Picone identities for half-linear elliptic operators with $p(x)$-Laplacians and applications to Sturmain comparison theory ☆

Norio Yoshida*

Department of Mathematics, University of Toyama, Toyama 930-8555, Japan

Abstract

Picone identities are established for a class of half-linear elliptic operators with $p(x)$-Laplacians, and Sturmian comparison theorems are obtained on the basis of the Picone identities. Generalizations to half-linear elliptic inequalities with mixed nonlinearities are discussed, and specializations to half-linear partial or ordinary differential inequalities with $p(x)$-Laplacians are shown.

Key words: $p(x)$-Laplacian, Picone identity, Picone-type inequality, half-linear, elliptic, Sturmian comparison theory
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1. Introduction

The operator $-\nabla \cdot \big(|\nabla u|^{p(x)-2}\nabla u\big)$ is said to be $p(x)$-Laplacian, and becomes p-Laplacian $-\nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big)$ if $p(x) = p$ (constant), where the dot $\cdot$ denotes the scalar product,$\nabla = (\partial/\partial x_1, ..., \partial/\partial x_n)$ and $|x|$ denotes the Euclidean length of $x \in \mathbb{R}^n$. There has been much current interest in studying various mathematical problems with variable exponent growth condition. The study of such problems arise from nonlinear elasticity theory, electrorheological fluids (cf. [20, 27]).

Existence of weak solutions of the elliptic equation with $p(x)$-Laplacian

$$-\nabla \cdot \big(a(x)|\nabla u|^{p(x)-2}\nabla u\big) + |u|^{p(x)-2}u = f(x, u) \quad \text{in } \mathbb{R}^n$$

were investigated by several authors, see, for example, [5, 7, 14, 25]. For the existence of weak solutions for $p(x)$-Laplacian Dirichlet problem, we refer to [8, 13, 15, 16].

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*Corresponding author
Email addresses: nori@sci.u-toyama.ac.jp (Norio Yoshida)

The paper [26] by Zhang seems to be the first paper dealing with oscillations of solutions of $p(x)$-Laplacian equations. In [26] oscillation problem for the $p(t)$-Laplacian equation

$$\left(|u'|^{p(t)-2}u'\right)' + t^{-\theta(t)}g(t,u) = 0, \quad t > 0$$

was treated. Motivated by Zhang [26], we establish Picone identities and Sturmian comparison theorems for half-linear elliptic inequalities.

Sturmain comparison theorems for half-linear elliptic equations

$$\nabla \cdot \left(a(x)|\nabla u|^{\alpha-1}\nabla u\right) + c(x)|u|^{\alpha-1}u = 0,$$
$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha-1}\nabla v\right) + C(x)|v|^{\alpha-1}v = 0,$$

where $\alpha > 0$, were derived by utilizing a Picone identity, where we means by *half-linear* that a solution multiplied by any constant is also a solution. We refer the reader to Allegretto [1], Allegretto and Huang [3, 4], Bognár and Došlý [6], Došlý [9], Dunninger [12], Kusano, Jaroš and Yoshida [19], Yoshida [21, 22, 23, 24] for Picone identities and Sturmian comparison theorems, and to Došlý [10], Došlý and Řehák [11] for half-linear ordinary differential equations.

It might be natural to consider more genaral elliptic equations

$$\nabla \cdot \left(a(x)|\nabla u|^{\alpha(x)-1}\nabla u\right) + c(x)|u|^{\alpha(x)-1}u = 0,$$
$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) + C(x)|v|^{\alpha(x)-1}v = 0,$$

where $\alpha(x) > 0$, but the above equations are not half-linear if $\alpha(x)$ is not a constant. In order to obtain some oscillation results such as Sturmian comparison theorems, etc., which are generalizations of those of linear differential equations, we first determine a class of half-linear elliptic equations with $p(x)$-Laplacians.

The objective of this paper is to establish Picone identities for half-linear elliptic inequalities

$$uq[u] \geq 0, \tag{1.1}$$
$$vQ[v] \leq 0, \tag{1.2}$$

where q and Q are defined by

$$\begin{aligned} q[u] \;&:=\; \nabla \cdot \left(a(x)|\nabla u|^{\alpha(x)-1}\nabla u\right) - a(x)(\log|u|)|\nabla u|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla u \\ &\qquad + |\nabla u|^{\alpha(x)-1}b(x)\cdot\nabla u + c(x)|u|^{\alpha(x)-1}u, \qquad (1.3) \\ Q[v] \;&:=\; \nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\qquad + |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v, \qquad (1.4) \end{aligned}$$

and derive Sturmian comparison theorems for q and Q by using the Picone identities. In Section 2 we first show that (1.1) and (1.2) are *half-linear* in the sense that a constant multiple of a solution u [resp. v] is also a solution of (1.1) [resp. (1.2)] (see Proposition 2.1), and then establish Picone identities for q and Q. We mention, in particular, the paper [2] by Allegretto in which Picone

Identity arguments are used, and the formulae that are closely related to Picone identities in Section 2 are established.

In Section 3 we derive Sturmian comparison theorems for q and Q, and Section 4 is devoted to specializations to the case $\alpha(x) = \alpha > 0$, and to half-linear ordinary differential equations with $p(t)$-Laplacians which seems to be unknown.

2. Picone identities

Let G be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G. It is assumed that $a(x), A(x) \in C(\overline{G}; (0,\infty))$, $b(x), B(x) \in C(\overline{G}; \mathbb{R}^n)$, $c(x), C(x) \in C(\overline{G}; \mathbb{R})$, and that $\alpha(x) \in C^1(\overline{G}; (0,\infty))$.

The domain $\mathcal{D}_q(G)$ of q is defined to be the set of all functions u of class $C^1(\overline{G}; \mathbb{R})$ such that $a(x)|\nabla u|^{\alpha(x)-1}\nabla u \in C^1(G; \mathbb{R}^n) \cap C(\overline{G}; \mathbb{R}^n)$. The domain $\mathcal{D}_Q(G)$ of Q is defined similarly.

We note in (1.1) that $\log|u|$ has singularities at zeros of $u(x)$, but $u \log|u|$ is continuous at every zero x_0 if we define $u \log|u| = 0$ at $x = x_0$, in view of the fact that $\lim_{\varepsilon \to +0} \varepsilon \log \varepsilon = 0$. We make the similar remarks in (1.2).

We consider the elliptic inequalities

$$uq[u] \geq 0 \quad \text{in } G, \tag{2.1}$$

$$vQ[v] \leq 0 \quad \text{in } G, \tag{2.2}$$

where q and Q are defined by (1.3) and (1.4).

