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Hamilton circuits of Cayley graphs of Weyl groupoids of generalized quantum groups

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Abstract. We study Hamilton circuits of the Cayley graphs of the Weyl groupoids of the generalized quantum groups, or the quantum double of the Nichols algebras of diagonal-type, with finite root systems. We prove the existence of a Hamilton circuit for any of them and explicitly draw one of them for rank 3 and 4 cases.

1. Introduction

For a finite graph $X = (Y, Z)$, where Y and Z are the sets of vertices and edges of X respectively, *a Hamilton circuit of* X means a bijection $h : \{1, 2, \ldots, |Y|\} \to Y$ such that

$$\{\{h(i), h(i+1)\}\,(1 \leq i \leq |Y| - 1),\ \{h(|Y|), h(1)\}\} \subset Z,$$

where $|Y|$ means the cardinal number of Y.

For a finite Coxeter system $W = (W, S)$, where W is a finite Coxeter group, *the Cayley graph of* W *with the generator system* S means the graph whose sets of vertices and edges are W and $\{\{w, sw\} | w \in W, s \in S\}$ respectively.

The Weyl groupoids [15], [8], [14, Definition 9.1.8] are a groupoid version of Coxeter systems.

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This paper studies Hamilton circuits of the Cayley graphs of the Weyl groupoids associated with the generalized root systems, or the quantum doubles of the finite-type Nichols algebras of diagonal type, over a field of characteristic zero. The importance of the Weyl groupoids for study of Hopf algebras and Lie superalgebras is consistently written in a recent book [14]. In particular, they play an important role in study of classification of Nichols algebras. Since the quantum doubles of the Nichols algebras of diagonal type form a generalization of the quantum groups, we call them *the generalized quantum groups*.

Let $\mathbb{K}$ be a field, and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$. Let $\mathbb{Z}\Pi$ be a finite-rank free $\mathbb{Z}$-module with a basis Π. Let $\chi : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \mathbb{K}^\times$ be the map satisfying

$$\chi(\lambda+\mu,\nu) = \chi(\lambda,\nu)\chi(\mu,\nu),\ \chi(\lambda,\mu+\nu) = \chi(\lambda,\mu)\chi(\lambda,\nu) \quad (\lambda,\mu,\nu \in \mathbb{Z}\Pi).$$

The Nichols algebra $U^+(\chi)$ of diagonal type has a Kharchenko's PBW-basis described by the generalized root system $R(\chi) = R^+(\chi) \cup (-R^+(\chi))$. I. Heckenberger [12] classified χ with $|R(\chi)| < \infty$ for $\mathbb{K}$ of characteristic zero. I. Heckenberger and the author [15] introduced an axiomatic definition of the generalized root systems, which include $R(\chi)$'s. M. Cuntz and I. Heckenbeger improved the axiomatic definition by [8] and completely classified the finite Weyl groupoids by [10]. The result of [10] tells that there are so many finite Weyl groupoids other than those associated with the Nichols algebras of diagonal type. In this paper, we concentrate on studying a Hamilton circuit of the Cayley graph of the Weyl groupoid of the Nichols algebras $U^+(\chi)$ of diagonal type with $|R(\chi)| < \infty$ over a field $\mathbb{K}$ of characteristic zero.

On the other hand, J.H. Conway, N.J.A. Sloane and Allan R. Wilks [7] proved the existence of a Hamilton circuit for the Cayley graph of every finite Coxeter group in a very concise way. Therefore it is natural to ask whether the Cayley graph of a finite Weyl groupoid has a Hamilton circuit. In general, to determine whether a Hamiton path or a Hamilton circuit exists for a given graph is one of the major themes in graph theory, and called the Hamiltonian path problem. For basic terminologies, see [11]. One important open problem is whether every Cayley graph contains a Hamilton path or a Hamilton circuit, see [19].

We also mention that the author with his collaborators achieved results [2], [3], [4], [5], [6], [16], [17], [22], [23], [24] concerning representation theory of the generalized quantum groups $U(\chi)$ defined as the quantum double of $U^+(\chi)$.

Since there are so many finite Weyl groupoids [10] (see also [9, ($|A|$ of) Table 1 and Table 2]) other than those of the generalized quantum groups, it would be interesting if we are able to obtain Hamilton circuits for them using computers (if any). At the present moment, this problem[1] looks very tough, especially for rank ≥ 4 cases.

This paper is organized as follows. Section 1 is Introduction. In Section 2, we recall the argument of [7] in a detailed way since we use arguments similar to it. Let χ be as above. In Section 3, we recall the positive part $U^+(\chi)$ of the generalized quantum group $U(\chi)$ (see [16]), the generalized root system $R(\chi)$ of $U^+(\chi)$ and the Weyl groupoid $\mathcal{W}(\chi)$ of $U^+(\chi)$, define the Cayley graph $\Gamma(\chi)$ of $\mathcal{W}(\chi)$, and give some technical lemmas and propositions used in Sections 4-6. In Sections 4-6, we assume that $|R(\chi)| < \infty$ and $R(\chi)$ is irreducible (see Subsection 3.1 for the term *irreducible*). By [15, Lemma 8(iii)], we see that $|R(\chi)| < \infty \Leftrightarrow |\mathcal{W}(\chi)| < \infty$. In Section 4, we exactly draw $\Gamma(\chi)$ with one of its Hamilton circuits for χ of rank 3. In Section 5, we exactly give Hamilton circuits (shown by symbols s_i^a's) of $\Gamma(\chi)$ for χ of rank 4. In Section 6, we prove the existence of a Hamilton circuit of $\Gamma(\chi)$ for χ whose rank is more than or equal to 5. In Appendix, we recall names for $U^+(\chi)$ given by [1].

Notation 1.1. (1) *For $n \in \mathbb{N}$, we let $\hat{\pi} : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ mean the canonical group epimorphism, i.e., $\hat{\pi}_n(t) := t + n\mathbb{Z}$ ($t \in \mathbb{Z}$).*
(2) *For x, $y \in \mathbb{R} \cup \{\infty, -\infty\}$, let $J_{x,y} := \{k \in \mathbb{Z} | x \leq k \leq y\}$.*

2. Hamilton circuits of Cayley graphs of finite Coxeter Groups

We begin with recalling the definition of a Cayley graph.

