By use of this equation, the stress concentration factor $k_b=(1-\lambda)M_{\max}/M_0$ is calculated and plotted versus ε in Fig. 5. The stress concentration factor k_b is also plotted against λ in Fig. 6. When λ tends to unity, the stress concentration factor k_b may be given approximately by the results obtained for the case of two symmetrically disposed hyperbolic notches. These approximate values also are shown by dotted lines in Fig. 6.

The distribution of the deflection at the middle plane of the strip is now calculated and shown in Figs. 7, 8, and 9. In these figures, the deflection at the point 0(0, 0) is assumed to be zero in every case.

Acknowledgement

The authors wish to express their hearty thanks to Professor Emeritus S. Higuchi of the Tohoku University for his advice and encouragement throughout the progress of the present investigation.

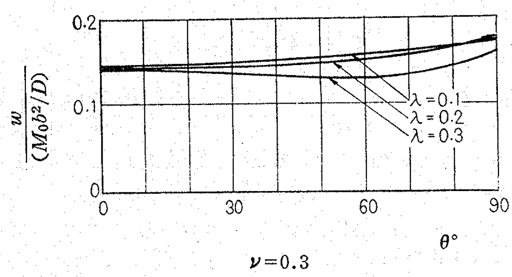

Fig. 9 Distribution of the deflection at the rim of the notch

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Stresses in a Circular Disk with an Eccentric Circular Hole Fixed at its Center and under a Tangential Force on the Outer Edge*

By Kazyu MIYAO**

In a previous paper, the author has reported "Stresses in a Circular Disk with an Eccentric Circular Hole under Radial Forces." In this paper, the stresses in a circular disk containing an eccentric circular hole fixed at its center and under the action of a tangential force were analysed. The method of solution was the same as that used in the previous paper, and the bipolar coordinates were used in the solution. The complete stress-function in the disk with a hole is constructed in the form, in which the basic stress-function is added to the auxiliary one. From the results obtained, the solution of the problem in a semi-infinite plate under a concentrated tangential force on its straight edge and containing an unstressed circular hole was derived.

Symbols

In this paper the following symbols are used:

x, y : rectangular coordinates
r_1, θ_1, r, θ : polar coordinates
a : real positive length
α, β : orthogonal curvilinear coordinates
$1/h$: stretch ratio
$\widehat{\alpha\alpha}, \widehat{\beta\beta}$: normal stress components in curvilinear coordinates
$\widehat{\alpha\beta}$: shearing stress in curvilinear coordinates
χ : complete stress function
χ_0 : basic stress function
χ_1 : auxiliary stress function
$\rho, \sigma, \tau, \alpha_1, \alpha_2$: constants
F : applied force, ν : Poisson's ratio,
ξ : $\alpha-\alpha_2$, ξ_1 : $\alpha_1-\alpha_2$

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Introduction

In the railway-vehicles, the disk-wheels containing several holes are used. In the previous paper[(1)], the author has reported the stresses in a circular disk with an eccentric circular hole under the action of the radial forces at the end of a radius through the center of a hole. The disk-wheel has the reaction and the frictional force by rolling from the rail. The former acts in the radial and the latter in the tangential direction. When the disk-wheel is used as the driving wheel, the frictional force for traction becomes larger.

Accordingly in this paper, the disk with an eccentric circular hole was fixed at its center and placed under a tangential force at the end of a radius through the center of a hole. As earlier investigations, we have the disk without a hole analysed by J. H. Michell[(2)] and K. Ikeda[(3)]. Finally, for a limiting case of our problem the semi-infinite plate containing a hole near to straight edge was discussed.

