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Global asymptotic stability for a nonautonomous Lotka-Volterra competition system

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Abstract. We consider nonautonomous N-dimensional generalized Lotka-Volterra competition systems. Under certain conditions we show that there exists a unique solution u^* whose components are bounded above and below by positive constants on $\mathbb{R}$, and u^* attracts any solution. If such system is periodic, so is u^*.

1. Introduction and statements of the main results

In this paper we consider the system of differential equations

$$u_i' = u_i \left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i,\ u_j) \right], \quad i = 1, \ldots, N,\ N \geq 2, \qquad \text{(GLV)}$$

where the functions $a_i(t)$, $1 \leq i \leq N$, and $b_{ij}(t)$, $1 \leq i, j \leq N$, are assumed to be continuous and bounded on $\mathbb{R}$. For a bounded function $g(t)$ on $\mathbb{R}$, we put $g_M := \sup_{t \in \mathbb{R}} g(t)$, $g_L := \inf_{t \in \mathbb{R}} g(t)$. We assume that

$$b_{ij}(t) \geq 0, \quad t \in \mathbb{R},\ 1 \leq i,\ j \leq N; \qquad (1.1)$$

$$a_{iL} > 0, \quad b_{iiL} > 0, \quad 1 \leq i \leq N. \qquad (1.2)$$

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Furthermore let the function $f_{ij}(x,\ y)$, $1 \le i,\ j \le N$, be continuously differentiable on $[0,\ \infty)^2$, and we impose the following condition on f_{ij}:

$$\begin{cases} f_{ij}(x,\ y) > 0, \quad (x,\ y) \in \mathbb{R}_+^2,\ 1 \le i,\ j \le N; \\ D_1 f_{ij}(x,\ y) \ge 0, \quad (x,\ y) \in \mathbb{R}_+^2,\ 1 \le i,\ j \le N; \\ D_2 f_{ij}(x,\ y) > 0, \quad (x,\ y) \in \mathbb{R}_+^2,\ 1 \le i,\ j \le N; \\ f_{ij}(0,\ 0) = 0, \quad 1 \le i,\ j \le N; \\ \lim_{x\to\infty} f_{ii}(x,\ x) = \infty, \quad 1 \le i \le N, \end{cases} \tag{1.3}$$

where $\mathbb{R}_+^2 = (0,\ \infty)^2$ and D_i, $i = 1,\ 2$, denotes the differentiation with respect to the i-th variable.

Throughout the paper we make use of the well-known fact (see e.g. [3], [11]) that if $u = (u_1, \ldots,\ u_N)$ is a local solution of system (GLV) with $u(t_0) \in \mathbb{R}_+^N$, then u can be extended to the interval $[t_0,\ \infty)$ and $u(t) \in \mathbb{R}_+^N$ for $t \in [t_0,\ \infty)$. Therefore in the sequel we may assume that all solutions of system (GLV) exist near $+\infty$ and are positive there.

System (GLV) is a generalization of the following nonautonomous N-dimensional Lotka-Volterra competition system that S. Ahmad and A. C. Lazer [2] considered:

$$u_i' = u_i \left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) u_j \right], \quad i = 1, \ldots,\ N,\ N \ge 2. \tag{LV}$$

In system (LV) the negative influence of the j-th species to the i-th species is regarded essentially as linear function, because the corresponding term in the classical system (LV) is written as $u_i u_j$. But, as is often the case with mathematical modeling in biology, such a hypothesis is too restrictive from the biological point of view. In fact, K. Gopalsamy [9] considered system

$$u_i' = u_i \left[a_i + \sum_{m=1}^{n} \sum_{j=1}^{N} b_{mij} u_j^m \right], \quad i = 1, \ldots, N, \tag{1.4}$$

where a_i, b_{mij}, $1 \le i,\ j \le N$, $1 \le m \le n$, are real constants. If $n = 1$, then (1.4) reduces to the form of system (GLV). Motivated by these facts, we propose less restrictive system (GLV) than system (LV), and make an attempt to analyze it in order to generalize previous results to system (GLV). In fourthcoming papers we will continue to consider system (GLV) further.

