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Stresses in a Circular Disk with an Eccentric Circular Hole under Radial Forces*

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This note gives a theoretical solution to a disk with an eccentric circular hole when it is compressed by two equal and opposite forces acting at a radius of the outer circle on the centre of inner circle. The method of solution adds to the given stress system a suitable biharmonic function, which gives no normal and tangential stress on the outer edge. The parametric coefficients involved in the solution are adjusted so as to satisfy the boundary conditions at the inner edge. Bipolar coordinates are used in the solution, by means of which explicit expressions are obtained for the parametric coefficients. Formulas of stress along the edge of circles are derived, and, in particular, the stress distributions of the inner hole are calculated.

Symbols

In this paper the following symbols are used:

$x,\ y$: rectangular coordinates
r_1, θ_1, r, θ : polar coordinates
a : real positive length
$\alpha,\ \beta$: orthogonal curvilinear coordinates
$1/h$: stretch ratio
$\widehat{\alpha\alpha},\ \widehat{\beta\beta}$: normal stress components in curvilinear coordinates
$\widehat{\alpha\beta}$: shearing stress in curvilinear coordinates
χ : complete stress function
χ_0 : basic stress function
χ_1 : auxiliary stress function
$\rho, \sigma, \tau, \alpha_1, \alpha_2$: constants
F : applied force
ν : Poisson's ratio
ξ : $\alpha-\alpha_2$
ξ_1 : $\alpha_1-\alpha_2$

Introduction

In the railway-car, the wheel disk having one or several holes is used. This paper gives the stresses of disk wheel with a hole in the state of repose and having concentrated load from its axle and rail. A disk problem involving an eccentric circular hole is best treated by means of bipolar coordinates. The stress problem due to diametral forces on a circular disk with an eccentric hole has been solved by A. M. Sen Gupta[1], but it is not suitable to railway-wheel.

The fundamental equation[2]

In this section we recall briefly Jeffery's general solution of the two-dimensional field equations referred to bipolar coordinates defined by

$$x+iy=-a\cot\frac{i}{2}(\alpha+i\beta),\quad (-\infty<\alpha<\infty,\ -\pi<\beta<\pi) \qquad (1)$$

The bipolar components in terms of χ have been given by

$$\left.\begin{aligned} a\widehat{\alpha\alpha}&=\left\{(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\beta^2}-\sinh\alpha\frac{\partial}{\partial\alpha}-\sin\beta\frac{\partial}{\partial\beta}+\cosh\alpha\right\}(h\chi)\\ a\widehat{\beta\beta}&=\left\{(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\alpha^2}-\sinh\alpha\frac{\partial}{\partial\alpha}-\sin\beta\frac{\partial}{\partial\beta}+\cos\beta\right\}(h\chi)\\ a\widehat{\alpha\beta}&=-(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\alpha\partial\beta}(h\chi)\end{aligned}\right\} \qquad (2)$$

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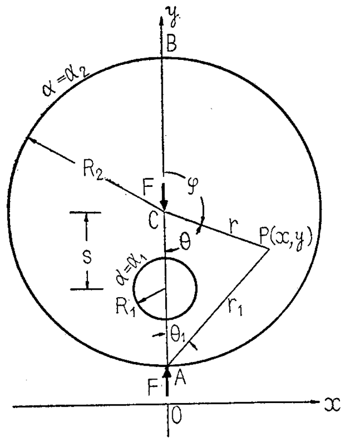

Fig. 1 Disk with an eccentric hole loaded at its centre and periphery

where the stress function must satisfy the biharmonic equation $\nabla^4\chi=0$. Let the outer boundary of the disk be defined by $\alpha=\alpha_2$, and the inner hole by $\alpha=\alpha_1$, $(\alpha_1>2\alpha_2>0)$. The necessary and sufficient conditions for the boundary α=const. to be free from stress are that on the boundary

$$(\partial/\partial\alpha)(h\chi)=\rho \qquad (3)$$

and

$$h\chi=\rho\tanh\alpha+\sigma(\cosh\alpha\cos\beta-1)+\tau\sin\beta \qquad (4)$$

