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Quasi-abelian varieties given by certain algebraic number fields

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Abstract. Let K_0 be a totally real algebraic number field. We consider an n-dimensional algebraic extension K of K_0 which has two complex conjugate fields over K_0 and $n-2$ real ones. We construct a quasi-abelian variety from K.

1. Introduction

In the previous paper [1] we defined $\mathfrak{o}_{K_0}$-quasi-abelian varieties for general algebraic number fields, and investigated their properties. It seems to us that it is not easy to use general $\mathfrak{o}_{K_0}$-quasi-abelian varieties practically. However, a usual quasi-abelian variety has a good projective algebraic compactification which will provide some useful tools for the progress of this subject. Then we treat algebraic number fields which give quasi-abelian varieties in this paper.

We consider a totally real algebraic number field K_0 of degree m. Let K be an n-dimensional extension of K_0 which has two complex conjugate fields and $n-2$ real ones over K_0. As in [1] we define a map $\Psi : K \longrightarrow \mathbb{C}^{m(n-1)}$ by embeddings of K over $\mathbb{Q}$. Let $\mathfrak{o}_K$ be the ring of integers of K. Then $X := \mathbb{C}^{m(n-1)}/\Psi(\mathfrak{o}_K)$ is a toroidal group ([3], see also [1] for a simple proof). We prove the following theorem which is a generalization of a result in [3].

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Theorem. *X is a quasi-abelian variety.*

We note that the word "$\mathfrak{o}_{K_0}$-" can be dropped in the results in [1] for these algebraic number fields.

2. Preliminaries

Let K_0 be a totally real algebraic number field of degree m, whose real embeddings over $\mathbb{Q}$ are $\varrho_i : K_0 \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$. We consider an n-dimensional algebraic extension K of K_0 with two complex conjugate fields over K_0 and $n-2$ real conjugate fields over K_0. Let $\tau_i, \overline{\tau}_i, \sigma_i^{(1)}, \ldots, \sigma_i^{(n-2)}$ be the extensions of ϱ_i to K for $i = 1, \ldots, m$ such that $\tau_i(K), \overline{\tau}_i(K) \not\subset \mathbb{R}$ and $\sigma_i^{(j)}(K) \subset \mathbb{R}$ for $j = 1, \ldots, n-2$. We define a map $\Psi : K \longrightarrow \mathbb{C}^m \times \mathbb{R}^{m(n-2)} \subset \mathbb{C}^{m(n-1)}$ by

$$\Psi(a) := (\tau_1(a), \ldots, \tau_m(a), \sigma_1^{(1)}(a), \ldots, \sigma_m^{(1)}(a), \ldots, \sigma_1^{(n-2)}(a), \ldots, \sigma_m^{(n-2)}(a))$$

for any $a \in K$. We set $\Gamma := \Psi(\mathfrak{o}_K)$. Then $X := \mathbb{C}^{m(n-1)}/\Gamma$ is a toroidal group. We refer to [2] for the definitions of toroidal groups and quasi-abelian varieties and their basic properties. We denote $X_{\mathfrak{o}} := \mathbb{C}^{m(n-1)}/\Psi(\mathfrak{o})$ for any order $\mathfrak{o}$ of K. Then $X_{\mathfrak{o}}$ is also a toroidal group. Since all $X_{\mathfrak{o}}$ are isogeneous, the following lemma is obvious.

Lemma 1. *If $X_{\mathfrak{o}}$ is a quasi-abelian variety for some order $\mathfrak{o}$, then so is any $X_{\mathfrak{o}'}$, especially X is a quasi-abelian variety.*

Let $1, \alpha_1, \ldots, \alpha_{m-1}$ be a basis of $\mathfrak{o}_{K_0}$, which are also a basis of K_0 over $\mathbb{Q}$. We take $x \in \mathfrak{o}_K$ such as $K = K_0(x)$. We set $y_i := \tau_i(x)$ for $i = 1, \ldots, m$. Then the imaginary part $\mathrm{Im}(y_i)$ is non-zero. The following lemma is due to Andreotti and Gherardelli [3].