When system (LV) is an autonomous case for $N = 2$, it is well known the following (see [10]): If positive constants $a_i,\ b_{ij},\ i,\ j = 1,\ 2$, satisfy

$$a_1 > \frac{b_{12}a_2}{b_{22}}, \quad a_2 > \frac{b_{21}a_1}{b_{11}},$$

then there exists a unique equilibrium point $(u_1^*,\ u_2^*) \in \mathbb{R}_+^2$ that attracts any solution curve $(u_1(t),\ u_2(t))$ of system (LV) with $(u_1(t_0),\ u_2(t_0)) \in \mathbb{R}_+^2$, that is,

$$u_1(t) \to u_1^* \text{ and } u_2(t) \to u_2^* \quad \text{as } t \to \infty.$$

For system (LV) S. Ahmad and A. C. Lazer [2] have shown a generalization of the above fact. They supposed conditions (1.1), (1.2) and the condition

$$a_{iL} > \sum_{j \neq i} b_{ijM} \left(\frac{a_j}{b_{jj}} \right)_M, \quad 1 \leq i \leq N. \tag{1.5}$$

Under these conditions they have shown the following: Let $u = (u_1, \ldots,\ u_N)$ be a solution of system (LV) with $u(t_0) \in \mathbb{R}_+^N$. Then there exists a unique solution $u^* = (u_1^*, \ldots,\ u_N^*)$ of system (LV) defined on $\mathbb{R}$ and the following statements (I)–(III) hold:

(I) $0 < \inf_{t\in\mathbb{R}} u_i^*(t) \leq \sup_{t\in\mathbb{R}} u_i^*(t) < \infty$ *for* $1 \leq i \leq N$;

(II) $\lim_{t\to\infty}(u_i(t) - u_i^*(t)) = 0$ *for* $1 \leq i \leq N$;

(III) *If* $a_i(t),\ b_{ij}(t),\ 1 \leq i,\ j \leq N$, *are periodic with period* $T > 0$, *then* u^* *is* T*-periodic.*

In [2] to see property (I) only the condition

$$\inf_{t\in\mathbb{R}} \frac{a_i(t) - \sum_{j\neq i} b_{ij}(t)(a_j/b_{jj})_M}{b_{ii}(t)} > 0, \quad 1 \leq i \leq N, \tag{1.6}$$

is required. Note that (1.6) is weaker condition than (1.5). In [1], [6]–[8] and [13] results related to properties (I)–(III) have been obtained under the condition

$$a_{iL} > \sum_{j\neq i} b_{ijM}\frac{a_{jM}}{b_{jjL}}, \quad 1 \le i \le N,$$

that is a stronger condition than (1.5).

To state our previous result in [12] we introduce the notation and the symbols: For a continuous and bounded function g on $\mathbb{R}$, we set

$$A[g,\ t_1,\ t_2] = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} g(s)ds,$$

where $t_1 < t_2$. We define the *upper average* $M[g]$ and the *lower average* $m[g]$, respectively, by

$$M[g] = \lim_{s\to\infty} \sup\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}; \text{ and}$$
$$m[g] = \lim_{s\to\infty} \inf\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}.$$

For $i = 1, \ldots,\ N$, we put

$$\tilde{f}_i(x) = f_{ii}(x,\ x), \quad x \in \mathbb{R}_+.$$

By (1.3), $\tilde{f}_i$, $i = 1, \ldots,\ N$, has the inverse function $\tilde{f}_i^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$. For $R > 0$ and $\delta > 0$, $\delta < R$, we define two constants $C^*(\delta,\ R)$ and $C_*(\delta,\ R)$, respectively, by

$$C^*(\delta,\ R) = \max\{D_k f_{ij}(x,\ y) \mid 1 \le i,\ j \le N, k = 1,\ 2, (x,\ y) \in [\delta,\ R]^2\},$$
$$C_*(\delta,\ R) = \min\{D_2 f_{ij}(x,\ y) \mid 1 \le i,\ j \le N, (x,\ y) \in [\delta,\ R]^2\}.$$