Fundamental equations[(4)]

In this paper, the solution followed the method of Jeffery's general solution of the two-dimensional field equations referred to bipolar coordinates defined by

$$x+iy=-a\cot\frac{i}{2}(\alpha+i\beta) \quad \cdots\cdots(1)$$

Let the outer edge of a disk be defined by $\alpha=\alpha_2$, and the inner hole by $\alpha=\alpha_1$, $(\alpha_1>2\alpha_2>0)$. The

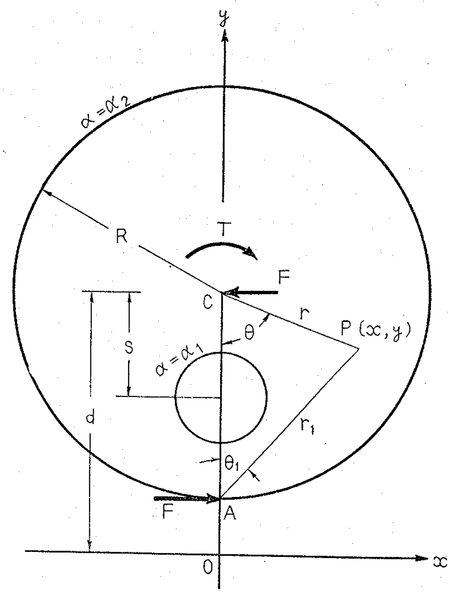

Fig. 1 Disk with an eccentric hole loaded at its center and periphery

complete stress-function in the disk with a hole is constructed in the form

$$\chi=\chi_0+\chi_1 \quad \cdots\cdots(2)$$

The basic stress-function, which is the single valued biharmonic function and having no dislocation, is the function in the action of a tangential force at the edge of the disk, and a reaction and a couple at the center of the disk. The auxiliary stress-function to be added to the basic one has no singular point in the region, and no resultant force and couple on the edge of the hole. The stress components in terms of χ have been given by

$$\left.\begin{aligned}
a\widehat{\alpha\alpha}&=\left\{(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\beta^2}-\sinh\alpha\frac{\partial}{\partial\alpha}-\sin\beta\frac{\partial}{\partial\beta}+\cosh\alpha\right\}(h\chi)\\
a\widehat{\beta\beta}&=\left\{(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\alpha^2}-\sinh\alpha\frac{\partial}{\partial\alpha}-\sin\beta\frac{\partial}{\partial\beta}+\cos\beta\right\}(h\chi)\\
a\widehat{\alpha\beta}&=-(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\alpha\partial\beta}(h\chi)
\end{aligned}\right\} \quad \cdots\cdots(3)$$

Method of solution

The basic stress-function χ_0 is given by[(3)]

$$\frac{2\pi}{F}\chi_0=2r_1\theta_1\cos\theta_1-r\theta\cos\theta-\frac{1-\nu}{2}r\sin\theta\log r-\frac{1+\nu}{4R^2}r^3\sin\theta+R\theta \quad \cdots\cdots(4)$$

or

$$\frac{2\pi}{F}\chi_0=(R-d+y)(2\theta_1+\theta)-\frac{1-\nu}{2}x\log r-\frac{1+\nu}{4}\frac{xr^2}{R^2} \quad \cdots\cdots(5)$$

where d is the distance between the center of the disk and x-axis. The necessary and sufficient conditions for the boundary $\alpha=$const. to be free from stresses are that on the boundary

$$(\partial/\partial\alpha)(h\chi)=\rho \quad \cdots\cdots(6)$$

and
$$h\chi=\rho\tanh\alpha+\sigma(\cosh\alpha\cos\beta-1)+\tau\sin\beta \qquad (7)$$

We assume a function, which produces no stresses on the outer boundary, that is

$$\begin{aligned}&-\frac{2\pi}{F}(h\chi_1)\\&=A_1(\cosh 2\xi-1)\sin\beta+\sum_{n=2}^{\infty}[A_n\{\cosh(n+1)\xi-\cosh(n-1)\xi\}\\&\quad+B_n\{(n-1)\sinh(n+1)\xi-(n+1)\sinh(n-1)\xi\}]\sin n\beta\end{aligned} \qquad (8)$$

In the previous paper, the following expanded equations were introduced, namely

$$\left.\begin{aligned}&\log r=\log(a\coth\alpha_2-a)-2\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\xi}\sinh n\alpha_2\cos n\beta,\quad \xi>\alpha_2\\&\theta=2\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\xi}\sinh n\alpha_2\sin n\beta,\quad \xi\geqq\alpha_2\\&\theta_1=\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\alpha}\{1-(-1)^n e^{n\alpha_2}\}\sin n\beta,\quad \alpha\geqq\alpha_2\end{aligned}\right\} \qquad (9)$$

