such a phenomenon not only in a electrical network but also in a wide field of science: nonlinear optics, laser physics[2] and plasma physics[3] etc..

A. L. Cullen examined the parametric amplification in the transmission line consisting of segments with a seriesed inductor and a shunt nonlinear capacitor.[4] The pulse satulation effect of this line ·was also reported by A. C. Scott et al.[5] This line is nondispersive, then there do not occur such phenomena as "backward" interactions and "oscillations". If we want to realize the backward operation in the electrical network, it is necessary to make the circuit model as the two modes system which is of course dispersive. For such a system, however, the resonant conditions can not be always satisfied then it becomes important to evaluate the effect of matching loss.

As shown in ref. 5 and 6, we can analyse the nonlinear behaviour of the parametric amplification to some content by using the basic equations of three wave interactions. These equations were also studied in laser physics[7] and plasma physics[8] etc.. Recently the initial value problem had been solved by the inverse scattering method[9~11] under the rapidly vanishing conditions at infinity.

In previous papers,[12,13] we analysed the wave propagation of nonlinear dispersive waves and derived the basic equations of three wave interactions in distributed networks with nonlinear capacitors. Our transmission line has two modes: high frequency mode (HF mode) and low frequency mode (LF mode). Then we can expect the parametric amplifications with both forward and backward types and are interested in the backward oscillations.

In this paper we analyse the parametric amplification from the view point of three wave interactions. Possible types of signal amplifications and the corresponding gain with dependency of matching loss are derived systematically by the linear approximation. The nonlinear behaviour of typical cases are made clear for both forward and backward states. The results of this paper are summarized in the following.

The coupled equations of three waves have two kinds of pumping states with constant amplitude: high and low frequency pump (HF and LF pump, respectively). Only the HF pump is unstable and decays into two low frequency waves (parametric decay instability). On the other hand the resonance conditions in our line allow two classes of mode coupling and there occur two kinds of interaction: forward and backward interactions. For the above each case we calculate two kinds of gain, the normal gain G_N and conversion gain G_C, which are defined between signals with same frequency and with different frequency, respectively. Generally the both gains satisfy $G_N|G_C-1|$. From the linear analysis we can get the dependency of frequency deviation (i.e., matching loss) and obtain the relations $G_N > 1$ and $G_N < 1$ for the case of HF pump and LF pump, respectively. The nonlinear analysis explains the difference between the forward and the backward states more exactly. In the forward state the gain is suppressed by the ratio of pumping power to input power, while the gain of backward state has no finite maximum if the input power is comparatively small. Furthermore there appears an "oscillating state" under certain boundary conditions.

Finally we estimate the half-power band width approximately.

2. Circuit Model and Basic Equations describing the Three Wave Interactions.

Our circuit model is given by taking a continuous limit of the nonlinear transmission line as shown in Fig. 1 where each circuit parameter is denoted with the distributed value. Each section of this line is constructed with a series inductor L_1 and the shunt circuit consisting of a seriesed
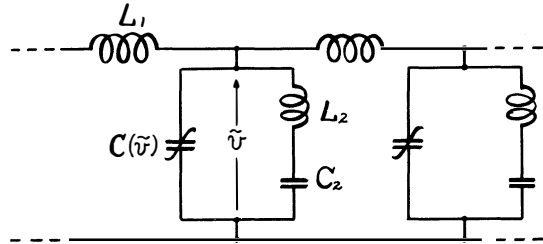

Fig. 1 Nonlinear transmission line.

$L_2 - C_2$ element and a nonlinear capacitor $C(\bar{v})$ where $\bar{v}$ is the line voltage.

In the following we list some necessary matters for this paper from our earlier report.[13] Taking the model of the nonlinear capacitor as $C(\bar{v}) = C_{10} + C_{11}(\bar{v}/v_{10})$, the dispersion relation of this line can be given by

$$\omega^4 - (\omega_p^2 + v_e^2 k^2)\omega^2 + v_e^2 \omega_i^2 k^2 = 0 \ , \tag{2.1}$$

where ω is the frequency and k is the wave number. The other coefficients are related to the circuit parameters as

$$\omega_p^2 = \frac{C_{10} + C_2}{L_2 C_{10} C_2} \ , \quad \omega_i^2 = \frac{1}{L_2 C_2} \ , \quad v_e^2 = \frac{1}{L_1 C_{10}}$$

$$c_s^2 = \frac{1}{L_1(C_{10} + C_2)} \ , \quad \frac{\omega_p^2}{\omega_i^2}\frac{v_e^2}{c_s^2} = 1 + \frac{C_2}{C_{10}} > 1.$$

Since eq. (2.1) allows two modes: high frequency mode (HF mode) and low frequency mode (LF moe), we can expect several mode coupling phenomena under certain conditions. The most basic process of these phenomena is three wave interaction. We consider three waves with characteristic frequencies ω_1, ω_2, ω_3 and wave numbers k_1, k_2, k_3 satisfying the resonance conditions,

$$\omega_1 + \omega_2 = \omega_3, \quad k_1 + k_2 = k_3, \tag{2.2}$$

where we assume all the sign of ω_n to be positive and impose the conditions $0 \leqq \omega_1 \leqq \omega_2 \leqq \omega_3$ and $k_3 \geqq 0$ without loss of generality. Following two classes of the mode coupling are realized in our case.

[class-1] "Two high frepuency waves with (k_3, ω_3) and (k_2, ω_2) belong to the HF mode, while the other with (k_1, ω_1) belongs to the LF mode."

In this class the following critical case is included,

$$\omega_3 = \omega_2 = \omega_c, \quad \omega_1 = 0, \quad v_{g3} = v_{g2} = v_{g1} = c_s \ , \tag{2.3}$$

where ω_c is determined from the relation $v_{g3} = c_s$. Such a case has been treated in ref. 14. Then we set

$$\omega_c < \omega_3, \quad v_{g2} < c_s < v_{g3}, \quad 0 < v_{g1} < c_s. \tag{2.4}$$

We remark that the sign of v_{g2} (or k_2) may be positive (called as "forward state") or negative ("backward state").

[class-2] "Two waves with (k_1, ω_1) and (k_2, ω_2) belong to the LF mode, while the other to the HF mode." To realize this case, we must impose the condition $\omega_p \leqq 2\omega_i$. Furthermore this class includes only the backward state $(v_{g1} < 0)$.

