On the Heat Transfer in Laminar Incompressible Boundary Layer on a Flat Plate with Fluid Injection or Suction

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1. Introduction

Previously, the author studied analytically the problems of the laminar boundary layer along a flat plate with uniform injection or suction by the method of v.Karman-Pohlhausen using the 5th degree polynomials and obtained some interesting results on the velocity distributions, friction coefficients, and critical suction Reynolds number etc. Now, the heat transfer problems under the same physical conditions have been tried to solve approximately in connection with the problems of transpiration cooling. But the energy equation has some difficult points even in the case of 'incompressible' problems, so attending to the analogous form of the equation with momentum boundary layer equation he has tried to approximate the solution using the results obtained previously in the case of momentum boundary layer problems assuming the value of Prandtl number be nearly unity.

2. Fundamental Fquations

Consider the two dimensional flow along a flat plate on which uniform fluid injection or suction of constant blowing velocity is distributed. As shown in fig. 1, let x-axis be taken along the boundary and y-axis normal to it with origin at the leading edge of the plate, then the basic equations in the boundary layer are the equation of continuity, of momentum and of energy as follows:

continuity $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

momentum $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

energy $$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=a\frac{\partial^2 t}{\partial y^2} \tag{3}$$

with the boundaty conditions:

$$\left.\begin{array}{lll} u=0, & v=v_o=\text{const}, \; t=t_w & \text{for } y=0 \\ u=U, & t=t_1 & \text{for } y\to\infty \end{array}\right\} \tag{4}$$

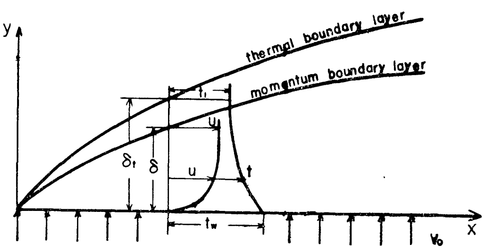

Fig — 1

where u, v are the velocity components in the x-, and y-directions, U the velocity of main stream, v_o the injection (positive) or suction (negative) velocity, t the temperature, t_w, t_1 the temperature of the wall and main flow respectively, $\nu=\mu/\rho$ the kinematic viscosity and $a=\kappa/\rho c_P$ the thermal diffusivity.

The 1st integration of the momentum equation (2) is the so-called v.Karman's momentum equation and it takes the following form in this case

$$\frac{\tau_o}{\rho}=\nu\left(\frac{\partial u}{\partial y}\right)_o=U^2\frac{d\Theta}{dx}-v_o U \tag{5}$$

Moreover, the integration of the energy equation (3) becomes

$$a\left(\frac{\partial t}{\partial y}\right)_o = \frac{d}{dx}\int_o^{\delta_t} (t_1-t)u\,dy - v_o\,(t_1-t_w) \tag{6}$$

where

$$\Theta = \int_o^{\delta} u(U-u)dy$$

δ, δ_t : the thickness of the momentum and thermal boundary layer.

Here, introducing the nondimensional variables

$$\xi = \left(\frac{v_o}{U}\right)^2 \frac{Ux}{\nu} = C_Q{}^2 Rex,\quad C_Q = \frac{v_o}{U},\quad Rex = \frac{Ux}{\nu},\quad \eta = \frac{y}{\delta},\quad \bar{u} = \frac{u}{U}$$

$$\bar{\Theta} = \frac{\Theta}{\delta} = \int_o^1 \bar{u}(1-\bar{u})d\eta,\quad R = \frac{v_o\,\delta}{\nu},\quad R_t = \frac{v_o\,\delta_t}{a} = \Delta\cdot P_r\cdot R$$

$$P_r = \frac{\nu}{a} : \text{Prandtl number},\quad \Delta = \frac{\delta_t}{\delta}$$