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[1]After this paper was submitted, the author noticed that for (most of) rank-3 cases, this problem can be solved using a command of Mathematica [21] concerning Hamilton circuits. However the Hamilton circuits for rank-3 cases we draw in Section 4 of this paper are necessary in order to use them in Sections 5 and 6.

Definition 2.1. *In general, a Cayley graph is defined as follows. Let G be a group with a unit e. Let S be a subset of G such that $e \notin S$, $\{x^{-1}|x \in S\} = S$ and $S = \{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} | n \in \mathbb{N}, x_i \in S, k_i \in \mathbb{Z}_{\geq 0} \, (i \in J_{1,n})\}$. Let $\Gamma(G,S)$ be the graph (V,E) such that the set V of vertices is identified with G and the set E of edges is formed by $\{g,h\}$ with $h^{-1}g \in S$. We call $\Gamma(G,S)$ the Cayley graph of G and S. For example, if G is the symmetric group $\mathfrak{S}(3)$ of degree 3 and $S = \{(12),(123),(132)\}$, then $\Gamma(G,S)$ is given by the figure below.*

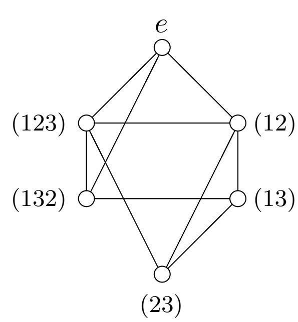


Fig.0: The Caylay graph of S_3 with generators $S = \{(12),(123),(132)\}$

In this section we recall the main ideas of the proof given in [7] of the existence of a Hamilton circuit for the Cayley graph of the finite Coxeter group. An adapted argument will be used later for finite Coxeter groupoids.

Recall that a Coxeter system consists of the following data:

(a) A non-empty finite set S.
(b) A map $\Psi = \Psi^{(W,S)} : S \times S \to \mathbb{N}$ such that

$$\Psi(s,s) = 1, \;\; \Psi(s,t) = \Psi(t,s) \geq 2, \quad s,t \in S, \, s \neq t.$$

(c) A group W defined by the generators $s \in S$ and relations

$$(st)^{\Psi(s,t)} = e, \quad s,t \in S,$$

Let (W,S) be a finite Coxeter system, i.e., the following (Cx1)-(Cx4) are fulfilled.

(Cx1) S is a non-empty finite set.
(Cx2) There exists a map $\Psi^{(W,S)} : S \times S \to \mathbb{N}$ such that $\Psi^{(W,S)}(s,s) = 1$ $(s \in S)$, $\Psi^{(W,S)}(s',s'') = \Psi^{(W,S)}(s'',s') \geq 2$ $(s', s'' \in S,\ s' \neq s'')$.
(Cx3) W is the group defined by the generators $s \in S$ and

$$\text{the relations } (s's'')^{\Psi^{(W,S)}(s',s'')} = e \text{ for all } s',\ s'' \in S.$$

where e means the unit element of W.
(Cx4) $|W| < \infty$.

Let X be an $|S|$-dimensional $\mathbb{R}$-linear space with a basis $\{x_s | s \in S\}$. Define the symmetric bilinear map $(\ ,\ ) : X \times X \to \mathbb{R}$ by

$$(x_{s'}, x_{s''}) := -2\cos\frac{\pi}{\Psi^{(W,S)}(s',s'')} \quad (s', s'' \in S).$$

Then we have a group monomorphism $f : W \to \mathrm{GL}(X)$ defined by $f(s)(y) := y - (y, x_s)x_s$ $(s \in S, y \in X)$. Hence

$$\forall s, \forall s' \in S, \Psi^{(W,S)}(s,s') = \min\{m \in \mathbb{N} | (ss')^m = e\}.$$

In this section, we explain the idea of [7] to obtain a Hamilton circuit $\mathcal{H}_{W,S}$ of the Cayley graph $\mathcal{C}_{W,S}$ of (W,S). Here $\mathcal{C}_{W,S}$ is the graph composed of the vertices bijectively corresponding to the elements of W and the edges labeled by $s \in S$ connecting two vertices corresponding to x, $y \in W$ with $y = sx$. Let $k := |W|$. If there exists a bijection $\theta : \mathbb{Z}/k\mathbb{Z} \to W$ such that $\theta(\hat{\pi}_k(t+1))\theta(\hat{\pi}_k(t))^{-1} \in S$ for all $t \in \mathbb{Z}$, then *the Hamilton circuit* $\mathcal{H}_{W,S}$ *of* $\mathcal{C}_{W,S}$ *defined by* θ is the subgraph of $\mathcal{C}_{W,S}$ composed of all vertices of $\mathcal{C}_{W,S}$ and the k-edges each of which connects two vertices labeled by $\theta(\hat{\pi}_k(t))$ and $\theta(\hat{\pi}_k(t+1))$ for some $t \in \mathbb{Z}$, and θ is called *a Hamilton circuit map of* $\mathcal{C}_{W,S}$.

We clearly have the following lemma.

Lemma 2.2. *Assume $|S| = 2$. Let $\{s_1, s_2\} := S$. Then a Hamilton circuit $\mathcal{H}_{W,S}$ uniquely exists. More precisely we have the following. Let $m := \Psi^{(W,S)}(s_1, s_2)$. Then $|W| = 2m$. Define the bijection $\theta : \mathbb{Z}/2m\mathbb{Z} \to W$ by $\theta(\hat{\pi}_k(0)) := e$, and $\theta(\hat{\pi}_k(2t+1)) := s_1\theta(\hat{\pi}_k(2t))$, $\theta(\hat{\pi}_k(2t+2)) := s_2\theta(\hat{\pi}_k(2t+1))$ $(t \in \mathbb{Z})$. Then $\mathcal{H}_{W,S}$ defined by θ coincides with $\mathcal{C}_{W,S}$.*

We shall mostly use the argument of the proof of the following theorem.

Theorem 2.3. ([7, Theorem]) *A Hamilton circuit $\mathcal{H}_{W,S}$ exists for any finite Coxeter system (W,S).*

Proof. Let $n := |S|$. If $n = 1$, the claim is obvious. If $n = 2$, the claim follows from Lemma 2.2.

