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Stresses in a Plate Containing a Circular Hole with a Notch*

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In this paper, the problem of determining stresses in a plate containing a circular hole with a circular notch, which has arbitrary shape, is taken up, when the plate is subjected to uniaxial tension. The stress function concerned is constructed by three parts, which are a basic function and two auxiliary ones. And they satisfy the boundary conditions along the rim of circular hole. The parametric coefficients involved in the solution are determined from the given conditions along the notch with the aid of Fourier's transforms. Three fundamental stress systems are discussed. Expressions of the stress along the rim of notch are derived, and the stress concentration factors are calculated.

1. Introduction

The values of multiple stress concentration factor are much greater than simple ones. Hence it is important for practical purpose that the effects of notch on the maximum stress in the plate containing a hole are estimated numerically. But their investigations and data are very few. The theoretical solutions concerned in an infinite plate containing a circular hole with an orthogonal circular notch have been discussed by Hirano[1] and with two symmetrical circular notches centered on the rim of hole by Mitchell[2]. An experimental result in a strip has been reported by Cheng[3].

In this paper a theoretical solution will be given for the two-dimensional multiple stress concentration in an infinite plate containing a circular hole with a circular notch which has arbitrary size and position. The bipolar coordinates are used, and by using the stress functions which satisfy the boundary conditions along the rim of circular hole, the necessary conditions for the determination of parametric coefficients included in the solution are reduced to a problem of two-dimensional linear equations. The crowded circular coordinates are used for the plate containing two contacting circular holes of different sizes. As special cases of this problem, the stresses in an infinite plate containing the hole whose rim is composed of two circular arcs of equal size or two contacting circular holes of equal size have been solved respectively by Ling[4] and the author[5].

* Received 5th November, 1968.

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2. Hole with a circular notch

2·1 Fundamental equations

For this problem we recall easily Jeffery's general solution of two-dimensional field equations referred to bipolar coordinates defined by

$$\left.\begin{array}{l} z=a\cot(\zeta/2),\ z\equiv x+iy,\ \zeta\equiv\beta+i\alpha,\ a>0 \\ ah=\cosh\alpha-\cos\beta,\ x+iy=(\sin\beta+i\sinh\alpha)/h \end{array}\right\} \quad\cdots\cdots\cdots(1)$$

The bipolar components of stress in terms of $h\chi$ have been given by[6]

$$\left.\begin{array}{l} a\sigma_\alpha=\left\{(\cosh\alpha-\cos\beta)\dfrac{\partial^2}{\partial\beta^2}-\sinh\alpha\dfrac{\partial}{\partial\alpha}\right. \\ \qquad \left.-\sin\beta\dfrac{\partial}{\partial\beta}+\cosh\alpha\right\}(h\chi) \\ a\sigma_\beta=\left\{(\cosh\alpha-\cos\beta)\dfrac{\partial^2}{\partial\alpha^2}-\sinh\alpha\dfrac{\partial}{\partial\alpha}\right. \\ \qquad \left.-\sin\beta\dfrac{\partial}{\partial\beta}+\cos\beta\right\}(h\chi) \\ a\tau_{\alpha\beta}=-(\cosh\alpha-\cos\beta)\dfrac{\partial^2(h\chi)}{\partial\alpha\partial\beta} \end{array}\right\} \quad\cdots\cdots(2)$$

and

$$a(\sigma_\alpha-\sigma_\beta)=(\cosh\alpha-\cos\beta)\times\left(\frac{\partial^2}{\partial\beta^2}-\frac{\partial^2}{\partial\alpha^2}+1\right)(h\chi) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

where the stress function χ must satisfy the biharmonic equation $\nabla^4\chi=0$. Let the boundary of the circular hole be difined by $\beta=-c, (c>0)$, and that of the circular notch by $\beta=\beta_1>0$. Since the region occupied by the plate is simply-connected domain, the necessary and sufficient conditions for the boundary β=constant to be free from stress are

$$h\chi=0,\ \partial(h\chi)/\partial\beta=0 \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

2·2 Stress function

If the notch is absent, the stress system in an elastic infinite plate, assumed homogeneous and isotropic, containing a circular hole of radius R_2 subjected to uniaxial tension T at infinity, will be supposed to be specified by stress function χ_0. When the notch is present, the complete stress function χ is constructed in the form

$$\chi=\chi_0+\chi_1+\chi_2 \quad\cdots\cdots(5)$$

The parametric coefficients involved are adjusted to satisfy the boundary conditions of the plate which are given in Eq. (4). The basic stress function χ_0 is given by

$$\frac{4\chi_0}{T}=\lambda(r_0^2-2R_2^2\log r_0)+\mu\frac{(r_0^2-R_2^2)^2}{r_0^2}\cos 2\theta_0 \quad\cdots\cdots(6)$$

where r_0 and θ_0 are polar coordinates of any point

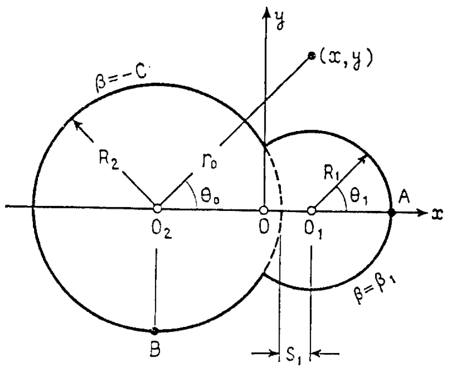

Fig. 1 Hole with a circular notch

$P(x, y)$ on the plate, as shown in Fig. 1, and $\lambda=2$, $\mu=0$ in an all-around tension case, $\lambda=1$, $\mu=-1$ in an x-directional one and $\lambda=1$, $\mu=1$ in a y-directional one. It is found that

$$\left.\begin{aligned}&R_2=\frac{a}{\sin c},\quad r_0^2=a^2\frac{\cosh\alpha-\cos(\beta+2c)}{(\cosh\alpha-\cos\beta)\sin^2 c},\\&\cos 2\theta_0=\cos 2c+\frac{\sin c}{\sin(\beta+c)}\left\{\frac{\sin^2\beta}{\cosh\alpha-\cos\beta}-\frac{\sin^2(\beta+2c)}{\cosh\alpha-\cos(\beta+2c)}\right\}\end{aligned}\right\} \quad\cdots\cdots(7)$$