We can derive the equations describing the three wave interactions under the conditions (2.2) by using a derivative expansion method. We take the lowest approximation of the normarised voltage as

$$\bar{v}/v_{10} = \sum_{n=1}^{3} A_n(x, t) \exp[i(k_n x - \omega_n t)] + c.c., \tag{2.5}$$

where c. c. stands for the complex conjugate of the preceeding term. The behaviour of the complex amplitude A_n can be well described by the following coupled equations,

$$\frac{\partial A_1}{\partial t} + v_{g1} \frac{\partial A_1}{\partial x} = i\delta M_1 A_3 A_2^* \quad , \tag{2.6a}$$

$$\frac{\partial A_2}{\partial t} + v_{g2} \frac{\partial A_2}{\partial x} = i\delta M_2 A_3 A_1^* \quad , \tag{2.6b}$$

$$\frac{\partial A_3}{\partial t} + v_{g3} \frac{\partial A_3}{\partial x} = i\delta M_3 A_1 A_2 \quad , \tag{2.6c}$$

where $\delta = C_{11} / C_{10}$ and A^* denotes the complex cojugate of A. The product δM_n is a coupling coefficient, where

$$M_n = \frac{\omega_n^2}{v_e^2 k_n} v_{gn} = \omega_n \frac{(\omega_n^2 - \omega_i^2)^2}{(\omega_n^2 - \omega_i^2)^2 + \omega_i^2 (\omega_p^2 - \omega_i^2)} \quad . \tag{2.7}$$

The group velocity v_{gn} is given by

$$v_{gn} = v_e \left(\frac{\omega_n^2 - \omega_p^2}{\omega_n^2 - \omega_i^2} \right)^{\frac{1}{2}} \frac{(\omega_n^2 - \omega_i^2)^2}{(\omega_n^2 - \omega_i^2)^2 + \omega_i^2 (\omega_p^2 - \omega_i^2)} \quad . \tag{2.8}$$

3. Parametric Instability.

The equations (2.6) of three wave interactions have constant solutions. In this section we analyse the instability of the following solutions,

$$A_1 = A_2 = 0 , \qquad A_3 = A_{3p} , \tag{3.1}$$

$$A_3 = A_1 = 0, \quad A_2 = A_{2p} \quad or \quad A_3 = A_2 = 0, \quad A_1 = A_{1p} , \tag{3.2}$$

where these are constant solutions of eqs. (2.6). From the fact that ω_3 is larger than ω_2 and ω_1, we call the states eqs. (3.1) and (3.2) as the high frequency pump (HF pump) and the low frequency pump (LF pump), respectively.

To analyse the linear stability of the HF pump, we take the solution of eqs. (2.6) as

$$A_1 = \widetilde{A}_1 , \qquad A_2 = \widetilde{A}_2 , \qquad A_3 = A_{3p} + \widetilde{A}_3 ,$$

then we get the linearised equations as

$$\begin{cases} \dfrac{\partial \widetilde{A}_1}{\partial t} + v_{g1} \dfrac{\partial \widetilde{A}_1}{\partial x} = i\delta M_1 A_{3p} \widetilde{A}_2^* , & \tag{3.3a} \\[2mm] \dfrac{\partial \widetilde{A}_2^*}{\partial t} + v_{g2} \dfrac{\partial \widetilde{A}_2^*}{\partial x} = - i\delta M_2 A_{3p}^* \widetilde{A}_1 , & \tag{3.3b} \\[2mm] \dfrac{\partial \widetilde{A}_3}{\partial t} + v_{g3} \dfrac{\partial \widetilde{A}_3}{\partial x} = 0 . & \tag{3.3c} \end{cases}$$

From eqs. (3.3a) and (3.3b) the dispersion relation can be obtained as

$$(\Omega - v_{g1} k)(\Omega - v_{g2} k) + \delta^2 M_1 M_2 |A_{3p}|^2 = 0 , \tag{3.4}$$

where we assumed the perturbations $\widetilde{A}_1$ and $\widetilde{A}_2^*$ as $\exp [i(Kx - \Omega t)]$. From the criterion that Ω becomes complex, we find that the waves with the following wave number region are excited,

$$|K| < K_m = 2\delta \sqrt{M_1 M_2} \; |A_{sp}|/|v_{g1} - v_{g2}| \; . \tag{3.5}$$

This means that the HF pump is unstable and decays into two low frequency waves, then we call this situation as the parametric decay instability. We can easily get the maximum growth rate Ω_{im} from eq. (3.4),

$$\Omega_{im} = \delta \sqrt{M_1 M_2} \, |A_{sp}| \; . \tag{3.6}$$

The stability of the LF pump can be made clear in a similar way as the case of HF pump. Taking the perturbations of this case as

$$A_m = \widetilde{A}_m, \quad A_s = \widetilde{A}_s \; , \quad A_n = A_{np} + \widetilde{A}_n \; ,$$

we get the linearised equations as

$$\begin{cases} \dfrac{\partial \widetilde{A}_m}{\partial t} + v_{gm} \dfrac{\partial \widetilde{A}_m}{\partial x} = i\delta M_m A_{np}^* \, \widetilde{A}_s \; , & \tag{3.7a} \\[2ex] \dfrac{\partial \widetilde{A}_s}{\partial t} + v_{gs} \dfrac{\partial \widetilde{A}_s}{\partial x} = i\delta M_s A_{np} \, \widetilde{A}_m \; . & \tag{3.7b} \end{cases}$$

where m and n are paired suffixes and are defined as

$$(m, \; n) = (1, \; 2) \; \text{or} \; (2, \; 1) \; . \tag{3.8}$$

The dispersion relation is obtained as

$$(\Omega - v_{gm} K)(\Omega - v_{gs} K) - \delta^2 M_m M_s |A_{np}|^2 = 0 \; . \tag{3.9}$$

On the contrary to the HF pump it can be easily shown that the LF pump is stable.

4. Parametric Amplification (Small Signal Theory).

In this section we examine the possibility of the parametric amplification by using a linear approximation. Using the general theory of Appendix A, we estimate the power gains with the dependency of the frequency deviation Ω in various cases.

For the case of HF pump we get the gain G_N (called a normal gain) between the signals with same carrier frequency, but do not get such a gain for the case of LF pump. However for the both cases it is possible to get the gain G_C (called a conversion gain) between the signals with the different carrier frequencies. In the forward state (i. e., two group velocities are positive), the gain of all the cases is bounded on condition that the distance d between input terminal and output terminal is constant. On the other hand, if $\Omega = 0$, the normal gain for the case of HF pump may be infinite in the backward state (one of the group velocity is negative), that is, there occurs an "oscillating" state.

4A). The case of HF pump.

Assuming a steady state response, we take the solution of eqs. (3.3a) and (3.3b) as

$$\widetilde{A}_1(x, \; t) = \tilde{a}_1(x)e^{-i\Omega t}, \quad \widetilde{A}_2^*(x, \; t) = \tilde{a}_2^*(x)e^{-i\Omega t}. \tag{4.1}$$

where Ω means the frequency deviation of waves with the amplitude A_1 and $|A_2|$ from the carrier frequency ω_1 and ω_2, respectively. Substituting eq. (4.1) into eqs. (3.3a) and (3.3b), we obtain the following ordinary differential equation with a vector form,

$$\begin{pmatrix} \tilde{a}_{1x} \\ \tilde{a}_{2x}^* \end{pmatrix} = i \begin{pmatrix} \Omega/v_{g1}, & \delta M_1 A_{3p}/v_{g1} \\ -\delta M_2 A_{3p}^*/v_{g2}, & \Omega/v_{g2} \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2^* \end{pmatrix} , \tag{4.2}$$

where suffix x denotes the differentiation as to x. The equation (4.2) has the same form as eq. (A.1), then we can entirely use the results of Appendix A.