Lemma 2. *We can take $x \in \mathfrak{o}_K$ such that $\mathrm{Im}(y_i) > 0$ for all $i = 1, \ldots, m$.*

Proof. By a map $K_0 \longrightarrow \mathbb{R}^m,\ a \longmapsto {}^t(\varrho_1(a), \ldots, \varrho_m(a))$ we can define an $\mathbb{R}$-isomorphism $\tilde{\varrho} : K_0 \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \mathbb{R}^m$. Let $\eta_i := \mathrm{Im}(y_i), i = 1, \ldots, m$. Take $\varepsilon > 0$ such that $\varepsilon < |\eta_i|$ for all $i = 1, \ldots, m$. Since $\varrho_i(K_0)$ is dense in $\mathbb{R}$, there exists $\xi_i \in \varrho_i(K_0)$ such that $|\xi_i - \eta_i| < \varepsilon/2$. Then we have $\xi \in K_0$ such that

$$|\tilde{\varrho}_i(\xi) - \xi_i| < \frac{\varepsilon}{2}, \quad i = 1, \ldots, m,$$

where $\tilde{\varrho}(\xi) = {}^t(\tilde{\varrho}_1(\xi), \dots, \tilde{\varrho}_m(\xi))$. This means that

$$\mathrm{Im}(\tau_i(\xi x)) = \tilde{\varrho}_i(\xi)\eta_i > 0, \quad i = 1, \dots, m.$$

Furthermore there exists $k \in \mathbb{N}$ such that $k\xi \in \mathfrak{o}_{K_0}$. If we newly take $k\xi x$ as x, then it has the desired properties. □

3. A lemma on polynomials

We define a polynomial $a_k^{(r)}(t_1, \dots, t_r)$ in r variables $t_1, \dots, t_r$ for $r \in \mathbb{N}$ and $k = -1, 0, 1, \dots$ by

$$a_k^{(r)}(t_1, \dots, t_r) := \begin{cases} \sum_{i_1+\cdots+i_r=k} t_1^{i_1} \cdots t_r^{i_r} & \text{if } k \geqq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k = -1. \end{cases}$$

Fixing $\xi_1, \dots, \xi_r, \xi_{r+1} \in \mathbb{C}$, we consider a polynomial $P_k^{(r)}(\xi_1, \dots, \xi_r; \xi_{r+1}; T)$ in a variable T of degree k defined by

$$P_k^{(r)}(\xi_1, \dots, \xi_r; \xi_{r+1}; T) := \sum_{j=0}^{k} a_j^{(r)}(\xi_1, \dots, \xi_r) \left(\sum_{\alpha+\beta=k-j} \xi_{r+1}^{\alpha} T^{\beta} \right).$$

Here we note $P_0^{(r)} = 1$ for any $r \in \mathbb{N}$.

Lemma 3. *For any $r \in \mathbb{N}$ and $k = 0, 1, \dots$ we have*

$$\begin{aligned} &P_k^{(r)}(\xi_1, \dots, \xi_r; \xi_{r+1}; T) - P_k^{(r)}(\xi_1, \dots, \xi_r; \xi_{r+1}; \xi_{r+2}) \\ &\quad = (T - \xi_{r+2}) P_{k-1}^{(r+1)}(\xi_1, \dots, \xi_{r+1}; \xi_{r+2}; T), \end{aligned}$$

where $\xi_{r+2} \in \mathbb{C}$.

Proof. First we have

$$\begin{aligned} &P_k^{(r)}(\xi_1, \dots, \xi_r; \xi_{r+1}; T) - P_k^{(r)}(\xi_1, \dots, \xi_r; \xi_{r+1}; \xi_{r+2}) \\ &\quad = \sum_{j=0}^{k} a_j^{(r)}(\xi_1, \dots, \xi_r) \left(\sum_{\alpha+\beta=k-j} \xi_{r+1}^{\alpha} \left(T^{\beta} - \xi_{r+2}^{\beta} \right) \right) \\ &\quad = (T - \xi_{r+2}) \sum_{j=0}^{k-1} a_j^{(r)}(\xi_1, \dots, \xi_r) \left(\sum_{\alpha+\beta=k-j} \xi_{r+1}^{\alpha} \left(\sum_{\substack{\gamma+\delta=\beta-1 \\ \beta\geqq 1}} \xi_{r+2}^{\gamma} T^{\delta} \right) \right). \end{aligned}$$