Assume $n \geq 3$. Then we can easily see from the classification of the irreducible finite Coxeter systems that there exists a bijection $\hat{s} : J_{1,n} \to S$ fulfilling that

$$\forall i \in J_{1,n-2},\ \hat{s}(i)\hat{s}(n) = \hat{s}(n)\hat{s}(i). \tag{2.1}$$

Let $\hat{S} := \{\hat{s}(i) | i \in J_{1,n-1}\}$. Let $\hat{W}$ be the subgroup of W generated by $\hat{S}$. Note that $(\hat{W}, \hat{S})$ is a Coxeter system. Let $h := |W|$, $\hat{h} := |\hat{W}|$, and $p := \frac{h}{\hat{h}} (\in J_{2,\infty})$. Then there exists $w_u \in W$ $(u \in J_{1,p})$ such that $w_1 = e$, $W = \cup_{x=1}^{p} \hat{W} w_x$ and $\hat{W} w_x \cap \hat{W} w_{x'} = \emptyset$ $(x \neq x')$, i.e., $W = \amalg_{x=1}^{p} \hat{W} w_x$ is the right coset decomposition of W by $\hat{W}$.

By induction, we have the Hamilton circuit $\mathcal{H}_{\hat{W},\hat{S}}$ of $\mathcal{C}_{\hat{W},\hat{S}}$ defined by a bijection $\hat{\theta} : \mathbb{Z}/\hat{h}\mathbb{Z} \to \hat{W}$. Since the elements of a non-empty proper subset of $\hat{S}$ can not generate $\hat{W}$, we see that

$$\forall k \in J_{1,n-1},\ \exists m_k \in \mathbb{Z},\ \hat{\theta}(\hat{\pi}_{\hat{h}}(m_k + 1)) = \hat{s}(k)\hat{\theta}(\hat{\pi}_{\hat{h}}(m_k)). \tag{2.2}$$

For $u \in \hat{W}$ and $k \in J_{1,n-1}$, fixing m_k of (2.2), define the bijection $\hat{\theta}_{u,k} : \mathbb{Z}/\hat{h}\mathbb{Z} \to \hat{W}$ by

$$\hat{\theta}_{u,k}(\hat{\pi}_{\hat{h}}(t)) := \hat{\theta}(\hat{\pi}_{\hat{h}}(m_k + t))\hat{\theta}(\hat{\pi}_{\hat{h}}(m_k))^{-1}u \quad (t \in \mathbb{Z}),$$

and define the surjection $\hat{\varphi}_{u,k} : \mathbb{Z}/\hat{h}\mathbb{Z} \to J_{1,n-1}$ by

$$(\hat{s} \circ \hat{\varphi}_{u,k})(\hat{\pi}_{\hat{h}}(t)) := \hat{\theta}(\hat{\pi}_{\hat{h}}(m_k + t))\hat{\theta}(\hat{\pi}_{\hat{h}}(m_k + t - 1))^{-1} \quad (t \in \mathbb{Z}).$$

Then we have the Hamilton circuit $\mathcal{H}^{u,k}_{\hat{W},\hat{S}}$ of $\mathcal{C}_{\hat{W},\hat{S}}$ defined by $\hat{\theta}_{u,k}$. Note that

$$\hat{\theta}_{u,k}(\hat{\pi}_{\hat{h}}(0)) = u \text{ and } \hat{\theta}_{u,k}(\hat{\pi}_{\hat{h}}(1)) = \hat{s}(k)u = (\hat{s} \circ \hat{\varphi}_{u,k})(\hat{\pi}_{\hat{h}}(1))u. \tag{2.3}$$

We shall construct a Hamilton circuit in an inductive way. Let $l \in J_{1,p}$. We shall show by induction on l that there exist an injection $g_l : J_{1,l} \to J_{1,p}$

and a bijection $\dot{\theta}_l : \mathbb{Z}/l\hat{h}\mathbb{Z} \to \dot{W}_l$, where $\dot{W}_l := \amalg_{z=1}^{l}\hat{W}w_{g_l(z)}$, such that $\dot{\theta}_l(\hat{\pi}_{l\hat{h}}(t+1))\dot{\theta}_l(\hat{\pi}_{l\hat{h}}(t))^{-1} \in S$ for all $t \in \mathbb{Z}$. Recall $p\hat{h} = |W|$. If $l = 1$, we may let $g_l(1) := 1$ and $\dot{\theta}_1 := \hat{\theta}$. Let $l \in J_{2,p}$, and assume that the hypothesis of the induction for $l-1$ in place of l holds. Define the surjection $\varphi_{l-1} : \mathbb{Z}/(l-1)\hat{h}\mathbb{Z} \to J_{1,n-1}$ by

$$\hat{s}(\varphi_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(t)))\dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(t-1)) = \dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(t)) \quad (t \in \mathbb{Z}).$$

Since W is a finite group generated by S, there exist $b_l \in J_{1,(l-1)\hat{h}}$ and $c \in J_{1,p} \setminus g_{l-1}(J_{1,l-1})$ such that

$$\hat{s}(n)\dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l-1)) \in \hat{W}w_c.$$

If $\varphi_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l)) = n-1$, then $\varphi_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l-1)) \in J_{1,n-2}$, which implies $\hat{s}(n)\dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l-2)) \in \hat{W}w_c$ by (2.1). Hence we may assume $\varphi_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l)) \in J_{1,n-2}$. Let $r := \varphi_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l))\,(\in J_{1,n-2})$. Let $v := \hat{s}(n)\dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l-1))w_c^{-1} \in \hat{W}$. Define the map $g_l : J_{1,l} \to J_{1,p}$ by by $(g_l)_{|J_{1,l-1}} := g_{l-1}$ and $g_l(l) := c$. Define the map $\dot{\theta}_l : \mathbb{Z}/l\mathbb{Z} \to \dot{W}_l(:= \amalg_{z=1}^{l}\hat{W}w_{g_l(z)})$ by

$$\dot{\theta}_l(\hat{\pi}_{l\hat{h}}(t)) := \begin{cases} \dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(t)) & \text{if } t \in J_{1,b_l-1}, \\ vw_c\,(= \hat{s}(n)\dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l-1))) & \text{if } t = b_l, \\ \hat{\theta}_{\hat{s}(r)v,r}(\hat{\pi}_{\hat{h}}(t+1-b_l))w_c & \text{if } t \in J_{b_l+1,b_l+\hat{h}-1}, \\ \dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(t-\hat{h})) & \text{if } t \in J_{b_l+\hat{h},l\hat{h}}, \end{cases}$$