Applying them to Eq. (5), we can expand the basic stress-function into the following Fourier series removing the terms, which produce no stress and strain,

$$\left.\begin{aligned}&-\frac{2\pi}{F}(h\chi_0)\\&=e^{-\xi}\{(1+\nu)\sinh^2\alpha_2-2(1+\cosh\alpha_2)\}\sinh\xi\sin\beta+\sinh\xi\sum_{n=2}^{\infty}\frac{1}{n}e^{-n\xi}K_n\sin n\beta\\&\quad-\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}e^{-n\xi}(n\sinh\xi+\cosh\xi)L_n\sin n\beta,\\&K_n=2\{(-1)^n-\cosh n\alpha_2\}+(1+\nu)n\sinh n\alpha_2\sinh\alpha_2,\\&L_n=2\{(-1)^n-\cosh n\alpha_2\}(\cosh\alpha_2-1)\operatorname{cosech}\alpha_2\\&\quad+2n\sinh n\alpha_2+(1+\nu)n(n\cosh n\alpha_2\sinh\alpha_2-\sinh n\alpha_2\cosh\alpha_2),\quad(\alpha>2\alpha_2)\end{aligned}\right\} \qquad (10)$$

Determination of coefficients

The boundary conditions at the rim of the hole are satisfied, provided that the coefficients involved in Eq. (8) take the following values on the edge of the hole,

$$\left.\begin{aligned}&A_1=-\frac{1}{2}e^{-2\xi_1}\{(1+\nu)\sinh^2\alpha_2-2(1+\cosh\alpha_2)\}\operatorname{cosech}2\xi_1,\\&A_n=\frac{K_n}{2n}\left(1-\frac{\sinh 2n\xi_1-n\sinh 2\xi_1}{2\varDelta_n}\right)+\frac{L_n\sinh^2\xi_1}{2\varDelta_n},\\&B_n=\frac{K_n\sinh^2\xi_1}{2\varDelta_n}+\frac{L_n}{2n(n^2-1)}\left(1-\frac{\sinh 2n\xi_1+n\sinh 2\xi_1}{2\varDelta_n}\right),\\&\varDelta_n=\sinh^2 n\xi_1-n^2\sinh^2\xi_1\end{aligned}\right\} \qquad (11)$$

To facilitate the computation of the stress-equation at the rim of the hole, we can reach the following result without difficulty.

$$\begin{aligned}&-\frac{2\pi}{F}(h\chi)\\&=\sinh\xi\sinh(2\xi_1-\xi)\{(1+\nu)\sinh^2\alpha_2-2(1+\cosh\alpha_2)\}\operatorname{cosech}2\xi_1\sin\beta\\&\quad+\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}[(n^2-1)\{\sinh\xi\sinh n\xi_1\sinh n(\xi_1-\xi)-n\sinh\xi_1\sinh n\xi\sinh(\xi_1-\xi)\}K_n\\&\quad+\{n^2\sinh\xi_1\cosh n\xi\sinh(\xi_1-\xi)+n\sinh\xi_1\sinh n\xi\cosh(\xi_1-\xi)\\&\quad-n\sinh\xi\sinh n\xi_1\cosh n(\xi_1-\xi)-\cosh\xi\sinh n\xi_1\sinh n(\xi_1-\xi)\}L_n]\frac{\sin n\beta}{\varDelta_n}\end{aligned} \qquad (12)$$

$\alpha_1\geqq\alpha>2\alpha_2$

From this equation, the stresses and strains can be calculated in the region $\alpha_1\geqq\alpha>2\alpha_2$, but in the other region Eqs. (4), (5), and (8) have to be used.

