ON THE LAMINAR INCOMPRESSIBLE BOUNDARY LAYER ON A FLAT PLATE WITH FLUID INJECTION OR SUCTION

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1. Introduction

Problems of the laminar boundary layer on a boundary with fluid injection, or suction have been studied theoretically, or experimentally by many investigators (1) (2) (3),(4),(5),(6),(7) in connection with the film cooling of the heated boundary, or with the problems of the boundary layer control to date. Though the exact solution was obtained by R.Iglisch (1) for a flat plate, it was the case of suction only. And also the brilliant results were obtained by S. W. Yuan (2) for the same case, using the method of v. Kramàn-Pohlhausen in 1949, but the presentation of the velocity distribution in the boundary layer with fourth degree polynomials exhibits the singularity when the suction Reynolds numbr $R=v_0\delta/\nu$ (v_0 : suction velocity and positive for injection) takes the value -6 for the sake of linear expression of R in the denominator. Then,T.P.Torda (3),(4) has tried to improve the fault by changing the boundary conditions with fourth degree polynomials and succeeded in making the denominator a quadratic form with no real roots. But, taking away the one of the outer edge conditions in Torda's method seems to have a great influence on the velocity distribution in the boundary layer. Hence, here, the velocity distribution has been expressed by the fifth degree polynomials and the outer edge boundary condition which was taken away in Torda's method, has been added, and then succeeded in obtaining a more improved solution. The results thus obtained will be compared with the exact solutions by R. Iglisch (1) and the experimental results of P. A. Libby, L. Kaufman and R. P. Harrington, (5) showing a good agreement.

2. Fundamental equations

Consider the two dimensional flow along a boundary on which a uniform fluid injection or suction of constant blowing velocity is distributed. Let x-axis be taken along the boundary and y-axis normal to it with origin at the stagnation point, then, the basic equations in the boundary layer are Prandtl's equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

and the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

with the boundary conditions

$$\left.\begin{aligned} y=0 \quad &: \quad u=0,\ v=v_0 \\ y=\infty \quad &: \quad u=U,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial^2 u}{\partial y^2}=0 \end{aligned}\right\} \tag{3}$$

where u, v are the velocity components in the x- and y-directions, U the velocity of main stream, and ν the kinematic viscosity.

The lst integration of the momentum equation (1) is the so-called v. Kármán's momentum equation and it takes the follwing form in this case

$$\frac{\tau_0}{\rho}=\nu\left(\frac{\partial u}{\partial y}\right)_0=U^2\frac{d\Theta}{dx}+(2\Theta+\delta^*)\frac{dU}{dx}-v_0U \tag{3}$$

The boundary condition can be taken by considering the eq. (1) as follows :

$$\left.\begin{array}{lll} y=0 & : & u=0,\ v_0\left(\frac{\partial u}{\partial y}\right)_0-\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_0=U\frac{dU}{dx},\quad v_0\left(\frac{\partial^2 u}{\partial y^2}\right)_0=\nu\left(\frac{\partial^3 u}{\partial y^3}\right)_0 \\ y=\infty & : & u=U,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial^2 u}{\partial y^2}=0 \end{array}\right\} \tag{4}$$

Now, assuming the velocity distribution in the boundary layer by the fifth degree polynomials

$$\bar{u}=\frac{u}{U}=a_1\eta+a_2\eta^2+a_3\eta^3+a_4\eta^4+a_5\eta^5$$

where $\eta=y/\delta$, and determining the coefficients a_1, a_2,$\cdots a_5$, by the boundary conditions (4) one obtains the velocity distribution as follows :

$$\bar{u}=\frac{1}{R^2+9R+36}\left[R^2F_1(\eta)+R\lambda F_2(\eta)+RF_3(\eta)+\lambda F_4(\eta)+F_5(\eta)\right] \tag{5}$$

where

$$\begin{aligned} &F_1(\eta)=10\eta^3-15\eta^4+6\eta^5 \\ &\left.\begin{aligned} &F_2(\eta)=\eta-6\eta^3+8\eta^4-3\eta^5 \\ &F_3(\eta)=30\eta^2-45\eta^4+24\eta^5 \\ &F^4(\eta)=9\eta-18\eta^2+18\eta^4-9\eta^5 \end{aligned}\right\} \\ &F_5(\eta)=60\eta-60\eta^4+36\eta^5 \end{aligned} \tag{6}$$

and

$$\lambda=\frac{\delta^2}{\nu}\frac{dU}{dx},\qquad R=\frac{\delta.v_0}{\nu} \tag{7}$$

As the denominator of the right hand side of eq. (5,) is a quadratic form with no real roots, the velocity distribution has no singularity on the whole range of the value of R.

