GROUPS OF KNOTS ON TEMPLATES

KAZUHIRO SASANO

1. Introduction

A template is a kind of branched surface with boundary equipped with smooth expansive semiflow. Birman and Williams [BW1] [BW2] showed that, if a hyperbolic chain-recurrent set of a flow on a three-manifold is given, there exists a template embedded in the manifold such that the link of periodic orbits of the flow is in bijective correspondence with the link of periodic orbits on the template. Furthermore on any finite sublink, this correspondence is via ambient isotopy. This implies that, to investigate knot types of periodic orbits of the flow, it is sufficient to investigate those of periodic orbits on the corresponding template.

In this paper we calculate groups of knots on some kinds of templates in the three dimensional sphere to show the complexity of periodic orbits of flows.

In §2 we prepare some basic notations and definitions about templates. In §3 we give some notations about the Lorenz knots, that is, the knots obtained as periodic orbits of the Lorenz template. Furthermore we calculate some fundamental group which will be necessary for our result. In §4 we state our main theorem and give its proof.

2. Templates and skeletons

In [BW2], Birman and Williams proved the existence of knot holders for hyperbolic chain recurrent sets of flows on three dimensional manifolds. Later knot holders were renamed templates. Thus we will call them templates in this paper.

Definition 2.1. A *template* is a compact branched surface with boundary in a three manifold build locally from two types of charts: *joining* and *splitting*. Each chart, as illustrated in Figure 1, carries a semiflow, endowing the template with an expanding semiflow, and the gluing maps between charts must respect the semiflow and act linearly on the edges.

Theorem 2.2. [BW2] *Given a flow ϕ_t on a three dimensional manifold M having a hyperbolic chain recurrent set, the link L_ϕ of periodic orbits in the chain recurrent set is in bijective correspondence with the link of periodic orbits L_T on a particular embedded template $T \subset M$ (with L_T containing at most two extraneous orbits). On any finite sublink, this correspondence is via ambient isotopy.*

In this paper, we will consider flows and templates in three dimensional sphere S^3. The simplest sort of templates is that build from one joining chart and one splitting chart. For example, the Lorenz template [BW1] is obtained from the Lorenz attractor,

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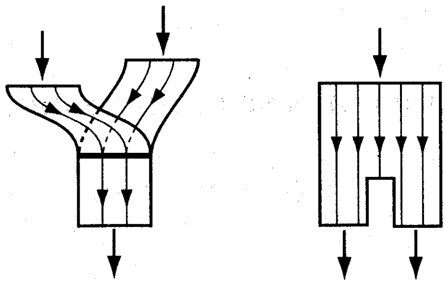
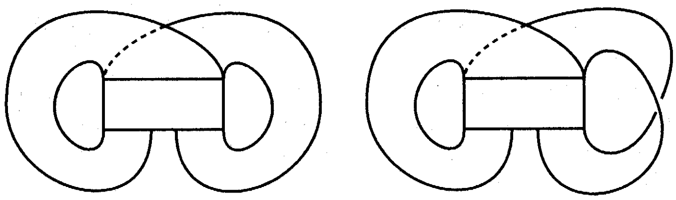


(a) (b)

FIGURE 1. (a) a joining chart; (b) a splitting chart.

and the horseshoe template [BW2] is obtained from the index one chain recurrent set of the suspension flow of Smale's horseshoe map. See Figure 2.



(a) (b)

FIGURE 2. (a) the Lorenz template; (b) the horseshoe template.

These templates are thought to be made of three parts: one rectangle, say the *base rectangle*, and two *arms* starting from the bottom edge of the base rectangle and arriving at its top edge. Note that no arm is "knotted" in these templates. The difference is whether the right hand arm is twisted or not. We generalize them by making arms knotted and twisted. To describe our generalization, we need some definitions.

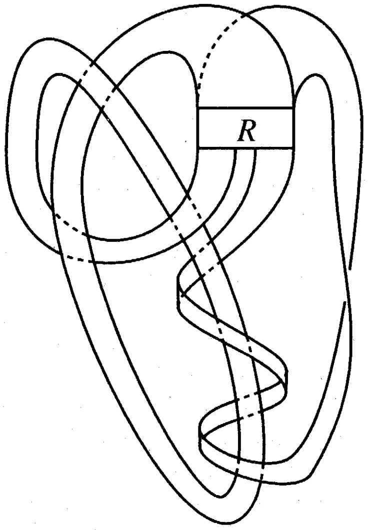
Definition 2.3. A template is called to be *with two arms* if it consists of one joining chart and one splitting chart, like the Lorenz template and the horseshoe template.

