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Proofs of operator monotonicity of some functions by using Löwner's integral representation

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Abstract. We give somewhat simple proofs of operator monotonicity of some functions by using Löwner's integral representation of an operator monotone function.

1. Introduction

A (bounded linear) operator A acting on a Hilbert space H is said to be positive, denoted by $A \geq 0$, if $(Av, v) \geq 0$ for all $v \in H$. The definition of positivity induces the order $A \geq B$ for self-adjoint operators A and B on H. A real-valued function f on $(0, \infty)$ is *operator monotone,* if $0 \leq f(A) \leq f(B)$ for operators A and B on H such that $0 \leq A \leq B$. For a positive operator monotone function f on $(0, \infty)$, by Löwner's integral representation theorem, we have:

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{x}{x+\lambda} d\mu(\lambda)$$

with nonnegative α, β and a positive measure μ on $(0, \infty)$. As a typical example, $x \mapsto x^p$ $(0 \leq p \leq 1)$ is an operator monotone function, which is well-known as *Löwner-Heinz theorem* (LH).

In this paper, applying Löwner's integral representation of an operator monotone function, we show an alternative simple proof of the known fact

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that the function $x \mapsto f(x^p)^{\frac{1}{p}}$ $(0 < p \leq 1)$ for an operator monotone function f is operator monotone and also give an alternative simple proof of a restricted case of a result in M. Uchiyama's theorem related to Petz-Hasegawa theorem.

We assume that all operator monotone functions f are defined on $(0, \infty)$ and strictly positive, and $f(0) = \lim_{x \to 0} f(x)$ if necessary.

2. Preliminaries

By Kubo-Ando theory [10], an operator mean σ is defined as a binary relation of positive operators, satisfying the following properties in common:

(monotonicity)	$A \leq C, B \leq D \Longrightarrow A\sigma B \leq C\sigma D$,
(transformer inequality)	$C(A\sigma B)C \leq (CAC)\sigma(CBC)$,
(normality)	$A\sigma A = A$,
(strong operator semi-continuity)	$A_n \downarrow A, B_n \downarrow B \Longrightarrow A_n\sigma B_n \downarrow A\sigma B$.

Sometimes for the definition of an operator mean we must assume operators to be invertible. Without any assumption for invertibility every mean is well-defined as the (strong operator) limits of $(A+\varepsilon I)\sigma(B+\varepsilon I)$ as $\varepsilon \downarrow 0$ instead of $A\sigma B$. (I is the identity operator.)

To every operator mean σ corresponds a unique operator monotone function, that is, its representing function f_σ which is defined by $f_\sigma(x) = 1\sigma x$. Conversely, if f is an operator monotone function with $f(1) = 1$, then the definition of the operator mean corresponding to f is given by

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for positive invertible operators A and B.

For our discussion, we use the following basic facts:

(I) For an operator mean σ and for two operator monotone functions g and h, if we define $g\sigma h$ by

$$(g\sigma h)(x) = g(x) f_\sigma\left(\frac{h(x)}{g(x)}\right),$$

then $g\sigma h$ is operator monotone.

(II) For a strictly positive function f on $(0,\infty)$, define $f^{\circ}(x) := xf(1/x)$ (transpose), $f^{*}(x) := 1/f(1/x)$ (adjoint) and $f^{\perp}(x) := x/f(x)$ (dual), then the four functions $f, f^{\circ}, f^{*}, f^{\perp}$ are equivalent to one another with respect to operator monotonicity ([10], [7]).