It is useful to define the following notations as shown in Appendix A,

$$\begin{cases} \theta_h = \dfrac{1}{2} \left(\dfrac{\Omega}{v_{g1}} - \dfrac{\Omega}{v_{g2}} \right), \ \beta_h = i\gamma_h = \delta \left(-\dfrac{M_1 M_2}{v_{g1} v_{g2}} \right)^{\frac{1}{2}} |A_{3p}| , \\ \eta_h = -i\zeta_h = \sqrt{1 + (\theta_h/\beta_h)^2} . \end{cases} \tag{4.3}$$

(1). Forward state.

In this state both waves with the envelopes $\widetilde{A}_1$ and $\widetilde{A}_2$ propagate to the positive direction. Then we must regard $\tilde{a}_1(0)$ and $\tilde{a}_2(0)$ as input signals. From eqs. (A.15) and (A.16) it is sufficient to deal with the following power gains,

$$G_{Nh} = |\tilde{a}_1(d)/\tilde{a}_1(0)|^2, \ G_{Ch} = |\tilde{a}_2^*(d)/\tilde{a}_1(0)|^2 , \tag{4.4}$$

where we assumed the casa $\tilde{a}_2^*(0) = 0$.

Now we remark that γ_h is positive. From eqs. (4.4), (A.17b-c) and (A.18b-c) we get

$$G_{Nh} = \begin{cases} 1 + \dfrac{\sinh^2(\gamma_h d\eta_h)}{\eta_h^2} , & (\gamma_h > |\theta|) \\ \\ 1 + \dfrac{\sin^2(\gamma_h d\zeta_h)}{\zeta_h^2} , & (\gamma_h < |\theta|) \end{cases} \tag{4.5}$$

$$G_{Ch} = Q_{21}(G_{Nh} - 1) , \tag{4.6}$$

where $\eta_h \leq 1$, $\zeta_h > 0$ and $Q_{21} = M_2 v_{g1}/M_1 v_{g2}$.
Noting the relation (3.5), we can express $\Omega = \pm \Omega_m$, which gives the condition $\gamma_h = |\theta|$, by the maximum excited wave number K_m,

$$\Omega_m = K_m \sqrt{v_{g1} v_{g2}} \quad . \tag{4.7}$$

From eq. (4.5) we can always get the gain between signals with same carrier frequency. If the frequency deviation is comparatively small ($|\Omega| < \Omega_m$), the normal gain becomes sufficiently large as $d \to \infty$. On the other hand this gain is bounded for comparatively large deviation ($|\Omega| > \Omega_m$). Examing the properties of functions $(\sinh x)/x$ and $(\sin x)/x$, we conclude that G_{Nh} takes the maximum at $\eta_h = 1$ i.e., $\Omega = 0$,

$$G_{Nh}(\Omega) \leq G_{Nh}(0) = \cosh^2(\gamma_h d) . \tag{4.8}$$

From eq. (4.6) we always get the gain for the case of frequency conversion.
(2). Backward state (only v_{g2} is negative).

It is sufficient to deal with the case that only v_{g2} is negative. Since $\widetilde{A}_2$ propagates

to the negative direction, we must regard $\tilde{a}_1(0)$ and $\tilde{a}_2^*(d)$ as the input signals. This situation corresponds to the two-points boundary value problem and the uniqueness of the solution is lost when there appears an eigen solution.

We define the power gains for the backward state,

$$\hat{G}_{Nh} = |\tilde{a}_1(d)/\tilde{a}_1(0)|^2, \quad \hat{G}_{Ch} = |\tilde{a}_2^*(0)/\tilde{a}_1(0)|^2, \tag{4.9}$$

where $\tilde{a}_2^*(d) = 0$.

Now we remark that β_h is positive. Using the relations (A.16), (A.17a) and (A18 a), we get

$$\hat{G}_{Nh} = \left[1 - \frac{\sin^2(\beta_h d \eta_h)}{\eta_h^2} \right]^{-1}, \tag{4.10}$$

$$\hat{G}_{Ch} = Q_{21}(G_{Nh} - 1), \tag{4.11}$$

where $\eta_h \geqq 1$. The normal gain is always larger than unit and may be sufficiently large as $\beta_h d \rightarrow (n + 1/2)\pi$ and $\eta_h \rightarrow 1$. If $\eta_h = 1(\Omega = 0)$ and $\beta_h d = (n + 1/2)\pi$, there appears an eigen solution i. e., an "oscillating" state,

$$\begin{cases} |\tilde{a}_1(x)|^2 = h|M_1/v_{g1}|\sin^2\left(\dfrac{2n+1}{2d}\pi x\right), \\[4mm] |\tilde{a}_2(x)|^2 = h|M_2/v_{g2}|\cos^2\left(\dfrac{2n+1}{2d}\pi x\right), \end{cases} \tag{4.12}$$

where h is an arbitrary constant and $n = 0, \pm 1, \ldots$. It is remarkable that the oscillating state can not be obtained when the frequency deviation is nonzero.

We can also get the gain for the case of frequency conversion. When $\eta_h = 1$ and $\beta_h d = n\pi$, each gain becomes minimum, $\hat{G}_{Nh} = 1$ and $\hat{G}_{Ch} = 0$.

4B). The case of LF pump.

To seek the amplification mechanism in the case of LF pump, we analyse eqs. (3.7 a) and (3.7b) in the same as the case of HF pump.
Taking the solutions as

$$\widetilde{A}_s(x, t) = \tilde{a}_s(x)e^{-i\Omega t}, \quad \widetilde{A}_m(x, t) = \tilde{a}_m(x)e^{-i\Omega t}, \tag{4.13}$$

we get

$$\begin{pmatrix} \tilde{a}_{3x} \\ \tilde{a}_{mx} \end{pmatrix} = i \begin{pmatrix} \Omega/v_{g3}, & \delta M_3 A_{nP}/v_{g3} \\ \delta M_m A_{nP}^*/v_{gm}, & \Omega/v_{gm} \end{pmatrix} \begin{pmatrix} \tilde{a}_3 \\ \tilde{a}_m \end{pmatrix}. \tag{4.14}$$