Definition 2.4. A template is called to be *orientable* if it is orientable as a branched surface. That is, if a direction transverse to the semiflow is always preserved along the semiflow.

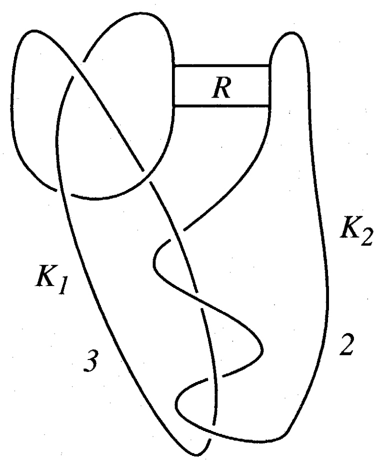
For example, the Lorenz template is orientable, but the horseshoe template is not.

Definition 2.5. Let T be an orientable template with two arms. It is considered to be made of one base rectangle R and two arms; the *left arm* which start from left half of the bottom edge of R, and the *right arm*. The orbit of the semiflow through the left edge (resp. right edge) of its base rectangle R will be a knot in S^3, say K_1 (resp. K_2). Let t_1 (resp. t_2) be the linking number of a closed curve on the left (resp. right) arm parallel to K_1 (resp. K_2) with K_1 (resp. K_2). Let Skel(T) be the figure in S^3 consisting of the base rectangle R and two knots K_1 and K_2, accompanied with two numbers t_1 and t_2. We call Skel(T) the *skeleton* of T.

Example 2.6. Figure 3 shows an example of a template and its skeleton. In this example, K_1 is a trefoil knot and K_2 is a trivial knot, and they are linked. Numbers "3" and "2" written near knots indicate t_1 and t_2 respectively.



(a)



(b)

FIGURE 3. (a) a template with two arms, and (b) its skeleton.

Remark 2.7. We always assume that K_1 and K_2 are oriented by the direction of the semiflow. We also assume that the total space S^3 is oriented by left-hand system.

Proposition 2.8. *Suppose we are given a figure in S^3, consisting of one rectangle R and a two-component link $K_1 \cup K_2$ such that $K_1 \cap R =$ "left edge of R" and $K_2 \cap R =$ "right edge of R", with two numbers t_1 and t_2 attached to K_1 and K_2 respectively. Then we can make an orientable template such that its skeleton is the given figure.*

Definition 2.9. The template as in Proposition 2.8 is denoted by Temp$(K_1 \cup K_2; t_1, t_2)$.

Proof of Proposition 2.8. Replace a curve $K_1 \setminus R$ by a narrow band B_1 such that its long edges consist of K_1 and a curve parallel to K_1 in a Seifert surface of K_1, and one of its short edges are glued to the top edge of R and the other to the bottom edge of R. Then expand linearly the short edge of B_1 attached to the top edge of R to the same length as the top edge of R. Finally cut R and B_1 along the top edge of R and twist B_1 t_1-times. Apply similar operations to K_2. Then glue B_1 to the top edge of R from the front, and glue B_2 to the top edge of R from the rear. Then we obtain the required template Temp$(K_1 \cup K_2; t_1, t_2)$. □

Although templates generally admit linking of K_1 and K_2, we need to restrict our attention to a special case for our main result.

Definition 2.10. A template with two arms is called *splittable* if there exists a three ball B^3 in S^3 such that K_1 is included in the interior of B^3, and K_2 is included in the interior of $S^3 \setminus B^3$, and $\partial B^3 \cap R =$ a "vertial line" of R.

For example, the Lorenz template and the horseshoe template are splittable, but the template shown in Figure 3 is not splittable since K_1 and K_2 are linked.

3. The Lorenz template

For the investigation of templates like $\mathrm{Temp}(K_1 \cup K_2; t_1, t_2)$, the most fundamental object is the Lorenz template. In this section we consider about the Lorenz template more precisely.

In this section, let T_0 be the Lorenz template. T_0 is thought to be $\mathrm{Temp}(K_1 \cup K_2; 0, 0)$, where $K_1 \cup K_2$ is the unlink. From the aspect of dynamical systems, we need to remember how T_0 is obtained in [BW1].