Using the velocity thus obtained one will be able to calculate the displacement thickness and the momentum thickness without any diifficulty. Namely, the nondimensional displacemet thickness $\bar{\delta}^*$ is

$$\begin{aligned} \bar{\delta}^*=\frac{\delta^*}{\delta}&=\int_0^1(1-\bar{u})d\eta \\ &=\frac{1}{R^2+9R+36}\left[\left(\frac{1}{2}R^2+4R+12\right)-\lambda\left(\frac{1}{10}R+\frac{3}{5}\right)\right] \end{aligned} \tag{8}$$

and the nondimensional momentum thickness ϑ becomes as follows :

$$\begin{aligned} \bar{\vartheta}=\frac{\vartheta}{\delta}&=\int_0^1\bar{u}(1-\bar{u})d\eta \\ &=\frac{1}{(R^2+9R+36)^2}\left[\left(\frac{25}{231}R^4+\frac{475}{231}R^3+\frac{1480}{77}R^2+\frac{6450}{77}R+\frac{12400}{77}\right)\right. \\ &\left.+\lambda\left(\frac{8}{231}R^3+\frac{9}{22}R^2+\frac{365}{462}R-\frac{150}{77}\right)+\lambda^2\left(-\frac{52}{3465}R^2-\frac{139}{770}R-\frac{423}{770}\right)\right] \end{aligned} \tag{9}$$

Substituting the eqs. (5), (6), (7), (8) and (9) into the momentum equation (3) one obtains the differential equation to determine the blowing Reynolds number R.

3. Solution for the case of a flat plate

When the boundary is a flat plate with uniform fluid injection, or suction distributed from the leading edge of the plate with constant blowing velocity, one obtains the differential equation in nondimensional form as follows, substituting eqs. (5), (6), (7), (8) and (9) into eq. (3) and putting $\lambda=0$

$$\frac{d\xi}{dR}=\frac{25}{231}-\frac{5}{231}\left[\frac{33R^5-267R^4-5004R^3+9180R^2+159840R+38800}{(R+3.64)(R^2+5.36R+16.5)(R^2+9R+36)^2}\right] \tag{10}$$

where ξ is a nondimensional coordinate of the following form

$$\xi=\left(\frac{v_0}{U}\right)^2\frac{Ux}{\nu}$$

Hence, integrating eq. (10) with the condition $R=0$, for $\xi=0$, the solution for the case of a flat plate with uniform fluid injection, or suction is given as follows :

$$\begin{aligned}\xi=&0.10823R-0.79065\ln(0.27473R+1)+0.2729\ln(0.060606R^2+0.32485R+1)\\&+0.12242\ln(0.027778R^2+0.25R+1)-2.6842\tan^{-1}\frac{R}{0.87797R+5.4055}\\&+2.7680\tan^{-1}\frac{R}{1.1339R+9.0712}+\frac{3.8186R-10.9061}{R^2+9R+36}+0.3029\end{aligned} \tag{11}$$

When the values of R are determined for ξ, the velocity distiribution, the thickness and any other charateristic values of the boundary layer may be calculated with some suitable equations. But, as in actual problems the values of ξ would be given previously, the following tables of R-ξ values will be helpful for the practical calculations.

In the tabtle the negative values of R correspond to the case of uniform suction, and positive to injection. Particularly, as the right hand side of eq. (11) diverges when R takes the value

$$R=-3.64$$

one can easily see that there must present an asymptotic value for the case of suction, and therefore the thickness of the boundary layer tends to a finite limit on the far downstream along the plate. Figs. (1) and (2) show the R-ξ curves and as the values of R show the thickness of the boundary