The method of solution

The complete stress function in the disk having a hole is constructed in the form

$$\chi=\chi_0+\chi_1 \qquad (5)$$

As the fundamental stress system it is considered that the equal and opposite forces act on a radius of the disk. The basic stress function χ_0 is given by[3]

$$\chi_0=-\frac{F}{2\pi}\left\{2r_1\theta_1\sin\theta_1+r\theta\sin\theta-\frac{1-\nu}{2}r\cos\theta\log r-\frac{1+\nu}{4R_2^2}r^3\cos\theta-\frac{r^2}{2R_2}\right\} \qquad (6)$$

χ_0 has the singularity on the disk, then the auxiliary stress function χ_1 must be the biharmonic function having no singular point on the region. For χ_1, we assume a function, which produces no stress on the outer boundary, that is

$$\begin{aligned}\frac{8\pi}{F}(h\chi_1)=&B_0\{\xi(\cosh\alpha-\cos\beta)\cosh\alpha_2+\sinh\xi(\cosh\alpha\cos\beta-\cosh^2\alpha_2)\}\\&+2A_1\sinh^2\xi\cos\beta\\&+\sum_{n=2}^{\infty}[A_n\{\cosh(n+1)\xi-\cosh(n-1)\xi\}\\&+B_n\{(n-1)\sinh(n+1)\xi-(n+1)\sinh(n-1)\xi\}]\cos n\beta \qquad (7)\end{aligned}$$

with the aid of the integrals[4],

$$\int_0^{\pi}\frac{\cos n\beta d\beta}{\cosh\alpha-\cos\beta}=\frac{\pi}{\sinh\alpha}e^{-n\alpha},\quad \alpha>0 \qquad (8)$$

$$\int_0^{\pi}\tan^{-1}\left(\frac{p\sin\beta}{1-p\cos\beta}\right)\sin n\beta d\beta=\frac{\pi}{2n}p^n,\quad p^2\leqq 1 \qquad (9)$$

we have the following equations reversed by means of Fourier's transforms.

$$\log r=\log(a\coth\alpha_2-a)-2\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\xi}\sinh n\alpha_2\cos n\beta,\quad \xi>\alpha_2 \qquad (10)$$

$$\theta=2\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\xi}\sinh n\alpha_2\sin n\beta,\quad \xi\geqq\alpha_2 \qquad (11)$$

$$\theta_1=\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\alpha}\{1-(-1)^n e^{n\alpha_2}\}\sin n\beta,\quad \alpha\geqq\alpha_2 \qquad (12)$$

Using the above equations, we expand the basic stress function removing the terms, which produce no stress and strain.

$$\begin{aligned}\frac{8\pi}{F}(h\chi_0)=&2(1-\nu)(\cosh\xi-\cos\beta)\operatorname{cosech}\alpha_2\log(a\coth\alpha_2-a)-4(\cosh\xi-\nu\cos\beta)\\&+(1+\nu)\{\cosh\xi\operatorname{cosech}\alpha_2-2\cosh(\xi-\alpha_2)\\&+(4e^{-\xi}\sinh\xi\sinh\alpha_2-2\cosh\alpha_2-\operatorname{cosech}\alpha_2)\cosh\alpha_2\cos\beta\\&+4\sinh\xi\sinh\alpha_2\sum_{n=2}^{\infty}e^{-n\xi}\cosh n\alpha_2\cos n\beta\}\\&-4\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}e^{-n\xi}(n\sinh\xi+\cosh\xi)R_n\cos n\beta,\quad \alpha>2\alpha_2 \qquad (13)\end{aligned}$$

where

$$\begin{aligned}R_n=&(1+\nu)n(n\sinh n\alpha_2\sinh\alpha_2-\cosh n\alpha_2\cosh\alpha_2)\\&-(1-\nu)\sinh n\alpha_2\operatorname{cosech}\alpha_2-2(-1)^n n \qquad (14)\end{aligned}$$

