The Bending of Uniformly Loaded Segmental Plate with a Clamped Circular Edge and a Supported Straight Edge

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This paper gives a theoretical solution to the bending of a segmental plate, clamped on a circular edge and supported on a straight edge, subjected to uniformly distributed load.

The parametric coefficients involved in the solution were adjusted so as to satisfy the boundary conditions at the edges of the plate.

Bipolar coordinates were used in the solution, by means of which explicit expressions were obtained for the parametric coefficients.

1. Symbols :

In this paper the following symbols are used:

x,y = rectangular coordinates
α,β = orthogonal curvilinear coordinates
a = a real positive length
$1/h$ = stretch ratio
p = applied load a unit area
ν = poisson's ratio
w = deflection
D = flexural rigidity of the plate
M_α, M_β = bending moments

2. Introduction

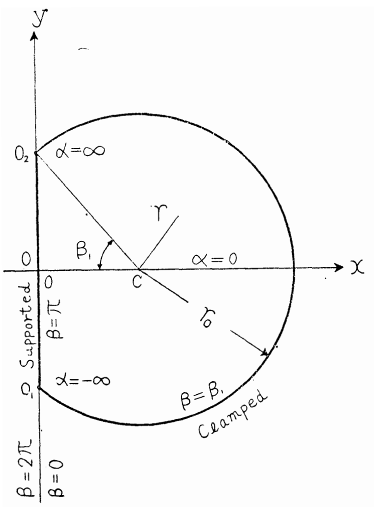

Fig.1 The Segmental Plate

The bending problems in the plate subjected to uniformly distributed load were analysed in the various shapes, but the almost all of these examples were the plate clamped on the all edges. In this paper, the segmental plate clamped on a circular edge and supported on a straight edge was dealt.

The bending problem of the segemental plate, clamped on the all edges under uniform load was investigated by OKADA.(1) And, the case of the semi-circular plate was analysed by KUNO,(2) NADAI,(3) WEINSTEIN(4) and etc.

In this paper, the bipolar coordinates were used and the deflection involved in the solution was determined from the given conditions with the aid of Fourier integral. And the bending moments on the circular edge were cal-

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culated from the deflection.

3. Method of solution

In this paper, a solution of this problem was induced by the Jeffery's method. The bi-polar coordinates (α, β) shall be defined by the equation of transformation

$$x+iy=-a\cot\frac{i}{2}(\alpha+i\beta), \tag{1}$$

such that the two poles of the coordinates are located on the y-axis at the points (o, $\pm a$), and

$$x=\sin\beta/h, \qquad y=\sinh\alpha/h, \tag{2}$$

where

$$h=(\cosh\alpha-\cos\beta)/a. \tag{3}$$

Let the circular edge of the plate be defined by $\beta=\beta_1$, $(\pi>\beta_1>0)$, and the straight edge by $\beta=\pi$.

The differential equation which must be satisfied by the flexure w, shows as follows.

$$D\Delta^2\Delta^2 w=p \tag{4}$$

The bending moments in the bipolar coordinates derived from (hw) are equal to eq. (5).

$$\left.\begin{aligned} aM_\alpha &= -D\left\{(\cosh\alpha-\cos\beta)\left(\frac{\partial^2}{\partial\alpha^2}+\nu\frac{\partial^2}{\partial\beta^2}-1\right)-(1+\nu)\left(\sinh\alpha\frac{\partial}{\partial\alpha}+\sin\beta\frac{\partial}{\partial\beta}-\cosh\alpha\right)\right\}(hw). \\ aM_\beta &= -D\left\{(\cosh\alpha-\cos\beta)\left(\frac{\partial^2}{\partial\beta^2}+\nu\frac{\partial^2}{\partial\alpha^2}+1\right)-(1+\nu)\left(\sinh\alpha\frac{\partial}{\partial\alpha}+\sin\beta\frac{\partial}{\partial\beta}-\cos\beta\right)\right\}(hw). \end{aligned}\right\} \tag{5}$$