Principal stress along the rim of hole

Now, it is of the greatest importance for practical purposes that the principal stresses along the rim of the hole are estimated numerically. By subtracting the first two equations in Eq. (3) we have

$$a(\widehat{\beta\beta}-\widehat{\alpha\alpha})=(\cosh\alpha-\cos\beta)\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\beta^2}-1\right)(h\chi) \qquad (13)$$

and consequently we have for $\alpha_1>2\alpha_2$,

$$\left.\begin{aligned}&\frac{\pi a}{F}[\widehat{\beta\beta}]_{\alpha_1}\\&\quad=(\cosh\alpha_1-\cos\beta)\Big[\{(1+\nu)\sinh^2\alpha_2-2(1+\cosh\alpha_2)\}\operatorname{cosech}2\xi_1\sin\beta+\sum_{n=2}^{\infty}(M_n/\Delta_n)\sin n\beta\Big],\\&M_n=2n\sinh\xi_1\sinh n\xi_1\sinh n\alpha_2+(1+\nu)n\{n\sinh\xi_1\sinh n(\xi_1-\alpha_2)\sinh\alpha_2\\&\qquad-\sinh n\xi_1\sinh(\xi_1-\alpha_2)\sinh n\alpha_2\}+2\{(-1)^n-\cosh n\alpha_2\}\\&\qquad\times\{(\sinh\alpha_1-\sinh\xi_1)\sinh n\xi_1-n\sinh\xi_1\cosh n\xi_1\sinh\alpha_2\}\operatorname{cosech}\alpha_2\end{aligned}\right\}\qquad(14)$$

And from Eqs. (4) and (8) the stress equation along the outer circular boundary is obtained as follows:

$$-\frac{\pi a}{F}[\widehat{\theta\theta}]_R=\frac{a}{2R}\left\{2+\nu+\frac{1+\cos\theta}{1-\cos\theta}\right\}\sin\theta+2(\cosh\alpha_2-\cos\beta)\left(A_1\sin\beta+\sum_{n=2}^{\infty}nA_n\sin n\beta\right) \qquad (15)$$

$$\alpha_2\geqq 0$$

$\cos\theta$ and $\sin\theta$ are connected to the bipolar coordinates $(\alpha,\ \beta)$ by

$$\left.\begin{aligned}&\cos\theta=(1-\cosh\alpha_2\cos\beta)/(\cosh\alpha_2-\cos\beta),\\&\sin\theta=\sinh\alpha_2\sin\beta/(\cosh\alpha_2-\cos\beta)\end{aligned}\right\}\qquad(16)$$

Applying Eq.(16) to (15), we have the following equation in bipolar coordinates

$$\begin{aligned}-\frac{\pi a}{F}[\widehat{\beta\beta}]_{\alpha_2}&=\frac{1}{2}\left(2+\nu+\coth^2\frac{\alpha_2}{2}\tan^2\frac{\beta}{2}\right)\frac{\sinh^2\alpha_2\sin\beta}{\cosh\alpha_2-\cos\beta}\\&\quad+2(\cosh\alpha_2-\cos\beta)\left(A_1\sin\beta+\sum_{n=2}^{\infty}nA_n\sin n\beta\right),\quad\alpha_2\geqq 0\end{aligned}\qquad(17)$$

The limiting case corresponds to the semi-infinite plate with a hole in the vicinity of the straight edge (Fig. 2). The case in which a normal force acts at the straight edge has been investigated already[(5)].