First of all we choose a small neighborhood of the Lorenz attractor in S^3, which has a local product structure by local stable manifolds and local unstable manifolds. Then we obtain the Lorenz template T_0 by shrinking each local stable manifold to a point. Conversely, when T_0 is given, we can re-construct a neighborhood as above by expanding T_0 to the direction normal to T_0 in S^3. We write this neighborhood as $N(T_0)$. Figure 4 shows $N(T_0)$ for the Lorenz template placed in $\mathbb{R}^3$. We introduce a coordinate system such that the cube at the center is $[-1,1]\times[-1,1]\times[-1,1]$ and the square B shadowed in Figure 4 at the center is $[-1,1] \times [-1,1] \times \{0\}$. The y-direction is the stable direction of Lorenz flow, and by shrinking each component of the intersection of $N(T_0)$ and a y-directional line to a point, we obtain the Lorenz template. The square $[-1,1] \times [-1,1] \times \{1\}$ corresponds to the top edge of R, and $[-1,1] \times [-1,1] \times \{-1\}$ corresponds to the bottom edge, and $B = [-1,1] \times [-1,1] \times \{0\}$ corresponds to one of the glue lines of the joining chart and the splitting chart. We may assume that the neighborhood of the left arm is glued to the bottom face of the center cube at $[-1,-1/3] \times [-1,1] \times \{-1\}$, and the neighborhood of the right arm is glued at $[1/3,1] \times [-1,1] \times \{-1\}$. Note that this coordinate is not the same as that in the original Lorenz equation.

Let K_0 be a periodic orbit of the semiflow. By applying an ambient isotopy if necessary, we may regard K_0 as an embedded circle in $N(T_0)$ which intersects B on the segment $[-1,1] \times \{0\} \times \{0\}$. Assume that $K_0 \cap B$ consists of n points. We name them by numbers $\{1, 2, \cdots, n\}$, and arrange them on the segment $[-1,1] \times \{0\} \times \{0\}$ from left to right. The orbit K_0 is divided into n segments by these points. Then, as in [BW1], K_0 is described by a permutation π on $\{1, 2, \cdots, n\}$ as follows: When the orbit segment starting from a point i meets B at a point j, we define $\pi(i) = j$. There exists a unique number p such that the orbit segment starting from i ($i = 1, 2, \cdots, p$) runs on the left arm, and the orbit segment starting from j ($j = p+1, \cdots, n$) runs on the right arm. Note that π enjoys a property : $\pi(1) < \pi(2) < \cdots < \pi(p) > \pi(n) > \pi(n-1) > \cdots > \pi(p+1)$. This property implies that $\pi(p) = n$, $\pi(p+1) = 1$, and that the set $\{\pi(1), \cdots, \pi(p)\}$ completely determines the permutation π. For $j = p+1, \cdots, n$, let $\rho_j = \min\{i = 1, 2, \cdots, p \mid \pi(i) > \pi(j)\}$. Note that $\pi(j) < \pi(\rho_j) < \pi(j+1)$, and if $\rho_j \geq 2$, $\pi(\rho_j - 1) < \pi(j) < \pi(\rho_j)$.

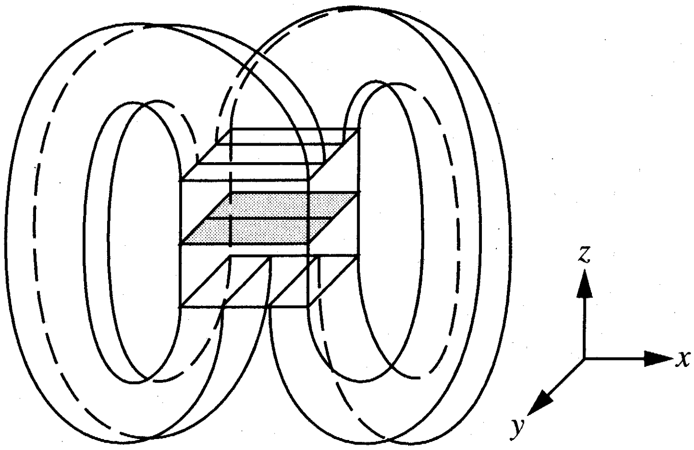


FIGURE 4

Remark 3.1. In general, if T is an orientable template with two arms, then any orbit is given by a permutation π which enjoys the same properties as above. But if T is not orientable, although any orbit corresponds to a permutation, its properties are different from those in orientable case.

In [BW1], the fundamental group of $S^3 \setminus K_0$ was calculated using these information. For our purpose their result is not sufficient and we have to calculate the fundamental groups of $N(T_0) \setminus K_0$ and its outer boundary.