By a *solution* u [resp. v] of (2.1) [resp. (2.2)] we mean a function $u \in \mathcal{D}_q(G)$ [resp. $v \in \mathcal{D}_Q(G)$] which satisfies (2.1) [resp. (2.2)] in G.

Proposition 2.1. *Elliptic inequalities* (2.1) *and* (2.2) *are half-linear in the sense that if u and v are solutions of* (2.1) *and* (2.2)*, then ku and kv are also solutions of* (2.1) *and* (2.2) *for any constant k, respectively.*

Proof. It suffices to show that (2.1) is half-linear. Let u be any solution of (2.1), and $k (\neq 0)$ be any constant. It is easy to see that

$$\begin{aligned} q[ku] &= \nabla \cdot \big(|k|^{\alpha(x)-1} k a(x)|\nabla u|^{\alpha(x)-1}\nabla u\big) \\ &\quad -a(x)\big(|k|^{\alpha(x)-1}k\big)(\log(|k||u|))|\nabla u|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla u \\ &\quad +\big(|k|^{\alpha(x)-1}k\big)|\nabla u|^{\alpha(x)-1} b(x)\cdot\nabla u \\ &\quad +\big(|k|^{\alpha(x)-1}k\big)c(x)|u|^{\alpha(x)-1}u. \end{aligned} \tag{2.3}$$

A simple computation shows that

$$\begin{aligned} &\nabla \cdot \big(|k|^{\alpha(x)-1} k a(x)|\nabla u|^{\alpha(x)-1}\nabla u\big) \\ = \; &\nabla\big(|k|^{\alpha(x)-1}k\big) \cdot \big(a(x)|\nabla u|^{\alpha(x)-1}\nabla u\big) \\ &+|k|^{\alpha(x)-1}k\nabla \cdot \big(a(x)|\nabla u|^{\alpha(x)-1}\nabla u\big). \end{aligned} \tag{2.4}$$

Since

$$\nabla\big(|k|^{\alpha(x)-1}k\big) = |k|^{\alpha(x)-1}k(\log|k|)\nabla\alpha(x),$$

we see that

$$\begin{aligned}
& \nabla\cdot\big(|k|^{\alpha(x)-1}ka(x)|\nabla u|^{\alpha(x)-1}\nabla u\big) \\
= \;& a(x)|k|^{\alpha(x)-1}k(\log|k|)|\nabla u|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla u \\
& +|k|^{\alpha(x)-1}k\nabla\cdot\big(a(x)|\nabla u|^{\alpha(x)-1}\nabla u\big). \qquad (2.5)
\end{aligned}$$

Combining (2.3) and (2.5) yields

$$(ku)q[ku] = |k|^{\alpha(x)+1}uq[u] \geq 0$$

for any constant $k(\neq 0)$. Since $(ku)\log|ku| = 0$ for $k = 0$, we easily see that $(ku)q[ku] = 0$ for $k = 0$. Hence, we conclude that (2.1) is half-linear.

Remark 2.1. We note that (2.1) and (2.2) are half-linear if and only if $uq[u]$ and $vQ[v]$ are "homogeneous" functions in u and v, respectively, which satisfy

$$\begin{aligned}
(ku)q[ku] &= |k|^{\alpha(x)+1}uq[u] \ (k\in\mathbb{R}), \\
(kv)Q[kv] &= |k|^{\alpha(x)+1}vQ[v] \ (k\in\mathbb{R}).
\end{aligned}$$

Theorem 2.1 (Picone identity for Q). *If $v \in \mathcal{D}_Q(G)$ and v has no zero in G, then we obtain the following Picone identity for any $u \in C^1(G;\mathbb{R})$:*

$$\begin{aligned}
& -\nabla\cdot\left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\right) \\
= \;& -A(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\
& +C(x)|u|^{\alpha(x)+1} \\
& +A(x)\Bigg[\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\
& \qquad +\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1} \\
& \qquad -(\alpha(x)+1)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}\left(\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\right. \\
& \qquad\qquad \left. - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right)\cdot\left(\frac{u}{v}\nabla v\right)\Bigg] \\
& -\frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}\big(vQ[v]\big) \quad \text{in } G, \qquad (2.6)
\end{aligned}$$

where $\varphi(u) = |u|^{\alpha(x)-1}u = |u(x)|^{\alpha(x)-1}u(x)$.

Proof. A direct calculation yields

$$
\begin{aligned}
& -\nabla \cdot \left(u\varphi(u) \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \right) \\
= \; & -\nabla(u\varphi(u)) \cdot \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \\
& -u\varphi(u)A(x)|\nabla v|^{\alpha(x)-1}\nabla\left(\frac{1}{\varphi(v)}\right) \cdot \nabla v \\
& -\frac{u\varphi(u)}{\varphi(v)}\nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right). \qquad (2.7)
\end{aligned}
$$

We easily see that

$$
\nabla(u\varphi(u)) = (\alpha(x)+1)\varphi(u)\nabla u + u\varphi(u)(\log|u|)\nabla\alpha(x), \qquad (2.8)
$$

$$
\nabla\left(\frac{1}{\varphi(v)}\right) = -\frac{\alpha(x)}{v\varphi(v)}\nabla v - \frac{\log|v|}{\varphi(v)}\nabla\alpha(x) \qquad (2.9)
$$

in view of the fact that

$$
\nabla\varphi(v) = \alpha(x)\frac{\varphi(v)}{v}\nabla v + (\log|v|)\varphi(v)\nabla\alpha(x).
$$

Hence, we observe from (2.8) and (2.9) that

$$
\begin{aligned}
& \nabla(u\varphi(u)) \cdot \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \\
= \; & (\alpha(x)+1)\frac{\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha(x)-1}\nabla u \cdot \nabla v \\
& +u\varphi(u)(\log|u|)\frac{A(x)|\nabla v|^{\alpha(x)-1}}{\varphi(v)}\nabla\alpha(x) \cdot \nabla v \\
= \; & (\alpha(x)+1)A(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}(\nabla u) \cdot \left(\frac{u}{v}\nabla v\right) \\
& +A(x)u(\log|u|)\frac{\varphi(u)}{\varphi(v)}|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v \qquad (2.10)
\end{aligned}
$$

and

$$
\begin{aligned}
& u\varphi(u)A(x)|\nabla v|^{\alpha(x)-1}\nabla\left(\frac{1}{\varphi(v)}\right) \cdot \nabla v \\
= \; & -\alpha(x)\frac{u\varphi(u)}{v\varphi(v)}A(x)|\nabla v|^{\alpha(x)+1} \\
& -\frac{u\varphi(u)}{\varphi(v)}A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v \\
= \; & -A(x)\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1} \\
& -\frac{u\varphi(u)}{\varphi(v)}A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v. \qquad (2.11)
\end{aligned}
$$