To keep $h\chi_0$ finite throughout the domain of plate $(\infty\geqq\alpha\geqq-\infty,\ \beta_1\geqq\beta\geqq-c)$ the basic stress function Eq. (6) is shown as follows in bipolar coordinates.

$$\begin{aligned}\frac{4a}{TR_2^2}h\chi_0=&\lambda\left\{2\sin^2 c\cos\beta-(\cosh\alpha-\cos\beta)\log\frac{\cosh\alpha-\cos(\beta+2c)}{\cosh\alpha-\cos\beta}\right\}\\&+2\mu\left[\frac{\sin^2\beta}{\cosh\alpha-\cos\beta}+\frac{\sin^2(\beta+2c)-2\sin^2 c}{\cosh\alpha-\cos(\beta+2c)}-\frac{2\sin(\beta+c)\sin^2(\beta+2c)\sin c}{\{\cosh\alpha-\cos(\beta+2c)\}^2}\right]\sin^2 c \quad\cdots\cdots(8)\end{aligned}$$

Now, let the first auxiliary function χ_1, which has a singular point at the center of circular hole of radius R_2 and produces no stress on the rim of hole $(r_0=R_2)$, be given by

$$4\chi_1/T=K(r_0^2-2R_2^2\log r_0) \quad\cdots\cdots(9)$$

where K is a constant. With the aid of the integrals[7]

$$\log\left(\sin\frac{p}{2}\right)=\int_0^\infty\frac{1-\cosh n(\pi-p)}{n\sinh n\pi}dn,\quad(2\pi>p>0) \quad\cdots\cdots(10)$$

$$\log\left(\cosh\frac{p}{2}\right)=-\log 2-\int_0^\infty\frac{\cos npdn}{n\sinh n\pi} \quad\cdots\cdots(11)$$

$$\sin p\log\left(\sin\frac{p}{2}\right)=-\sin p\log 2-\frac{\pi-p}{2}\cos p+\int_0^\infty\frac{\sinh n(\pi-p)}{(n^2+1)\sinh n\pi}dn,\quad(2\pi>p>0) \quad\cdots\cdots(12)$$

$$\cos p\log\left(\sin\frac{p}{2}\right)=-\frac{1}{2}-\cos p\log 2+\frac{\pi-p}{2}\sin p-\int_0^\infty\frac{n\cosh n(\pi-p)}{(n^2+1)\sinh n\pi}dn,\quad(2\pi>p>0) \quad\cdots\cdots(13)$$

and of $\cosh\alpha-\cos\beta=2\sin\frac{\zeta}{2}\sin\frac{\bar{\zeta}}{2}=\cos\zeta\cos\beta+\sin\zeta\sin\beta-\cos\beta=\cos\bar{\zeta}\cos\beta+\sin\bar{\zeta}\sin\beta-\cos\beta$, it can be shown that

$$\log(\cosh\alpha-\cos\beta)=-\log 2-2\int_0^\infty\frac{\cosh n(\pi-\beta)}{n\sinh n\pi}\cos n\alpha dn,\quad(2\pi>\beta>0) \quad\cdots\cdots(14)$$

$$\log\frac{\cosh\alpha-\cos(\beta+2c)}{\cosh\alpha-\cos\beta}=4\int_0^\infty\frac{\sinh n(\pi-\beta-c)\sinh nc}{n\sinh n\pi}\cos n\alpha dn,\quad(2\pi>\beta>0) \quad\cdots\cdots(15)$$

$$\begin{aligned}&(\cosh\alpha-\cos\beta)\log(\cosh\alpha-\cos\beta)=\alpha\sinh\alpha-\cos\beta-(\cosh\alpha-\cos\beta)\log 2\\&\quad+2\cos\beta\int_0^\infty\frac{\cosh n(\pi-\beta)}{n\sinh n\pi}\cos n\alpha dn-2\int_0^\infty\frac{n\cos\beta\cosh n(\pi-\beta)-\sin\beta\sinh n(\pi-\beta)}{(n^2+1)\sinh n\pi}\cos n\alpha dn,\\&\quad(2\pi>\beta>0) \quad\cdots\cdots(16)\end{aligned}$$

$$\begin{aligned}&(\cosh\alpha-\cos\beta)\log\{\cosh\alpha-\cos(\beta+2c)\}=\alpha\sinh\alpha-\cos(\beta+2c)\\&\quad-(\cosh\alpha-\cos\beta)\log 2+2\cos\beta\int_0^\infty\frac{\cosh n(\pi-\beta-2c)}{n\sinh n\pi}\cos n\alpha dn\\&\quad-2\int_0^\infty\frac{n\cos(\beta+2c)\cosh n(\pi-\beta-2c)-\sin(\beta+2c)\sinh n(\pi-\beta-2c)}{(n^2+1)\sinh n\pi}\cos n\alpha dn,\quad(2\pi>\beta>0) \quad\cdots\cdots(17)\end{aligned}$$

$$\frac{\sin\beta}{\cosh\alpha-\cos\beta}=2\int_0^\infty\frac{\sinh n(\pi-\beta)}{\sinh n\pi}\cos n\alpha dn,\ (2\pi>\beta>0) \quad\cdots(18)$$

$$\frac{\sin^3\beta}{(\cosh\alpha-\cos\beta)^2}=2\int_0^\infty\frac{n\sin\beta\cosh n(\pi-\beta)+\cos\beta\sinh n(\pi-\beta)}{\sinh n\pi}\cos n\alpha dn,\ (2\pi>\beta>0) \quad\cdots(19)$$