Let $R > 0$ and $\delta > 0$. For system (GLV) we introduce the condition

$$m[a_i] > \frac{C^*(\delta,\ R)}{C_*(\delta,\ R)} \sum_{j\neq i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \quad 1 \le i \le N. \tag{GA}$$

Condition (GA) is a generalization of the *average condition* that S. Ahmad and A. C. Lazer [3, 4] supposed for system (LV):

$$m[a_i] > \sum_{j\neq i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \quad 1 \le i \le N. \tag{A}$$

Remark 1.1. When b_{ii}, $1 \leq i \leq N$, is positive constants, it is clear that (1.5) implies condition (A).

Conditions (A) and (GA) have played an important role in studying systems (LV) and (GLV) (see [3]–[5] and [12]). For example, in [12] we have shown that condition (GA) and some additional conditions imply $\lim_{t\to\infty}(u_i(t) - v_i(t)) = 0$, $1 \leq i \leq N$, where $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$ are arbitrary solutions of system (GLV) (see [12], Theorem 2.3). Therefore when system (GLV) satisfies condition (GA) and similar condition to (1.6), we naturally expect that (I)–(III) hold for system (GLV). In this paper we will show that this conjecture is true; the following is the main result of this paper.

Theorem 1.2. *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Suppose that for some* $\delta > 0$ *and* R *satisfying*

$$R > \max\{\tilde{f}_i^{-1}((a_i/b_{ii})_M) \mid 1 \leq i \leq N\}, \tag{1.7}$$

condition (GA) *and the condition*

$$\inf_{t\in\mathbb{R}} \frac{a_i(t) - \sum_{j\neq i} f_{ij}(R,\ R)b_{ij}(t)}{b_{ii}(t)} > \max_{1\leq j\leq N}\{f_{jj}(\delta,\ \delta)\}, \quad 1 \leq i \leq N \tag{1.8}$$

hold. Then there exists a unique solution $u^* = (u_1^*, \dots, u_N^*)$ *of system* (GLV) *defined on* $\mathbb{R}$ *such that* (I)–(III) *hold for any solution* $u = (u_1, \dots, u_N)$ *of system* (GLV) *with* $u(t_0) \in \mathbb{R}_+^N$.

We give an example of system (GLV) for which above conditions hold (see [12, Example 1.2]).

Example 1.3. We consider the following competition system for two species:

$$u_1' = u_1 \left[(\cos t + 7) - (\sin t + 7) \cdot \left(\frac{u_1}{2} + 1\right) u_1 - \left\{\frac{1}{10}(\sin t + 1)\right\} \cdot u_1 u_2\right]$$
$$u_2' = u_2 \left[(\cos t + 9) - \left\{\frac{1}{6}(\sin t + 1)\right\} \cdot \frac{3u_2 u_1}{u_2 + 1} - (\sin t + 9) \cdot \left(\frac{u_2}{3} + 1\right) u_2\right],$$

where $f_{11}(x,\ y) = (y/2 + 1)y$, $f_{12}(x,\ y) = xy$, $f_{21}(x,\ y) = 3xy/(x + 1)$ and $f_{22}(x,\ y) = (y/3 + 1)y$.

In [12] we have shown that for $\delta = 9/20$ and $R = 1$, this system satisfies conditions (GA) and (1.8). So this system has only one periodic solution u^* satisfying (I)–(III).

The paper is organized as follows. In Section 2 we present preparatory results that are employed in proving Theorem 1.2. The proof of Theorem 1.2 are given in Section 3.

2. Preliminary results

In this section we give preliminary results to prove Theorem 1.2. The proofs of Proposition 2.1 and Theorem 2.3 are based on [12].