It follows from (1.2) that

$$
\begin{aligned}
&\frac{u\varphi(u)}{\varphi(v)}\nabla\cdot\big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big)\\
=\;&\frac{u\varphi(u)}{\varphi(v)}\Big(Q[v]+A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v\\
&\qquad\qquad -|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v-C(x)|v|^{\alpha(x)-1}v\Big)\\
=\;&\frac{u\varphi(u)}{\varphi(v)}Q[v]+\frac{u\varphi(u)}{\varphi(v)}A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v\\
&-\frac{u\varphi(u)}{\varphi(v)}|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v-C(x)|u|^{\alpha(x)+1}.
\end{aligned}
\tag{2.12}
$$

Combining (2.7), (2.10)–(2.12), we arrive at

$$
\begin{aligned}
&-\nabla\cdot\left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\right)\\
=\;&C(x)|u|^{\alpha(x)+1}\\
&+A(x)\left[\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1}-(\alpha(x)+1)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}(\nabla u)\cdot\left(\frac{u}{v}\nabla v\right)\right]\\
&-A(x)u(\log|u|)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}(\nabla\alpha(x))\cdot\left(\frac{u}{v}\nabla v\right)\\
&+u\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}B(x)\cdot\left(\frac{u}{v}\nabla v\right)\\
&-\frac{u}{\varphi(v)}\big(\varphi(u)Q[v]\big)\\
=\;&C(x)|u|^{\alpha(x)+1}\\
&+A(x)\Bigg[\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1}\\
&\qquad-(\alpha(x)+1)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}\Bigg(\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\\
&\qquad\qquad-\frac{u}{(\alpha(x)+1)A(x)}B(x)\Bigg)\cdot\left(\frac{u}{v}\nabla v\right)\Bigg]\\
&-\frac{u\varphi(u)}{v\varphi(v)}\big(vQ[v]\big),
\end{aligned}
$$

which is equivalent to the desired identity (2.6).

Now we consider the first-order differential system

$$
\nabla w=H(x),
\tag{2.13}
$$

where $H(x)=\big(h_1(x),h_2(x),...,h_n(x)\big)$ is a vector function of class C^1, and we define the sequence of functions $\{g_k(x)\}_{k=1}^n$ by

$$
g_1(x)=\int h_1(x)dx_1,
\tag{2.14}
$$

$$g_k(x) = g_{k-1}(x) + \int \left(h_k(x) - \frac{\partial}{\partial x_k} g_{k-1}(x) \right) dx_k \quad (k = 2, 3, ..., n). \quad (2.15)$$

Proposition 2.2. *The system* (2.13) *has a* C^1*-solution if and only if*

$$\frac{\partial}{\partial x_{j-1}} \left(h_k(x) - \frac{\partial}{\partial x_k} g_{k-1}(x) \right) = 0 \quad (j = 2, 3, ..., k; k = 2, 3, ..., n). \quad (2.16)$$

Then any C^1*-solution* w *of* (2.13) *has the form*

$$w = g_n(x) + C_n \quad (2.17)$$

for some constant C_n*.*

Proof. Assume that (2.13) has a C^1-solution w, then we obtain

$$\frac{\partial w}{\partial x_1} = h_1(x)$$

and

$$\begin{aligned} w &= \int h_1(x) dx_1 + C_1(x_2, ..., x_n) \\ &= g_1(x) + C_1(x_2, ..., x_n) \end{aligned}$$

for some function $C_1(x_2, ..., x_n)$. Since we have

$$\frac{\partial w}{\partial x_2} = h_2(x),$$

we find that $C_1(x_2, ..., x_n)$ must satisfy

$$\frac{\partial C_1}{\partial x_2} = h_2(x) - \frac{\partial}{\partial x_2} g_1(x).$$

It is necessary that

$$\frac{\partial}{\partial x_1} \left(h_2(x) - \frac{\partial}{\partial x_2} g_1(x) \right) = 0$$

and we obtain

$$C_1 = \int \left(h_2(x) - \frac{\partial}{\partial x_2} g_1(x) \right) dx_2 + C_2(x_3, ..., x_n)$$

for some function $C_2(x_3, ..., x_n)$, and hence

$$\begin{aligned} w &= g_1(x) + \int \left(h_2(x) - \frac{\partial}{\partial x_2} g_1(x) \right) dx_2 + C_2(x_3, ..., x_n) \\ &= g_2(x) + C_2(x_3, ..., x_n). \end{aligned}$$

Repeating the above procedure, we observe that (2.15) is necessary that the solution w can be written in the form (2.17). It can be shown from the above consideration that the condition (2.16) is sufficient for (2.13) to have a C^1-solution.

Theorem 2.2 (Picone identity for q and Q). *Let $\alpha(x) \in C^2(G;(0,\infty))$ and $b(x)/a(x) \in C^1(G;\mathbb{R}^n)$. Assume that $u \in C^1(G;\mathbb{R})$, u has no zero in G, and that:*

(H$_1$) *there is a function $f \in C(\overline{G};\mathbb{R})$ such that $f \in C^1(G;\mathbb{R})$ and*

$$\nabla f = \frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{b(x)}{(\alpha(x)+1)a(x)} \quad \text{in } G.$$

If $e^f u \in \mathcal{D}_q(G)$, $v \in \mathcal{D}_Q(G)$ and v has no zero in G, then we obtain the following Picone identity:

$$\begin{aligned}
&\nabla \cdot \Big(e^{-(\alpha(x)+1)f}(e^f u)a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) \\
&\qquad - \frac{u\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha(x)-1}\nabla v \Big) \\
= \;& a(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1} \\
& -A(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\
& +\big(C(x)-c(x)\big)|u|^{\alpha(x)+1} \\
& +A(x)\Bigg[\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\
&\qquad +\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1} \\
&\qquad -(\alpha(x)+1)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}\Big(\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) \\
&\qquad\qquad - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\Big(\frac{u}{v}\nabla v\Big)\Bigg] \\
& +e^{-(\alpha(x)+1)f}(e^f u)q[e^f u] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}\big(vQ[v]\big) \quad \text{in } G. \qquad (2.18)
\end{aligned}$$