Using the above equations, we transform the stress function $h(\chi_0+\chi_1)$ into Fourier's integrals as follows.

$$(\sin^2c/aT)h(\chi_0+\chi_1)=\int_0^\infty[\phi_1(n^2+1)\sin\xi\sinh n(\pi-\xi)-\omega_1\{n\sin\xi\cosh n(\pi-\xi)$$
$$+\cos\xi\sinh n(\pi-\xi)\}]\frac{\cos n\alpha dn}{n(n^2+1)\sinh n\pi},\quad \xi\equiv\beta+c,\quad (2\pi>\beta>0) \quad\cdots(20)$$

$$\left.\begin{aligned}\phi_1&\equiv(\lambda+K)\sin c\sinh nc+2\mu n(n\sin c\sinh nc+\cos c\cosh nc)\sin^2c,\\ \omega_1&\equiv(\lambda+K)(n\sin c\cosh nc-\cos c\sinh nc)+2\mu n(n^2+1)\sin^3c\cosh nc\end{aligned}\right\} \quad\cdots(21)$$

Here the terms which produce no stress and strain were omitted. As the second auxiliary function $h\chi_2$ which is an integral solution of biharmonic equation, has no singular point in the domain of plate and produces no stress on the boundary $\beta=-c$, we take the following equation

$$(\sin^2c/aT)h\chi_2=\int_0^\infty\{A_n(n\sin\xi\cos c\sinh n\beta+\cos\beta\sinh n\xi\sinh nc)$$
$$+B_n(n\sin\xi\cos c\cosh n\beta-\cos\beta\sinh n\xi\cosh nc)\}\cos n\alpha dn \quad\cdots(22)$$

The complete stress function $h\chi$ must satisfy the boundary conditions Eq. (4) for the boundary $\beta=\beta_1$ or $\xi=\beta_1+c\equiv\xi_1$. Consequently, the values of the constants A_n and B_n in Eq. (22) are determined to be:

$$A_n=[\phi_1(n^2+1)[n(n\cos c\sinh nc+\sin c\cosh nc)\sin^2\xi_1\sinh n\pi$$
$$-\{n^2\sin^2\xi_1\cosh n\pi-n\sin\xi_1\cos\xi_1\sinh n\pi+\sinh n\xi_1\sinh n(\pi-\xi_1)\}\cos c\cosh nc]$$
$$-\omega_1[(n\cos c\sinh nc+\sin c\cosh nc)\{n^2\sin^2\xi_1\cosh n\pi+n\sin\xi_1\cos\xi_1\sinh n\pi+\sinh n\xi_1\sinh n(\pi-\xi_1)\}$$
$$-n(n^2+1)\sin^2\xi_1\cos c\cosh nc\sinh n\pi]]/n^2(n^2+1)\cos^2c\sinh n\pi\Delta_n \quad\cdots(23)$$

$$B_n=[\phi_1(n^2+1)[n(n\cos c\cosh nc+\sin c\sinh nc)\sin^2\xi_1\sinh n\pi$$
$$-\{n^2\sin^2\xi_1\cosh n\pi-n\sin\xi_1\cos\xi_1\sinh n\pi+\sinh n\xi_1\sinh n(\pi-\xi_1)\}\cos c\sinh nc]$$
$$-\omega_1[(n\cos c\cosh nc+\sin c\sinh nc)\{n^2\sin^2\xi_1\cosh n\pi+n\sin\xi_1\cos\xi_1\sinh n\pi+\sinh n\xi_1\sinh n(\pi-\xi_1)\}$$
$$-n(n^2+1)\sin^2\xi_1\cos c\sinh nc\sinh n\pi]]/n^2(n^2+1)\cos^2c\sinh n\pi\Delta_n \quad\cdots(24)$$

where $\Delta_n\equiv\sinh^2n\xi_1-n^2\sin^2\xi_1$, $\xi_1\equiv\beta_1+c$. For convenience of calculating the stresses in the vicinity of the notch, we reach the following result without difficulty.

$$(\sin^2c/aT)h\chi=\int_0^\infty\phi_1\{n\sin(\xi-\xi_1)\sin\xi_1\sinh n\xi-\sin\xi\sinh n(\xi-\xi_1)\sinh n\xi_1\}\cos n\alpha dn/n\Delta_n$$
$$-\int_0^\infty\omega_1\{n(n\sin\xi_1\cosh n\xi+\cos\xi_1\sinh n\xi)\sin(\xi-\xi_1)$$
$$-(n\sin\xi\cosh n\xi_1+\cos\xi\sinh n\xi_1)\sinh n(\xi-\xi_1)\}\cos n\alpha dn/n(n^2+1)\Delta_n,\ (2\pi>\beta_1\geqq\beta>0) \quad\cdots(25)$$