Proposition 2.1. *Let $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$ be solutions of* (GLV). *Suppose that there exist constants $A, B > 0$ and $T = T_{u,\,v} \geq t_0$ such that for $j = 1, \dots, N$ and $t \geq T$,*

$$A \leq u_j(t),\ v_j(t) \leq B. \tag{2.1}$$

Suppose moreover that for system (GLV) *there exist positive constants $\alpha_1, \dots, \alpha_N$ such that for $j = 1, \dots, N$,*

$$\liminf_{t\to\infty} \left[\alpha_j b_{jj}(t) - \frac{C^*(A,\ B)}{C_*(A,\ B)} \sum_{i \neq j} \alpha_i b_{ij}(t) \right] > 0. \tag{2.2}$$

Then there exist some constants $\tilde{T} \geq T$, $C = C_{A,\,B} > 0$ and $\gamma = \gamma_{A,\,B} > 0$ such that for $t \geq \tilde{T}$,

$$\sum_{i=1}^{N} \mid u_i(t) - v_i(t) \mid \leq \left(\sum_{i=1}^{N} \mid u_i(\tilde{T}) - v_i(\tilde{T}) \mid \right) C e^{-\gamma(t-\tilde{T})}. \tag{2.3}$$

Proof. Firstly, by the mean value theorem, there exist $0 < w_{ij} < 1$, $1 \leq i,\ j \leq N$, satisfying for $i,\ j = 1, \dots, N$,

$$\begin{aligned} f_{ij}(u_i,\ u_j) - f_{ij}(v_i,\ v_j) \\ =&(u_i - v_i) D_1 f_{ij}(w_{ij}u_i + z_{ij}v_i,\ w_{ij}u_j + z_{ij}v_j) \\ &+ (u_j - v_j) D_2 f_{ij}(w_{ij}u_i + z_{ij}v_i,\ w_{ij}u_j + z_{ij}v_j), \end{aligned} \tag{2.4}$$

where $z_{ij} = 1 - w_{ij}$, $1 \le i,\ j \le N$. By (2.1), we have

$$C_*(A,\ B) \le D_k f_{ij}(w_{ij}u_i + z_{ij}v_i,\ w_{ij}u_j + z_{ij}v_j) \le C^*(A,\ B) \qquad (2.5)$$

for $k = 1,\ 2$ and $i,\ j = 1, \ldots,\ N$. By (2.5), we can have (2.3) as in the proof of [12, Proposition 1]. □

When condition (GA) holds, we can reduce (2.2) to a simpler one (see [12, Proposition 2]).

Proposition 2.2. *Suppose that condition* (GA) *holds with* $\delta = A$ *and* $R = B$; *that is*

$$m[a_i] > \frac{C^*(A,\ B)}{C_*(A,\ B)} \sum_{j \ne i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \quad 1 \le i \le N.$$

Then there exist some positive constants $\alpha_1, \ldots,\ \alpha_N$ *such that for* $j = 1, \ldots,\ N$,

$$\alpha_j b_{jjL} - \frac{C^*(A,\ B)}{C_*(A,\ B)} \sum_{i \ne j} \alpha_i b_{ijM} > 0.$$

Therefore, if condition (GA) *holds, so does* (2.2) *with* $A = \delta$ *and* $B = R$.

By Proposition 2.1 and 2.2, the following theorem holds (see [12, Theorem 2]):

Theorem 2.3. *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Suppose that for some* $\delta > 0$ *and* $R > 0$ *satisfying* (1.7), *condition* (GA) *and the condition*

$$\liminf_{t \to \infty} \frac{a_i(t) - \sum_{j \ne i} f_{ij}(R,\ R) b_{ij}(t)}{b_{ii}(t)} > \max_{1 \le j \le N} \{f_{jj}(\delta,\ \delta)\}, \qquad 1 \le i \le N, \qquad (2.6)$$

hold. Then for solutions $u = (u_1, \ldots,\ u_N)$ *and* $v = (v_1, \ldots,\ v_N)$ *of system* (GLV) *with* $u(t_0),\ v(t_0) \in \mathbb{R}^N_+$, *the following statements* (i) *and* (ii) *hold*:

(i) $0 < \liminf_{t \to \infty} u_i(t) \le \limsup_{t \to \infty} u_i(t) < \infty, \quad i = 1, \ldots,\ N$;

(ii) $\lim_{t \to \infty} (u_i(t) - v_i(t)) = 0, \quad i = 1, \ldots,\ N$.