By a straight calculation we obtain

$$\begin{aligned}
&\sum_{j=0}^{k-1} a_j^{(r)}(\xi_1,\ldots,\xi_r)\left(\sum_{\alpha+\beta=k-j}\xi_{r+1}^{\alpha}\left(\sum_{\substack{\gamma+\delta=\beta-1\\ \beta\geqq 1}}\xi_{r+2}^{\gamma}T^{\delta}\right)\right)\\
&\quad=\sum_{j=0}^{k-1} a_j^{(r)}(\xi_1,\ldots,\xi_r)\left(\sum_{\alpha=0}^{k-1-j}\xi_{r+1}^{\alpha}\left(\sum_{\gamma+\delta=k-1-j-\alpha}\xi_{r+2}^{\gamma}T^{\delta}\right)\right)\\
&\quad=\sum_{j=0}^{k-1}\sum_{\alpha=0}^{k-1-j} a_j^{(r)}(\xi_1,\ldots,\xi_r)\xi_{r+1}^{\alpha}\left(\sum_{\gamma+\delta=k-1-(j+\alpha)}\xi_{r+2}^{\gamma}T^{\delta}\right)\\
&\quad=\sum_{s=0}^{k-1}\sum_{j+\alpha=s} a_j^{(r)}(\xi_1,\ldots,\xi_r)\xi_{r+1}^{\alpha}\left(\sum_{\gamma+\delta=k-1-s}\xi_{r+2}^{\gamma}T^{\delta}\right)\\
&\quad=\sum_{s=0}^{k-1} a_s^{(r+1)}(\xi_1,\ldots,\xi_r,\xi_{r+1})\left(\sum_{\gamma+\delta=k-1-s}\xi_{r+2}^{\gamma}T^{\delta}\right)\\
&\quad=P_{k-1}^{(r+1)}(\xi_1,\ldots,\xi_{r+1};\xi_{r+2};T).
\end{aligned}$$

Thus the proof completes. □

4. Proof of the theorem

We use the notations in the previous sections. Let $1,\alpha_1,\ldots,\alpha_{m-1}$ be a basis of $\mathfrak{o}_{K_0}$. We may assume that $K=K_0(x)$ with $x\in\mathfrak{o}_K$ and x has the property in Lemma 2. Then the following is a basis of K over $\mathbb{Q}$

$$1,\alpha_1,\ldots,\alpha_{m-1},x,x\alpha_1,\ldots,x\alpha_{m-1},\ldots,x^{n-1},x^{n-1}\alpha_1,\ldots,x^{n-1}\alpha_{m-1}.$$

We set $\alpha_{ij}:=\varrho_i(\alpha_j)\in\mathbb{R}$ and $x_i^{(j)}:=\sigma_i^{(j)}(x)\in\mathbb{R}$ for $i=1,\ldots,m$ and $j=1,\ldots,n-2$.

We denote by $\mathfrak{o}$ the order of K generated by the above basis. It suffices to show that $X_{\mathfrak{o}}$ is quasi-abelian, by Lemma 1. Let P be the period matrix of $X_{\mathfrak{o}}$ given by the above basis of $\mathfrak{o}$. Then we have

$$P=\begin{pmatrix} A & YA & Y^2A & \cdots & Y^{n-1}A\\ A & X^{(1)}A & (X^{(1)})^2A & \cdots & (X^{(1)})^{n-1}A\\ \vdots & \vdots & \vdots & & \vdots\\ A & X^{(n-1)}A & (X^{(n-2)})^2A & \cdots & (X^{(n-2)})^{n-1}A\end{pmatrix},$$

where

$$A := \begin{pmatrix} 1 & \alpha_{11} & \cdots & \alpha_{1,m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_{m1} & \cdots & \alpha_{m,m-1} \end{pmatrix},$$

$$X^{(\ell)} := \begin{pmatrix} x_1^{(\ell)} & & \\ & \ddots & \\ & & x_m^{(\ell)} \end{pmatrix}, \quad \ell = 1, \dots, n-2$$

and

$$Y := \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_m \end{pmatrix}.$$

We shall transform P into the standard form of a period matrix of a quasi-abelian variety. If we set

$$Q := \begin{pmatrix} I & Y & Y^2 & \cdots & Y^{n-1} \\ I & X^{(1)} & (X^{(1)})^2 & \cdots & (X^{(1)})^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ I & X^{(n-2)} & (X^{(n-2)})^2 & \cdots & (X^{(n-2)})^{n-1} \end{pmatrix},$$

then

$$P = Q \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix},$$

where I is the unit matrix of degree m. Therefore it is sufficient to consider the transformation of Q.