The following notations are introduced,

$$\theta_l = \frac{1}{2}\left(\frac{\Omega}{v_{g3}} - \frac{\Omega}{v_{gm}}\right), \quad \beta_l = \delta\left(\frac{M_3 M_m}{v_{g3} v_{gm}}\right)^{\frac{1}{2}}|A_{nP}|,$$

$$\eta_l = -i\zeta_l = \sqrt{1 + (\theta_l/\beta_l)^2}. \tag{4.15}$$

(1). Forward state. (v_{g3} and v_{gm} are positive)

Assuming $\tilde{a}_3(0)=0$, we define the gains,

$$G_{Nl} = |\tilde{a}_m(d)/\tilde{a}_m(0)|^2, \quad G_{Cl} = |\tilde{a}_3(d)/\tilde{a}_m(0)|^2. \tag{4.16}$$

Paying attention to that β_t is positive, we obtain

$$G_{Nl} = 1 - \frac{\sin^2(\beta_t\, d\eta_t)}{\eta_t^2}, \tag{4.17}$$

$$G_{Cl} = Q_{3m}(1-G_{Nl}). \tag{4.18}$$

where $\eta_t \geqq 1$ and $Q_{3m} = M_3\, v_{gm}/M_m\, v_{g3}$.

we remark that the gain can not be obtained between the signals with the same frequency.

Now from eqs. (2.7) and (2.8) we obtain

$$Q_{mn} = |M_m\, v_{gn}/M_n\, v_{gm}|$$

$$= \frac{\omega_m}{\omega_n}\left[\frac{(\omega_m^2-\omega_t^2)(\omega_n^2-\omega_p^2)}{(\omega_n^2-\omega_t^2)(\omega_m^2-\omega_p^2)}\right]^{\frac{1}{2}}. \tag{4.19}$$

The quantity Q_{3m} is estimated as follows in each class of the mode coupling,

$$\begin{cases} Q_{32} < \omega_3/\omega_2, \quad Q_{31} > \omega_3/\omega_1, \quad \text{(in class-1)} \\ Q_{3m} > \omega_3/\omega_m. \quad\quad\quad\quad \text{(in class-2)}. \end{cases} \tag{4.20}$$

When the wave with middle carrier frequency ω_2 are pumped in the state of class-1, we can expect the amplification as frequency conversion i.e., "up-conversion".

(2). Backward state. (v_{gm} is negative)

In this case the input signals are $\tilde{a}_3(0)$ and $\tilde{a}_m(d)$. Assuming $\tilde{a}_3(0)=0$, we define

$$\hat{G}_{Nl} = |\tilde{a}_m(0)/\tilde{a}_m(d)|^2, \quad \hat{G}_{Cl} = |\tilde{a}_3(d)/\tilde{a}_m(d)|^2. \tag{4.21}$$

Because γ_t becomes positive, we obtain

$$\hat{G}_{Nl} = \begin{cases} \left[1+\dfrac{\sinh^2(\gamma_t\, d\eta_t)}{\eta_t^2}\right]^{-1}, & (\gamma_t > |\theta_t|) \\[3mm] \left[1+\dfrac{\sin^2(\gamma_t\, d\zeta_t)}{\zeta_t^2}\right]^{-1}, & (\gamma_t < |\theta_t|) \end{cases} \tag{4.22}$$

$$\hat{G}_{Cl} = Q_{3m}(1-\hat{G}_{Nl}). \tag{4.23}$$

We also remark that $\hat{G}_{Nl} < 1$.

5. Parametric Amplification (Large Signal Theory)

It is desirable to analyse the nonlinear behaviour of eqs. (2.6). In this section we give a nonlinear analysis for the case of HF pump especially. However we impose a cetain assumption; each envelope A_n is independent of the time. This corresponds to the

important case $\Omega = 0$ for the linear analysis.

The basic equations are given as

$$\frac{\partial A_1}{\partial x} = i\alpha_1 (A_2^* A_3 + A_p A_2^*) \; , \tag{5.1a}$$

$$\frac{\partial A_2}{\partial x} = i\alpha_2 (A_3 A_1^* + A_p A_1^*) \; , \tag{5.1b}$$

$$\frac{\partial A_3}{\partial x} = i\alpha_3 A_1 A_2 \; , \tag{5.1c}$$

where we changed the notations as $\widetilde{A}_n \to A_n$, $A_{3p} \to A_p$ and α_n is a real constant,

$$\alpha_n = \delta M_n / v_{gn} = \delta \omega_n^2 / v_e^2 k_n \; . \tag{5.2}$$

Introducing functions $a_n(x) = |A_n(x)|^2$, we get the following equations with closed foms (see Appendix B),

$$\frac{1}{2} \frac{\partial^2 a_1}{\partial x^2} = -3\alpha_3 \alpha_2 a_1^2 + 2\left[c_1 \alpha_2 - \alpha_3 (\alpha_1 a_{20} - \alpha_2 a_{10}) \right] a_1 \\ + c_1 (\alpha_1 a_{20} - \alpha_2 a_{10}) \; , \tag{5.3a}$$

$$\frac{1}{2} \frac{\partial^2 a_2}{\partial x^2} = -3\alpha_3 \alpha_1 a_2^2 + 2\left[c_2 \alpha_1 - \alpha_3 (\alpha_2 a_{10} - \alpha_1 a_{20}) \right] a_2 \\ + c_2 (\alpha_2 a_{10} - \alpha_1 a_{10}) \; , \tag{5.3b}$$

where c_1 and c_2 are constants,

$$c_1 = \alpha_3 a_{10} + \alpha_1 a_{30} + 2\alpha_1 X_0 + \alpha_1 |A_p|^2 \; , \tag{5.4a}$$

$$c_2 = \alpha_3 a_{20} + \alpha_2 a_{30} + 2\alpha_2 X_0 + \alpha_2 |A_p|^2 \; , \tag{5.4b}$$

and $a_{n0} = a_n(0)$, $X_0 = X(0)$ where

$$2X(x) = A_p A_3^*(x) + A_p^* A_3(x) \; . \tag{5.5}$$

It is rather difficult to solve eqs. (5.3) generally. Then we take the simple but important case,

$$A_2(0) = 0 \; (\text{and } a_{20} = 0), \tag{5.6}$$

which corresponds to limit the input signals at $x = 0$ to only $a_1(0)$.

From eqs. (B.2) of Appendix B, the following functions,

$$b_1 = \partial a_1 / \partial x \; , \qquad b_2 = \partial a_2 / \partial x \; , \tag{5.7}$$

vanish at $x = 0$ under the condition (5.6). Then equations (5.3) can be reduced to

$$\frac{1}{4} b_1^2 = -\alpha_3 \alpha_2 a_1 (a_1 - a_{10})\left(a_1 - \frac{c_1}{\alpha_3}\right) \; , \tag{5.8a}$$

$$\frac{1}{4} b_2^2 = -\alpha_3 \alpha_1 a_2 \left(a_2 + \frac{\alpha_2}{\alpha_1} a_{10}\right)\left(a_2 - \frac{c_2}{\alpha_3}\right) \; . \tag{5.8b}$$

From eqs. (5.4), (5.5) and (5.6), we obtain

$$\frac{c_1}{\alpha_3} = a_{10} + \frac{\alpha_1}{\alpha_3}|A_{30} + A_p|^2 , \quad \frac{c^2}{\alpha_3} = \frac{\alpha_2}{\alpha_3}|A_{30} + A_p|^2 , \tag{5.9}$$

where $A_{30} = A_3(0)$.