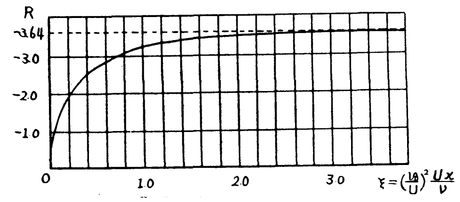

Fig. - 1

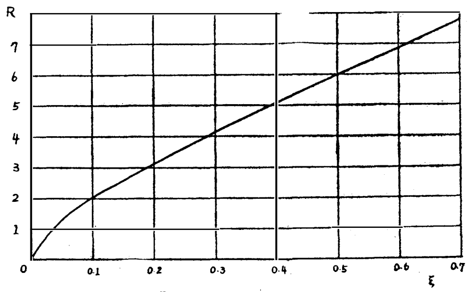

Fig. - 2

R	ξ	R	ξ	R	ξ	R	ξ
−3.64	∞	−0.1	0.0003	3.5	0.24078	7.1	0.62137
−3.6	2.7546	0.0	0.00000	3.6	0.25083	7.2	0.63219
−3.5	1.7950	0.1	0.00039	3.7	0.26092	7.3	0.64300
−3.4	1.3998	0.2	0.00151	3.8	0.27109	7.4	0.65384
−3.3	1.1546	0.3	0.00329	3.9	0.28129	7.5	0.66465
−3.2	0.9801	0.4	0.00568	4.0	0.29153	7.6	0.67549
−3.1	0.8465	0.5	0.00864	4.1	0.30182	7.7	0.68631
−3.0	0.7397	0.6	0.01214	4.2	0.31215	7.8	0.69703
−2.9	0.6513	0.7	0.01551	4.3	0.32254	7.9	0.70799
−2.8	0.5767	0.8	0.01999	4.4	0.33292	8.0	0.71881
−2.7	0.5124	0.9	0.02554	4.5	0.34338	8.1	0.72966
−2.6	0.4563	1.0	0.03085	4.6	0.35387	8.2	0.74050
−2.5	0.4067	1.1	0.03657	4.7	0.36436	8.3	0.75135
−2.4	0.3626	1.2	0.04258	4.8	0.37489	8.4	0.76502
−2.3	0.3229	1.3	0.04902	4.9	0.38543	8.5	0.77305
−2.2	0.2871	1.4	0.05565	5.0	0.39598	8.6	0.78390
−2.1	0.2546	1.5	0.06274	5.1	0.40663	8.7	0.79474
−2.0	0.2248	1.6	0.07001	5.2	0.41643	8.8	0.80559
−1.9	0.1983	1.7	0.07753	5.3	0.42787	8.9	0.81646
−1.8	0.1738	1.8	0.08551	5.4	0.43851	9.0	0.82730
−1.7	0.1514	1.9	0.09327	5.5	0.44921	9.1	0.83751
−1.6	0.1312	2.0	0.10144	5.6	0.45987	9.2	0.84900
−1.5	0.1127	2.1	0.10980	5.7	0.47058	9.3	0.85984
−1.4	0.0661	2.2	0.11833	5.8	0.48130	9.4	0.87072
−1.3	0.0810	2.3	0.12679	5.9	0.49203	9.5	0.88156
−1.2	0.0675	2.4	0.13589	6.0	0.50276	9.6	0.89239
−1.1	0.0544	2.5	0.14473	6.1	0.51306	9.7	0.90306
−1.0	0.0449	2.9	0.15403	6.2	0.52425	9.8	0.91410
−0.9	0.0354	2.7	0.16146	6.3	0.53503	9.9	0.92495
−0.8	0.0274	2.8	0.17212	6.4	0.54579	10.0	0.93530
−0.7	0.0204	2.9	0.18212	6.5	0.55656		
−0.6	0.0147	3.0	0.19168	6.6	0.56736		
−0.5	0.0099	3.1	0.20135	6.7	0.57814		
−0.4	0.0061	3.2	0.21109	6.8	0.58895		
−0.3	0.0033	3.3	0.22092	6.9	0.59975		
−0.2	0.0014	3.4	0.23084	7.0	0.61054		

layer for a given value of blowing velocity v_0, the fact described above may easily be confirmed.