To calculate them, we fix a base point $b_0 = (0, 1, 0)$ on B. We choose n loops on B and 4 loops on $\partial N(T_0)$ as follows.

Definition 3.2. For $i = 1, 2, \cdots, n$, let x_i be a loop on B which starts from b_0, goes around the point i clockwise, and returns to b_0. See Figure 5. Let m_1, m_2, l_1, l_2 be loops on $\partial N(T_0)$ as in Figure 6. That is, m_1 is a loop which starts from b_0, passes through $(-1, 1, 0)$, $(-1, -1, -1)$, $(-1/3, -1, -1)$, $(-1/3, 1, -1)$, and returns to b_0. m_2 starts from b_0, passes through $(1/3, 1, -1)$, $(1/3, -1, -1)$, $(1, -1, -1)$, $(1, 1, 0)$, and returns to b_0. On the other hand, l_1 and l_2 are loops which go along the arms on the "front face" of $\partial N(T_0)$.

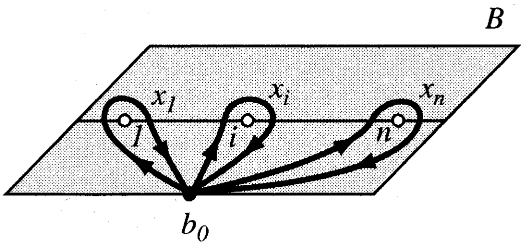


FIGURE 5

By identifying these loops as elements of fundamental groups with the base point b_0, we have:

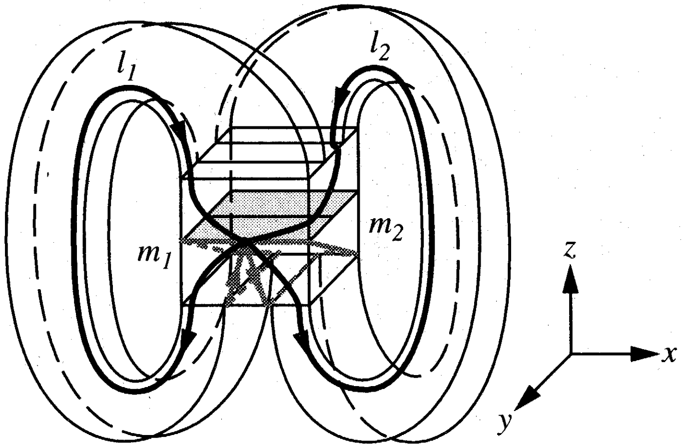


FIGURE 6

Theorem 3.3. *For the Lorenz template T_0 and a periodic orbit K_0 on T_0 given by a permutation π on $\{1, 2, \cdots, n\}$,*

(1) *$\pi_1(N(T_0) \setminus K_0)$ has a presentation as follows:*

Generators: $x_1, x_2, \cdots, x_n, l_1, l_2$.

Relations: $l_1^{-1} x_i l_1 = x_{\pi(i)}$ $(i = 1, 2, \cdots, p)$,

$$l_2^{-1} x_j l_2 = x_{\pi(p)}^{-1} x_{\pi(p-1)}^{-1} \cdots x_{\pi(\rho_j)}^{-1} x_{\pi(j)} x_{\pi(\rho_j)} \cdots x_{\pi(p-1)} x_{\pi(p)} \quad (j = p+1, \cdots, n).$$

(2) *$\pi_1(\partial N(T_0))$ is generated by l_1, l_2, m_1, m_2 with a relation*

$$m_1^{-1} l_1^{-1} m_1 l_1 = m_2 l_2^{-1} m_2^{-1} l_2.$$

(3) *In $\pi_1(N(T_0) \setminus K_0)$, $m_1 = x_1 x_2 \cdots x_p$ and $m_2 = x_{p+1} \cdots x_n$.*

Proof. (1) It is obvious that $\pi_1(N(T_0) \setminus K_0)$ is generated by $x_1, x_2, \cdots, x_n, l_1, l_2$. To obtain the relations among them, it is sufficient to know how the "push-forward" of each x_i along the orbit segment is represented by x_i's. Note that the "push-forward" is given by $l_1^{-1} x_i l_1$ if $i \leq p$ and $l_2^{-1} x_j l_2$ if $j > p$. For $i = 1, 2, \cdots, p$, the "push-forward" of x_i is just equal to $x_{\pi(i)}$ since there is no knot segment in front of the segment from x_i to $x_{\pi(i)}$. This gives the first p relations. For $j = p+1, \cdots, n$, the "push-forward" of x_j is a loop on B as in Figure 7, and it is represented as $x_{\pi(p)}^{-1} x_{\pi(p-1)}^{-1} \cdots x_{\pi(\rho_j)}^{-1} x_{\pi(j)} x_{\pi(\rho_j)} \cdots x_{\pi(p-1)} x_{\pi(p)}$. This completes the proof of (1).