Proof. A direct calculation shows that

$$\begin{aligned}
&\nabla \cdot \Big(e^{-(\alpha(x)+1)f}(e^f u)a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) \Big) \\
= \;& (e^f u)\nabla(e^{-(\alpha(x)+1)f}) \cdot \Big(a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) \Big) \\
& +e^{-(\alpha(x)+1)f}\nabla(e^f u) \cdot \Big(a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) \Big) \\
& +e^{-(\alpha(x)+1)f}(e^f u)\nabla \cdot \Big(a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) \Big). \qquad (2.19)
\end{aligned}$$

Since

$$\nabla(e^{-(\alpha(x)+1)f}) = e^{-(\alpha(x)+1)f}\Big(-(\nabla\alpha(x))f - (\alpha(x)+1)\nabla f\Big),$$

we observe, using the hypothesis (H_1), that

$$\begin{aligned}
& (e^f u)\nabla(e^{-(\alpha(x)+1)f}) \\
= \;& e^{-(\alpha(x)+1)f}\Big[-(e^f u)(\nabla\alpha(x))f - (\alpha(x)+1)e^f u\nabla f\Big] \\
= \;& e^{-(\alpha(x)+1)f}\Big[-(e^f u)(\nabla\alpha(x))f - e^f(u\log|u|)\nabla\alpha(x) + e^f\frac{u}{a(x)}b(x)\Big] \\
= \;& e^{-(\alpha(x)+1)f}(e^f u)\Big[-\log|e^f u|\nabla\alpha(x) + \frac{b(x)}{a(x)}\Big]
\end{aligned}$$

and therefore

$$\begin{aligned}
& (e^f u)\nabla(e^{-(\alpha(x)+1)f}) \cdot \Big(a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u)\Big) \\
= \;& e^{-(\alpha(x)+1)f}(e^f u)\Big[-a(x)\log|e^f u||\nabla(e^f u)|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla(e^f u) \\
& \qquad + |\nabla(e^f u)|^{\alpha(x)-1}b(x)\cdot\nabla(e^f u)\Big]. \qquad (2.20)
\end{aligned}$$

It is clear that

$$\begin{aligned}
& e^{-(\alpha(x)+1)f}\nabla(e^f u)\cdot a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) \\
= \;& e^{-(\alpha(x)+1)f}a(x)|\nabla(e^f u)|^{\alpha(x)+1} \\
= \;& a(x)|e^{-f}\nabla(e^f u)|^{\alpha(x)+1} \\
= \;& a(x)|\nabla u + u\nabla f|^{\alpha(x)+1} \\
= \;& a(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1} \qquad (2.21)
\end{aligned}$$

in view of the hypothesis (H_1). From (2.19)–(2.21) it follows that

$$\begin{aligned}
& \nabla\cdot\Big(e^{-(\alpha(x)+1)f}(e^f u)a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u)\Big) \\
= \;& e^{-(\alpha(x)+1)f}(e^f u)\Big[\nabla\cdot\Big(a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u)\Big) \\
& \qquad - a(x)\log|e^f u||\nabla(e^f u)|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla(e^f u) \\
& \qquad + |\nabla(e^f u)|^{\alpha(x)-1}b(x)\cdot\nabla(e^f u)\Big] \\
& + a(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1} \\
= \;& a(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1} \\
& + e^{-(\alpha(x)+1)f}(e^f u)\Big[q[e^f u] - c(x)|e^f u|^{\alpha(x)-1}e^f u\Big] \\
= \;& a(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1} \\
& - c(x)|u|^{\alpha(x)+1} + e^{-(\alpha(x)+1)f}(e^f u)q[e^f u]. \qquad (2.22)
\end{aligned}$$

Combining (2.6) with (2.22) yields the desired identity (2.18).

Remark 2.2. In order to explain the role of the function f in the hypothesis (H_1), we treat the ordinary differential operaor ℓ and the variation $V[y]$ defined by

$$\ell[y] = \big(a(t)y'\big)' + b(t)y', \quad t \in (t_1, t_2),$$
$$V[y] = \int_{t_1}^{t_2} a(t) \left| y' - \frac{b(t)}{2a(t)} y \right|^2 dt,$$

where $a(t) \in C^1([t_1, t_2]; (0, \infty))$ and $b(t) \in C([t_1, t_2]; \mathbb{R})$. Letting

$$f(t) = -\int \frac{b(t)}{2a(t)}\, dt,$$

we observe that

$$\begin{aligned} V[y] &= \int_{t_1}^{t_2} a(t) \left| e^{-f(t)} \big(e^{f(t)} y\big)' \right|^2 dt \\ &= \int_{t_1}^{t_2} e^{-2f(t)} a(t) \Big(\big(e^{f(t)} y\big)' \Big)^2 dt \\ &= -\int_{t_1}^{t_2} e^{-2f(t)} \big(e^{f(t)} y\big) \ell[e^{f(t)} y]\, dt \end{aligned}$$

if $y(t_1) = y(t_2) = 0$. Introducing the function $f(t)$, we can consider the function $e^{f(t)} y$ to be a new unknown function.

Remark 2.3. We give an example which illustrates the hypothesis (H_1). Let $n = 1$, $G = (0, \pi)$, $u = \sin x$, $\alpha(x) = e^{\sin x + 1} - 1$, $a(x) = 1$, $b(x) = -(\cos x) e^{\sin x + 1}$. Defining $f(x)$ by

$$f(x) = \begin{cases} (\sin x) \log \sin x, & x \in (0, \pi) \\ 0 & \text{at } x = 0, \pi, \end{cases}$$

we conclude that

$$\begin{aligned} f'(x) &= (\cos x) \log \sin x + \cos x \\ &= \frac{\log |u|}{\alpha(x) + 1} \alpha'(x) - \frac{b(x)}{(\alpha(x) + 1) a(x)} \quad \text{in } (0, \pi). \end{aligned}$$

Moreover, we see that $f(x)$ is a continuous function on $[0, \pi]$ in view of the fact that $\lim_{\varepsilon \to +0} \varepsilon \log \varepsilon = 0$.