Consequently from boundary conditions (3) and (4), the parametric functions of n in Eq. (7) are solved as follows:

$$P\{\sinh\xi_1\sinh\alpha_1+\cosh\xi_1\sinh(\alpha_1+\alpha_2)\sinh\alpha_2\}\sinh 2\xi_1$$

$$=(1+\nu)\{\sinh(\xi_1-\alpha_2)-2\cosh\xi_1\sinh\alpha_2\}\sinh\alpha_2-2(1+\cosh\alpha_2)\cosh\xi_1 \quad \cdots(15)$$

$$\varDelta_n=\sinh^2 n\xi_1-n^2\sinh^2\xi_1 \quad \cdots(16)$$

$$B_0=4P\cosh\xi_1 \quad \cdots(17)$$

$$A_1=-(1+\nu)e^{-2\xi_1}\operatorname{cosech}2\xi_1\sinh 2\alpha_2-2P\sinh\alpha_1 \quad \cdots(18)$$

$$A_n=2(1+\nu)\cosh n\alpha_2\sinh\alpha_2 - \{(1+\nu)(\sinh 2n\xi_1-n\sinh 2\xi_1)\cosh n\alpha_2\sinh\alpha_2-2R_n\sinh^2\xi_1\}/\varDelta_n \quad \cdots(19)$$

$$B_n=2(1+\nu)n\cosh n\alpha_2\sinh^2\xi_1\sinh\alpha_2/\varDelta_n + \{2-(\sinh 2n\xi_1+n\sinh 2\xi_1)/\varDelta_n\}R_n/n(n^2-1) \quad \cdots(20)$$

For convenience' sake to calculate the stresses in the neighbourhood of the hole, we reach the following result without difficulty.

$$\begin{aligned}\frac{8\pi}{F}(h\chi)=&4P[\xi(\cosh\alpha-\cos\beta)\cosh\xi_1\\&-\{\cosh\xi_1\cosh\alpha_2-\cosh(\alpha_1-\alpha)\cos\beta\}\sinh\xi]\cosh\alpha_2\\&-4(\cosh\xi-\nu\cos\beta)+2(1-\nu)(\cosh\xi-\cos\beta)\operatorname{cosech}\alpha_2\log(a\coth\alpha_2-a)\\&+(1+\nu)\{\cosh\xi\operatorname{cosech}\alpha_2-2\cosh(\xi-\alpha_2)-(1+\sinh 2\alpha_2)\coth\alpha_2\cos\beta\\&+2\sinh\xi\sinh(2\xi_1-\xi)\operatorname{cosech}2\xi_1\sinh 2\alpha_2\cos\beta\}\\&-4\sum_{n=2}^{\infty}\frac{\left[\begin{array}{l}\{n\sinh\xi\cosh n(\alpha_1-\alpha)+\cosh\xi\sinh n(\alpha_1-\alpha)\}\sinh n\xi_1\\-n\{n\cosh n\xi\sinh(\alpha_1-\alpha)+\sinh n\xi\cosh(\alpha_1-\alpha)\}\sinh\xi_1\end{array}\right]}{n(n^2-1)\varDelta_n}R_n\cos n\beta\\&+4(1+\nu)\sinh\alpha_2\sum_{n=2}^{\infty}\frac{\left[\begin{array}{l}\sinh\xi\sinh n(\alpha_1-\alpha)\sinh n\xi_1\\-n\sinh n\xi\sinh(\alpha_1-\alpha)\sinh\xi_1\end{array}\right]}{\varDelta_n}\cosh n\alpha_2\cos n\beta,\end{aligned}$$

$$\alpha_1\geqq\alpha>2\alpha_2 \quad \cdots(21)$$

Principal stress along the boundary

The principal stress along the hole is the most important in this problem. To find it, we have, by subtracting the first two equations in (2) and by virtue of $[\widehat{\alpha\alpha}]_{\alpha_1}=0$.

$$\begin{aligned}\frac{\pi a}{F}[\widehat{\beta\beta}]_{\alpha_1}=&(\cosh\alpha_1-\cos\beta)\Big[-\frac{1+\nu}{2}\operatorname{cosech}2\xi_1\sinh 2\alpha_2\cos\beta\\&+P(\cosh\xi_1\sinh\alpha_1+\sinh\xi_1\cos\beta)\cosh\alpha_2+\sum_{n=2}^{\infty}M_n\cos n\beta\Big],\end{aligned}$$

$$\alpha_1>2\alpha_2 \quad \cdots(22)$$

where
$$\begin{aligned}M_n\varDelta_n=&(1+\nu)n\{n\sinh\xi_1\sinh\alpha_2\cosh n(\xi_1-\alpha_2)\\&+\sinh n\xi_1\cosh n\alpha_2\sinh(\xi_1-\alpha_2)\}\\&+\{2(-1)^n n+(1-\nu)\sinh n\alpha_2\operatorname{cosech}\alpha_2\}\sinh n\xi_1\sinh\xi_1\end{aligned} \quad \cdots(23)$$