4. Comparison with the previous results.

(a) Suction

For the case of uniform snction with constant blowing velocity distributed along a flat plate there has been given an asymptotic solution obtained by putting $\partial u/\partial x \equiv 0$ in eqs. (1) and (2) directly. As its velocity distribution is given in the present notaions as follows

$$\bar{u} = 1 - e^{R\eta} \qquad (R < O)$$

then, the displacement thickness and momentum thickness becomes

$$\delta^* = \frac{\nu}{-v_0},\quad \vartheta = \frac{1}{2}\,\frac{\nu}{-v_0} \qquad (u_0 < 0)$$

Meanwhile, putting $\lambda=0$ in eqs. (8) and (9), and substituting the value $R=-3,64$, the corresponding thickness may be obtained respectively

$$(\delta^*)_{R=-3,64} = 0.897\frac{\nu}{-V_0},\quad (\vartheta)_{R=-3,64} = 0.41\frac{\nu}{-v_0}$$

These values show satisfactory agreement.

Next, R.Iglisch has obtained an exact solution by means of the v. Mises' transformation, hence the displacement thickness and the velocity distribution will be compared with his results in Figs. (3) and (4). In the neighbourhood of the leading edge of the plate the both results are in good agreement but in far down stream region, though Iglisch's results coincide with the asymptotic solutions described above the present results give some lower values.

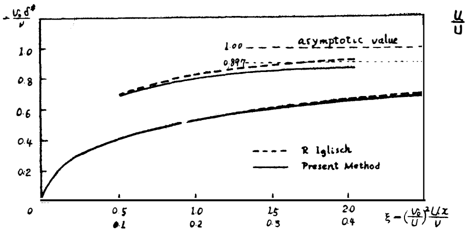

Fig. — 3

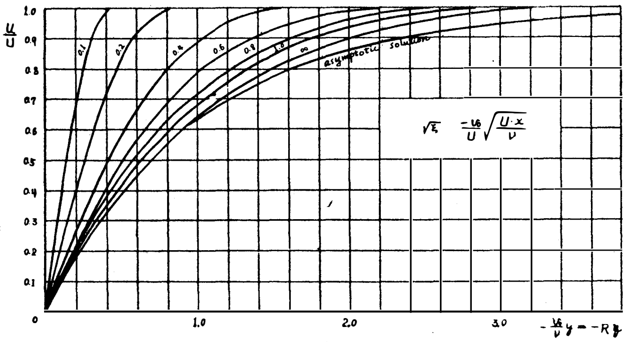

Fig. 4

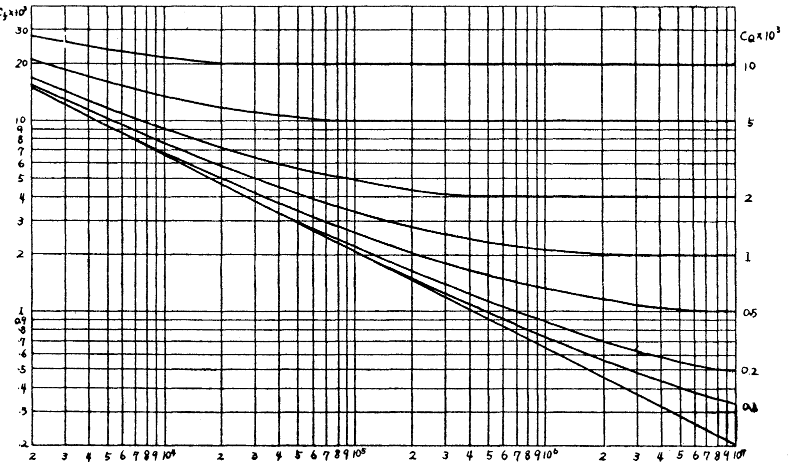

Fig. — 5

The coefficient of friction drag c_f is given by the following expression and the results are shown in Fig. 5.

$$c_f = \frac{\tau_0}{\frac{1}{2}\rho U^2} = 120\frac{v_0}{U}\,\frac{1}{R(R^2+9R+36)} \tag{12}$$

The experiments of this problem have been studied, for example, by P. A. Libby etc., replacing a part of the side wall of wind tunnel by a porous wall. They measured the velocity distribution in the boundary layer with a hot wire anemometer by the constant temperature method, compareing the results with the theoretical distributions calculated by the Yuan's method of fourth degree polynomials. In Fig.6 these results are also compared with the calculated results by the present method. In the case of suction these two calculated results seem to give nearly eaqual distribution.