$$\xi_t = P_r\cdot\xi,\quad \eta_t = \frac{y}{\delta_t} = \frac{\eta}{\Delta},\quad \bar{\theta} = \frac{t-t_w}{t_1-t_w}$$

the equations (5) and (6) take the following nondimensional forms:

$$\frac{d}{d\xi}(R\cdot\bar{\Theta}) - 1 = \frac{1}{R}\left(\frac{\partial\bar{u}}{\partial\eta}\right)_o \tag{5a}$$

$$\frac{d}{d\xi_t}\left[R_t\int_o^1 (1-\bar{\theta})\bar{u}\,d\eta_t\right] - 1 = \frac{1}{R_t}\left(\frac{\partial\bar{\theta}}{\partial\eta_t}\right)_o \tag{6a}$$

And the boundary conditions are

$$\left.\begin{array}{l}
\bar{u}=0,\quad R\left(\frac{\partial\bar{u}}{\partial\eta}\right)_o = \left(\frac{\partial^2\bar{u}}{\partial\eta^2}\right)_o,\quad R\left(\frac{\partial^2\bar{u}}{\partial\eta^2}\right)_o = \left(\frac{\partial^3\bar{u}}{\partial\eta^3}\right)_o \\
\bar{\theta}=0,\quad R_t\left(\frac{\partial\bar{\theta}}{\partial\eta_t}\right)_o = \left(\frac{\partial^2\bar{\theta}}{\partial\eta_t^2}\right)_o,\quad R_t\left(\frac{\partial^2\bar{\theta}}{\partial\eta_t^2}\right)_o = \left(\frac{\partial^3\bar{\theta}}{\partial\eta_t^3}\right)_o \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \eta=0,\ \eta_t=0 \\
\bar{u}=1,\quad \frac{\partial\bar{u}}{\partial\eta}=0,\quad \frac{\partial^2\bar{u}}{\partial\eta^2}=0 \\
\bar{\theta}=1,\quad \frac{\partial\bar{\theta}}{\partial\eta_t}=0,\quad \frac{\partial^2\bar{\theta}}{\partial\eta_t^2}=0 \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \eta=1,\ \eta_t=1
\end{array}\right\} \tag{7}$$

3. Approximate Solutions

Assuming the velocity and temperature distributions in the boundary layer take the form of 5th degree of polynomials

$$\bar{u} = a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + a_5\eta^5 \tag{8}$$

$$\bar{\theta} = b_1\eta_t + b_2\eta_t{}^2 + b_3\eta_t{}^3 + b_4\eta_t{}^4 + b_5\eta_t{}^5 \tag{9}$$

and determining the coefficients a_i and b_i by the boundary conditions (7), we obtain the velocity and temperature distributions as follows: (cf.fig.2)

$$\bar{u} = \frac{1}{R^2+9R+36}\Big[60\eta + 30R\eta^2 + 10R^2\eta^3 - (15R^2 + 45R + 60)\eta^4 + (6R^2+24R+36)\eta^5\Big] \tag{10}$$

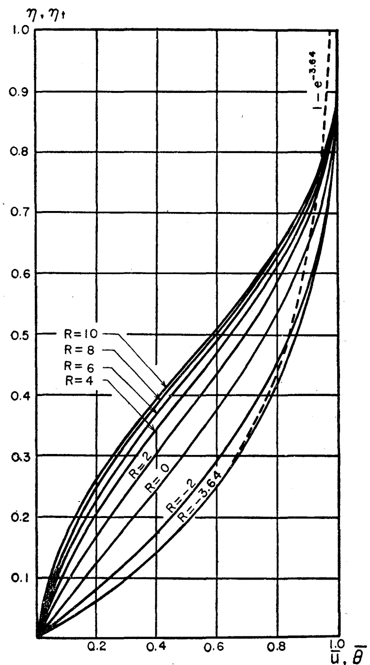

Fig — 2

$$\overline{\theta}=\frac{1}{R_t^2+9R_t+36}\Big[60\eta_t+30R_t\eta_t^2+10R_t^2\eta_t^3-(15R_t^2+45R_t+60)\eta_t^4$$
$$+(6R_t^2+24R_t+36)\eta_t^5\Big] \tag{11}$$