The method of solution for this segmental plate is to construct the required flexure w in the form:

$$w=(a^3p/16D)(w_0+w_1) \tag{6}$$

4. Analysis

The particular solution w_0 is given by

$$w_o=x^2(r^2-r_o^2), \tag{7}$$

or

$$hw_o=-\sin^2\beta\sin(\beta-\beta_1)/(\cosh\alpha-\cos\beta)^2\sin\beta_1, \tag{7$_1$}$$

and the auxiliary solution w_1 is therefore

$$hw_1=\int_o^\infty\left\{A_n\sin(\beta-\beta_1)\sinh n(\pi-\beta)+B_n\sin\beta\sinh n(\beta-\beta_1)\right\}\cos n\alpha dn. \tag{8}$$

For this plate under consideration, the boundary conditions are as follows;

(1) along the edge $\beta=\beta_1$,

$$\left[hw\right]_{\beta_1}=\left[\frac{\partial}{\partial\beta}(hw)\right]_{\beta_1}=0, \tag{9}$$

(2) along the edge $\beta=\pi$,

$$[hw]_\pi=[M_\beta]_\pi=0. \tag{10}$$

In the eq. (10) the latter is

$$\left\{(1+\cosh\alpha)(1+\nu\frac{\partial^2}{\partial\alpha^2})-(1+\nu)(1+\sinh\alpha\frac{\partial}{\partial\alpha})\right\}[hw]_\pi+(1+\cosh\alpha)\left[\frac{\partial^2}{\partial\beta^2}(hw)\right]_\pi=0. \tag{11}$$

From eqs. (7) (8) and (11) we get for the condition (10)

$$[hw]_\pi=\left[\frac{\partial^2}{\partial\beta^2}(hw)\right]_\pi=0. \tag{12}$$

For the equations determining the coefficients A_n and B_n, it requires that

$$\int_o^\infty\{A_n\sinh n(\pi-\beta_1)+nB_n\sin\beta_1\}\cos n\alpha dn-\frac{\sin\beta_1}{(\cosh\alpha-\cos\beta_1)^2}=0, \tag{13}$$

and

$$\int_o^\infty n\{A_n\cos\beta_1-B_n\cosh n(\pi-\beta_1)\}\cos n\alpha dn-\frac{1}{(1+\cosh\alpha)^2}=0. \tag{14}$$

By means of the following integral formula[5]

$$\int_o^\infty\frac{\cos n\alpha d\alpha}{\cosh\alpha-\cos\beta}=\frac{\pi\sinh n(\pi-\beta)}{\sin\beta\sinh n\pi},\qquad (0<\beta<2\pi) \tag{15}$$

we get

$$\frac{\sin\beta}{(\cosh\alpha-\cos\beta)^2}=\frac{2}{\sin^2\beta}\int_o^\infty\frac{n\sin\beta\cosh n(\pi-\beta)+\cos\beta\sinh n(\pi-\beta)}{\sinh n\pi}\cos n\alpha dn, \tag{16}$$

and

$$\frac{1}{(1+\cosh\alpha)^2}=\frac{2}{3}\int_o^\infty\frac{n(n^2+1)}{\sinh n\pi}\cos n\alpha dn. \tag{17}$$

Consequently, A_n and B_n are solved as follows:

$$\left.\begin{aligned}A_n&=\frac{2}{3\sin^2\beta_1\sinh n\pi}\,\frac{6n\sin\beta_1\cosh^2 n(\pi-\beta_1)+3\cos\beta_1\sinh 2n(\pi-\beta_1)+2n(n^2+1)\sin^3\beta_1}{\sinh 2n(\pi-\beta_1)+n\sin 2\beta_1},\\ B_n&=\frac{2}{3\sin^2\beta_1\sinh n\pi}\,\frac{3n\sin 2\beta_1\cosh n(\pi-\beta_1)-2(n^2\sin^2\beta_1-4\cos^2\beta_1+1)\sinh n(\pi-\beta_1)}{\sinh 2n(\pi-\beta_1)+n\sin 2\beta_1}.\end{aligned}\right\} \tag{18}$$