At infinity $\alpha=\beta=0$, the stresses derived from the auxiliary stress functions must all vanish. This condition requires that $h(\chi_1+\chi_2)=0$ when $\alpha=\beta=0$, then we have

$$K\sin^2c+\int_0^\infty\{2A_n\sinh^2nc-B_n(\sinh 2nc-n\sin 2c)\}dn=0 \quad\cdots(26)$$

Using Eq. (23) and (24) to (26), we find

$$(\lambda+K)\int_0^\infty\{(\sinh^2n\beta_1-n^2\sin^2\beta_1)(\sinh 2nc-n\sin 2c)$$
$$+(\sinh^2nc-n^2\sin^2c)(\sinh 2n\beta_1-n\sin 2\beta_1)\}dn/n(n^2+1)\Delta_n$$
$$=(\lambda-\mu)\sin^2c+2\mu\sin^2c\int_0^\infty n\{(\sinh 2nc-n\sin 2c)\sin^2\beta_1$$
$$+(\sinh 2n\beta_1-n\sin 2\beta_1)\sin^2c\}dn/\Delta_n,\quad(2\pi>\beta_1>0,\quad \pi>c>0) \quad\cdots(27)$$

Equation (27) supplies the necessary condition for the determination of the constant K involved in the stress function χ_1.

2·3 Principal stress along rim of notch

Since $\sigma_\beta=0$, we have on the boundary $\beta=\beta_1$

$$[\sigma_\alpha]_{\beta_1}/T=2(\lambda+K)(\cosh\alpha-\cos\beta_1)\mathrm{cosec}^2c$$
$$\times\int_0^\infty(\sin\beta_1\sinh n\xi_1\sinh nc-n\sin\xi_1\sin c\sinh n\beta_1)\cos n\alpha dn/\Delta_n$$
$$-4\mu(\cosh\alpha-\cos\beta_1)\int_0^\infty n\{n(n\sin c\sinh n\beta_1-\cos c\cosh n\beta_1)\sin\xi_1$$
$$-(n\sin\beta_1\sinh nc-\cos\beta_1\cosh nc)\sinh n\xi_1\}\cos n\alpha dn/\Delta_n,\quad(2\pi>\beta_1>0) \quad\cdots(28)$$

We have some limiting cases mentioned below.

(1) Taking $c\rightarrow 0$ in Eq. (27) and (28), we have the case of a circular hole with an infinitesimal circular notch. In this case $K=0$ and

$$[\sigma_\alpha]_{\beta_1}/(\lambda+2\mu)T=2(\cosh\alpha-\cos\beta_1)$$
$$\times\int_0^\infty n(n\sin\beta_1\cosh n\beta_1-\cos\beta_1\sinh n\beta_1)\cos n\alpha dn/(\sinh^2 n\beta_1-n^2\sin^2\beta_1),\quad (2\pi>\beta_1>0)\cdots\cdots(29)$$

Equation (29) is equal to the stress equation, which was reported by Ling[6] or Udoguchi[9], along the rim of a circular notch in the semi-infinite plate under tension of $(\lambda+2\mu)T$.

(2) Taking $c=\beta_1=\pi/2$, we have the case of a notchless circular hole of radius a with center at the point 0 in an infinite plate. In this case $K=0$ and with the aid of some such integrals[10]

$$\int_0^\infty\frac{\cos n\alpha dn}{\cosh(n\pi/2)}=\frac{1}{\cosh\alpha},\quad \int_0^\infty\frac{n^2\cos n\alpha dn}{\cosh(n\pi/2)}=\frac{1-\sinh^2\alpha}{\cosh^3\alpha}\cdots\cdots(30)$$

and of $\operatorname{sech}\alpha=\cos\theta_1$, $\tanh\alpha=-\sin\theta_1$, we have

$$\frac{[\sigma_\alpha]_{\beta_1}}{T}=\cosh\alpha\int_0^\infty\frac{(\lambda+2\mu n^2)\cos n\alpha dn}{\cosh(n\pi/2)}=\lambda+2\mu\cos 2\theta_1\cdots\cdots(31)$$

Moreover by putting $c=\pi-\beta_1$, we have the case of a notchless circular hole of radius R_2 centered at the point 0_1. Then we obtain the same equation as Eq. (31) without difficulty.

(3) Taking $c=\beta_1$, we have the case of a hole bounded by two equal circular arcs in an infinite plate. In this case, Eqs. (27) and (28) become the following equations.

$$\frac{\lambda+K}{\sin^2\beta_1}\int_0^\infty\frac{(\sinh^2 n\beta_1-n^2\sin^2\beta_1)dn}{n(n^2+1)(\sinh 2n\beta_1+n\sin 2\beta_1)}=\frac{\lambda-\mu}{2}+2\mu\sin^2\beta_1\int_0^\infty\frac{ndn}{\sinh 2n\beta_1+n\sin 2\beta_1}\cdots\cdots(32)$$

$$\frac{[\sigma_\alpha]_{\beta_1}}{T}=4(\cosh\alpha-\cos\beta_1)\sin\beta_1\int_0^\infty\left\{\frac{\lambda+K}{2\sin^2\beta_1}+\mu n(n-\cot\beta_1\coth n\beta_1)\right\}\frac{\sinh n\beta_1\cos n\alpha dn}{\sinh 2n\beta_1+n\sin 2\beta_1}\cdots\cdots(33)$$

They are equal to the Ling's results[4].