The behaviour of eqs. (5.8) depends on the sign of the coupling coefficient α_n.   In the following we discuss the forward and backward case, respectively.

(1). Forward case (each α_n is positive).

We can integrate eqs. (5.8) using elliptic functions.   Substituting eq. (5.7) into eqs. (5.8) and integrating as to a_1 and a_2, respectively, we get

$$2\sqrt{\alpha_3\,\alpha_2}\cdot x = \pm\int_{a_{10}}^{a_1} \frac{da}{\sqrt{-a(a-a_{10})(a-c_1/\alpha_3)}} = \pm J_1(a_1), \tag{5.10a}$$

$$2\sqrt{\alpha_3\,\alpha_1}\cdot x = \pm\int_{0}^{a_2} \frac{da}{\sqrt{-a(a+\alpha_2\,a_{10}/\alpha_1)(a-c_2/\alpha_3)}} = \pm J_2(a_2) . \tag{5.10b}$$

From eqs. (5.9) the next estimations are obtained,

$$0 < a_{10} < c_1/\alpha_3 , \quad -(\alpha_2/\alpha_1)a_{10} < 0 < c_2/\alpha_3 .$$

Accordingly the range of a_1 and a_2 becomes

$$a_{10} < a_1 < c_1/\alpha_3 , \quad 0 < a_2 < c_2/\alpha_3 . \tag{5.11}$$

From the integrel formulae of elliptic functions, we can calaulate J_1 and J_2 as

$$J_1(a_1) = 2\sqrt{\alpha_2\,\alpha_3}\ \beta F\left[\arcsin\frac{1}{k}\left(\frac{a_1-a_{10}}{a_1}\right)^{\frac{1}{2}},\ k\right] ,$$

$$J_2(a_2) = 2\sqrt{\alpha_1\,\alpha_3}\ \beta F\left[\arcsin\frac{1}{k}\left(\frac{a_2}{a_2+a_{10}\,\alpha_2/\alpha_1}\right)^{\frac{1}{2}},\ k\right] ,$$

where $F(\theta,\ k)$ is the first kind elliptic function with modulus k and

$$\beta = [\alpha_2(\alpha_3\,a_{10} + \alpha_1|A_{30} + A_p|^2)]^{\frac{1}{2}}, \tag{5.12}$$

$$k = \left(\frac{\alpha_1|A_{30} + A_p|^2}{\alpha_3\,a_{10} + \alpha_1|A_{30} + A_p|^2}\right)^{\frac{1}{2}} < 1 . \tag{5.13}$$

Introducing the normal gain $G_N(= a_1(d)/a_1(0))$ and the conversion gain $G_c(= a_2(d)/a_1(0))$ as well as linear analysis, we finally obtain

$$G_N = \mathrm{dn}^{-2}(\beta\mathrm{d},\ k)\leqq 1 + \frac{\alpha_1}{\alpha_3}\frac{|A_{30} + A_p|^2}{a_{10}} , \tag{5.14a}$$

$$G_c = (\alpha_2/\alpha_1)(G_N - 1) , \tag{5.14b}$$

where $\mathrm{dn}(\cdot)$ is Jacobi's elliptic function and the other functions $\mathrm{sn}(\cdot)$ and $\mathrm{cn}(\cdot)$ will be also used in the later discriptions.

If input signal a_{10} is sufficiently small, our results (5.14) must coincide with the results of linear analysis.   Making the approximation,

$$\beta\to\beta_0 = \sqrt{\alpha_1\,\alpha_2}\ |A_{30} + A_p| , \quad k\to 1\ \text{as}\ a_{10}\to 0 ,$$

we get

$$G_N = \cosh^2(\beta_0 d) , \qquad G_C = (\alpha_2/\alpha_1) \sinh^2(\beta_0 d) . \tag{5.15}$$

These coincide with the results of linear analysis. If we make the size d sufficiently large in the linear approximation, the gain becomes as large as we want. But by the exact theory we can remark that the gain has a finite maximum at $d = K(k)/\beta$ where $K(k)$ is the first kind complete elliptic integral.

(2). Backward case (only α_1 is negative).

We set only the coefficient α_1 to be negative. Then the quantity a_{10} is regarded as output signal. If we assume the existence of unique solution, the treatment of this case can be brought out as same as the forward case.

Corresponding to eqs. (5.10), we get

$$2\sqrt{\alpha_2 \alpha_3}\, x = \pm \int_{a_{10}}^{a_1} \frac{da}{\sqrt{-a(a-a_{10})(a-c_1/\alpha_3)}} , \tag{5.16a}$$

$$2\sqrt{-\alpha_1 \alpha_3}\, x = \pm \int_0^{a_1} \frac{da}{\sqrt{a(a-c_2/\alpha_3)(a+a_{10}\,\alpha_2/\alpha_1)}} . \tag{5.16b}$$

To calculate the right hand sides of eqs. (5.16), we prepare the following two cases.

(2A) $(-\alpha_1/\alpha_3)|A_{30} + A_p|^2 < a_{10}$ (large signal case).

The followings are obtained,

$$a_1(x) = a_{10}\, \mathrm{dn}^2(\sqrt{\alpha_2 \alpha_3 a_{10}}\, x,\ k) , \tag{5.17a}$$

$$a_2(x) = -(\alpha_2/\alpha_1)a_{10}\, k^2\, \mathrm{sn}^2(\sqrt{\alpha_2 \alpha_3 a_{10}}\, x,\ k) , \tag{5.17b}$$

where

$$k = \sqrt{-(\alpha_1/\alpha_3 a_{10})}\ |A_{30} + A_p| < 1 . \tag{5.18}$$

From eq. (5.17a) we remark that the normal gain $\hat{G}_N (= a_1(0)/a_1(d))$ has a finite maximum,

$$\hat{G}_N = \mathrm{dn}^{-2}(\sqrt{\alpha_2 \alpha_3 a_{10}}\ d,\ k) \leq \left(1 + \frac{\alpha_1}{\alpha_3}\frac{|A_{30} + A_p|^2}{a_{10}}\right)^{-1} . \tag{5.19}$$

(2B) $a_{10} < (-\alpha_1/\alpha_3)|A_{30} + A_p|^2$ (small signal case).