for $t \in J_{1,l\hat{h}}$. Let $u := \hat{s}(r)v\,(\in \hat{W}w_c)$. For $t \in J_{b_l+1,b_l+\hat{h}-1}$, we have

$$\begin{aligned} &\dot{\theta}_l(\hat{\pi}_{l\hat{h}}(t)) \\ &\quad = \hat{\theta}_{u,r}(\hat{\pi}_{\hat{h}}(t+1-b_l))w_c \\ &\quad = \underbrace{\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(t+1-b_l)))\cdots\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(2)))\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(1)))}_{t+1-b_l}\cdot uw_c \\ &\quad = \underbrace{\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(t+1-b_l)))\cdots\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(1)))}_{t-b_l+1}\cdot\hat{s}(r)vw_c \\ &\quad = \underbrace{\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(t+1-b_l)))\cdots\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(3)))\hat{s}(\hat{\varphi}_{u,r}(\hat{\pi}_{\hat{h}}(2)))}_{t-b_l}\cdot vw_c. \end{aligned}$$

We also have

$$\begin{aligned}
&\hat{s}(n)\dot{\theta}_l(\hat{\pi}_{l\hat{h}}(b_l+\hat{h}-1)) \\
&\quad = \hat{s}(n)\hat{\theta}_{\hat{s}(r)v,r}(\hat{\pi}_{\hat{h}}(\hat{h}))w_c \\
&\quad = \hat{s}(n)\hat{\theta}_{\hat{s}(r)v,r}(\hat{\pi}_{\hat{h}}(0))w_c \\
&\quad = \hat{s}(n)\hat{s}(r)vw_c \quad \text{(by (2.3))} \\
&\quad = \hat{s}(n)\hat{s}(r)\hat{s}(n)\dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l-1)) \\
&\quad = \hat{s}(r)\dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l-1)) \\
&\quad = \dot{\theta}_{l-1}(\hat{\pi}_{(l-1)\hat{h}}(b_l)) \\
&\quad = \dot{\theta}_l(\hat{\pi}_{l\hat{h}}(b_l+\hat{h})).
\end{aligned}$$

Then g_l and $\dot{\theta}_l$ are the desired ones, where see also Fig.1. □

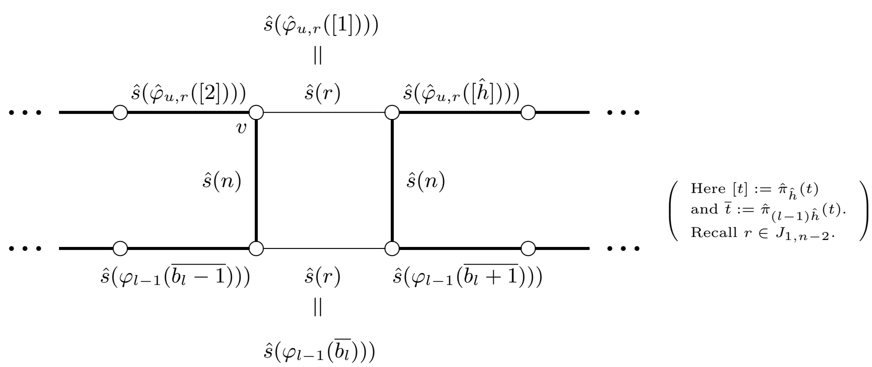


Fig.1: Joint to get a Hamilton circuit in Proof of Theorem 2.3

Example 2.4. Here let (W,S) be a Coxeter system of type B_n ($n \in J_{2,\infty}$). It means that W is the group defined by generators s_i ($i \in J_{1,n}$) and relations $s_i^2 = e$, $s_is_j = s_js_i$ ($|i-j| \geq 2$), $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ ($i \in J_{2,n-1}$), and $s_1s_2s_1s_2 = s_2s_1s_2s_1$, where we let $S = \{s_i | i \in J_{1,n}\}$. It is well-known that $|W| = 2^n \cdot n!$. If $n = 2$, a Hamilton circuit $\mathcal{H}_{W,S}$ is $s_1s_2s_1s_2s_1s_2s_1s_2$ by Lemma 2.2. If $n = 3$, a Hamilton circuit $\mathcal{H}_{W,S}$ is drawn Fig.2 below, where we use the argument of Proof of Theorem 2.3 for $m_{k_l} = 0$, $\hat{s}(k_l) = s_1$ ($l \in J_{2,6}$) and $b_2 = 7$, $b_3 = 13$, $b_4 = 17$, $b_5 = 21$, $b_6 = 23$. If $n = 4$, see Fig.4

below.

$$s_1s_2s_1s_2s_1s_2(s_3s_2s_1s_2s_1s_2$$
$$\cdot(s_3s_2s_1s_2(s_3s_2s_1s_2(s_3s_2(s_3s_2s_1s_2s_1s_2s_1s_2s_3)s_2s_1s_2s_1s_2$$
$$\cdot s_3)s_2s_1s_2s_3)s_2s_1s_2s_3)s_2s_3)s_2,$$

Fig.2: Hamilton circuit of the Weyl group of type B_3.
See also Fig.3 below.