3. Contacted circular holes

3·1 Fundamental equations[5]

The boundary conditions are considerably simplified when the crowded circular coordinates (α,β) are introduced as defined by the mapping

$$\left.\begin{array}{l} z=a/\bar{\zeta},\ z\equiv x+iy,\ \zeta\equiv\beta+i\alpha,\ a>0\\ ah=\alpha^2+\beta^2,\ x+iy=(\beta+i\alpha)/h\end{array}\right\}\cdots\cdots(34)$$

The curves of $\alpha=$constant are tangential circles to x-axis at (0, 0), and those of $\beta=$constant are tangential circles to y-axis, being a family of coaxial circles having (0, 0) as limiting point. And we have $\alpha=0$ on the x-axis and $\beta=0$ on the y-axis.

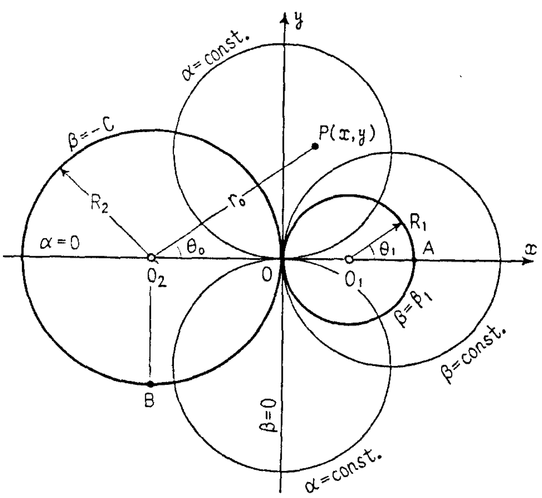

Fig. 2 Two contacted holes

In this coordinates system, the stress components in terms of stress function $h\chi$ are given by

$$\left.\begin{array}{l} a\sigma_\alpha=\left\{(\alpha^2+\beta^2)\dfrac{\partial^2}{\partial\beta^2}-2\alpha\dfrac{\partial}{\partial\alpha}-2\beta\dfrac{\partial}{\partial\beta}+2\right\}(h\chi)\\ a\sigma_\beta=\left\{(\alpha^2+\beta^2)\dfrac{\partial^2}{\partial\alpha^2}-2\alpha\dfrac{\partial}{\partial\alpha}-2\beta\dfrac{\partial}{\partial\beta}+2\right\}(h\chi)\\ a\tau_{\alpha\beta}=-(\alpha^2+\beta^2)\dfrac{\partial^2(h\chi)}{\partial\alpha\partial\beta}\end{array}\right\}\cdots\cdots(35)$$

and

$$a(\sigma_\alpha-\sigma_\beta)=(\alpha^2+\beta^2)\left(\frac{\partial^2}{\partial\beta^2}-\frac{\partial^2}{\partial\alpha^2}\right)(h\chi)\cdots\cdots(36)$$

The biharmonic equation $\nabla^4\chi=0$, which must be satisfied by the stress function χ, transforms to

$$\left(\frac{\partial^4}{\partial\alpha^4}+2\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+\frac{\partial^4}{\partial\beta^4}\right)(h\chi)=0\cdots\cdots(37)$$

It can be readily shown that an integral solution of this equation, which is even in α, may be obtained in the following form.

$$h\chi=\int_0^\infty(A_n\beta\sinh n\beta+B_n\beta\cosh n\beta+C_n\sinh n\beta+D_n\cosh n\beta)\cos n\alpha dn\cdots\cdots(38)$$

The necessary and sufficient conditions that a boundary $\beta=$constant is stress-free, are

$$\partial(h\chi)/\partial\beta=\rho,\ h\chi=\rho\beta+\sigma(\alpha^2-\beta^2)+\tau\alpha\cdots\cdots(39)$$

where ρ, σ and τ are constants of the boundary.

3·2 Stress function

Let the boundary of a large circular hole be defined by $\beta=-c$ $(c>0)$ and that of a contacted small circular hole by $\beta=\beta_1>0$, as shown Fig. 2. It is found that

$$R_2=\frac{a}{2c},\quad r_0{}^2=a^2\frac{\alpha^2+(\beta+2c)^2}{4c^2(\alpha^2+\beta^2)},\quad \cos 2\theta_0=1+\frac{2c}{\beta+c}\left\{\frac{\beta^2}{\alpha^2+\beta^2}-\frac{(\beta+2c)^2}{\alpha^2+(\beta+2c)^2}\right\}\quad\cdots\cdots(40)$$

Then Eqs. (6) and (9) are represented in the following equation.

$$\frac{4a}{TR_2{}^2}h(\chi_0+\chi_1)=(\lambda+K)\left\{4c^2-(\alpha^2+\beta^2)\log\frac{\alpha^2+(\beta+2c)^2}{\alpha^2+\beta^2}\right\}$$
$$+8\mu\left[\frac{\beta^2}{\alpha^2+\beta^2}+\frac{(\beta+2c)^2-2c^2}{\alpha^2+(\beta+2c)^2}-\frac{4(\beta+c)(\beta+2c)^2c}{\{\alpha^2+(\beta+2c)^2\}^2}\right]c^2\quad\cdots\cdots(41)$$