In this case the power of signals is smallsr than the pumping power, then this situation includes the case of linear analysis.

We get the followings,

$$a_1(x) = a_{10}\, \mathrm{cn}^2(\beta x,\ k) , \tag{5.20a}$$

$$a_2(x) = (-\alpha_2/\alpha_1)a_{10}\, \mathrm{sn}^2(\beta x,\ k) , \tag{5.20b}$$

where

$$\beta = \sqrt{-\alpha_1 \alpha_2}|A_{30} + A_p| , \qquad k = \sqrt{(-\alpha_3/\alpha_1)a_{10}}\ /\ |A_{30} + A_p| . \tag{5.21}$$

From eq. (5.20a) the normal gain,

$$\hat{G}_N = \mathrm{cn}^{-2}(\beta d,\ k) , \tag{5.22}$$

has no finite maximum against the case (2A).   In other words there appears an eigen solution when k satisfies the relation,

$$d = (2m+1) k(k) \qquad (m=0,\ 1,\ ...). \tag{5.23}$$

If a_{10} and a_{30} are sufficiently small, equations (5.20) can be approximated as

$$a_1(x) = a_{10} \cos^2 \beta_0 x, \qquad a_2(x) = (-\alpha_2/\alpha_1) a_{10} \sin^2 \beta_0 x, \tag{5.24}$$

where $\beta_0 = \sqrt{-\alpha_1 \alpha_2}$.   These also coincide with the results of linear analysis.

Finally we remark thet for the case of $a_{10} = -(\alpha_1/\alpha_3)|A_{30} + A_p|^2$ there appear solitary pulses[8] by solving eq. (B. 7).

6. Concluding Remarks.

In this section we give some remarks for the case of HF pump.

We examine the physical meaning of the quantities defined in eq. (4.3).   Taking Ω to be real, we write the solution K of eq. (3.4) as $K(\Omega) = K_r(\Omega) + iK_i(\Omega)$.   That is, we get $K_i(\Omega) = \sqrt{\gamma_h^2 - \theta_h^2}$.   If $K(\Omega)$ is complex, we note that $K_i(\Omega)$ represents the spatial growth rate of waves.   Furthermore the function $K_i(\Omega)$ has a maximum $K_{im} = \gamma_h$ at $\Omega = 0$ and vanishes at $\Omega = \Omega_m$, where Ω_m is defined in eq. (4.7).   We also get

$$\theta_h = K_{im}(\Omega/\Omega_m) . \tag{6.1}$$

Using the results of linear analysis, we can estimate the half-power band width Ω_0 as follows.

(1) Forward state.

If we assume that the normal gain G_{Nh} of eq. (4.5) is sufficiently large ($\gamma_h d \gg 1$) i. e.,

$$G_{Nh}(\eta_h) \cong \exp(2\gamma_h d\eta_h)/4\eta_h^2,$$

we can estimate the half-power band width approximately.   Setting the relation $G_{Nh}(\eta_h) = G_{Nh}(1)/2$, we get $\eta \theta_h = \sqrt{\gamma_h/2d}$ from eq. (4.3).   Using eq. (6.1), we obtain

$$\Omega_0 = \Omega_m / \sqrt{2dK_{im}} \tag{6.2}$$

(2) Backward state.

Assuming $\hat{G}_{Nh} \gg 1$ in eq. (4.10), we also get

$$\theta_h \cong \sqrt{2}\ \beta_h / \sqrt{\hat{G}_{Nh}(1)} \quad,$$

where we regard $\hat{G}_{Nh}$ as $\hat{G}_{Nh}(\eta_h)$.

Now, changing the group velocities as $|v_{g1}| \to v_{g1}$ and $|v_{g2}| \to v_{g2}$, we can define a certain forward state.   Then we introduce the spatial growth rate $K_i(\Omega)$ which vanishes at $\Omega = \hat{\Omega}_m$, where $\hat{\Omega}_m = |\Omega_m|$.   Accordingly we can express the band width Ω_0 as

$$\Omega_0 = \sqrt{2}\ \hat{\Omega}_m / \sqrt{\hat{G}_{Nh}(1)} \quad. \tag{6.3}$$

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Appendix A. Linear boundary value problems.

We consider the following equations with a vector form,

$$\begin{pmatrix} u_{1x} \\ u_{2x} \end{pmatrix} = i \begin{pmatrix} z_{11}, z_{12} \\ z_{21}, z_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \tag{A.1}$$

where z_{11}, z_{22} and a product $z_{12} z_{21}$ are real. At the boundaries $x = 0$ and $x = d$ we assume two paired values $[u_1(0), u_2(0)]$ and $[u_1(d), u_2(d)]$ to be specified. In this appendix we derive the following relations,

$$\begin{pmatrix} u_1(d) \\ u_2(d) \end{pmatrix} = \begin{pmatrix} f_{11}, f_{12} \\ f_{21}, f_{22} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} , \tag{A.2}$$

$$\begin{pmatrix} u_1(d) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} b_{11}, b_{12} \\ b_{21}, b_{22} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(d) \end{pmatrix} . \tag{A.3}$$

The characteristic equation of matrix $[z_{ij}]$ is

$$\lambda^2 - (z_{11} + z_{22})\lambda + (z_{11} z_{22} - z_{12} z_{21}) = 0 , \tag{A.4}$$

where two roots λ_1 and λ_2 are assumed to be different each other. The matrix $[z_{ij}]$ can be expressed as

$$\begin{pmatrix} z_{11}, z_{12} \\ z_{21}, z_{22} \end{pmatrix} = \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix} \begin{pmatrix} \lambda_1, 0 \\ 0, \lambda_2 \end{pmatrix} \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix}^{-1} , \tag{A.5}$$

If we introduce the transformation,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{n}$$

equation (A. 1) is reduced to the diagonal form,

$$\begin{pmatrix} v_{1x} \\ v_{2x} \end{pmatrix} = \begin{pmatrix} i\lambda_1, 0 \\ 0, i\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} .$$

General solutions of this equation are given as

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{i\lambda_1 x}, 0 \\ 0, e^{i\lambda_2 x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} ,$$

where c_1 and c_2 are arbitrary constants.

After all we get the general solutions of eq. (A. 1) as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix} \begin{pmatrix} e^{i\lambda_1 x}, 0 \\ 0, e^{i\lambda_2 x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} , \qquad (A. 6)$$

From eq. (A. 5) we get

$$P_1 = \frac{\lambda_2 - z_{22}}{z_{21}} , \qquad P_2 = \frac{\lambda_1 - z_{11}}{z_{12}} . \qquad (A. 7)$$

At this stage, we define the following quantities,

$$\lambda_1 = \alpha + \beta , \qquad \lambda_2 = \alpha - \beta \qquad (A. 8a)$$

where

$$\beta = \sqrt{\beta_0^2 + \theta^2} = \beta_0 \eta , \qquad \eta = \sqrt{1 + (\theta/\beta_0)^2} \qquad (A. 8b)$$

$$\alpha = \frac{1}{2}(z_{11} + z_{22}), \quad \beta_0 = \sqrt{z_{12} z_{21}}, \quad \theta = \frac{1}{2}(z_{11} - z_{22}) . \qquad (A. 8c)$$

Three quantities β, β_0 and η may be real or pure imaginary. Then it is useful to introduce the following three cases.