Remark 2.4. We note that (1.8) implies (2.6). Therefore, by Theorem 2.3, in order to prove Theorem 1.2, it is sufficient to prove the existence of a unique solution of system (GLV) satisfying (I) and (III).

Lemma 2.5. *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Let* $u = (u_1, \ldots, u_N)$ *and* $v = (v_1, \ldots, v_N)$ *be solutions of system* (GLV) *defined on* $(-\infty, T]$. *Suppose that there exist constants* $A, B > 0$ *satisfying*

$$A \leq u_j(t),\ v_j(t) \leq B \tag{2.7}$$

for $j = 1, \ldots, N$ *and* $t \leq T$. *Suppose moreover that for system* (GLV) *there exist positive constants* $\alpha_1, \ldots, \alpha_N$ *satisfying*

$$\inf_{t \leq T} \left[\alpha_j b_{jj}(t) - \frac{C*(A,\ B)}{C_*(A,\ B)} \sum_{i \neq j} \alpha_i b_{ij}(t) \right] > 0 \tag{2.8}$$

for $j = 1, \ldots, N$. *Then* $u \equiv v$.

Proof. Step 1. Firstly, from (2.8), there exists some $\varepsilon > 0$ such that for $j = 1, \ldots, N$ and $t \leq T$,

$$\alpha_j b_{jj}(t) - \frac{C^*(A,\ B)}{C_*(A,\ B)} \sum_{i \neq j} \alpha_i b_{ij}(t) \geq \varepsilon. \tag{2.9}$$

Next let

$$\theta(t) = \sum_{i=1}^{N} \left| \log \left(\frac{u_i(t)}{v_i(t)} \right) \right|, \quad t \leq T.$$

Then we can claim the following:
Claim. There exists some $\eta > 0$ such that for $t < T$,

$$\theta(T) \leq \theta(t) - \eta \int_t^T \sum_{i=1}^{N} \mid u_i(s) - v_i(s) \mid ds. \tag{2.10}$$

In fact, since $\mid \log(u_i(t)/v_i(t)) \mid$, $1 \leq i \leq N$, is absolutely continuous on every finite interval $I \subset (-\infty,\ T]$, for almost $t \le T$,

$$\begin{aligned}\theta'(t) &= \sum_{i=1}^{N} \alpha_i \left[\frac{u_i'}{u_i} - \frac{v_i'}{v_i}\right] \mathrm{sgn}(u_i - v_i)\\ &= \sum_{i=1}^{N} \alpha_i \left[-\sum_{j=1}^{N} b_{ij}(t)(f_{ij}(u_i,\ u_j) - f_{ij}(v_i,\ v_j))\right] \mathrm{sgn}(u_i - v_i)\\ &= \sum_{j=1}^{N} \Bigg[- \alpha_j b_{jj}(t)(\tilde{f}_j(u_j) - \tilde{f}_j(v_j))\mathrm{sgn}(u_j - v_j)\\ &\qquad - \sum_{i \neq j} \alpha_i b_{ij}(t)(f_{ij}(u_i,\ u_j) - f_{ij}(v_i,\ v_j))\mathrm{sgn}(u_i - v_i)\Bigg],\end{aligned}$$