In fig 2 the asymptotic solution is shown by the broken line in comparison with above distributions. Hence, the displacement thickness δ^* and momentum thickness $\overline{\Theta}$ become

$$\delta^*=\frac{\frac{1}{2}R^2+4R+12}{R^2+9R+36} \tag{12}$$

$$\overline{\Theta}=\frac{1}{R^2+9R+36}\left[\frac{25}{231}R^4+\frac{475}{231}R^3+\frac{1480}{77}R^2+\frac{6450}{77}R+\frac{12400}{77}\right] \tag{13}$$

Then, substituting (10) and (13) into (5a), we obtain the following basic ordinary differential equation of the 1st order

$$\frac{d\xi}{dR}=\frac{25}{231}-\frac{5}{231}\,\frac{33R^5-267R^4-5004R^3+9180R^2+159840R+388800}{(R^3+9R^2+36R+60)(R^2+9R+36)} \tag{14}$$

The solution of the above equation may be obtained easily and takes the following form under the leading edge condition $R=0$ for $x=0$:

$$\begin{aligned}\xi=&\,0.10823R-0.79065\ln(1+0.27473R)\\&+0.2729\ln(1+0.32485R+0.060606R^2)+0.12242\ln(1+0.25R+0.027778R^2)\\&-2.6842\tan^{-1}\frac{R}{5.4055+0.87797R}+2.7680\ln\frac{R}{9.0712+1.1339R}\\&+\frac{3.8186R-10.9061}{R^2+9R+36}+0.3029\end{aligned} \tag{15}$$

And the local friction coefficient c_f becomes

$$c_f=\frac{\tau_0}{\frac{1}{2}\rho U^2}=\frac{120C_Q}{R(R^2+9R+36)} \tag{16}$$

Specially, for impermeable wall ($R\to 0$)

$$C_{f0}=0.6475\,(Re_x)^{-\frac{1}{2}} \tag{16a}$$

The coefficient 0,6475 is a desirable value in comparison with 0,664 of exact solution and 0,686 of 4th degree approximation.

The integration of energy equation (6a) is rather difficult as the integrand contains the velocity function $\bar{u}$ besides $\overline{\theta}$, but compairing the velocity with temperature polynomials it is obvious that the types of those polynomials take the same form for the variables (R, η) and (R_t, η_t). Then, the functions $\bar{u}$ and $\overline{\theta}$ may be connected as follows. As there must be an identical distribution of $\overline{\theta}$ as $\bar{u}$ for suitable value of R or R_t and in that distribution we must be able to find the same value of $\overline{\theta}$ as $\bar{u}$ by a suitable transformation of the ordinates, the function $\bar{u}$ contained in the integrand of integrated energy equation may be substituted with $\overline{\theta}$ by the following relation (cf. fig. 3)

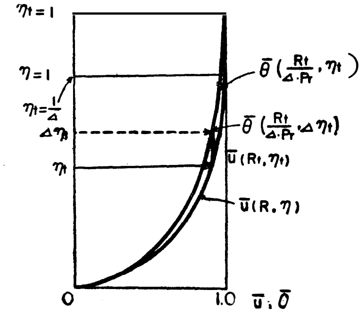

Fig — 3

$$\bar{u}(R_t,\ \eta_t)=\overline{\theta}\left(\frac{R_t}{\Delta\cdot P_r},\ \Delta\cdot\eta_t\right) \tag{17}$$