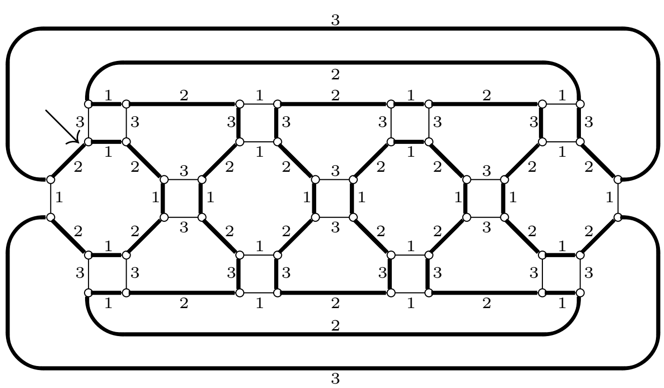


Fig.3: The (2-convenient (see Definition 3.10)) Hamilton circuit Fig.2 of the Weyl group of type B_3,
where ↘ shows the initial and end point of Fig.2

$s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}$

$\cdot s_2^{\overline{0}}s_3^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_1^{\overline{0}}s_2^{\overline{0}}s_4^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}$

$\cdot s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_4^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}$

$\cdot s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_4^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}$

$\cdot s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_4^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}$

$\cdot s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_3^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_4^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}$

$\cdot s_2^{\overline{3}}s_3^{\overline{3}}s_4^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}$

$\cdot s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_3^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_2^{\overline{5}}s_1^{\overline{5}}s_4^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_4^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}$

$\cdot s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_3^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_2^{\overline{2}}s_1^{\overline{2}}s_4^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_3^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_2^{\overline{3}}s_1^{\overline{3}}s_4^{\overline{6}}s_3^{\overline{6}}$

$\cdot s_2^{\overline{6}}s_1^{\overline{6}}s_2^{\overline{6}}s_1^{\overline{6}}s_4^{\overline{7}}s_3^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_2^{\overline{7}}s_1^{\overline{7}}s_4^{\overline{4}}s_3^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_2^{\overline{4}}s_1^{\overline{4}}s_4^{\overline{1}}s_3^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_1^{\overline{1}}s_2^{\overline{1}}s_4^{\overline{0}}$

Fig.4: This is a Hamilton circuit of Weyl group of type B_4 with $s_i^2=e$, $(s_1s_2)^4=(s_2s_3)^3=(s_3s_4)^3=e$, $(s_is_j)^2=2$ $(|i-j|\geq 2)$.
(Length $=384$) This is obtained by the argument of Proof of Theorem 2.3 for Fig.2 (that is Fig.3) and $\hat{W}:=\langle s_1,s_2,s_3\rangle$.
Here $s_{i_{r+1}}^{\overline{k(r)}}s_{i_r}^{\overline{k(r-1)}}\cdots s_{i_2}^{\overline{k(1)}}s_{i_1}^{\overline{k(0)}}$ means $s_{i_r}\cdots s_{i_2}s_{i_1}\in\hat{W}_{k(r)}$
$(i_1=4,i_2=2,i_3=1,\ldots,k(0)=0,k(1)=1,k(2)=1,\ldots)$,
where $\hat{W}_0:=\hat{W}$, $\hat{W}_1:=\hat{W}_0s_4$, $\hat{W}_2:=\hat{W}_1s_3$, $\hat{W}_3:=\hat{W}_2s_2$, $\hat{W}_4:=\hat{W}_3s_1$, $\hat{W}_5:=\hat{W}_4s_2$, $\hat{W}_6:=\hat{W}_5s_3$ and $\hat{W}_7:=\hat{W}_5s_4$. Notice $W=\amalg_{z=0}^7\hat{W}_z$.
See also Fig.5 below.

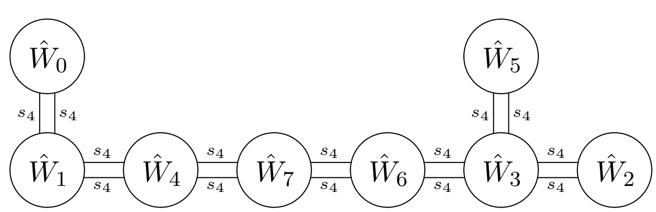


Fig.5: Visual explanation to obtain the Hamilton circuit of Fig.4 of the Weyl group of type B_4

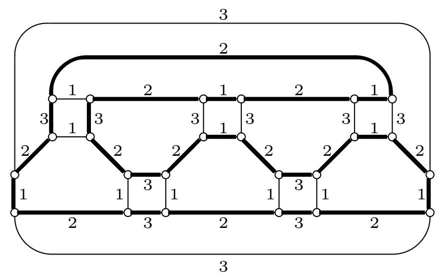


Fig.6: The Caylay graph and a (2-convenient (see Definition 3.10)) Hamilton circuit of the Weyl group of type-A_3

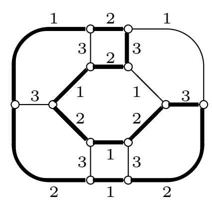


Fig.7: The Caylay graph and a (2-convenient (see Definition 3.10)) Hamilton circuit of the Weyl group of type-$A_2 \times A_1$

$s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_1s_4s_3s_2s_3s_2s_3s_4$

$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2$

$\cdot s_3s_4s_3s_4s_1s_4s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_1s_4s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2$

$\cdot s_3s_4s_1s_4s_3s_2s_3s_2s_3s_4$

$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_1s_2s_3s_2s_3s_4s_3s_4s_3s_2s_1s_2s_3s_2$

$\cdot s_3s_4s_3s_4s_3s_2s_1s_2s_1s_4$

Fig.8: Hamilton circuit of Weyl group of type A_4 with $s_i^2 = e$, $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, $s_is_{i+2} = s_{i+2}s_i$ and $s_1s_4 = s_4s_1$ (Length = 120)

$s_2s_3s_2s_3s_2s_4s_2s_4s_2s_3s_2s_4s_2s_3s_2s_3s_2s_4s_2s_4s_2s_3s_2s_1s_2s_3s_2s_3s_2s_4$
$\cdot s_2s_4s_2s_3s_2s_4s_2s_3s_2s_3s_2s_4s_2s_4s_2s_1s_2s_4s_2s_3$
$\cdot s_2s_3s_2s_4s_2s_4s_2s_3s_2s_4s_2s_3s_2s_3s_2s_1s_2s_3s_2s_3s_2s_4s_2s_4s_2s_3s_2s_4s_2s_3$
$\cdot s_2s_3s_2s_4s_2s_1s_2s_3s_2s_3s_2s_4s_2s_4s_2s_3s_2s_4s_2s_3$
$\cdot s_2s_3s_2s_1s_2s_3s_2s_3s_2s_4s_2s_4s_2s_3s_2s_4s_2s_3s_2s_3s_2s_4s_2s_1s_2s_3s_2s_3s_2s_4$
$\cdot s_2s_4s_2s_1s_2s_4s_2s_3s_2s_3s_2s_4s_2s_4s_2s_3s_2s_4s_2s_3$
$\cdot s_2s_3s_2s_4s_2s_4s_2s_1s_2s_4s_2s_3s_2s_3s_2s_4s_2s_4s_2s_3s_2s_1s_2s_3s_2s_1s_2s_4s_2s_3$
$\cdot s_2s_1s_2s_3s_2s_1s_2s_4s_2s_1$
$\cdot s_2s_1$