With the aid of the integrals[11]

$$\left.\begin{aligned}
&\log p=\int_0^\infty (e^{-n}-e^{-np})\frac{dn}{n},\ (p>0)\\
&p\log p=p+\int_0^\infty\left\{pe^{-n}-\frac{1}{n}(1-e^{-np})\right\}\frac{dn}{n},\ (p>0)\\
&p^2\log p=\frac{3}{2}p^2+\int_0^\infty\left\{p^2e^{-n}-\frac{2p}{n}+\frac{2}{n^2}(1-e^{-np})\right\}\frac{dn}{n},\ (p>0)
\end{aligned}\right\}\quad\cdots\cdots(42)$$

and of the formulas $\alpha^2+\beta^2=\zeta\bar{\zeta}=-\zeta_1{}^2+2(\beta+2c)\zeta_1-4c(\beta+c)$, $\zeta_1\equiv\beta+2c+i\alpha$, it can be shown that

$$\log(\alpha^2+\beta^2)=\int_0^\infty(2e^{-n}-e^{-n\zeta}-e^{-n\bar{\zeta}})\frac{dn}{n},\ (\beta>0)\quad\cdots\cdots(43)$$

$$(\alpha^2+\beta^2)\log(\alpha^2+\beta^2)=3\alpha^2+\beta^2+2\int_0^\infty\left\{(\alpha^2+\beta^2)e^{-n}+\frac{\beta}{n}(e^{-n\zeta}+e^{-n\bar{\zeta}})-\frac{1}{n^2}(2-e^{-n\zeta}-e^{-n\bar{\zeta}})\right\}\frac{dn}{n},\ (\beta>0)\quad\cdots\cdots(44)$$

$$(\alpha^2+\beta^2)\log\{\alpha^2+(\beta+2c)^2\}=3\alpha^2+(\beta+2c)^2+2\int_0^\infty\left\{(\alpha^2+\beta^2)e^{-n}+2c(\beta+c)(e^{-n\zeta_1}+e^{-n\bar{\zeta}_1})\right.$$
$$\left.+\frac{\beta+2c}{n}(e^{-n\zeta_1}+e^{-n\bar{\zeta}_1})-\frac{1}{n^2}(2-e^{-n\zeta_1}-e^{-n\bar{\zeta}_1})\right\}\frac{dn}{n},\ (\beta>0)\quad\cdots\cdots(45)$$

$$\frac{\beta}{\alpha^2+\beta^2}=\int_0^\infty e^{-n\beta}\cos n\alpha dn,\ (\beta>0)\quad\cdots\cdots(46)$$

$$\frac{2\beta^3}{(\alpha^2+\beta^2)^2}=\int_0^\infty(1+n\beta)e^{-n\beta}\cos n\alpha dn,\ (\beta>0)\quad\cdots\cdots(47)$$

Using above equations, we transform the stress function $h(\chi_0+\chi_1)$, which is shown in Eq. (41), into Fourier's integrals, omitting the terms which produce no stress and strain.

$$(2c^2/aT)h(\chi_0+\chi_1)=\int_0^\infty\{\phi_2n^2\xi-\omega_2(1+n\xi)\}(e^{-n\xi}/n^3)\cos n\alpha dn,\quad \xi\equiv\beta+c,\quad(\beta>0)\quad\cdots\cdots(48)$$

$$\left.\begin{aligned}
&\phi_2\equiv(\lambda+K)c\sinh c+2\mu nc^2(nc\sinh nc+\cosh nc)\\
&\omega_2\equiv(\lambda+K)(nc\cosh nc-\sinh nc)+2\mu n^3c^3\cosh nc
\end{aligned}\right\}\quad\cdots\cdots(49)$$

Now, let an auxiliary function $h\chi_2$ which produces no stress on the boundary $\beta=-c$ be given by

$$(2c^2/aT)h\chi_2=\int_0^\infty\{A_n(n\xi\sinh n\beta+\sinh n\xi\sinh nc)$$
$$+B_n(n\xi\cosh n\beta-\sinh n\xi\cosh nc)\}\cos n\alpha dn\quad\cdots\cdots(50)$$

For $\beta=\beta_1$, the complete stress function $\chi=\chi_0+\chi_1+\chi_2$ must also satisfy the boundary conditions $\partial(h\chi)/\partial\beta=0$ and $h\chi=0$ which are obtained from Eq. (39). From these, the values of the constants involved in Eq. (50) are determined to be

$$\left.\begin{aligned}
&A_n=[-n\phi_2\{n\xi_1(n\xi_1e^{-nc}-\cosh nc)+e^{-n\xi_1}\sinh n\xi_1\cosh nc\}\\
&\quad+\omega_2\{n\xi_1(n\xi_1e^{-nc}-\sinh nc)-e^{-n\xi_1}\sinh n\xi_1\sinh nc\}]/n^3\Delta_n\\
&B_n=[n\phi_2\{n\xi_1(n\xi_1e^{-nc}+\sinh nc)-e^{-n\xi_1}\sinh n\xi_1\sinh nc\}\\
&\quad-\omega_2\{n\xi_1(n\xi_1e^{-nc}+\cosh nc)+e^{-n\xi_1}\sinh n\xi_1\cosh nc\}]/n^3\Delta_n\\
&\Delta_n\equiv\sinh^2n\xi_1-n^2\xi_1{}^2,\quad \xi_1\equiv\beta_1+c
\end{aligned}\right\}\quad\cdots\cdots(51)$$

Consequently, we reach the following result.

$$(2c^2/aT)h\chi=\int_0^\infty[n^2\phi_2\{n\xi_1(\xi-\xi_1)\sinh n\xi-\xi\sinh n(\xi-\xi_1)\sinh n\xi_1\}$$
$$-\omega_2\{n(\xi-\xi_1)(n\xi_1\cosh n\xi+\sinh n\xi)-(n\xi\cosh n\xi_1+\sinh n\xi_1)\sinh n(\xi-\xi_1)\}]$$
$$\times\cos n\alpha dn/n^3\Delta_n,\ (\beta_1\geqq\beta>0)\quad\cdots\cdots(52)$$

The necessary condition for the determination of K is shown in the following equation from the condition at infinity $(\alpha=\beta=0)$.