To find the bending moment along the circular edge, we have in the virture of eq. (9)

$$-\frac{16}{a^2p}[M_\beta]_{\beta_1}=(\cosh\alpha-\cos\beta_1)\left[\frac{\partial^2}{\partial\beta^2}(hw)\right]_{\beta_1}. \tag{19}$$

Namely, we have

$$\frac{4}{r_o^2\,p}[M_\beta]_{\beta_1}=\frac{\cosh\alpha\cos\beta_1-1}{(\cosh\alpha-\cos\beta_1)^2}\sin^2\beta_1+\frac{2}{3}(\cosh\alpha-\cos\beta_1)\int_o^\infty N_n\cos n\alpha dn, \tag{20}$$

where

$$\left.\begin{aligned}N_n&=n\{n\sin\beta_1\cosh n(\pi-\beta_1)+\cos\beta_1\sinh n(\pi-\beta_1)\}\\ &\quad\times\{3\cosh^2 n(\pi-\beta_1)+n^2\sin^2\beta_1-4\cos^2\beta_1+1\}/\triangle_n\sinh n\pi,\\ \triangle_n&=\sinh 2n(\pi-\beta_1)+n\sin 2\beta_1.\end{aligned}\right\} \tag{21}$$

5. Numerical-results

The most important is the maximum bending moment along the clamped edge of the plate. From $\partial[M_\beta]_{\beta_1}/\partial\alpha=0$, we get $\alpha=0$, and from the earlier paper[6] in which "The Bending of a Plate Shaped by Two Circular Arcs" was analysed by one of the authers, the bending moment at the poles O_1 or O_2 becomes null.

Therefore the bending moment at $\alpha=0$ along the edge may well be taken as the maxi mum one without any appreciable error

For numerical integration, Nn converges slowly for large values of β_1.
To make it rapid convergence, we put

$$Q_n\triangle_n\sinh n\pi=n^2\sin^2\beta_1\{\sin\beta_1\cosh n(\pi-\beta_1)-n\cos\beta_1\sinh n(\pi-\beta_1)\}, \tag{22}$$

and

$$R_n \triangle_n \sinh n\pi = n(n^2\sin^2\beta_1 - 4\cos^2\beta_1 + 1)\{n\sin\beta_1\cosh n(\pi-\beta_1) + \cos\beta_1\sinh n(\pi-\beta_1)\}. \quad (23)$$

Noting that

$$2\int_o^\infty n^2 \frac{\sinh n(\pi-\beta)}{\sinh n\pi} dn = \frac{\sin\beta}{(1-\cos\beta)^2}, \quad (o<\beta<2\pi), \quad (24)$$

and

$$2\int_o^\infty n \frac{\cosh n(\pi-\beta)}{\sinh n\pi} dn = \frac{1}{1-\cos\beta}, \quad (o<\beta<2\pi). \quad (25)$$

we have

$$-\frac{8}{r_o^2 p}[M_\beta]_{\substack{\alpha=0\\ \beta=\beta_1}} = 1 - 4(1-\cos\beta_1)\int_o^\infty (Q_n + \frac{1}{3}R_n)dn. \quad (26)$$

In Fig. 2 and 3 the numerical curves of the coefficients Q_n and R_n respectively, for the different values of β_1, are shown.

In these graphs the numerical integrations were carried out by Simpson' s rule for the approximate quadrature.