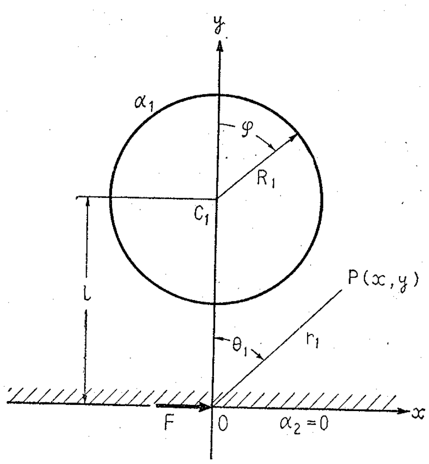

Fig. 2 Semi-infinite plate with a hole loaded at its straight edge

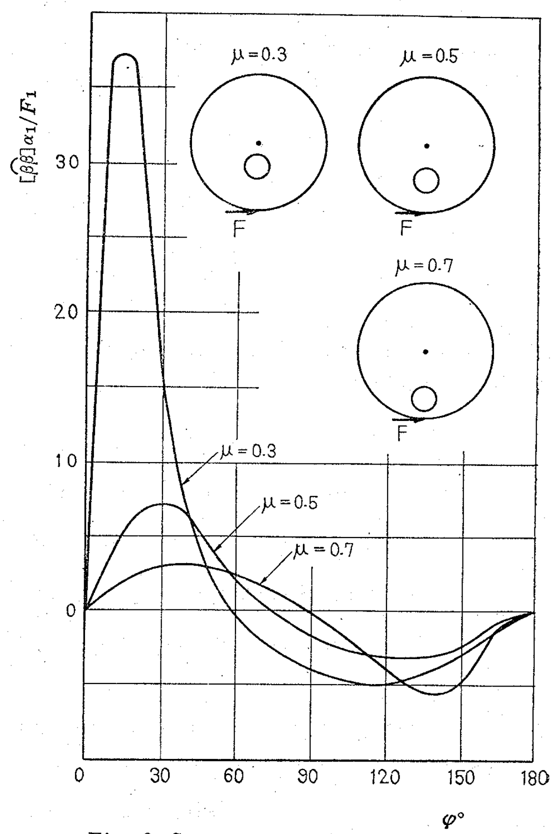

Fig. 3 Stress curves along hole when $\lambda=0.2$, where $F_1=4F/\pi a\operatorname{cosech}\alpha_2$

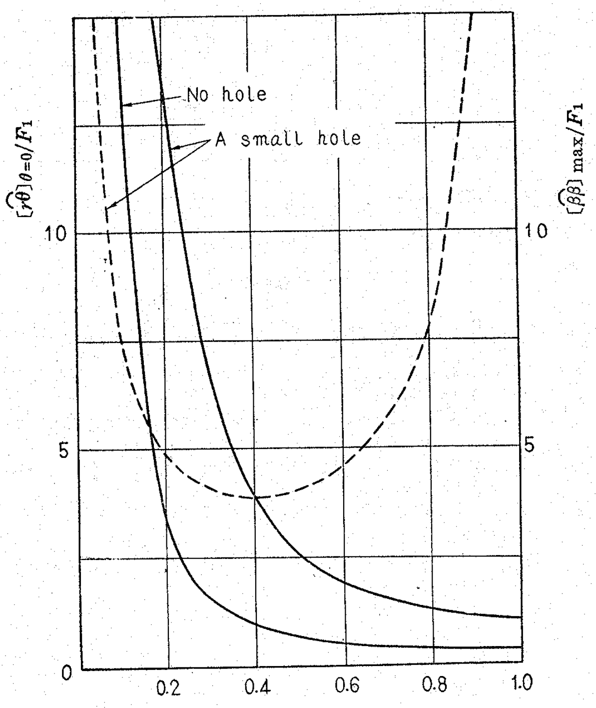

Fig. 4 Maximum stress curves in very small hole

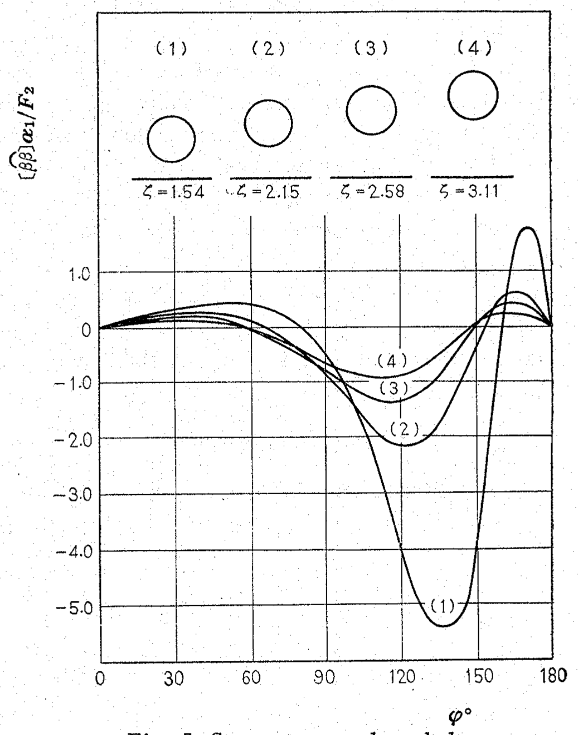

Fig. 5 Stress curves along hole in semi-infinite plate

In our case, the stress-equations along the rim of the hole and the straight edge are given respectively in Eqs.(18) and (19).