$$Kc^2+\int_0^\infty\{2A_n\sinh^2nc-B_n(\sinh 2nc-2nc)\}dn=0\quad\cdots\cdots(53)$$

Accordingly

$$(\lambda+K)\int_0^\infty\{(\sinh^2 n\beta_1-n^2\beta_1^2)(\sinh 2nc-2nc)+(\sinh^2 nc-n^2c^2)(\sinh 2n\beta_1-2n\beta_1)\}dn/n^3\Delta_n$$
$$=(\lambda-\mu)c^2+2\mu c^2\int_0^\infty n\{\beta_1^2(\sinh 2nc-2nc)+c^2(\sinh 2n\beta_1-2n\beta_1)\}dn/\Delta_n,\ (\beta_1>0,\ c>0) \quad\cdots\cdots(54)$$

3·3 Principal stress along rim of hole

Since $\sigma_\beta=0$, we have on the boundary $\beta=\beta_1$

$$[\sigma_\alpha]_{\beta_1}/T=(\lambda+K)(\alpha^2+\beta_1^2)(1/c^2)\int_0^\infty(\beta_1\sinh n\xi_1\sinh nc-nc\xi_1\sinh n\beta_1)\cos n\alpha dn/\Delta_n$$
$$-2\mu(\alpha^2+\beta_1^2)\int_0^\infty n\{n\xi_1(nc\sinh n\beta_1-\cosh n\beta_1)-(n\beta_1\sinh nc-\cosh nc)\sinh n\xi_1\}\cos n\alpha dn/\Delta_n,$$
$$(\beta_1>0) \quad\cdots\cdots(55)$$

We have some limiting cases in Eq. (54) and (55).

(1) The case $c\to 0$ corresponds to a circular hole with a contacted infinitesimal circular hole. Then we have $K=0$ and

$$[\sigma_\alpha]_{\beta_1}/(\lambda+2\mu)T=(\alpha^2+\beta_1^2)\int_0^\infty n(n\beta_1\cosh n\beta_1-\sinh n\beta_1)\cos n\alpha dn/(\sinh^2 n\beta_1-n^2\beta_1^2) \quad\cdots\cdots(56)$$

This equation agrees with the stress equation, which has been reported by the author[12], along the rim of a hole contacted to the straight edge in semi-infinite plate under uniaxial tension $(\lambda+2\mu)T$.

(2) The case $c=\beta_1$ corresponds to two contacted circular holes of equal size, then we have

$$\frac{\lambda+K}{\beta_1^2}\int_0^\infty\frac{(\sinh^2 n\beta_1-n^2\beta_1^2)dn}{n^3(\sinh 2n\beta_1+2n\beta_1)}=\frac{\lambda-\mu}{2}+2\mu\beta_1^2\int_0^\infty\frac{ndn}{\sinh 2n\beta_1+2n\beta_1} \quad\cdots\cdots(57)$$

$$\frac{[\sigma_\alpha]_{\beta_1}}{T}=2(\alpha^2+\beta_1^2)\beta_1\int_0^\infty\left\{\frac{\lambda+K}{2\beta_1^2}+\mu n\left(n-\frac{\coth n\beta_1}{\beta_1}\right)\right\}\frac{\sinh n\beta_1\cos n\alpha dn}{\sinh 2n\beta_1+2n\beta_1} \quad\cdots\cdots(58)$$

They are in agreement with the author's solutions[5].

4. Numerical results

Numerical computations were carried out to determine the stress concentration factors for various shapes of notch. Let us represent the size of notch by $\rho=R_1/R_2$ and the depth by $\eta=s_1/R_1$, as shown in Fig. 1. Hence, $\rho=0$ is an infinitesimal circular notch, $\rho=1$ is two equal circular arcs, $\eta=-1$ is a notchless circle, and $\eta=1$ is two contacted circles. The stress equations were integrated numerically by means of Simpson's formula using an electronic computer.

(1) Figure 3 shows the case of all-around tension. The maximum stress occurs at the bottom on the notch, and the smaller or deeper the notch is, the greater are the values of maximum stress.

(2) In the x-directional tension case, for the sake of simplicity, the stress at $\theta_0=\pi/2$ on the rim of hole may well be taken as the maximum stress without any appreciable errors. Figure 4 shows the values of stress at this point. From this graph, we

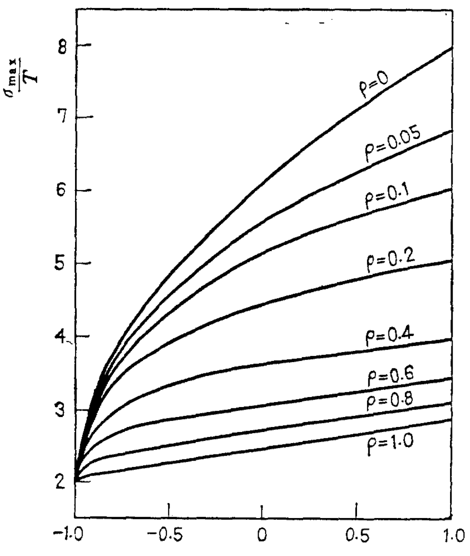

Fig. 3 Stresses at point A under all-around tension

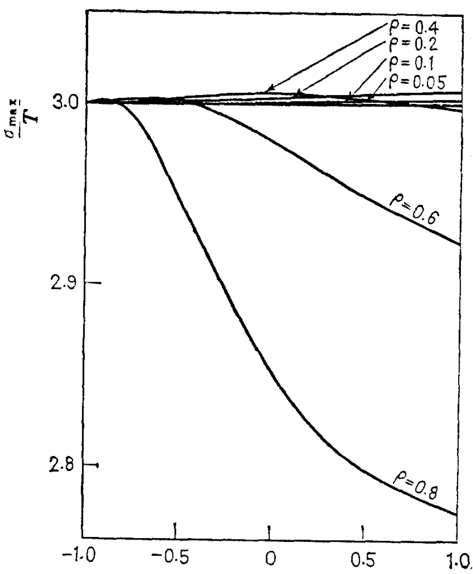

Fig. 4 Stresses at point B under x-directional tension

note that the effects of notch appear remarkably in $\rho>0.6$. Figure 5 shows the maximum compressive stress at the point A in Fig. 1. In the case of contacted infinitesimal hole, we have $\sigma_{min}/T=-3.966$.