Hence, eq. (6a) may be rewritten

$$\frac{d}{d\xi_t}\left[R_t\int_0^1\left\{1-\overline{\theta}(R_t\,,\ \eta_t\,)\right\}\overline{\theta}\left(\frac{R_t}{\Delta\cdot P_r},\ \Delta\eta_t\right)d\eta_t\right]-1=\frac{1}{R_t}\left(\frac{\partial\overline{\theta}}{\partial\eta_t}\right)_0 \quad \text{(6b)}$$

For $P_r \simeq 1$,

$$\overline{\theta}\left(\frac{R_t}{\Delta P_r},\ \Delta\eta_t\right)\simeq\overline{\theta}(R_t\,,\ \eta_t\,) \tag{18}$$

Here, defining the heat transfer coefficient and local Nusselt number by

$$q_{(x)}=-\kappa\left(\frac{\partial t}{\partial y}\right)_0=\frac{\rho c_p\, v_o\,(t_w-t_1)}{R_t}\left(\frac{\partial\overline{\theta}}{\partial\eta_t}\right)_0 \tag{19}$$

and

$$Nu_x=\frac{x}{\kappa}\ \frac{q_{(x)}}{t_w-t_1} \tag{20}$$

and substituting the temperature polynomial (11), we obtain the local Nusselt number for the case of a flat plate with uniform injection or suction

$$Nu_x=C_Q\cdot P_r\cdot Re_x\frac{60}{R_t\,(R_t^{\,2}+9R_t+36)} \tag{21}$$

Therefore, for impermeable wall, taking the limit of $R \to 0$, and assuming the momentum boundary layer thickness be

$$\delta=5.0\sqrt{\frac{\nu x}{U}}$$

the local Nusselt number takes the form of

$$(Nu_x\,)_{imp}=\frac{1}{3}\sqrt[3]{P_r}\cdot\sqrt{Re_x}$$

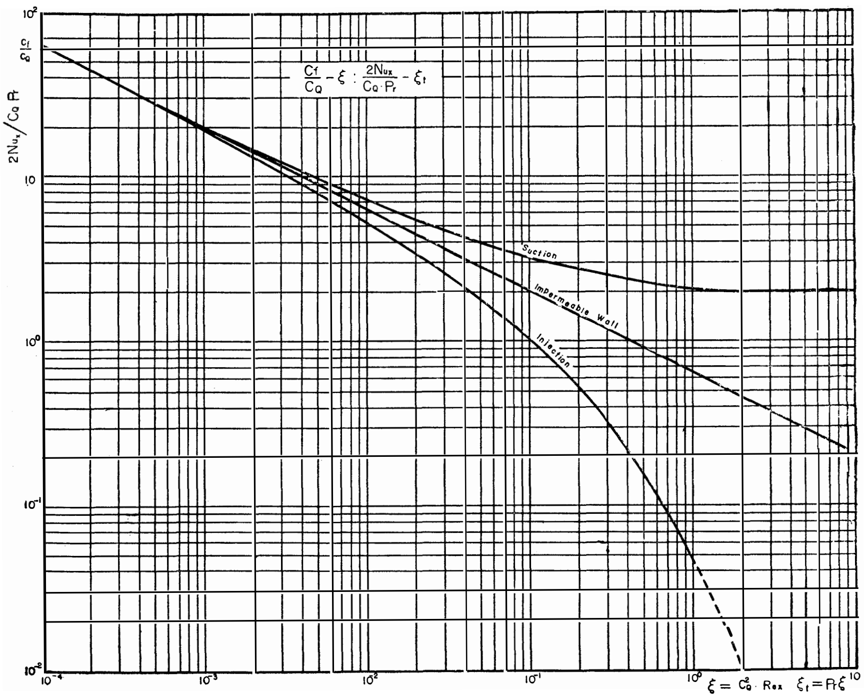

Fig — 4

Thus the coefficient 0,357 for the case of 3rd degree approximation (cf. ref. 2)has been replaced by 1/3 in this case.