$$-\frac{\pi a}{F}[\widehat{\beta\beta}]_{\alpha_1}=4(\cosh\alpha_1-\cos\beta)\left(\operatorname{cosech}2\alpha_1\sin\beta+\sum_{n=3,5\cdots}^{\infty}\frac{\sinh n\alpha_1\cosh\alpha_1-n\sinh\alpha_1\cosh n\alpha_1}{\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1}\sin n\beta\right),\quad \alpha_1>0 \quad\cdots\cdots(18)$$

$$-\frac{\pi a}{F}[\widehat{\beta\beta}]_{\alpha_2=0}=2(1-\cos\beta)\left(\operatorname{cosec}\beta-2e^{-2\alpha_1}\operatorname{cosech}2\alpha_1\sin\beta+2\sinh^2\alpha_1\sum_{n=3,5\cdots}^{\infty}\frac{n\sin n\beta}{\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1}\right)\quad\cdots\cdots(19)$$

Numerical results and discussions

The foregoing stress-equations shall be worked out in some cases numerically. The summation terms in the solution converge slowly for small values of ξ_1, and to make them rapidly convergent the slow convergent terms were separated from the summation terms with the aid of the following equation (20).

$$\sum_{n=1}^{\infty}e^{-np}\sin n\beta=\frac{1}{2}\frac{\sin\beta}{\cosh p-\cos\beta}\quad\cdots\cdots(20)$$

In consequence, the calculations of summation terms suited sufficiently until $n=3$. The numerical computations were carried out to determine the stress-distributions along the rim of the hole for three position ratios. In accordance with the previous paper, the points on the rim of the hole were given by φ which was the angle between the radius of the hole and the positive direction of y-axis, the sizes of hole by $\lambda=\frac{R_1}{R}$, and the positions by $\mu=\frac{s}{R}$. Fig. 3 shows the stress-distributions along the hole. From these curves it is observed that the stresses in the neighbourhood of $\varphi=0$ increase suddenly when the hole locates near the center of the disk. This is the contrary result to that of the previous problem. In Fig. 4, the maximum stresses in the very small hole and the stress-curve along the radius $A-C$ of the disk without a hole as Fig. 1, were shown graphically by full lines, and the maximum stresses in the very small hole under radial forces were added as broken line. It is obvious that even a small hole has the larger stresses than a disk without a hole, and as we compare this case with the previous one, the inclination of stress concentration changes at $\mu>0.4$. Finally the stress-curves along the rim of the hole in the semi-infinite plate were shown in Fig. 5, where F_2 equal to $F/2a\pi\,\mathrm{cosech}\,\alpha_1$. The positions of the hole were specified by $\zeta = l/R_1$. In this graph, we can see that the small tensile stresses occur in the neighbourhood of $\varphi = 180°$ and when the position of the hole is near the straight edge the value of the maximum compressive stress increases. And the values of the maximum stress along the rim of the hole under radial forces were shown in Table 1. Comparing the both cases, the values in the case of radial forces are found larger than those in this case.

In the above numerical computations we took $\nu = 0.3$.

Table 1 Max. values of $\widehat{\beta\beta}$ on boundary $\alpha = \alpha_1$ under radial forces

ζ	1.54	2.15	2.58	3.11
$[\widehat{\beta\beta}]_{\alpha_1}/F_2$	19.748	6.570 4	4.226 0	2.853 9

Acknowledgement

Appreciation goes to Prof. O. Tamate, Tōhoku University, who gave the author the personal advices and strong encouragements throughout the progress of this investigation.

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Errata

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