Remark 2.4. It follows from Proposition 2.2 that if (H_1) holds, then the function

$$\frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{b(x)}{(\alpha(x) + 1) a(x)} \tag{2.23}$$

must satisfy (2.16) in G with $H(x)$ replaced by (2.23). It is necessary that $\alpha(x) \in C^2$ and $b(x)/a(x) \in C^1$. For example, we treat the case where $n = 2$, $G = (0, \pi) \times (0, \pi)$, $u = \sin x_1 \sin x_2$, $\alpha(x) = e^{\sin x_1 \sin x_2 + 1} - 1$, $a(x) = 1$, and

$$b(x) = \big(-(\cos x_1 \sin x_2) e^{\sin x_1 \sin x_2 + 1}, -(\sin x_1 \cos x_2) e^{\sin x_1 \sin x_2 + 1} \big).$$

Then we have

$$\frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{b(x)}{(\alpha(x)+1)a(x)} = \big(h_1(x_1,x_2), h_2(x_1,x_2)\big),$$

where

$$\begin{aligned} h_1(x_1,x_2) &= \big(\cos x_1 \log\sin x_1 + \cos x_1\big)\sin x_2 + \cos x_1 \sin x_2 \log\sin x_2, \\ h_2(x_1,x_2) &= \big(\cos x_2 \log\sin x_2 + \cos x_2\big)\sin x_1 + \cos x_2 \sin x_1 \log\sin x_1. \end{aligned}$$

It is easy to check that

$$\frac{\partial}{\partial x_1}\left(h_2(x_1,x_2) - \frac{\partial}{\partial x_2}\int h_1(x_1,x_2)dx_1\right) = 0,$$

and the solution f of

$$\nabla f = \big(h_1(x_1,x_2), h_2(x_1,x_2)\big) \quad \text{in } G$$

is written in the form

$$f = \big(\sin x_1 \log\sin x_1\big)\sin x_2 + \big(\sin x_2 \log\sin x_2\big)\sin x_1,$$

which is continuous on $\overline{G} = [0,\pi]\times[0,\pi]$ by defining $f=0$ on ∂G.

3. Sturmian comparison theorems

On the basis of the Picone identity in Section 2 we present Sturmian comparison theorems for the half-linear elliptic operators q and Q.

Lemma 3.1. *The inequality*

$$|\xi|^{\alpha(x)+1} + \alpha(x)\,|\eta|^{\alpha(x)+1} - (\alpha(x)+1)|\eta|^{\alpha(x)-1}\xi\cdot\eta \geq 0 \tag{3.1}$$

is valid for $x \in G$, $\xi,\eta \in \mathbb{R}^n$, where the equality holds if and only if $\xi = \eta$.

PROOF. For any fixed $x \in G$, the inequality (3.1) holds for any $\xi,\eta \in \mathbb{R}^n$ by Hardy, Littlewood and Pólya [17, Theorem 41] and Kusano, Jaroš and Yoshida [19, Lemma 2.1].

Theorem 3.1 (Sturmian comparison theorem). *Let $\alpha(x) \in C^2(G;(0,\infty))$ and $b(x)/a(x), B(x)/A(x) \in C^1(G;\mathbb{R}^n)$. Assume that there exists a function $u \in C^1(\overline{G};\mathbb{R})$ such that $u=0$ on ∂G, u has no zero in G, the hypothesis* (H$_1$) *of Theorem* 2.2 *holds and that*:

(H$_2$) *there is a function $F \in C(\overline{G};\mathbb{R})$ such that $F \in C^1(G;\mathbb{R})$ and*

$$\nabla F = \frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{B(x)}{(\alpha(x)+1)A(x)} \quad \textit{in } G.$$

If the following conditions are satisfied:

(i) $e^f u \in \mathcal{D}_q(G)$ *and*

$$(e^f u)q[e^f u] \geq 0 \quad \textit{in } G;$$

(ii)

$$\begin{aligned} V_G[u] \quad := \quad & \int_G \Bigg[a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)a(x)} b(x) \right|^{\alpha(x)+1} \\ & -A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \\ & + \big(C(x) - c(x)\big)|u|^{\alpha(x)+1} \Bigg] dx \geq 0, \end{aligned}$$

then every solution $v \in \mathcal{D}_Q(G)$ *of* (2.2) *must vanish at some point of* $\overline{G}$.

Proof. Suppose to the contrary that there exists a solution $v \in \mathcal{D}_Q(G)$ of (2.2) such that v has no zero on $\overline{G}$. Integrating the Picone identity (2.18) over G and using the divergence theorem, we obtain

$$0 \geq V_G[u] + \int_G W(u,v)\, dx \geq 0,$$

which yields the following

$$\int_G W(u,v)\, dx = 0,$$

where

$$\begin{aligned} & W(u,v) \\ := \quad & A(x) \Bigg[\left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \\ & \quad + \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \\ & \quad -(\alpha(x)+1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \bigg(\nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) \\ & \qquad \qquad - \frac{u}{(\alpha(x)+1)A(x)} B(x) \bigg) \cdot \Big(\frac{u}{v} \nabla v. \Big) \Bigg]. \end{aligned}$$

It follows from Lemma 3.1 that

$$\nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \equiv \frac{u}{v} \nabla v \quad \text{in } G,$$

that is,

$$\nabla u + u \nabla F \equiv \frac{u}{v} \nabla v \quad \text{in } G,$$

from which we have

$$e^{-F} v \nabla \left(e^F \frac{u}{v} \right) \equiv 0 \quad \text{in } G.$$

Therefore, there exists a constant k_0 such that $e^F u/v = k_0$ in G and hence on $\overline{G}$ by continuity. Since $u = 0$ on ∂G, we see that $k_0 = 0$, which contradicts the hypothesis that u is nontrivial. The proof is complete.

Corollary 3.1. *Let $\alpha(x) \in C^2(G; (0, \infty))$, $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$. Assume that*:

(i) $\dfrac{b(x)}{a(x)} = \dfrac{B(x)}{A(x)}$ *in G*;

(ii) $a(x) \geq A(x)$, $C(x) \geq c(x)$ *in G*.

If there exists a function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G, u has no zero in G, the hypothesis (H_1) *of Theorem* 2.2 *holds and* (i) *of Theorem* 3.1 *is satisfied, then every solution $v \in \mathcal{D}_Q(G)$ of* (2.2) *must vanish at some point of $\overline{G}$.*

Proof. The conditions (i), (ii) imply that $V_G[u] \geq 0$ for any $u \in C^1(\overline{G}; \mathbb{R})$ and (H_2) is the same as (H_1). The conclusion follows from Theorem 3.1.