Let P_1 and P_2 be square matrices of degree k. We write $P_1 \simeq P_2$ if there exists $M \in \mathrm{GL}(k, \mathbb{C})$ such that $MP_1 = P_2$. We first obtain

$$Q \simeq \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & Q_0 & \\ 0 & & & \\ \hline I & X^{(n-2)} & \cdots & (X^{(n-2)})^{n-1} \end{array}\right),$$

where

$$Q_0 = \begin{pmatrix} B_{0,1} & B_{0,2} & \cdots & B_{0,n-1} \\ C_{0,1}^{(1)} & C_{0,2}^{(1)} & \cdots & C_{0,n-1}^{(1)} \\ \vdots & \vdots & & \vdots \\ C_{0,1}^{(n-3)} & C_{0,2}^{(n-3)} & \cdots & C_{0,n-1}^{(n-3)} \end{pmatrix},$$

$$B_{0,k} = (Y - X^{(n-2)}) \sum_{i+j=k-1} (X^{(n-2)})^i Y^j, \quad k = 1, \dots, n-1,$$

$$C_{0,k}^{(\ell)} = (X^{(\ell)} - X^{(n-2)}) \sum_{i+j=k-1} (X^{(n-2)})^i (X^{(\ell)})^j$$

for $k = 1, \dots, n-1$ and $\ell = 1, \dots, n-3$.

Next we consider the transformation of Q_0. We note that $C_{0,1}^{(n-3)}$ is a non-singular matrix for $C_{0,1}^{(n-3)} = X^{(n-3)} - X^{(n-2)}$. Substracting the $(n-2)$-nd row multiplied by $B_{0,1}(C_{0,1}^{(n-3)})^{-1}$ from the first row and the $(n-2)$-nd row multiplied by $C_{0,1}^{(\ell)}(C_{0,1}^{(n-3)})^{-1}$ from the $(1+\ell)$-th row, we obtain

$$Q_0 \simeq \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & Q_1 & \\ 0 & & & \\ \hline C_{0,1}^{(n-3)} & C_{0,2}^{(n-3)} & \cdots & C_{0,n-1}^{(n-3)} \end{array}\right),$$

where

$$Q_1 = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,n-2} \\ C_{1,1}^{(1)} & C_{1,2}^{(1)} & \cdots & C_{1,n-2}^{(1)} \\ \vdots & \vdots & & \vdots \\ C_{1,1}^{(n-4)} & C_{1,2}^{(n-4)} & \cdots & C_{1,n-2}^{(n-4)} \end{pmatrix}.$$

Since

$$(C_{0,1}^{(n-3)})^{-1} C_{0,k+1}^{(n-3)} = \sum_{i+j=k} (X^{(n-2)})^i (X^{(n-3)})^j,$$

we have

$$\begin{aligned} B_{1,k} &= B_{0,k+1} - B_{0,1}(C_{0,1}^{(n-3)})^{-1} C_{0,k+1}^{(n-3)} \\ &= (Y - X^{(n-2)}) \sum_{i+j=k} (X^{(n-2)})^i \left(Y^j - (X^{(n-3)})^j \right) \end{aligned}$$

$$= \prod_{p=n-3}^{n-2}(Y-X^{(p)}) \sum_{i+j=k}(X^{(n-2)})^i \left(\sum_{\substack{\alpha+\beta=j-1 \\ \alpha\geqq 1}} (X^{(n-3)})^{\alpha} Y^{\beta} \right)$$

$$= \prod_{p=n-3}^{n-2}(Y-X^{(p)}) \sum_{\substack{i+j=k \\ j\geqq 1}}(X^{(n-2)})^i \left(\sum_{\alpha+\beta=j-1} (X^{(n-3)})^{\alpha} Y^{\beta} \right)$$

$$= \prod_{p=n-3}^{n-2}(Y-X^{(p)}) \sum_{j'=0}^{k-1}(X^{(n-2)})^{j'} \left(\sum_{\alpha+\beta=k-1-j'} (X^{(n-3)})^{\alpha} Y^{\beta} \right)$$

$$= \prod_{p=n-3}^{n-2}(Y-X^{(p)}) P^{(1)}_{k-1}(X^{(n-2)}; X^{(n-3)}; Y)$$

for $k = 1, \dots, n-2$. Similarly we have

$$C^{(\ell)}_{1,k} = \prod_{p=n-3}^{n-2}(X^{(\ell)}-X^{(p)}) P^{(1)}_{k-1}(X^{(n-2)}; X^{(n-3)}; X^{(\ell)})$$

for $k = 1, \dots, n-2$ and $\ell = 1, \dots, n-4$.