(1). $1 \leq \eta^2 <$; three quantities β, β_0 and η are real.

(2). $0 < \eta^2 \leq 1$: β and β_0 are pure imaginary. Then introducing the notations $\gamma = -i\beta$ and $\gamma_0 = -i\beta_0$, we may discuss with real numbers,

$$\gamma = \sqrt{\gamma_0^2 - \theta^2} = \gamma_0 \eta , \qquad \eta = \sqrt{1 - (\theta/\gamma_0)^2} \qquad (A. 8d)$$

(3). $-\infty < \eta^2 < 0$; β is real but β_0 and η are pure imaginary. If we also define $\zeta = i\eta$, we may use the real numbers,

$$\beta = \sqrt{\theta^2 - \gamma_0^2} = \gamma_0 \zeta, \quad \zeta = \sqrt{(\theta/\gamma_0)^2 - 1} . \qquad (A. 8e)$$

From eqs. (A. 2) and (A. 6) we can obtain

$$[f_{ij}] = \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix} \begin{pmatrix} e^{i\lambda_1 d}, 0 \\ 0, e^{i\lambda_2 d} \end{pmatrix} \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix}^{-1} , \qquad (A. 9a)$$

or

$$\begin{cases} f_{11} = \dfrac{1}{\Delta_p}(e^{i\lambda_1 d} - P_1 P_2 e^{i\lambda_2 d}), & f_{12} = \dfrac{P_1}{\Delta}(e^{i\lambda_1 d} - e^{i\lambda_2 d}) , \\[3mm] f_{21} = \dfrac{P_2}{\Delta_p}(e^{i\lambda_1 d} - e^{i\lambda_2 d}), & f_{22} = \dfrac{1}{\Delta_p}(e^{i\lambda_2 d} - P_1 P_2 e^{i\lambda_1 d}) . \end{cases} \qquad (A.\,9b)$$

where $\Delta_p = 1 - P_1 P_2 \, (\neq 0)$.
Substituting eqs. (A. 7) and (A. 8a-c) into eq. (A. 9b)
  we obtain

$$\begin{cases} f_{11} = e^{iad}(\cos\beta d + i\dfrac{\theta}{\beta}\sin\beta d) , & f_{12} = ie^{iad}\dfrac{z_{12}}{\beta}\sin\beta d , \\[3mm] f_{21} = ie^{iad}\dfrac{z_{21}}{\beta}\sin\beta d , & f_{22} = e^{iad}(\cos\beta d - i\dfrac{\theta}{\beta}\sin\beta d) , \\[3mm] \det[f_{ij}] = e^{2iad} . \end{cases} \qquad (A.\,10)$$

We can express the matrix $[b_{ij}]$ of eq. (A. 3) by the elements f_{ij},

$$\begin{pmatrix} b_{11} , & b_{12} \\ b_{21} , & b_{22} \end{pmatrix} = \dfrac{1}{f_{22}} \begin{pmatrix} \det[f_{ij}], & f_{12} \\ - f_{21} , & 1 \end{pmatrix} , \qquad (A.\,11)$$

where we must assume the condition,

$$f_{22} \neq 0 , \qquad (A.\,12)$$

whicn is equivalent to $f_{11} \neq 0$. Using eq. (A. 10), we obtain

$$b_{11} / b_{22} = e^{2iad}, \qquad b_{12} / b_{21} = - z_{12} / z_{21} , \qquad (A.\,13)$$

$$b_{11} = \dfrac{e^{iad}}{\cos\beta d - i\dfrac{\theta}{\beta}\sin\beta d} , \qquad b_{12} = \dfrac{z_{12}}{\beta} \cdot \dfrac{\sin\beta d}{\cos\beta d - i\dfrac{\theta}{\beta}\sin\beta d} \qquad (A.\,14)$$

We obtain the following relations,

$$|f_{11}/f_{22}| = |b_{11} / b_{22}| = 1 , \qquad (A.\,15a)$$

$$|f_{12} / f_{21}| = |d_{12} / d_{21}| = |z_{12} / z_{21}| , \qquad (A.\,15b)$$

$$|b_{11}| = |f_{11}|^{-1}, \qquad |b_{12}| = |f_{12} / f_{11}| . \qquad (A.\,16)$$

Especially we calculate the quantities $|f_{11}|^2$ and $|f_{12}|^2$ in the following cases.
(1) $1 \leqq \eta < \infty$;

$$0 \leqq |f_{11}|^2 = 1 - \dfrac{\sin^2(\beta_0 d\eta)}{\eta^2} \leqq 1 . \qquad (A.\,17a)$$

$$|f_{12}|^2 = \left|\dfrac{z_{12}}{z_{21}}\right| \dfrac{\sin^2(\beta_0 d\eta)}{\eta^2} \qquad (A.\,18a)$$

(2) $0 < \eta \leqq 1$;

$$1 \leqq |f_{11}|^2 = 1 + \dfrac{\sinh^2(\gamma_0 d\eta)}{\eta^2} \leqq \cosh^2(\gamma_0 d) . \qquad (A.\,17b)$$

$$|f_{12}|^2 = \left|\frac{z_{12}}{z_{21}}\right| \frac{\sinh^2(\gamma_0 d\eta)}{\eta^2} \quad . \tag{A.18b}$$

(3) $0 < \zeta < \infty$;

$$1 < |f_{11}|^2 = 1 + \frac{\sin^2(\gamma_0 d\zeta)}{\zeta^2} \quad . \tag{A.17c}$$

$$|f_{12}|^2 = \left|\frac{z_{12}}{z_{21}}\right| \cdot \frac{\sin^2(\gamma_0 d\zeta)}{\zeta^2} \quad . \tag{A.18c}$$

We remark that the condition (A.12) holds except for the conditions,

$$\theta = 0, \quad \beta_0 d = (n+1/2)\pi, \quad (n=0, \pm 1, ...) \tag{A.19}$$

where β_0 is of course real.

Furthermore we supplement the following discussions. When $[u_1(0), u_2(d)]$ are specified, equation (A.1) can be treated from the viewpoint of two point boundary value problem.