For convenience' sake, the points on the circular boundary shall be specified by φ, as shown in Fig. 1. They are connected to bipolar coordinates (α, β) by

$$\cos\beta=(\cosh\alpha\cos\varphi+1)/(\cosh\alpha+\cos\varphi) \quad \cdots(24)$$

By adding the equation (25) to the equation (26), the stress along the outer circular boundary is obtained.

$$\frac{\pi}{F}[\widehat{\varphi\varphi}]_{r=R_2}=-\frac{1}{2R_2}\Big[\frac{\sin^2\varphi}{1+\cos\varphi}-(2+\nu)\cos\varphi-1\Big\} \quad \cdots(25)$$

$$\begin{aligned}\frac{\pi a}{F}[\widehat{\beta\beta}]_{\alpha_2}=&(\cosh\alpha_2-\cos\beta)\Big[\frac{1+\nu}{2}\Big\{\frac{\cosh\alpha_2\cos\beta-1}{(\cosh\alpha_2-\cos\beta)^2}-2\frac{\cosh(2\xi_1-\alpha_2)}{\sinh 2\xi_1}\cos\beta\Big\}\sinh\alpha_2\\&+P(\cosh\xi_1\sinh\alpha_2-\sinh\xi_1\cos\beta)\cosh\alpha_2-\sum_{n=2}^{\infty}nN_n\cos n\beta\Big],\end{aligned}$$

$$\alpha_2\geqq 0 \quad \cdots(26)$$

where
$$\begin{aligned}N_n\varDelta_n=&(1+\nu)\{n\sin\xi_1\cosh n\alpha_2\sinh(\xi_1-\alpha_2)\\&+\sinh n\xi_1\sinh\alpha_2\cosh n(\xi_1-\alpha_2)\}\\&+\{2(-1)^n n+(1-\nu)\sinh n\alpha_2\operatorname{cosech}\alpha_2\}\sinh^2\xi_1\end{aligned} \quad \cdots(27)$$

For the very small hole, the stress along the hole becomes

$$\frac{\pi a}{2F}[\widehat{\varphi\varphi}]_{r=R_1}=-1+2\cos 2\varphi \quad \cdots(28)$$

It is equal to the equation of stress-distribution along a hole in the infinite plate.

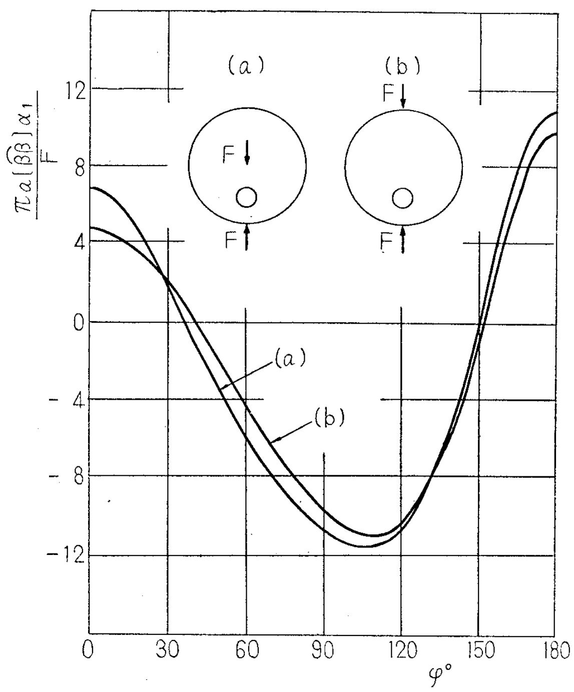

Fig. 2 Stress curves along hole, under radial or diametral forces, when $\lambda=0.176$ and $\mu=0.525$