Comparing the equations (16) with (21), it may easily be known that the nondimensional friction parameter C_f/C_Q and heat transfer parameter $2N_{ux}/C_QP_r$ take the same functional dependence on the variables R and R_t respectively. Fig. 4 shows their form indicating the deviation from the case of well known impermeable flat plate. Friction coefficient C_f and Nusselt number Nu_x depend on Reynolds number Re_x with injection or suction coefficient C_Q as a parameter and these families of curves may easily be derived from above two curves. Fig. 5 shows C_f - Re_x family as an example.

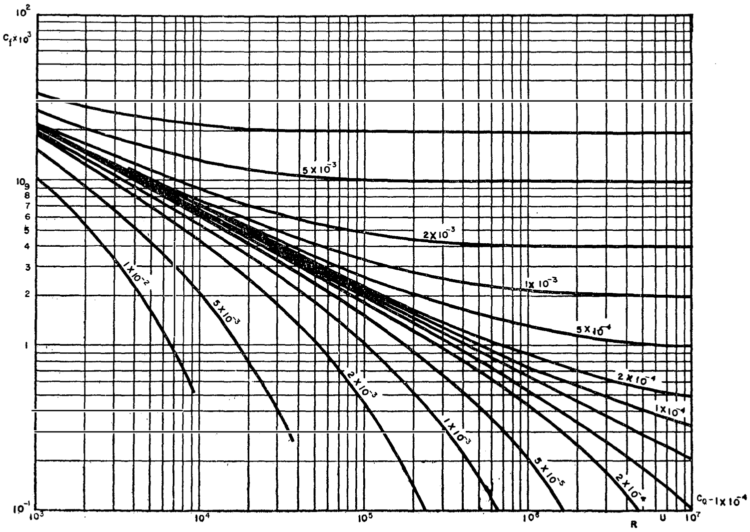

Fig — 5

4. The Thickness Ratio of the Thermal Boundary Layer to the Momentum Boundary Layer

As has been shown above $R_t(\xi_t)$ exhibits the nondimensional thickness of the thermal boundary layer at the point ξ_t, but by the relation

$$R_t(\xi_t)=R_t(P_r\cdot\xi)=R(P_r\cdot\xi)$$

we know that as a matter of fact $R_t(\xi_t)=R(P_r\cdot\xi)$ is the nondimensional thickness of the thermal boundary layer at the point ξ. Because R_t and R take the same numerical values for the same numerical values of independent variables ξ and ξ_t by their analogous form of defining equations and boundary conditions when $P_r \simeq 1$. Hence the thickness ratio Δ of the thermal to the momentum boundary layer at point ξ may be calculated by

$$\Delta=\frac{1}{P_r}\frac{R_t}{R}=\frac{1}{P_r}\frac{R(P_r\xi)}{R(\xi)} \tag{22}$$

Again, for the impermeable wall $R\to 0$, by eq. (15)

$$(\xi)_{R\ll1}=0.03701R^2$$

Hence,

$$\Delta=\frac{1}{P_r}\frac{R(P_r\xi)}{R(\xi)}\longrightarrow\frac{1}{\sqrt{P_r}} \text{ when } R\to 0$$

This ratio takes a different form in contrast with 3rd degree approximation (cf. ref. 2)

$$\Delta=\frac{0.977}{\sqrt[3]{P_r}}$$

Fig. 6 shows the calculated value of Δ for air taking $P=0.71$ in the case of injection. It is an interesting fact that the thickness ratio decreases towards the down stream sides of the plate.