3. Main results

Applying (I) to the operator mean σ_{a_p} corresponding to the operator monotone function $a_p(x) = (\frac{1+x^p}{2})^{\frac{1}{p}}$ $(-1 \le p \le 1, p \neq 0, a_0(x) = x^{\frac{1}{2}})$ (*notice* $a_p(1) = 1$), we showed in [8]:

Lemma 3.1 (cf. [8, Lemma 3.1], [11])**.** *Let f, g be operator monotone functions, then $f\sigma_{a_p}g = \left(\frac{f^p+g^p}{2}\right)^{\frac{1}{p}}$ (or equivalently, $(f^p + g^p)^{\frac{1}{p}}$) is operator monotone for $-1 \le p \le 1,\ p \neq 0$. Further, if $f_1, ..., f_n$ are operator monotone functions, then $(\sum_{i=1}^{n} f_i^p)^{\frac{1}{p}}$ is operator monotone. In particular, $(\sum_{i=1}^{n}(\alpha_i + \beta_i x)^p)^{\frac{1}{p}}$ $(\alpha_i, \beta_i \ge 0)$ is operator monotone.*

The following theorem was shown first by T. Ando [1], next by Y. Nakamura [11], and recently by J.I. Fujii-M. Fujii [3], by T. Sano-S. Tachibana [13]. We give an alternative proof to the theorem, applying Löwner's integral representation of an operator monotone function. (We can see that *the theorem is valid for a wider interval* $-1 \le p \le 1,\ p \neq 0$ by the proof.)

Theorem 3.2. *For an operator monotone function f, the function $x \mapsto (f(x^p))^{\frac{1}{p}}$ for $0 < p \le 1$ is operator monotone.*

Proof. From the integral representation of $f(x)$, we have:

$$(f(x^p))^{\frac{1}{p}} = \left(\alpha + \beta x^p + \int_0^\infty \frac{x^p}{x^p+\lambda} d\mu(\lambda)\right)^{\frac{1}{p}} \ (-1 \le p \le 1,\ p \neq 0).$$

Note that the integral $\int_0^\infty \frac{x}{x+\lambda} d\mu(\lambda)$ *is approximated* by $J_{\epsilon,E}(x) := \int_\epsilon^E \frac{x}{x+\lambda} d\mu(\lambda)$ for $0 < \epsilon < E < \infty$, so that $\int_0^\infty \frac{x^p}{x^p+\lambda} d\mu(\lambda)$ by $J_{\epsilon,E}(x^p) := \int_\epsilon^E \frac{x^p}{x^p+\lambda} d\mu(\lambda)$. Let

$$\Sigma_{\epsilon,E}(x^p) := \sum_{i=1}^{n} \frac{x^p}{x^p+\lambda_i} m_i \quad (\epsilon = \lambda_0 < \lambda_1 < ... < \lambda_n = E)$$

with $m_i = \mu((\lambda_{i-1}, \lambda_i])$ be an approximate sum of $J_{\epsilon,E}(x^p)$. Then we have to show that

$$\phi_n(x) := (\alpha + \beta x^p + \Sigma_{\epsilon,E}(x^p))^{\frac{1}{p}}$$

is operator monotone. Now if we put $f_{-1} = \alpha^{\frac{1}{p}}, f_0 = \beta^{\frac{1}{p}} x$ and $f_i = \frac{x}{(x^p+\lambda_i)^{\frac{1}{p}}} m_i^{\frac{1}{p}}$ for $i = 1, ..., n$, then all f_i $(-1 \leq i \leq n)$ are operator monotone and $\phi_n(x) = (\sum_{i=-1}^{n} f_i^p)^{\frac{1}{p}}$, so that from Lemma 3.1, we see that $\phi_n(x)$ is operator monotone.

Assuming Löwner's integral representation of the operator monotone function again, by using the approximate sum $\Sigma_{\epsilon,E}(x)$ of the integral $J_{\epsilon,E}(x)$, we show the following (modified) Bendat-Sherman theorem (cf. [2], [11], [4], [14]):

Theorem 3.3. *If f is a (non-constant) operator monotone function, then $F(x) := \frac{x-a}{f(x)-f(a)}$ for $a \geq 0$ is operator monotone.*

Proof. If we put $\psi_n(x) := \alpha + \beta x + \Sigma_{\epsilon,E}(x) = \alpha + \beta x + \sum_{i=1}^{n} \frac{x}{x+\lambda_i} m_i$ instead of $f(x)$ in the proof of Theorem 3.2, then we have

$$F_n(x) := \frac{x-a}{\psi_n(x) - \psi_n(a)} = \left(\beta + \sum_{i=1}^{n} \frac{\lambda_i m_i}{(a+\lambda_i)(x+\lambda_i)}\right)^{-1}.$$

This function is operator monotone since $F_n^{\perp}(x) = \beta x + \sum_{i=1}^{n} \frac{\lambda_i m_i x}{(a+\lambda_i)(x+\lambda_i)}$ is operator monotone. Hence the limit $F(x)$ of $F_n(x)$ is operator monotone.