Fig.9: Hamilton circuit of Weyl group of type D_4 with $s_i^2 = e$, $s_is_js_i = s_js_is_j$ $((i,j) \in \{(1,2),(2,3),(2,4)\})$ and $s_is_j = s_js_i$ $((i,j) \in \{(1,3),(1,4),(3,4)\})$ (Length $= 192$)

$s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_4$
$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_1s_2s_3s_4s_3s_4s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4$
$\cdot s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2$

Fig.10′: S' of Fig.10

$S' \cdot s_3s_4s_3s_2s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_1s_2s_3s_4$
$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2$
$\cdot s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_1s_2s_3s_4$
$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_1s_2$
$\cdot s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2$
$\cdot s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4$
$\cdot s_1s_2s_3s_4s_3s_4s_1s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2$
$\cdot s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_1s_2s_3s_4s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_3s_4s_1s_2$
$\cdot s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2$
$\cdot s_1s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2s_3s_4s_3s_4s_1s_2s_3s_4s_3s_4$
$\cdot s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_3s_2$
$\cdot s_3s_4s_3s_4s_3s_2s_3s_4s_3s_2s_3s_4s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_4s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_1s_2s_3s_2s_3s_4$
$\cdot s_3s_2s_3s_2s_1s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_1s_2s_3s_2$
$\cdot s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_1s_2s_3s_2s_1s_2s_3s_4s_3s_2$
$\cdot s_3s_2s_1s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_1s_2s_3s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_1s_2$
$\cdot s_3s_2s_1s_2s_3s_4s_3s_2s_3s_2s_1s_2s_3s_2s_1s_2s_3s_2s_1s_2$
$\cdot s_1s_2$

Fig.10: Hamilton circuit of Weyl group of type F_4 with $s_i^2 = e$, $(s_1s_2)^3 = (s_2s_3)^4 = (s_3s_4)^3 = e$ and $(s_is_j)^2 = 2$ $(|i-j| \geq 2)$ (Length = 1152)

3. Weyl groupoids associated with Nichols algebras of diagonal type

3.1. Basic of Nichols algebras of diagonal-type

Let $\mathbb{K}$ be a field of characteristic zero, and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$. For $x, y \in \mathbb{K}$ and $k \in \mathbb{Z}_{\geq 0}$, let $(k)_x := \sum_{r=1}^{k} x^{r-1}$, $(k)_x! := \prod_{r=1}^{k} (r)_x$, $(k; x, y) := 1 - x^{k-1}y$ and $(k; x, y)! := \prod_{r=1}^{k} (r; x, y)$. Let I be a non-empty finite set. Let $\mathbb{Z}\Pi (= \oplus_{i \in I} \mathbb{Z}\alpha_i)$ be a free $\mathbb{Z}$-module with a basis $\Pi := \{\alpha_i | i \in I\}$, where $|\Pi| = |I|$. Let $\mathbb{Z}_{\geq 0}\Pi := \oplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ $(\subset \mathbb{Z}\Pi)$. Let $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)$ be the group formed by $\mathbb{Z}$-module automorphism of $\mathbb{Z}\Pi$.

Let $\mathcal{X}_\Pi$ be the set formed by all maps $\chi' : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \mathbb{K}^\times$ with

$$\begin{aligned}&\chi'(\lambda, \mu+\nu) = \chi'(\lambda,\mu)\chi'(\lambda,\nu),\ \chi'(\lambda+\mu,\nu) = \chi'(\lambda,\nu)\chi'(\mu,\nu) \\ &(\lambda,\mu,\nu \in \mathbb{Z}\Pi).\end{aligned} \tag{3.1}$$

We call an element of $\mathcal{X}_\Pi$ *a bi-character associated with* I.

Let $\chi \in \mathcal{X}_\Pi$. We call χ *reducible* if there exists a non-empty proper subset I' of I such that $\chi(\alpha_i,\alpha_j)\chi(\alpha_j,\alpha_i) = 1$ for every $(i,j) \in I' \times (I \setminus I')$. If this is not the case, we call χ *irreducible*.

In this subsection, let $\chi \in \mathcal{X}_\Pi$, and let $q_{ij} := \chi(\alpha_i,\alpha_j)$ for $i,\ j \in I$.

Let $U^+(\chi)$ be the Nichols algebra of diagonal-type associated with χ, that is to say, $U^+(\chi)$ is the unital associative $\mathbb{K}$-algebra with the generators E_i $(i \in I)$ satisfying the conditions (Nic1)-(Nic2) below.

(Nic1) There exist subspaces $U^+(\chi)_\lambda$ $(\lambda \in \mathbb{Z}_{\geq 0}\Pi)$ such that

$$\begin{aligned}&U^+(\chi) = \oplus_{\lambda\in\mathbb{Z}_{\geq 0}\Pi} U^+(\chi)_\lambda \text{ (as a } \mathbb{K}\text{-linear space)}, \\ &U^+(\chi)_\lambda U^+(\chi)_\mu \subset U^+(\chi)_{\lambda+\mu}\ (\lambda,\mu \in \mathbb{Z}_{\geq 0}\Pi), \\ &\mathbb{K}1 = U^+(\chi)_0,\ \dim U^+(\chi)_0 = 1,\ \mathbb{K}E_i = U^+(\chi)_{\alpha_i},\ \dim U^+(\chi)_{\alpha_i} = 1\ (i \in I).\end{aligned}$$

(Nic2) There exist $\mathbb{K}$-linear maps $\partial_i^L : U^+(\chi) \to U^+(\chi)$ $(i \in I)$ such that

- $\partial_i^L(1) = 0$,
- $\forall i, j \in I, \forall\lambda \in \mathbb{Z}_{\geq 0}\Pi, \forall X \in U^+(\chi)_\lambda$,
  $\partial_i^L(E_jX) = \delta_{i,j}X + q_{ji}E_j\partial_i^L(X)$,
- $\cap_{i\in I} \ker \partial_i^L = U^+(\chi)_0$.