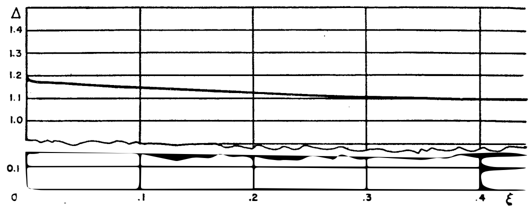

Fig — 6

5. Approximate Solution of Eq. (6a)

As shown in fig. 7 the variation of $\theta(R_t\,,\ \eta_t)$ is very small with respect to R_t , then, it is considered reasonable to retain the variation by η_t . only, or

$$\bar{\theta}\left(\frac{R_t}{\Delta P_r},\ \Delta\eta_t\right)=\bar{\theta}(R_t\,,\ \eta_t)+\delta\bar{\theta}$$

and

$$R_t=R_{to}+\delta R_t$$

Therefore, substituting them into eq. (6a) we obtain

$$\frac{d}{d\xi_t}\left[(R_{to}+\delta R_t)\int_o^1(1-\bar{\theta})(\bar{\theta}+\delta\bar{\theta})\,d\eta_t\right]-1$$

$$=\frac{1}{R_{to}}\left(\frac{\partial\bar{\theta}}{\partial\eta_t}\right)_o-\frac{\delta R_t}{R_{to}}\left(\frac{\partial\bar{\theta}}{\partial\eta_t}\right)_o$$

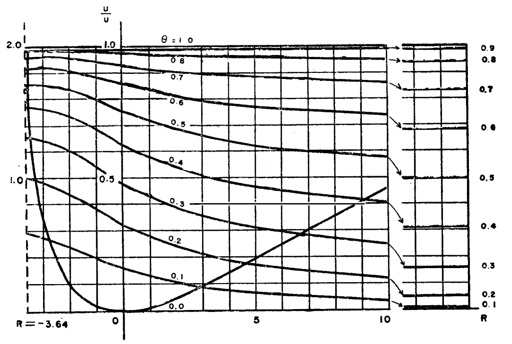

Fig — 7

Hence, neglecting the higher order term with respect to δR_t and $\delta\bar{\theta}$, and taking into consideration the relation (6a) this equation may be simplified as follows

$$\delta R_t=-R_{to}\frac{\int_o^1\left[1-\bar{\theta}(R_{to},\ \eta_t)\right]\delta\bar{\theta}(R_{to},\ \eta_t)\,d\eta_t}{\int_o^1\left[1-\bar{\theta}(R_{to},\ \eta_t)\right]\bar{\theta}(R_{to},\ \eta_t)\,d\eta_t} \tag{23}$$

where

$$\delta\bar{\theta}=\bar{\theta}(R_{to},\ \Delta\eta_t)-\bar{\theta}(R_{to},\ \eta_t)=(\Delta-1)\eta_t\left(\frac{\partial\bar{\theta}}{\partial\eta_t}\right)$$

And the integration contained in numerator may be deformed in following way

$$\int_o^1\left[1-\bar{\theta}(R_{to},\ \eta_t)\right]\delta\bar{\theta}\,d\eta_t$$

$$=(\Delta-1)\int_o^1\left[1-\bar{\theta}(R_{to},\ \eta_t)\right]\eta_t\left(\frac{\partial\bar{\theta}}{\partial\eta_t}\right)d\eta_t=(\Delta-1)\cdot I$$

and

$$I=\int_o^1\left[1-\bar{\theta}(R_{to},\ \eta_t)\right]\eta_t\left(\frac{\partial\bar{\theta}}{\partial\eta_t}\right)d\eta_t$$

$$=-\int_o^1\left[1-\bar{\theta}(R_{to},\ \eta_t)\right]\bar{\theta}(R_{to},\ \eta_t)\,d\eta_t+\int_o^1\bar{\theta}(R_{to},\ \eta_t)\eta_t\frac{\partial\bar{\theta}}{\partial\eta_t}\,d\eta_t$$

where

$$\int_0^1 \bar{\theta}(R_{to},\ \eta_t)\frac{\partial\bar{\theta}}{\partial\eta_t}\eta_t\, d\eta_t = \Big[\eta_t\, \bar{\theta}(R_{to},\ \eta_t)\Big]_0^1 - \int_0^1 \bar{\theta}(R_{to},\ \eta_t)d\eta_t - I$$