Further with a similar method as the above, we show the following theorem (which is a restricted case of a result in [14, Theorem 2.7]):

Theorem 3.4. *If f is a (non-constant) operator monotone function, then for $a \geq 0$*

$$G(x) := \frac{x-a}{f(x)-f(a)} \cdot \frac{x-a}{f(x)^{\perp} - f^{\perp}(a)}$$

is operator monotone.

Proof. Put $\gamma_i = \frac{\lambda_i}{a+\lambda_i}$ and $\delta_i = \frac{a}{a+\lambda_i}$. Then

$$G_n(x) = \frac{x-a}{\psi_n(x)-\psi_n(a)} \cdot \frac{x-a}{\psi_n^{\perp}(x)-\psi_n^{\perp}(a)}$$

$$= \psi_n(a) \cdot \frac{\alpha + \beta x + \sum_{i=1}^{n} \frac{m_i x}{x+\lambda_i}}{\left(\beta + \sum_{i=1}^{n} \frac{m_i \gamma_i}{x+\lambda_i}\right)\left(\alpha + \sum_{i=1}^{n} \frac{m_i \delta_i x}{x+\lambda_i}\right)}$$

$$= \psi_n(a) \cdot \left(\frac{1}{\beta + \sum_{i=1}^{n} \frac{m_i \gamma_i}{x+\lambda_i}} + \frac{x}{\alpha + \sum_{i=1}^{n} \frac{m_i \delta_i x}{x+\lambda_i}} \right) = \psi_n(a) \cdot (I + II).$$

Here,

$$I = \frac{1}{\beta + \sum_{i=1}^{n} \frac{m_i \gamma_i}{x+\lambda_i}}, \quad II = \frac{x}{\alpha + \sum_{i=1}^{n} \frac{m_i \delta_i x}{x+\lambda_i}}.$$

Then we obtain

$$I^* = \beta + \sum_{i=1}^{n} \frac{m_i \gamma_i}{\frac{1}{x} + \lambda_i} = \beta + \sum_{i=1}^{n} \frac{m_i \gamma_i}{\lambda_i} \cdot \frac{x}{x + \frac{1}{\lambda_i}},$$

which is operator monotone, so that I is also operator monotone. For II we see that:

$$II^{\perp} = \alpha + \sum_{i=1}^{n} \frac{m_i \delta_i x}{x+\lambda_i},$$

which is operator monotone, so that II is also operator monotone. Therefore, $G_n(x)$ is operator monotone. Hence $G(x)$ is operator monotone as the limit of $G_n(x)$, tending n to ∞.

If $f(x) = x^p$ $(0 < p < 1)$, then $f^{\perp}(x) = x^{1-p}$. Hence as an application of Theorem 3.4 we at once obtain the following :

Corollary 3.5. *For* $0 < p < 1,\ a \geq 0$

$$\frac{(x-a)^2}{(x^p - a^p)(x^{1-p} - a^{1-p})}$$

is operator monotone.

An extension of the above theorem is known [9] as follows:

Theorem 3.6. *For* $-1 \leq p \leq 2,\ a, b \geq 0$

$$H_p(x) := \frac{p(1-p)(x-a)(x-b)}{(x^p - a^p)(x^{1-p} - b^{1-p})},\ p \neq 0, 1 \ \left(H_0(x) = H_1(x) = \frac{x-1}{\log x} \right)$$

is operator monotone.

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