where $u = u(t),\ v = v(t)$. By (2.4), we have

$$\begin{aligned}\theta'(t) = \sum_{j=1}^{N} \Bigg[&- \alpha_j b_{jj}(t)(D_1 f_{jj}(w_{jj}u_j + z_{jj}v_j,\ w_{jj}u_j + z_{jj}v_j)\\ &+ D_2 f_{jj}(w_{jj}u_j + z_{jj}v_j,\ w_{jj}u_j + z_{jj}v_j))(u_j - v_j)\mathrm{sgn}(u_j - v_j)\\ &- \sum_{i \neq j} \alpha_i b_{ij}(t) D_1 f_{ij}(w_{ij}u_i + z_{ij}v_i,\ w_{ij}u_j + z_{ij}v_j)(u_i - v_i)\mathrm{sgn}(u_i - v_i)\\ &- \sum_{i \neq j} \alpha_i b_{ij}(t) D_2 f_{ij}(w_{ij}u_i + z_{ij}v_i,\ w_{ij}u_j + z_{ij}v_j)(u_j - v_j)\mathrm{sgn}(u_i - v_i)\Bigg]\\ \le \sum_{j=1}^{N} \Bigg[&-C_*(A,\ B)\alpha_j b_{jj}(t) \mid u_j - v_j \mid + \sum_{i \neq j} C^*(A,\ B)\alpha_i b_{ij}(t) \mid u_j - v_j \mid\Bigg].\end{aligned}$$

By (2.9), we have

$$\theta'(t) \le -\varepsilon C_*(A,\ B) \sum_{j=1}^{N} \mid u_j - v_j \mid \quad \text{a.e. } t \le T.$$

Hence, putting $\eta = \varepsilon C_*(A,\ B)$, we can obtain (2.10).

Step 2. By (2.7), there exists some $C > 0$ such that $\theta(t) \le C,\ t \le T$. Since (2.10) can be rewritten as

$$\eta \int_t^T \mid u_i(s) - v_i(s) \mid ds \le \theta(t) - \theta(T) \le \theta(t) \le C$$

for $t < T$ and $i = 1, \ldots, N$, we find

$$\int_{-\infty}^{T} | u_i(s) - v_i(s) | \, ds < \infty.$$

Here let

$$m(t) = \max\{| u_i(t) - v_i(t) | \mid 1 \leq i \leq N\}.$$

Since $m(t) \leq | u_1(t) - v_1(t) | + \cdots + | u_N(t) + v_N(t) |$, we have

$$\int_{-\infty}^{T} m(s) ds < \infty$$

and so we may assume that $\liminf_{t \to -\infty} m(t) = 0$. Thus there exists some sequence $\{t_n\}_{n=1}^{\infty} \subset (-\infty, T]$ such that

$$t_n \to -\infty \quad \text{and} \quad m(t_n) \to 0 \quad \text{as } n \to \infty.$$

Since for $i = 1, \ldots, N$,

$$\left| \frac{u_i(t_n)}{v_i(t_n)} - 1 \right| = \left| \frac{u_i(t_n) - v_i(t_n)}{v_i(t_n)} \right| \leq \frac{m(t_n)}{A},$$

it follows that $\theta(t_n) \to 0$ as $n \to \infty$. Furthermore, since θ is nonincreasing by (2.10),

$$0 \leq \theta(T) \leq \theta(t_n).$$

Hence $\theta(T) = 0$, that is, $u(T) = v(T)$ and $u \equiv v$. □

Remark 2.6. By Proposition 2.2, we note that condition (GA) implies (2.8).

The following lemma has been employed in [11, 12]. However, its proof has not been given explicitly therein. Therefore we give the sketch of the proof.

Lemma 2.7. *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Let* $u = (u_1, \ldots, u_N)$ *be a solution of system* (GLV) *with* $u(t_0) \in \mathbb{R}_+^N$. *Suppose that for some* $\delta > 0$ *and* R *satisfying* (1.7), (1.8) *hold. Then the following statements* (i) *and* (ii) *hold*:

(i) *For* $1 \le i \le N$ *and* $t \ge t_0$,

$$u_i(t) \le \max\{u_i(t_0),\ \tilde{f}_i^{-1}((a_i/b_{ii})_M)\};$$

(ii) *For* $1 \le i \le N$ *and* $t \ge t_0$,

$$u_i(t) \ge \min\{u_i(t_0),\ \delta\}$$

if $0 < u_i(t_0) < \tilde{f}_i^{-1}((a_i/b_{ii})_M)$, $1 \le i \le N$.

Proof. (i) If there exist some T and $i \in \{1, \dots,\ N\}$ such that $\tilde{f}_i(u_i(T)) > (a_i/b_{ii})_M$, then

$$u_i'(T) \le u_i(T)[a_i(T) - b_{ii}(T)\tilde{f}_i(u_i(T))] < 0.$$

This proves (i).