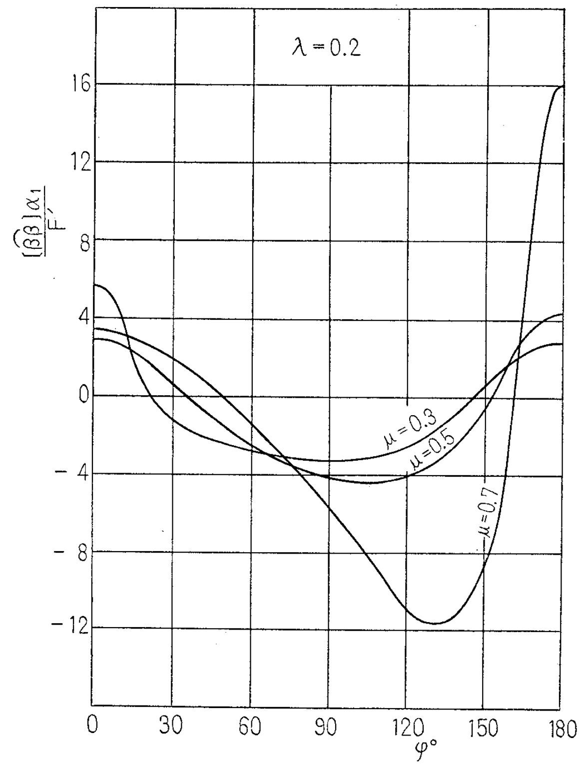

Fig. 3 Stress curves along hole when $\lambda=0.2$, where $F'=F/(^1/_4\pi R_2)$

Numerical results-discussion

Numerical computations were carried out to determine the stress distributions along the hole for four size ratios and three position ratios. We take

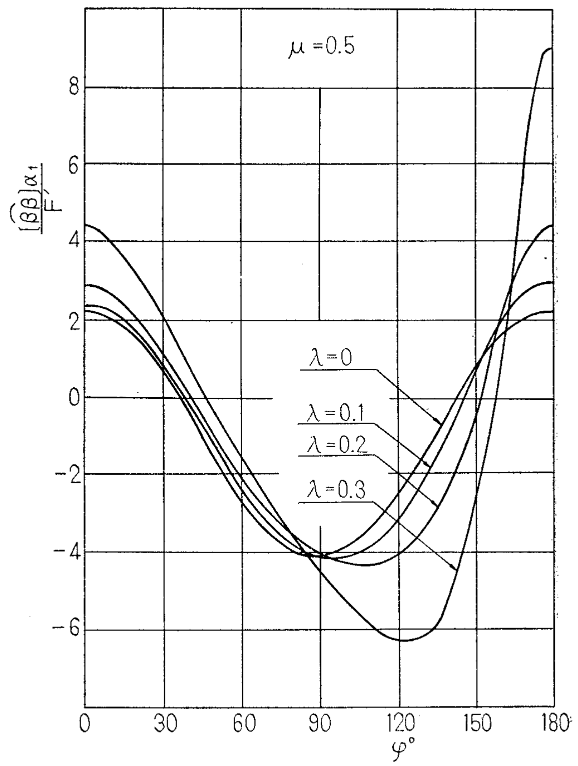

Fig. 4 Stress curves along hole when $\mu=0.5$ where $F'=F/(^1/_4\pi R_2)$

$\nu=0.3$.

The points on the hole were given by φ, the size of the hole by $\lambda=R_1/R_2$, and the position by $\mu=s/R_2$. Fig. 2 shows the stress distributions along the hole, for $\lambda=0.176$ and $\mu=0.525$, in the case of this problem and the diametral forces.

Fig. 3 and 4 show the stress-distributions along the hole in the several cases. It is mentioned that the maximum stress becomes larger in the case of the large hole or of the hole in the neighbourhood of outer edge.

We observe also that this problem can be applied to the disk containing several holes when they are not close to one another, and have the practical applications for disk-wheel except the parts which are very near to the centre.

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References

(1) A.M. Sen Gupta: Jour. Appl. Mech., Vol. 22, No. 2(1955), p. 263.
(2) G.B. Jeffery: Phil. Trans. Roy. Soc., London, Vol. 221(1921), p. 265.
(3) J.H. Michell: Proc. London Math. Soc., Vol. 32 (1900), p. 44.
(4) D. Bierens: Nouvelles Tables d'Integrales, p. 100 and p. 485.