Suppose that we have already obtained the matrix Q_r for $1 \leqq r < n-3$ such that

$$Q_{r-1} \simeq \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & Q_r & \\ 0 & & & \\ \hline C^{(n-r-2)}_{r-1,1} & C^{(n-r-2)}_{r-1,2} & \cdots & C^{(n-r-2)}_{r-1,n-r} \end{array} \right),$$

$$Q_r = \begin{pmatrix} B_{r,1} & B_{r,2} & \cdots & B_{r,n-r-1} \\ C^{(1)}_{r,1} & C^{(1)}_{r,2} & \cdots & C^{(1)}_{r,n-r-1} \\ \vdots & \vdots & & \vdots \\ C^{(n-r-3)}_{r,1} & C^{(n-r-3)}_{r,2} & \cdots & C^{(n-r-3)}_{r,n-r-1} \end{pmatrix},$$

$$B_{r,k} = \prod_{p=n-r-2}^{n-2}(Y-X^{(p)}) P^{(r)}_{k-1}(X^{(n-2)}, \dots, X^{(n-r-1)}; X^{(n-r-2)}; Y)$$

for $k = 1, \dots, n-r-1$ and

$$C^{(\ell)}_{r,k} = \prod_{p=n-r-2}^{n-2}(X^{(\ell)}-X^{(p)}) P^{(r)}_{k-1}(X^{(n-2)}, \dots, X^{(n-r-1)}; X^{(n-r-2)}; X^{(\ell)})$$

for $k = 1, \dots, n-r-1$ and $\ell = 1, \dots, n-r-3$. We note that the matrix

$$C_{r,1}^{(n-r-3)} = \prod_{p=n-r-2}^{n-2} (X^{(n-r-3)} - X^{(p)})$$

is non-singular. Then we can carry out the same procedure as in the case $r = 0$. Hence we obtain Q_{r+1} such that

$$Q_r \simeq \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & Q_{r+1} & \\ 0 & & & \\ \hline C_{r,1}^{(n-r-3)} & C_{r,2}^{(n-r-3)} & \cdots & C_{r,n-r-1}^{(n-r-3)} \end{array}\right),$$

$$Q_{r+1} = \begin{pmatrix} B_{r+1,1} & B_{r+1,2} & \cdots & B_{r+1,n-r-2} \\ C_{r+1,1}^{(1)} & C_{r+1,2}^{(1)} & \cdots & C_{r+1,n-r-2}^{(1)} \\ \vdots & \vdots & & \vdots \\ C_{r+1,1}^{(n-r-4)} & C_{r+1,2}^{(n-r-4)} & \cdots & C_{r+1,n-r-2}^{(n-r-4)} \end{pmatrix},$$

where

$$\begin{cases} B_{r+1,k} = B_{r,k+1} - B_{r,1}(C^{(n-r-3)})^{-1} C_{r,k+1}^{(n-r-3)}, \\ C_{r+1,k}^{(\ell)} = C_{r,k+1}^{(\ell)} - C_{r,1}^{(\ell)} (C_{r,1}^{(n-r-3)})^{-1} C_{r,k+1}^{(n-r-3)}. \end{cases}$$

Since

$$B_{r,1} = \prod_{p=n-r-2}^{n-2} (Y - X^{(p)}),$$

$$B_{r,k+1} = \prod_{p=n-r-2}^{n-2} (Y - X^{(p)}) P_k^{(r)}(X^{(n-2)}, \dots, X^{(n-r-1)}; X^{(n-r-2)}; Y)$$

and

$$(C_{r,1}^{(n-r-3)})^{-1} C_{r,k+1}^{(n-r-3)} = P_k^{(r)}(X^{(n-2)}, \dots, X^{(n-r-1)}; X^{(n-r-2)}; X^{(n-r-3)}),$$

we have

$$\begin{aligned} B_{r+1,k} \quad = \quad & \prod_{p=n-r-2}^{n-2} (Y - X^{(p)}) \times \\ & \Big(P_k^{(r)}(X^{(n-2)}, \dots, X^{(n-r-1)}; X^{(n-r-2)}; Y) \\ & \quad - P_k^{(r)}(X^{(n-2)}, \dots, X^{(n-r-1)}; X^{(n-r-2)}; X^{(n-r-3)}) \Big) . \end{aligned}$$