Using eq. (A.6), we construct two solutions E_0 and E_d which satjsfy the boundary conditions at $x = 0$ and $x = d$, respectively,

$$E_0(x) = \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\lambda_1 x}, 0 \\ 0, e^{i\lambda_2 x} \end{pmatrix} \begin{pmatrix} u_1(0) - P_1 h_1 \\ h_1 \end{pmatrix},$$

$$E_d(x) = \begin{pmatrix} 1, P_1 \\ P_2, 1 \end{pmatrix} \begin{pmatrix} e^{i\lambda_1(x-d)}, 0 \\ 0, e^{i\lambda_2(x-d)} \end{pmatrix} \begin{pmatrix} h_2 \\ u_2(d) - P_2 h_2 \end{pmatrix},$$

where h_1 and h_2 are constants.

Connecting these solutions at a point $x = \xi$, we get

$$\begin{pmatrix} u_1(0) \\ u_2(d) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda_1 d}, P_1 \\ P_2, e^{+i\lambda_2 d} \end{pmatrix} \begin{pmatrix} h_2 \\ h_1 \end{pmatrix}, \tag{A.20}$$

To determine h_1 and h_2 uniquely from eq. (A.20), we must impose the condition,

$$\Delta = e^{i(\lambda_2 - \lambda_1)d} - P_1 P_2 \neq 0 ,$$

which is just equivalent to eq. (A.12). If eq. (A.19) holds, there appears an eigen solution,

$$E_{0n}(x) \propto \begin{pmatrix} i\sqrt{z_{12}}, 0 \\ 0, \sqrt{z_{21}} \end{pmatrix} \begin{pmatrix} \sin[(n+1/2)\pi x] \\ \cos[n+1/2)\pi x] \end{pmatrix} . \tag{A.21}$$

Appendix B. Analytical treatment of the time independent coupling eqvations.

The time independent coupling equations with a constant pumping can be reduced to the solvable forms (5.3).

Equations (5.1) are altered to the various forms,

$$\begin{cases} A_2 A_3^* \dfrac{\partial A_1}{\partial x} = i\alpha_1 \left(|A_2 A_3|^2 + A_p A_3^* |A_2|^2 \right) , \\[2mm] A_3^* A_1 \dfrac{\partial A_2}{\partial x} = i\alpha_2 \left(|A_3 A_1|^2 + A_p A_3^* |A_1|^2 \right) , \\[2mm] A_1^* A_2^* \dfrac{\partial A_3}{\partial x} = i\alpha_3 |A_1 A_2|^2 , \end{cases} \tag{B.1}$$

$$\begin{cases} \dfrac{\partial |A_1|^2}{\partial x} = i\alpha_1 \left(A_1^* A_2^* A_3 - A_1 A_2 A_3^* \right) + i\alpha_1 \left(A_p A_1^* A_2^* - A_p^* A_1 A_2 \right) , \\[2mm] \dfrac{\partial |A_2|^2}{\partial x} = i\alpha_2 \left(A_1^* A_2^* A_3 - A_1 A_2 A_3^* \right) + i\alpha_2 \left(A_p A_1^* A_2^* - A_p^* A_1 A_2 \right) , \\[2mm] \dfrac{\partial |A_3|^2}{\partial x} = - i\alpha_3 \left(A_1^* A_2^* A_3 - A_1 A_2 A_3^* \right) . \end{cases} \tag{B.2}$$

For the briefness, we define the notations,

$$\begin{cases} a_n(x) = |A_n(x)|^2 , \quad 2X(x) = A_p A_3^*(x) + A_p^* A_3(x) , \\[2mm] F(a_1, a_2, a_3) = \alpha_1 a_2 a_3 + \alpha_2 a_3 a_1 - \alpha_3 a_1 a_2 , \\[2mm] f(a_1, a_2) = \alpha_1 a_2 + \alpha_2 a_1 . \end{cases}$$

Using eqs. (5. 1a), (5. 1b) and (B. 1), we get

$$\frac{\partial}{\partial x}\left(A_p A_1^* A_2^* - A_p^* A_1 A_2 \right) = - 2i \left(X + |A_p|^2 \right) f(a_1, a_2) , \tag{B.3a}$$

$$\frac{\partial}{\partial x}\left(A_1^* A_2^* A_3 - A_1 A_2 A_3^* \right) = - 2i F(a_1, a_2, a_3) - 2i X f(a_1, a_2) . \tag{B.3b}$$

Differetiating eqs. (B. 2) with relations (B. 3), we obtain the coupled equations with real variables,

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial^2 a_1}{\partial x^2} = \alpha_1 \left[F(a_1, a_2, a_3) + 2X f(a_1, a_2) + |A_p|^2 f(a_1, a_2) \right] , \\[2mm] \dfrac{1}{2}\dfrac{\partial^2 a_2}{\partial x^2} = \alpha_2 \left[F(a_1, a_2, a_3) + 2X f(a_1, a_2) + |A_p|^2 f(a_1, a_2) \right] , \\[2mm] \dfrac{1}{2}\dfrac{\partial^2 a_3}{\partial x^2} = - \alpha_3 \left[F(a_1, a_2, a_3) + X f(a_1, a_2) \right] , \end{cases} \tag{B.4}$$

The conservation laws can be obtained from eqs. (B. 2),

$$\alpha_2 (a_1 - a_{10}) = \alpha_1 (a_2 - a_{20}) , \tag{B.5a}$$

$$a_3 - a_{30} = - \frac{\alpha_3}{\alpha_2} (a_2 - a_{20}) + J(x) = - \frac{\alpha_3}{\alpha_1} (a_1 - a_{10}) + J(x) , \tag{B.5b}$$

where

$$J(x) = i\alpha_3 \int_0^x \left(A_p A_1^* A_2^* - A_p^* A_1 A_2 \right) dy = - 2(x - x_0),$$

are obtained from eq. (5. 1c).

From eq. (B. 5b) the function $a_3(x)$ can be expressed as

$$a_3 = -(\alpha_3/\alpha_1)(a_1 - a_{10}) - 2(X - X_o) + a_{30} \; ,$$

$$= -(\alpha_3/\alpha_2)(a_2 - a_{20}) - 2(X - X_o) + a_{30} \; . \tag{B.6}$$

We remark that the functions $a_3(x)$ and $X(x)$ can be excluded from the term,

$$F(a_1, a_2, a_3) + 2Xf(a_1, a_2) + |A_p|^2 f(a_1, a_2) \; ,$$

by substituting eq. (B.6). That is, the first two equations of eqs. (B.4) are closed as to a_1 and a_2. Furthermore using eq. (B.5a), we can get eqs. (5.3).

On the other hand the function $X(x)$ satisfy

$$\frac{\partial^2 X}{\partial x^2} + \alpha_3 (X + |A_p|^2)f(a_1, a_2) = 0. \tag{B.7}$$

If a_1 and a_2 are determined from eqs. (5.3), the function $X(x)$ can be first obtained from eq. (B.7). Then the function $a_3(x)$ is also obtained from eq. (B.6).