(3) Figure 6 shows the maximum stress at the bottom of notch (point A) in y-directional tension. The smaller or deeper the notch is, the greater are the values of maximum stress.

(4) From Figs. 3 ~ 6, we observe that the stress concentrations are remarkably great at the bottom of notch in all-around and y-directional tension, and that the maximum values occur in the case of contacted holes for a certain ratio of radius of two arcs. For various values of ρ, they are given in Table 1.

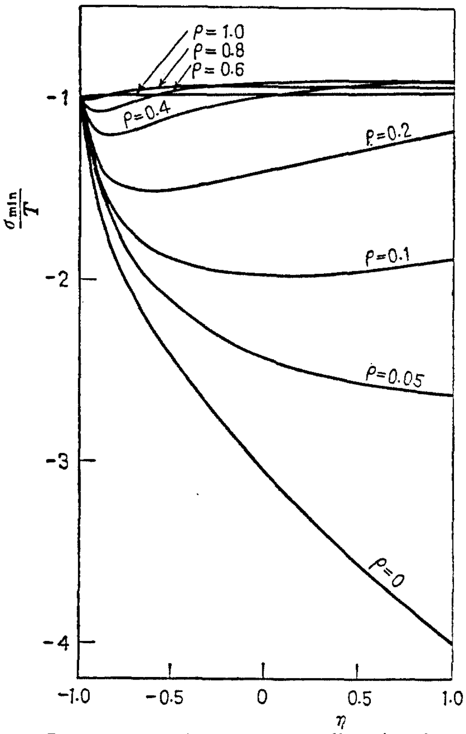

Fig. 5 Stresses at point A under x-directional tension

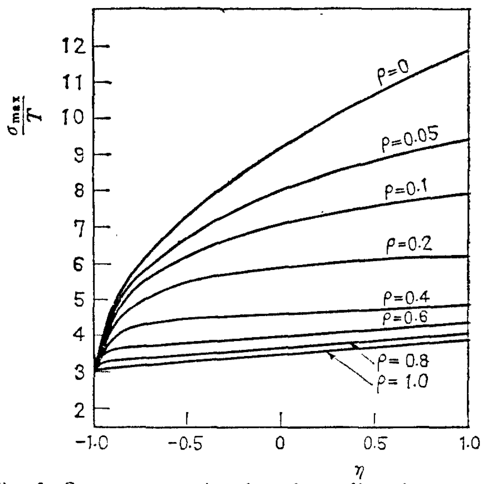

Fig. 6 Stresses at point A under y-directional tension

(5) When the center of notch is on the rim of hole, we have comparisons with Mitchell's results[2] in Fig. 7. The curves σ_A show the stresses at point A subjected to y-directional tension and the curves σ_B at point B subjected to x-directional one. And the full-lines show the case of a circular notch and the broken lines that of doubly symmetrical circular notches.

5. Conclusions

Two-dimensional stress concentrations in an infinite plate which had a circular hole with a circular notch, were discussed. It is noted that the values of maximum stress in this plate are much greater than those in the case of notchless hole.

Acknowledgements

The author wishes to express his hearty thanks to professor Y. Suezawa of the Tokyo Institute of Technology for his advices and encouragements throughout the progress of the present investigation, and to Mr. T. Iwaki and Mr. N. Taniguchi for their great cooperations.

Table 1 Maximum stress concentration factors at point A on contacted circular hole

ρ	All-around tension σ_{max}/T	y-directional tension σ_{max}/T
0	7.992	11.989
0.05	6.862	9.489
0.1	6.073	7.942
0.2	5.048	6.232
0.4	3.982	4.871
0.6	3.439	4.335
0.8	3.111	4.050
1.0	2.894	3.869

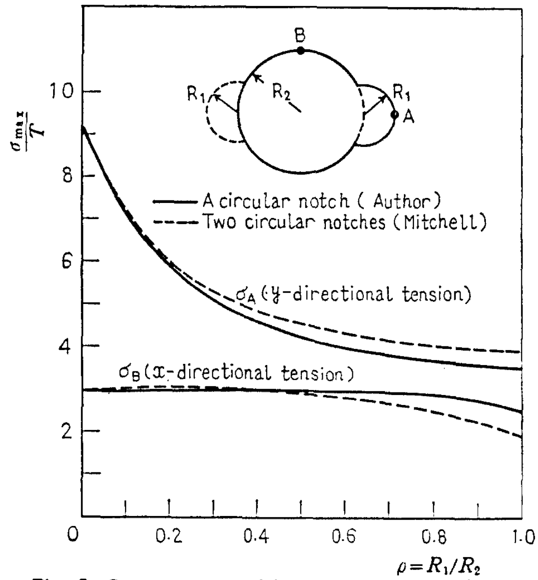

Fig. 7 Comparisons with two circular notches

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