Corollary 3.2. *Let $\alpha(x) \in C^2(G; (0, \infty))$, $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$. Assume that the hypotheses* (i), (ii) *of Corollary* 3.1 *are satisfied, and that there exists a nontrivial function $u \in C^1(\overline{G}; \mathbb{R})$ which satisfies $u = 0$ on ∂G and the following*:

($\tilde{\mathrm{H}}_1$) *there is a function $f \in C(\overline{G}; \mathbb{R})$ such that $f \in C^1(N_u; \mathbb{R})$ and*

$$\nabla f = \frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{b(x)}{(\alpha(x) + 1) a(x)} \quad \text{in } N_u,$$

where

$$N_u := \{x \in G;\ u(x) \neq 0\}.$$

If $e^f u \in \mathcal{D}_q(N_u)$, $(e^f u) q[e^f u] \geq 0$ in N_u, then every solution $v \in \mathcal{D}_Q(G)$ of (2.2) *must vanish at some point of $\overline{G}$.*

Proof. Since u is nontrivial and $u = 0$ on ∂G, there is a domain $G_0 \subset G$ for which $u = 0$ on ∂G_0 and u has no zero in G_0. Applying Corollary 3.1 with G replaced by G_0, we conclude that every solution $v \in \mathcal{D}_Q(G)$ of (2.2) must vanish at some point of $\overline{G_0} \subset \overline{G}$, that is, v has a zero on $\overline{G}$.

Next we deal with the case where G is the annular domain $A(r_1, r_2)$ defined by

$$A(r_1, r_2) = \{x \in \mathbb{R}^n;\ r_1 < |x| < r_2\}\ (r_1 < r_2).$$

We use the notation:

$$A[r_1, r_2] = \{x \in \mathbb{R}^n;\ r_1 \leq |x| \leq r_2\},$$
$$S_r = \{x \in \mathbb{R}^n;\ |x| = r\}.$$

Let $\bar{A}(r)$ and $\bar{C}(r)$ denote the spherical means of $A(x)$ and $C(x)$ over the sphere S_r, respectively, that is,

$$\bar{A}(r) = \frac{1}{\omega_n r^{n-1}} \int_{S_r} A(x)\, dS = \frac{1}{\omega_n} \int_{S_1} A(r,\theta)\, d\omega,$$
$$\bar{C}(r) = \frac{1}{\omega_n r^{n-1}} \int_{S_r} C(x)\, dS = \frac{1}{\omega_n} \int_{S_1} C(r,\theta)\, d\omega,$$

where ω_n is the surface area of the unit sphere S_1, (r,θ) is the hyperspherical coordinates in $\mathbb{R}^n$ and ω is the measure on S_1.

We assume that:

(H_3) $\alpha(x) \equiv \alpha(|x|)$ in $A(r_1,r_2)$;

(H_4) $\dfrac{B(x)}{A(x)} = B_0(|x|)\dfrac{x_i}{|x|}$ in $A(r_1,r_2)$ for some function $B_0(r) \in C[r_1,r_2]$.

Associated with (2.2) we treat the half-linear elliptic operator $\tilde{q}$ defined by

$$\begin{aligned} \tilde{q}[u] \quad := \quad & \nabla \cdot \Big(\bar{A}(|x|)|\nabla u|^{\alpha(|x|)-1}\nabla u\Big) - \bar{A}(|x|)(\log |u|)|\nabla u|^{\alpha(|x|)-1}\nabla\alpha(|x|)\cdot\nabla u \\ & +|\nabla u|^{\alpha(|x|)-1}\bar{A}(|x|)B_0(|x|)\frac{x_i}{|x|}\cdot\nabla u + \bar{C}(|x|)|u|^{\alpha(|x|)-1}u. \end{aligned}$$

We define the half-linear ordinary differential operator q_0 by

$$\begin{aligned} q_0[y] \quad := \quad & \Big(r^{n-1}\bar{A}(r)|y'|^{\alpha(r)-1}y'\Big)' - r^{n-1}\bar{A}(r)(\log|y|)|y'|^{\alpha(r)-1}\alpha'(r)y' \\ & +r^{n-1}\bar{A}(r)B_0(r)|y'|^{\alpha(r)-1}y' + r^{n-1}\bar{C}(r)|y|^{\alpha(r)-1}y, \end{aligned}$$

and the domain $\mathcal{D}_{q_0}((r_1,r_2))$ of q_0 is defined to be the set of all functions y of class $C^1[r_1,r_2]$ such that $r^{n-1}\bar{A}(r)|y'|^{\alpha(r)-1}y' \in C^1(r_1,r_2)\cap C[r_1,r_2]$. If $y(r)$ is a solution of $yq_0[y] \geq 0$, then $u(x) = y(|x|)$ is a radially symmetric solution of $u\tilde{q}[u] \geq 0$.

Theorem 3.2. *Assume that the hypotheses* (H_3), (H_4) *hold. If there exists a function $z = z(r) \in \mathcal{D}_{q_0}((r_1,r_2))$ such that:*

(i) *$z(r_1) = z(r_2) = 0$ and $z(r) > 0$ in (r_1,r_2);*

(ii) *there is a function $f_0 = f_0(r) \in C[r_1,r_2]$ such that $f_0 \in C^1(r_1,r_2)$ and*

$$f_0'(r) = \frac{\log|z(r)|}{\alpha(r)+1}\alpha'(r) - \frac{B_0(r)}{\alpha(r)+1} \quad \textit{in } (r_1,r_2);$$

(iii) *$e^{f_0}z \in \mathcal{D}_{q_0}((r_1,r_2))$ and*

$$(e^{f_0}z)q_0[e^{f_0}z] \geq 0 \quad \textit{in } (r_1,r_2),$$

then every solution $v \in \mathcal{D}_Q(G)$ of (2.2) *must vanish at some point of $\overline{G}$.*

PROOF. Suppose on the contrary, that there is a solution $v \in \mathcal{D}_Q(G)$ of (2.2) such that v has no zero on $\overline{G}$. Defining

$$u(x) := z(|x|),$$

we compare $u\tilde{q}[u] \geq 0$ with (2.2). The condition (ii) implies that the hypotheses (H_1), (H_2) are satisfied for the case where $u(x) = z(|x|)$, $f = F = f_0(|x|)$ and

$$\bar{A}(|x|)B_0(|x|)\frac{x_i}{|x|}\Big/\bar{A}(|x|) = B(x)/A(x) = B_0(|x|)\frac{x_i}{|x|}.$$