(2) $\partial N(T_0)$ is a closed surface of genus 2, and loops m_1, m_2, l_1, l_2 form a system of generators, which is standard except for their orientations. By taking account of orientation, the result is obvious.

(3) is immediate from definitions. □

4. GROUPS OF KNOTS ON SPLITTABLE AND ORIENTABLE TEMPLATES

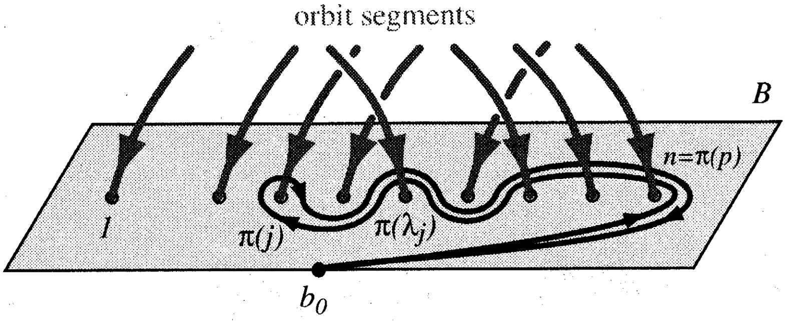


FIGURE 7

WITH TWO ARMS.

Periodic orbits of the semiflow on a template are considered as knots. In this section we state and prove our main result about groups of these knots on splittable and orientable templates with two arms.

Theorem 4.1. *Let $T = Temp(K_1 \cup K_2; t_1, t_2)$ be an splittable and orientable template with two arms. Let G_i be the knot group of K_i; that is, $G_i = \pi_1(S^3 - K_i)$ $(i = 1, 2)$, and suppose that G_i is presented as*

Generators: $\lambda_i, \mu_i, y_i(j)$ $(j = 1, 2, \cdots, n_i)$,

Relations: $R_i(k)$ $(k = 1, 2, \cdots, r_i)$,

where λ_i is the longitude of K_i and μ_i is the meridian of K_i, and $R_i(k)$ is a relation in λ_i, μ_i, and $y_i(j)$'s.

Let K be the knot of a periodic orbit on T given by a permutation π on $\{1, 2, \cdots, n\}$. Then $\pi_1(S^3 \setminus K)$ is presented as follows:

Generators: $x_1, x_2, \cdots, x_n, l_1, l_2, m_1, m_2,$
$\lambda_i, \mu_i, y_i(j)$ $(i = 1, 2;\ j = 1, 2, \cdots, n_i)$.

Relations:
$$l_1^{-1} x_i l_1 = x_{\pi(i)} \quad (i = 1, 2, \cdots, p),$$
$$l_2^{-1} x_j l_2 = x_{\pi(p)}^{-1} x_{\pi(p-1)}^{-1} \cdots x_{\pi(\rho_j)}^{-1} x_{\pi(j)} x_{\pi(\rho_j)} \cdots x_{\pi(p-1)} x_{\pi(p)} \qquad (j = p+1, \cdots, n),$$
$$m_1 = x_1 x_2 \cdots x_p,\ m_2 = x_{p+1} \cdots x_n,$$
$$m_1^{-1} l_1^{-1} m_1 l_1 = m_2 l_2^{-1} m_2^{-1} l_2,$$
$$\lambda_1^{-1} \mu_1 \lambda_1 \mu_1^{-1} = \lambda_2^{-1} \mu_2^{-1} \lambda_2 \mu_2,$$
$$l_i = \mu_i^{-t_i} \lambda_i,\ m_i = \mu_i \quad (i = 1, 2),$$
$$R_i(k) \ (i = 1, 2;\ k = 1, 2, \cdots, r_i).$$

Proof. Let K_0 be a Lorenz knot given by π on the Lorenz template T_0. Then $S^3 \setminus K$ is regarded as $(N(T_0) \setminus K_0) \cup_\phi (S^3 \setminus N(T))$. They are glued by a homeomorphism $\phi : \Sigma_2 = \partial N(T_0) \to \Sigma_2 = \partial(S^3 \setminus N(T))$, which respects t_1 and t_2 as twisting. Since T is splittable, $S^3 \setminus N(T)$ is thought to be $M_1 \cup M_2$, where $M_i = B^3 \setminus (N(K_i) \cup P_i)$ and $P_i \cong D^2 \times D^1$ is a neighborhood of left or right half of the base rectangle R which looks like a "plug" connecting ∂B^3 and $\partial N(K_i)$ $(i = 1, 2)$ as in Figure 8.