It follows from Lemma 3 that

$$\begin{aligned}\Big(P_k^{(r)}&(X^{(n-2)},\dots,X^{(n-r-1)};X^{(n-r-2)};Y)\\ &-P_k^{(r)}(X^{(n-2)},\dots,X^{(n-r-1)};X^{(n-r-2)};X^{(n-r-3)})\Big)\\ &\quad=(Y-X^{(n-r-3)})P_{k-1}^{(r+1)}(X^{(n-2)},\dots,X^{(n-r-2)};X^{(n-r-3)};Y).\end{aligned}$$

Then we have

$$B_{r+1,k}=\prod_{p=n-r-3}^{n-2}(Y-X^{(p)})P_{k-1}^{(r+1)}(X^{(n-2)},\dots,X^{(n-r-2)};X^{(n-r-3)};Y).$$

Similarly we obtain

$$C_{r+1,k}^{(\ell)}=\prod_{p=n-r-3}^{n-2}(X^{(\ell)}-X^{(p)})P_{k-1}^{(r+1)}(X^{(n-2)},\dots,X^{(n-r-2)};X^{(n-r-3)};X^{(\ell)}).$$

Repeating this procedure to $r=n-3$, we finally obtain a matrix

$$Q_{n-3}=\begin{pmatrix} B_{n-3,1} & B_{n-3,2}\end{pmatrix}$$

such that

$$Q\simeq\begin{pmatrix}0 & Q_{n-3}\\ * & **\end{pmatrix},$$

where

$$B_{n-3,1}=\prod_{p=1}^{n-2}(Y-X^{(p)}),$$

$$\begin{aligned}B_{n-3,2}&=\prod_{p=1}^{n-2}(Y-X^{(p)})P_1^{(n-3)}(X^{(n-2)},\dots,X^{(2)};X^{(1)};Y)\\ &=\prod_{p=1}^{n-2}(Y-X^{(p)})(Y+\sum_{\ell=1}^{n-2}X^{(\ell)}).\end{aligned}$$

Then we need only to show that

$$\begin{pmatrix} B_{n-3,1}A & B_{n-3,2}A\end{pmatrix}$$

is a period matrix of an abelian variety of dimension m. Let $d(K_0)$ be the discriminant of K_0. Noting that $|\det A|^2=|d(K_0)|\geqq 1$, we obtain

$$\begin{pmatrix} B_{n-3,1}A & B_{n-3,2}A\end{pmatrix}\simeq\begin{pmatrix} {}^tAA & {}^tA\left(Y+\sum_{\ell=1}^{n-2}X^{(\ell)}\right)A\end{pmatrix}.$$

Any entry of ${}^tAA$ is

$$\sum_{k=1}^{m} \alpha_{ki}\alpha_{kj} = \sum_{k=1}^{m} \varrho_k(\alpha_i\alpha_j) = \mathrm{Tr}_{K_0}(\alpha_i\alpha_j) \in \mathbb{Z},$$

where we set $\alpha_0 = 1$ and $\alpha_{k0} = 1$. It is obvious that ${}^tA\left(Y + \sum_{\ell=1}^{n-2} X^{(\ell)}\right)A$ is symmetric. Furthermore we have

$$\mathrm{Im}\left({}^tA\left(Y + \sum_{\ell=1}^{n-2} X^{(\ell)}\right)A\right) = {}^tA\mathrm{Im}(Y)A > 0$$

by Lemma 2. Thus we complete the proof. □

Remark. If K is a CM-field of degree $2n$, then the period matrix of $X = \mathbb{C}^n/\Psi(\mathfrak{o}_K)$ in our argument is $P = (A \quad YA)$. Then it is obvious that $P \simeq ({}^tAA \quad {}^tAYA)$ is a period matrix of an abelian variety. This is another way to show that any CM-field gives an abelian variety.

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