Noting that

$$\begin{aligned}
& \left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)\bar{A}(|x|)}\bar{A}(|x|)B_0(|x|)\frac{x_i}{|x|}\right|^{\alpha(x)+1} \\
= & \left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\
= & \left|z'(r) + \frac{z(r)\log|z(r)|}{\alpha(r)+1}\alpha'(r) - \frac{z(r)}{\alpha(r)+1}B_0(r)\right|^{\alpha(r)+1} \qquad \text{on } S_r,
\end{aligned}$$

we easily arrive at

$$\begin{aligned}
& V_{A(r_1,r_2)}[u] \\
= & \int_{A(r_1,r_2)} \Bigg[\big(\bar{A}(|x|) - A(x)\big) \times \\
& \qquad \times \left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{\alpha(x)+1}B_0(|x|)\frac{x_i}{|x|}\right|^{\alpha(x)+1} \\
& \qquad\qquad + \big(C(x) - \bar{C}(|x|)\big)|u|^{\alpha(x)+1}\Bigg]dx \\
= & \int_{r_1}^{r_2}\int_{S_r} \Bigg[\big(\bar{A}(|x|) - A(x)\big) \times \\
& \qquad \times \left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{\alpha(x)+1}B_0(|x|)\frac{x_i}{|x|}\right|^{\alpha(x)+1} \\
& \qquad\qquad + \big(C(x) - \bar{C}(|x|)\big)|u|^{\alpha(x)+1}\Bigg]dSdr \\
= & \int_{r_1}^{r_2}\int_{S_1} \Bigg[\big(\bar{A}(r) - A(r,\theta)\big) \times \\
& \qquad \times \left|z'(r) + \frac{z(r)\log|z(r)|}{\alpha(r)+1}\alpha'(r) - \frac{z(r)}{\alpha(r)+1}B_0(r)\right|^{\alpha(r)+1} \\
& \qquad\qquad + \big(C(r,\theta) - \bar{C}(r)\big)|z(r)|^{\alpha(r)+1}\Bigg]r^{n-1}d\omega dr
\end{aligned}$$

$$
\begin{aligned}
&= \omega_n \int_{r_1}^{r_2} \Bigg[\left(\bar{A}(r) - \frac{1}{\omega_n} \int_{S_1} A(r,\theta) d\omega \right) \times \\
&\qquad\qquad \times \left| z'(r) + \frac{z(r) \log |z(r)|}{\alpha(r)+1} \alpha'(r) - \frac{z(r)}{\alpha(r)+1} B_0(r) \right|^{\alpha(r)+1} \\
&\qquad\qquad + \left(\frac{1}{\omega_n} \int_{S_1} C(r,\theta) d\omega - \bar{C}(r) \right) |z(r)|^{\alpha(r)+1} \Bigg] r^{n-1} dr \\
&= 0.
\end{aligned}
$$

Therefore, all hypotheses of Theorem 3.1 are satisfied, and the conclusion follows from Theorem 3.1. The proof is complete.

4. Specializations

In this Section we give some specializations to the case where $\alpha(x) = \alpha > 0$, and the case where $n = 1$, $b(x) = B(x) \equiv 0$.

Theorem 4.1. *Let $\alpha(x) = \alpha > 0$ and $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$. Assume that there exists a nontrivial function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G, and that the following hypotheses are satisfied:*

($\hat{\mathrm{H}}_1$) *there is a function $f \in C(\overline{G}; \mathbb{R})$ such that $f \in C^1(G; \mathbb{R})$ and*

$$
\nabla f = -\frac{b(x)}{(\alpha+1)a(x)} \quad \text{in } G;
$$

($\hat{\mathrm{H}}_2$) *there exists a function $F \in C(\overline{G}; \mathbb{R})$ such that $F \in C^1(G; \mathbb{R})$ and*

$$
\nabla F = -\frac{B(x)}{(\alpha+1)A(x)} \quad \text{in } G.
$$

If $e^f u \in \mathcal{D}_q(G)$,

$$
(e^f u) q[e^f u] \geq 0 \quad \text{in } G,
$$

and

$$
\begin{aligned}
V_G[u] &= \int_G \Bigg[a(x) \left| \nabla u - \frac{u}{(\alpha+1)a(x)} b(x) \right|^{\alpha+1} \\
&\qquad - A(x) \left| \nabla u - \frac{u}{(\alpha+1)A(x)} B(x) \right|^{\alpha+1} \\
&\qquad + \big(C(x) - c(x) \big) |u|^{\alpha+1} \Bigg] dx \geq 0,
\end{aligned}
$$

then every solution $v \in \mathcal{D}_Q(G)$ of (2.2) must vanish at some point of $\overline{G}$.

PROOF. Since $\nabla\alpha(x) \equiv 0$ on $\overline{G}$, the Picone identity (2.18) holds without the hypothesis that u has no zero in G. Therefore, the conclusion follows from Theorem 3.1.

The following corollary was established by Dunninger [12], Kusano, Jaroš and Yoshida [19].

Corollary 4.1. *Let $\alpha(x) = \alpha > 0$ and $b(x) = B(x) \equiv 0$ in G. If there exists a nontrivial function $u \in \mathcal{D}_q(G)$ such that $u = 0$ on ∂G, $uq[u] \geq 0$ in G, and*

$$V_G[u] = \int_G \left[\big(a(x) - A(x)\big)|\nabla u|^{\alpha+1} + \big(C(x) - c(x)\big)|u|^{\alpha+1}\right] dx \geq 0,$$

then every solution $v \in \mathcal{D}_Q(G)$ of (2.2) must vanish at some point of $\overline{G}$.

PROOF. Since $b(x) = B(x) \equiv 0$ on $\overline{G}$, we can choose $f = F \equiv 0$ on $\overline{G}$. Hence, the conclusion follows from Theorem 4.1.

Next we consider the special case where $n = 1$, $b(x) = B(x) \equiv 0$, that is, we let $x_1 = t$, $G = (t_1, t_2)$, and define q_1 and Q_1 by

$$\begin{aligned} q_1[y] \quad &:= \quad \Big(a(t)|y'|^{\alpha(t)-1}y'\Big)' - a(t)(\log|y|)|y'|^{\alpha(t)-1}\alpha'(t)y' \\ &\qquad +c(t)|y|^{\alpha(t)-1}y, \qquad\qquad (4.1) \\ Q_1[z] \quad &:= \quad \Big(A(t)|z'|^{\alpha(t)-1}z'\Big)' - A(t)(\log|z|)|z'|^{\alpha(t)-1}\alpha'(t)z' \\ &\qquad +C(t)|z|^{\alpha(t)-1}z, \qquad\qquad (4.2) \end{aligned}$$

where the coefficients appearing in (4.1) and (4.2) are supposed to satisfy the same conditions as in Section 2. The domains $\mathcal{D}_{q_1}(I)$, $\mathcal{D}_{Q_1}(I)$ are defined as in Section 2, where $I = (t_1, t_2)$.