(ii) From (i), it follows that for $i = 1, \dots,\ N$ and $t \ge t_0$,

$$u_i(t) \le R.$$

Therefore, by (1.8), if there exist some T and $i \in \{1, \dots,\ N\}$ such that $u_i(T) \le \delta$, then

$$u_i'(T) \ge u_i(T)\left[a_i(T) - \sum_{j \ne i} b_{ij}(T) f_{ij}(R,\ R) - b_{ii}(T)\tilde{f}_i(u_i(T))\right] > 0.$$

This proves (ii). □

Remark 2.8. Suppose that for some $\delta > 0$ and R satisfying (1.7), (1.8) hold. By Lemma 2.7, if u is a solution of system (GLV) such that for $i = 1, \dots,\ N$,

$$\delta < u_i(t_0) < \tilde{f}_i^{-1}((a_i/b_{ii})_M),$$

then for $i = 1, \dots,\ N$ and $t \ge t_0$,

$$\delta \le u_i(t) \le \tilde{f}_i^{-1}((a_i/b_{ii})_M).$$

Note that the proof of Theorem 1.2 is essentially based on this simple consideration.

3. Proof of theorem 1.2

In this section we prove Theorem 1.2 by employing the results in the previous section.

Proof of Theorem 1.2. (I) Step 1. For $m = 1, 2, \ldots$, let $u_m = (u_{m1}, \ldots, u_{mN})$ be a solution of system (GLV) such that $\delta < u_{mi}(-m) < \tilde{f}_i^{-1}((a_i/b_{ii})_M)$, $1 \leq i \leq N$. By Lemma 2.7, it follows that for $i = 1, \ldots, N$ and $t \geq -m$,

$$\delta \leq u_{mi}(t) \leq \tilde{f}_i^{-1}((a_i/b_{ii})_M). \tag{3.1}$$

Step 2. We put $I_1 = [-1, 1]$. By (3.1), a sequence $\{u'_m\}$ are uniformly bounded on I_1. Therefore, by Arzelà-Ascoli theorem, there exists a subsequence $\{u_{m,1}\} \subset \{u_m\}$ such that $\{u_{m,1}\}$ converges to a solution of system (GLV) uniformly on I_1

For $n = 1, 2, \ldots$, we put $I_n = [-n, n]$. Similarly to the above argument, $\{u_{m,1}\}$ has a subsequence $\{u_{m,2}\}$ that converges to a solution of system (GLV) uniformly on I_2. More generally $\{u_{m,n-1}\}$ has a subsequence $\{u_{m,n}\}$ that converges to a solution of system (GLV) uniformly on I_n. Therefore the diagonal sequence $\{u_{m,m}\}$ converges to a solution u^* of system (GLV) uniformly on every finite interval $I \subset \mathbb{R}$.

Moreover, since for $i = 1, \ldots, N$ and $t \in \mathbb{R}$,

$$\delta \leq u_i^*(t) \leq \tilde{f}_i^{-1}((a_i/b_{ii})_M),$$

it follows from Proposition 2.2 and Lemma 2.5 that u^* is the unique solution satisfying property (I).

(II) This is a direct consequence of Theorem 2.3.

(III) From (I), $u^*(t)$ and $u^*(t+T)$ satisfy (I). Since $a_i(t)$, $b_{ij}(t)$, $1 \leq i, j \leq N$, are periodic with period $T > 0$, it follows that for $i = 1, \ldots, N$,

$$\begin{aligned} u_i^{*'}(t+T) &= u_i^*(t+T)\left[a_i(t+T) - \sum_{j=1}^{N} b_{ij}(t+T) f_{ij}(u_i^*(t+T), u_j^*(t+T))\right] \\ &= u_i^*(t+T)\left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i^*(t+T), u_j^*(t+T))\right]. \end{aligned}$$

Hence, by Proposition 2.2 and Lemma 2.5, $u^*(t) \equiv u^*(t+T)$, that is, u^* is T-periodic. □

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