Namely $U^+(\chi)$ is *the Nichols algebra of diagonal-type associated with* χ, see [13, Proposition 5.4], [16, Section 3] (see also [20, CHAPTER 38]).

Theorem 3.1. (*Kharchenko PBW Theorem* [18]) (*We emphasize that* χ *be anyone satisfying* (3.1).) *There exists a unique pair* $(R^+(\chi), \varphi_+^\chi)$ *of a subset* $R^+(\chi)$ *of* $\mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$ *and a map* $\varphi_+^\chi : R^+(\chi) \to \mathbb{N}$ *satisfying the following condition* $(*)$, *where notice* $\Pi \subset R^+(\chi) \subset \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$.

$(*)$ *There exist* $E_{\beta,t} \in U^+(\chi)_\beta$ $(\beta \in R^+(\chi), t \in J_{1,\varphi_+^\chi(\beta)})$ *and a total order* $\preceq$ *on* $X := \{(\beta,t)|\beta \in R^+(\chi), t \in J_{1,\varphi_+^\chi(\beta)}\}(\subset R^+(\chi) \times \mathbb{N})$ *such that for* *every* $\lambda \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$*, a* $\mathbb{K}$*-basis of* $U^+(\chi)_\lambda$ *is formed by the elements* $E^{k_1}_{\beta_1,t_1} E^{k_2}_{\beta_2,t_2} \cdots E^{k_m}_{\beta_2,t_m}$ *with* $m \in \mathbb{N}$*,* $(\beta_1, t_1) \prec (\beta_2, t_2) \prec \cdots \prec (\beta_m, t_m)$*,* $(k_t)_{\chi(\beta_t,\beta_t)}! \neq 0$ $(t \in J_{1,m})$ *and* $\sum_{t=1}^m k_t\beta_t = \lambda$*.*

Definition 3.2. *(1) Let* $i, j \in I$ *with* $i \neq j$*. For* $k \in \mathbb{Z}_{\geq 0}$*, let* $E^{\chi,+}_{0;i,j} := E^{\chi,-}_{0;i,j} := E_i$ *(if* $k = 0$*) and* $E^{\chi,+}_{k;i,j} := E_i E^{\chi,+}_{k-1;i,j} - q^{k-1}_{ii} q_{ij} E^{\chi,+}_{k-1;i,j} E_i$*,* $E^{\chi,-}_{k;i,j} := E^{\chi,-}_{k-1;i,j} E_i - q^{k-1}_{ii} q_{ji} E_i E^{\chi,-}_{k-1;i,j}$ *(if* $k \geq 1$*). Let* $N^\chi_{ij} := |\{k \in \mathbb{N} | E^{\chi,+}_{k;i,j} \neq 0\}|$ $(\in \mathbb{Z}_{\geq 0} \cup \{\infty\})$*.*
(2) For $i \in I$ *and* $\epsilon \in \{+, -\}$*, let* $U^+(\chi)[\epsilon; i]$ *be the non-unital* $\mathbb{K}$*-subalgebra of* $\oplus_{\lambda \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}} U^+(\chi)_\lambda$ *generated by* $E^{\chi,\epsilon}_{k;i,j}$ *with* $j \in I \setminus \{i\}$ *and* $k \in \mathbb{Z}_{\geq 0}$*.*
(3) Let $U^+(\chi)[\epsilon; i]_\lambda := U^+(\chi)_\lambda \cap U^+(\chi)[\epsilon; i]$ *for* $\lambda \in \mathbb{Z}_{\geq 0}\Pi$*, where notice* $U^+(\chi)[\epsilon; i]_{k\alpha_i+\alpha_j} = \mathbb{K}E^{\chi,\epsilon}_{k;i,j}$ $(\epsilon \in \{+, -\}, \{i, j\} \subset I, i \neq j, k \in \mathbb{Z}_{\geq 0})$*.*

Lemma 3.3. *Let* i*,* $j \in I$ *with* $i \neq j$*. Let* $k \in \mathbb{Z}_{\geq 0}$*. Then*

$$(k)_{q_{ii}}!(k; q_{ii}, q_{ij}q_{ji})! = 0 \Leftrightarrow E^{\chi,+}_{k;i,j} = 0 \Leftrightarrow E^{\chi,-}_{k;i,j} = 0 \Leftrightarrow k\alpha_i + \alpha_j \in R^+(\chi).$$

Proof. Use [2, Lemma 4.9 (1)] , and also use Ω and Υ of [2]. □

Lemma 3.4. ([13, Proposition 5.10], [16, Lemma 3.6 (ii)]) *Let* $i \in I$*. Let* $Y := \{y \in \mathbb{Z}_{\geq 0} | (y)_{q_{ii}}! \neq 0\}$*, and let* Z *be the* $|Y|$*-dimensional* $\mathbb{K}$*-linear space with a basis* $\{z_y | y \in Y\}$*. Then we have:*

(1) *We have* $Y = \{y \in \mathbb{Z}_{\geq 0} | E^y_i \neq 0\}$*.*
(2) *We have* $\ker \partial^L_i = U^+(\chi)[-; i]$*.*
(3) *For* $\epsilon \in \{+, -\}$*, we have the* $\mathbb{K}$*-linear isomorphism:*

$$m : Z \otimes U^+(\chi)[\epsilon; i] \to U^+(\chi) \text{ defined by}$$
$$m(z_y \otimes x) := E^y_i x \, (y \in Y, x \in U^+(\chi)[\epsilon; i]).$$

Definition 3.5. *Let* $i \in I$*. Assume* $N^\chi_{ij} < \infty$ *for all* $j \in I \setminus \{i\}$*. Define* $s^\chi_i \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)$ *by* $s^\chi_i(\alpha_i) := -\alpha_i$ *and* $s^\chi_i(\alpha_j) := \alpha_j + N^\chi_{ij}\alpha_i$ $(j \in I \setminus \{i\})$*. Define* $\tau_i\chi \in \mathcal{X}_\Pi$ *by* $\tau_i\chi(\lambda, \mu) := \chi(s^\chi_i(\lambda), s^\chi_i(\mu))$ $(\lambda, \mu \in \mathbb{Z}\Pi)$*.*

Lemma 3.6. *Let* $i \in I$*. Assume* $N^\chi_{ij} < \infty$ *for all* $j \in I \setminus \{i\}$*. Then we have the following.*

(1) *We have* $N^{\tau_i\chi}_{ij} = N^\chi_{ij}$ *for all* $j \in I \setminus \{i\}$*. In particular,* $s^{\tau_i\chi}_i = s^\chi_i = (s^\chi_i)^{-1}$

and $\tau_i\tau_i\chi=\chi$.