We first have to calculate $\pi_1(M_i)$, and then $\pi_1(S^3 \setminus N(T))$.

Lemma 4.2. $\pi_1(M_i) = \pi_1(S^3 \setminus N(K_i))$ $(i = 1, 2)$

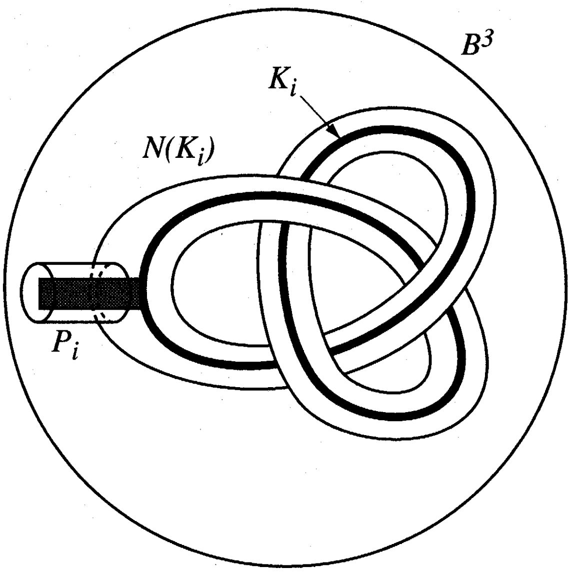


FIGURE 8

Proof. Since $M_i \cup P_i = B^3 \setminus N(K_i)$ and $M_i \cap P_i \cong S^1 \times D^2$ and the generator of $\pi_1(S^1 \times D^1)$ is null homotopic in M_i, we obtain $\pi_1(M_i) = \pi_1(B^3 \setminus N(K_i))$ by van Kampen's theorem. $\pi_1(B^3 \setminus N(K_i))$ is apparently equal to $\pi_1(S^3 \setminus N(K_i))$.

$\square$

Lemma 4.3.

(1) $\pi_1(S^3 - N(T))$ *is given by:*
Generators: $\lambda_i, \mu_i, y_i(j)$ $(i = 1, 2;\ j = 1, 2, \cdots, n_i)$,
Relations: $R_i(k)$ $(i = 1, 2;\ k = 1, 2, \cdots, r_i)$.

(2) $\pi_1(\partial(S^3 \setminus N(T)))$ *is generated by* λ_1, λ_2, μ_1, μ_2 *with a relation* $\lambda_1^{-1}\mu_1\lambda_1\mu_1^{-1} = \lambda_2^{-1}\mu_2^{-1}\lambda_2\mu_2$.

Proof. Since $S^3 \setminus N(T) = M_1 \cup_{\partial B^3 \setminus D^2} M_2$ and $\partial B^3 \setminus D^2$ is contractible, van Kampen's theorem implies (1). (2) is easy because $\partial(S^3 \setminus N(T))$ is a surface of genus 2 obtained by glueing $\partial N(K_1) \setminus D^2$ and $\partial N(K_2) \setminus D^2$ along a circle representing $\lambda_1^{-1}\mu_1\lambda_1\mu_1^{-1} = \lambda_2^{-1}\mu_2^{-1}\lambda_2\mu_2$. This completes the proof. $\square$

Proof of Theorem 4.1 (continued). Since $S^3 \setminus K = (N(T_0) \setminus K_0) \cup_\phi (S^3 \setminus N(T))$, and both of $\pi_1(N(T_0) \setminus K_0)$ and $\pi_1(S^3 \setminus N(T))$ are already calculated, our result is easily obtained by van Kampen's theorem because the glueing map ϕ satisfies $\phi_\#(l_i) = \mu_i^{-t_i}\lambda_i$ and $\phi_\#(m_i) = \mu_i$ $(i = 1, 2)$, where $\phi_\#$ is the map induced by ϕ on the fundamental groups. $\square$

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DEPARTMENT OF MATHEMATICS, TOYAMA MEDICAL AND PHARMACEUTICAL UNIVERSITY, TOYAMA, 930-0194 JAPAN

E-mail address: ksasano@toyama-mpu.ac.jp