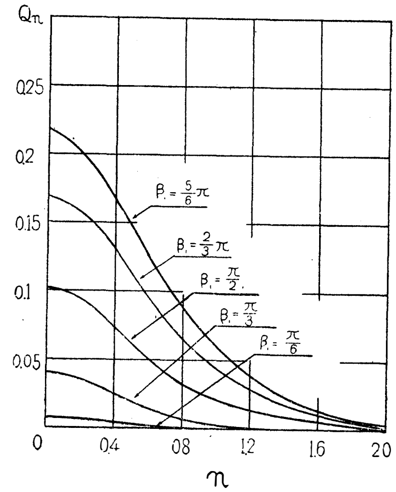

Fig.2 Curves of Coefficients Q_n

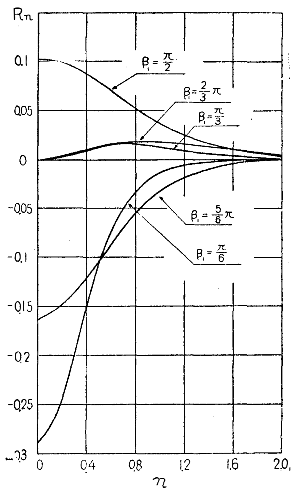

Fig.3 Curves of Coefficients R_n

For the check of these values the theoretical integrations were computed to $\beta_1 = \frac{\pi}{2}$. Namely, we have

$$\int_o^\infty [N_n]_{\beta_1=\frac{\pi}{2}} dn = \frac{1}{4}\int_o^\infty n^2 \left\{ (n^2+4)\frac{\cosh\frac{n\pi}{2}}{\sinh^2\frac{n\pi}{2}} - (n^2+1)\frac{1}{\cosh\frac{n\pi}{2}} \right\} dn. \quad (27)$$

From the integral equation(7)

$$\int_o^\infty \frac{\sin n\alpha dn}{\sinh qn} = \frac{\pi}{2q}\tanh\frac{\pi\alpha}{2q}, \quad (28)$$

we have

$$\int_o^\infty n^2 \frac{\cosh\frac{n\pi}{2}}{\sinh^2\frac{n\pi}{2}} dn = \frac{4}{\pi}, \quad \text{and} \quad \int_o^\infty n^4 \frac{\cosh\frac{n\pi}{2}}{\sinh^2\frac{n\pi}{2}} dn = \frac{16}{\pi}. \quad (29)$$

And from the integral equation(8)

$$\int_0^\infty \frac{\cos n\alpha dn}{\cosh\frac{n\pi}{2}} = \frac{1}{\cosh\alpha}, \tag{30}$$

we have

$$\int_0^\infty \frac{n^2 dn}{\cosh\frac{n\pi}{2}} = 1, \quad \text{and} \quad \int_0^\infty \frac{n^4 dn}{\cosh\frac{n\pi}{2}} = 5. \tag{31}$$

Therefore we have

$$\int_0^\infty [N_n]_{\beta_1 = \frac{\pi}{2}} dn = 1.04645. \tag{32}$$

On the other hand, by the numerical integration we got 1.05368, therefore the error in the numerical integration is 0.691%.

The curves of bending moment at $\alpha=0$, $\beta=\beta_1$ for the different values of β_1 were shown in Fig.4.

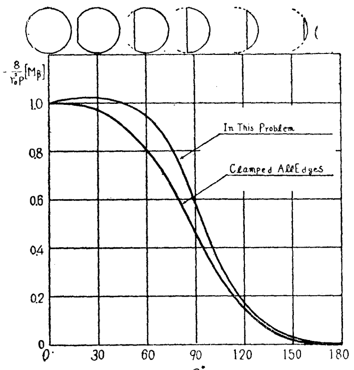

Fig.4 Bending Moment Curves At $\alpha=0$, $\beta=\beta_1$

In this graph, the axis of ordinates shows the ratio to the bending moment in the circular plate clamped on the edge under uniform load.

It is the remarkable result that the curve rises above 1·0-line in the range of $0°\sim45°$ in the values of β_1.

6. Acknowledgement

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7. References

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