Theorem 4.2. *Let $\alpha(x) \in C^2(I; (0,\infty)) \cap C^1(\overline{I}; (0,\infty))$. Assume that there exists a function $y \in C^1(\overline{I}; \mathbb{R})$ such that $y(t_1) = y(t_2) = 0$, y has no zero in I, and the following hypothesis is satisfied*:

($\bar{\mathrm{H}}_1$) *there is a function $f \in C(\overline{I}; \mathbb{R})$ such that $f \in C^1(I; \mathbb{R})$ and*

$$f'(t) = \frac{\log|y|}{\alpha(t)+1}\alpha'(t) \quad \text{in } I.$$

If $e^f y \in \mathcal{D}_{q_1}(I)$,

$$(e^f y)q_1[e^f y] \geq 0 \quad \text{in } I,$$

and

$$\begin{aligned} V_I[u] \quad &= \quad \int_I \Bigg[\big(a(t) - A(t)\big)\left|y' + \frac{y\log|y|}{\alpha(t)+1}\alpha'(t)\right|^{\alpha(t)+1} \\ &\qquad + \big(C(t) - c(t)\big)|y|^{\alpha(t)+1}\Bigg] dt \geq 0, \end{aligned}$$

then every solution $z \in \mathcal{D}_{Q_1}(I)$ of $zQ_1[z] \leq 0$ must vanish at some point of $\overline{I}$.

Proof. The conclusion follows from Theorem 3.1.

We state the analogue of Corollary 3.1.

Corollary 4.2. *Let $\alpha(x) \in C^2(I;(0,\infty)) \cap C^1(\overline{I};(0,\infty))$. Assume that there is a function $y \in C^1(\overline{I};\mathbb{R})$ such that $y(t_1) = y(t_2) = 0$, y has no zero in I, and the hypothesis $(\bar{\mathrm{H}}_1)$ of Theorem 4.2 holds. If $e^f y \in \mathcal{D}_{q_1}(I)$,*

$$(e^f y)q_1[e^f y] \geq 0 \quad \text{in } I,$$

and

$$a(t) \geq A(t), \quad C(t) \geq c(t) \quad \text{in } I,$$

then every solution $z \in \mathcal{D}_{Q_1}(I)$ of $zQ_1[z] \leq 0$ must vanish at some point of $\overline{I}$.

References

[1] W. Allegretto, Sturm theorems for degenerate elliptic equations, Proc. Amer. Math. Soc. 129 (2001) 3031–3035.

[2] W. Allegretto, Form estimates for the $p(x)$-Laplacian, Proc. Amer. Math. Soc. 135 (2007) 2177–2185.

[3] W. Allegretto, Y.X. Huang, A Picone's identity for the p-Laplacian and applications, Nonlinear Anal. 32 (1998) 819–830.

[4] W. Allegretto, Y.X. Huang, Principal eigenvalues and Sturm comparison via Picone's identity, J. Differential Equations 156 (1999) 427–438.

[5] C. Alves, Existence of solution for a degenerate $p(x)$-Laplacian equation in $\mathbb{R}^N$, J. Math. Anal. Appl. 345 (2008) 731–742.

[6] G. Bognár, O. Došlý, The application of Picone-type identity for some nonlinear elliptic differential equations, Acta Math. Univ. Comenian. 72 (2003) 45–57.

[7] G. Dai, Infinitely many solutions for a $p(x)$-Laplacian equation in $\mathbb{R}^N$, Nonlinear Anal. 71 (2009) 1133–1139.

[8] G. Dai, Infinitely many non-negative solutions for a Dirichlet problem involving $p(x)$-Laplacian, Nonlinear Anal. 71 (2009) 5840–5849.

[9] O. Došlý, The Picone identity for a class of partial differential equations, Math. Bohem. 127 (2002) 581–589.

[10] O. Došlý, Half-linear Differential Equations, Handbook of Differential Equations: Ordinary Differential Equations, Volume 1, Elsevier B. V., Amsterdam, 2004.

[11] O. Došlý and P. Řehák, Half-linear Differential Equations, North-Holland Mathematics Studies, 202, Elsevier Science B.V., Amsterdam, 2005.

[12] D. R. Dunninger, A Sturm comparison theorem for some degenerate quasilinear elliptic operators, Boll. Un. Mat. Ital. A (7) 9 (1995), 117–121.

[13] X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, Nonlinear Anal. 52 (2003) 1843–1852.

[14] Y. Fu and X. Zhang, A multiplicity result for $p(x)$-Laplacian problem in $\mathbb{R}^N$, Nonlinear Anal. 70 (2009) 2261–2269.

[15] M. Galewski, On the existence and stability of solutions for Dirichlet problem with $p(x)$-Laplacian, J. Math. Anal. Appl. 326 (2007) 352–362.

[16] M. Galewski, On a Dirichlet problem with $p(x)$-Laplacian, J. Math. Anal. Appl. 337 (2008) 281–291.

[17] G. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Second Edition, Cambridge University Press, 1988.

[18] J. Jaroš, T. Kusano and N. Yoshida, Picone-type inequalities for half-linear elliptic equations and their applications, Adv. Math. Sci. Appl. 12 (2002), 709–724.

[19] T. Kusano, J. Jaroš and N. Yoshida, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order, Nonlinear Anal. 40 (2000) 381–395.

[20] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.

[21] N. Yoshida, Oscillation criteria for half-linear partial differential equations via Picone's identity, Proceedings of Equadiff 11 (Bratislava, 2005), pp. 589–598, Bratislava, 2007.

[22] N. Yoshida, Oscillation Theory of Partial Differential Equations, World Scientific Publishing Co. Pte. Ltd., 2008.

[23] N. Yoshida, Sturmian comparison and oscillation theorems for a class of half-linear elliptic equations, Nonlinear Anal. 71 (2009) e1354–e1359.

[24] N. Yoshida, A Picone identity for half-linear elliptic equations and its applications to oscillation theory, Nonlinear Anal. 71 (2009) 4935–4951.

[25] Q. H. Zhang, Existence of radial solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, J. Math. Anal. Appl. 315 (2006) 506–516.

[26] Q. H. Zhang, Oscillatory property of solutions for $p(x)$-Laplacian equations, J. Inequal. Appl. 2007 (2007), Article ID 58548, 8 pages.

[27] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR Izv. 29 (1987) 33–66.