(b) Injection

In Fig. 7 the experimental, calculated by the fourth degree polynomials and the present results are compared with each other. In the case of injection, as the effect of the boundary

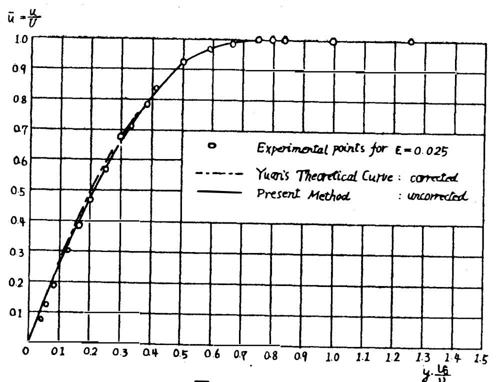

Fig. 6

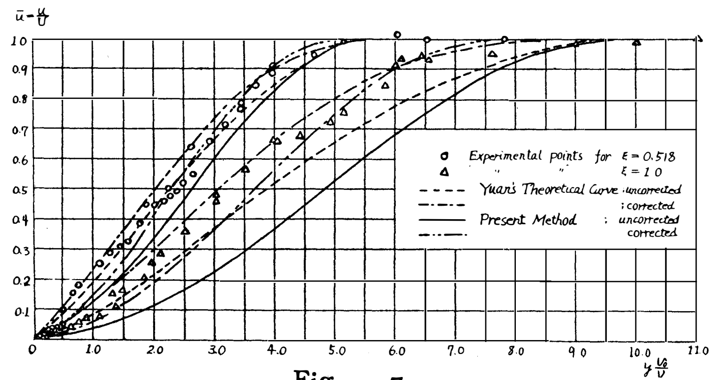

Fig. — 7

layer along the up stream range of the side wall without injection becomes remarkable, then, on the present calculations the following corrections have been made.

Namely, let the injection initiate from the point lying at x_i downstream of the leading edge of the plate and the thickness of the boundary layer be δ_0 which would be taken if the injection were distributed from the leading edge (*cf.* Fig. 8). Here, we assume that as the wall without injection lies ahead of the porous wall the boundary layer along this wall takes the form of a usual laminar layer with thickness of δ_1 and then, it becomes the layer of thickness δ_2 along the porous wall from the point at which the condition of the wall changes. Thus, determining the velocity distributions in the layer δ_2 the wanted correction may be obtained.

Meanwhile, the boundary layer δ_2 may be determined by obtaining the equivalent length of the wall with uniform injection for which the thickness of both layers, without injection and with injection have the same value δ_i at x_i.

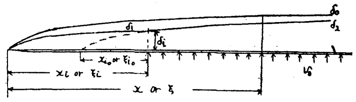

Fig. -8

Therefore, at first, calculating δ_i (x_i) by Blasius' solution as follows

$$\delta_i = 5.0\sqrt{\frac{\nu x_i}{U}}$$

one obtain the blowing Reynolds number R_i by

$$R_i = \frac{v_0 \delta_i}{\nu} = 5.0\sqrt{\left(\frac{v_0}{U}\right)^2 \frac{U x_i}{\nu}} = 5.0\sqrt{\xi_i}$$

By the interpolation, the value of ξ_{i_0} may be obtained in the R-ξ table corresponding to R_i, and the corrected velocity distribution will be able to calculate by the value of R which is interpolated by

$$\xi_r = \xi - \xi_i + \xi_{i_0}$$

As in the paper of P. A. Libby etc. the equivalent length ξ_i of the curved inlet bell is given, the corrected distributions calculated by above method are also shown in Fig. 7.

In the vicinity of the wall the velocity gradients are qreater by fifth degree polynomials than those by the fourth degree. By the present method the separation cannot also be determined, but in practice, there exists the separation at the point given by the following critical blowing parameter

$$\frac{v_0}{U}\sqrt{R_{ex}} = 0.619$$

obtained experimentallyby H. W. Emmons and D.C. Leigh.(8) Thus, the present solution may be regarded as one considerably improved.

5. Concluding Remarks

The laminar boundary layer along a flat plate with uniform injection, or suction of constant blowingvelocity has been studied and a considerably improved sloution has been obtained.

The velocity gradients near the wall become larger by the present method than by the method of fourth degree polynomials, but the separation that must exist in the case of uniform injection cannot be obtained. Hence, this problem is to be left for more advanced investigations. As the blowing Reynolds number R does not exceed the value $-3{,}64$, the anxiety that the velocity distribution diverges at $R=-6$, occurred in the solution by the fourth degree polynomials has vanished.

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