hence,

$$I = \frac{1}{2}\int_0^1 \Big[1-\bar{\theta}(R_{to},\ \eta_t)\Big]^2 d\eta_t$$

Therefore, substituting into eq. (23)

$$\frac{R_t}{R_{to}} = \frac{R_{to}+\delta R_t}{R_{to}} = 1-\frac{\Delta-1}{2}\ \frac{\int_0^1 \Big[1-\bar{\theta}\ (R_{to},\ \eta_t)\Big]^2 d\eta_t}{\int_0^1 \Big[1-\bar{\theta}(R_{to},\ \eta_t)\Big]\bar{\theta}\ (R_{to},\ \eta_t)d\eta_t}$$

Substituting the temperature polynomial (11) into eq. (24) we obtain the following expression for R_t

$$\frac{R_t}{R_{to}} = 1+\left(\frac{1}{\Delta P_r}-1\right)\frac{\frac{281}{924}R_{to}^5+\frac{9633}{924}R_{to}^4+\frac{44172}{462}R_{to}^3+\frac{29651}{77}R_{to}^2+\frac{45396}{77}R_{to}}{\left(\frac{25}{231}R_{to}^4+\frac{475}{231}R_{to}^3+\frac{1480}{77}R_{to}^2+\frac{6450}{77}R_{to}+\frac{12400}{77}\right)(R_{to}^2+9R_{to}+36)}$$

$$-(\Delta-1)\frac{\frac{181}{924}R_{to}^4+\frac{2977}{924}R_{to}^3+\frac{1801}{77}R_{to}^2+\frac{6477}{77}R_{to}+\frac{10432}{77}}{\frac{25}{231}R_{to}^4+\frac{475}{231}R_{to}^3+\frac{1480}{77}R_{to}^2+\frac{6450}{77}R_{to}+\frac{12400}{77}} \qquad (25)$$

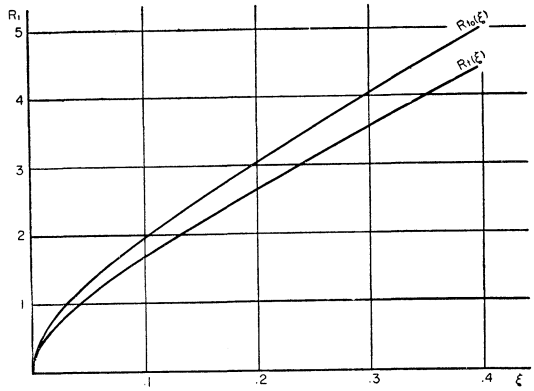

Fig — 8

By eq. (25) and above thickness ratio $\triangle$ the corrected value of R_t may be obtained using R_{to} which relates to ξ_t by the same expression as R to ξ (15). In fig.8 the curve of the corrected R_t by $P_r=0,71$ are compared with one of the uncorrected R_{to} which corresponds to $Pr=1$.

6. Conclusion

The heat transfer in a boundary layer along a flat plate with uniform injection or suction has been studied approximately, and (1) the behavior of the nondimensional heat transfer number $2Nu_x/C_Q\cdot P_r$ with $\xi_t=P_r\cdot\xi$ are the same as that of the nondimensional friction number C_f/C_Q with $\xi=C_Q{}^2\cdot R_{ex}$, (2) in the case of uniform suction these numbers tend to an asymptotic value 2,0, (3) in the case of uniform injection these numbers decrease rapidly for $\xi>0,1$ and in real flow, the separation will occur at some value of ξ which must be an asymptotic value in the numerical operation but in this case even with 5th degree polynomials this value has not been able to find out, (4) with the behavior of $\bar{u}$ (R, η)in mind the approximate solution of the energy equation has been obtained and the first order approximation has been compared with the zeroth order approximation which is the same as the solution of momentum boundary layer equation.

References

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