(2) *There exists a unique* (*non-unital*) $\mathbb{K}$*-algebra isomorphism*

$$T_i^\chi : U^+(\chi)[-;i]\to U^+(\tau_i\chi)[+;i]$$

such that

$$T_i^\chi(E^{\chi,-}_{k;i,j}) \\ = \acute{q}_{ji}^{-k} q_{ii}^{\frac{k(k-1-2N^\chi_{ij})}{2}} (\textstyle\prod_{r=0}^k (N^\chi_{ij}-r+1)_{q_{ii}}(N^\chi_{ij}-r+1;q_{ii},\acute{q}_{ij}\acute{q}_{ji}))E^{\tau_i\chi,+}_{N^\chi_{ij}-k;i,j}$$

for $j\in I\setminus\{i\}$ *and* $k\in J_{0,N^\chi_{ij}}$, *where* $\acute{q}_{xy}:=\tau_i\chi(\alpha_x,\alpha_y)$ $(x,y\in I)$.

By Theorem 3.1 and Lemmas 3.4 (3) and 3.6 (2), we have:

Proposition 3.7. *Let* $i\in I$. *Assume* $N^\chi_{ij}<\infty$ *for all* $j\in I\setminus\{i\}$. *Then we have*

$$s_i^\chi(R^+(\chi)\setminus\{\alpha_i\})=R^+(\tau_i\chi)\setminus\{\alpha_i\}$$
$$\text{and } (\varphi_+^{\tau_i\chi}\circ s_i^\chi)_{|R^+(\chi)\setminus\{\alpha_i\}}=\varphi^\chi_{+|R^+(\chi)\setminus\{\alpha_i\}}.$$

3.2. Cayley graph associated with χ satisfying $|R^+(\chi)|<\infty$

In this subsection, let $\chi\in\mathcal{X}_\Pi$, and assume $|R^+(\chi)|<\infty$. For a finite sequence $i_1,i_2,\ldots,i_k$ in I, let $\chi_{(i_k,\ldots,i_2,i_1)}:=\tau_{i_k}\cdots\tau_{i_2}\tau_{i_1}\chi\,(\in\mathcal{X}_\Pi)$, and let $s^\chi_{(i_k,\cdots,i_2,i_1)}:=s^{\chi_k}_{i_k}\cdots s^{\chi_2}_{i_2}s^{\chi_1}_{i_1}\,(\in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi))$, where $\chi_1:=\chi$ and $\chi_t:=\chi_{(i_{t-1},\ldots i_2,i_1)}$ $(t\in J_{2,k})$. Let $\chi_{()}:=\chi$ and $s^\chi_{()}:=\mathrm{id}_{\mathbb{Z}\Pi}$. If $k=0$, let $\chi_{(i_k,\cdots,i_2,i_1)}$ and $s^\chi_{(i_k,\cdots,i_2,i_1)}$ mean $\chi_{()}$ and $s^\chi_{()}$ respectively.

Definition 3.8. *(Cayley graph associated with* χ*) Let*

$$\begin{aligned}\mathcal{V}(\chi) &:= \{s^\chi_{(i_k,\cdots,i_2,i_1)}\in\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)\,|\,k\in\mathbb{Z}_{\geq0},\,i_t\in I\,(t\in J_{1,k})\},\\ \mathcal{G}(\chi) &:= \{\chi_{(i_k,\ldots,i_2,i_1)}\in\mathcal{X}_\Pi\,|\,k\in\mathbb{Z}_{\geq0},\,i_t\in I\,(t\in J_{1,k})\}.\end{aligned}$$

Let $\mathfrak{p}_2(\chi)$ *be the subset of the power set* $\mathfrak{P}(\mathcal{V}(\chi))$ *of* $\mathcal{V}(\chi)$ *formed by all the sets having two different elements of* $\mathcal{V}(\chi)$. *Let*

$$\mathcal{E}(\chi):=\{\{s^\chi_{(i_k,\ldots,i_2,i_1)},s^\chi_{(i_{k+1},i_k,\ldots,i_2,i_1)}\}\in\mathfrak{p}_2(\chi)\,|\,k\in\mathbb{Z}_{\geq0},\,i_t\in I\,(t\in J_{1,k+1})\}.$$

For $\chi',\,\chi''\in\mathcal{G}(\chi)$, *we write* $\chi'\equiv\chi''$ *if*

$$\chi'(\alpha_i,\alpha_i)=\chi''(\alpha_i,\alpha_i) \text{ and } \chi'(\alpha_i,\alpha_j)\chi'(\alpha_j,\alpha_i)=\chi''(\alpha_i,\alpha_j)\chi''(\alpha_j,\alpha_i)$$

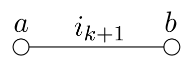
for all i, $j \in I$. Clearly $\equiv$ is an equivalent relation on $\mathcal{G}(\chi)$. Then for χ', $\chi'' \in \mathcal{G}(\chi)$, we have

$$\chi' \equiv \chi'' \;\Rightarrow\; R^+(\chi') = R^+(\chi''), s_i^{\chi'} = s_i^{\chi''}, \tau_i\chi' \equiv \tau_i\chi'' \tag{3.2}$$

(see [2, (4.28), Lemma 4.22] for example). Let $\bar{\mathcal{G}}(\chi)$ be the quotient set of $\mathcal{G}(\chi)$ by $\equiv$. Let $\pi_\chi : \mathcal{G}(\chi) \to \bar{\mathcal{G}}(\chi)$ be the canonical projection map, i.e.,

$$\pi_\chi(\chi') := \{\chi'' \in \mathcal{G}(\chi) | \chi'' \equiv \chi'\} \quad (\chi' \in \mathcal{G}(\chi)).$$

Let $\Gamma(\chi)$ be the graph $(\mathcal{V}(\chi), \mathcal{E}(\chi))$, see also $(\star)$ below. When we draw the graph $\Gamma(\chi)$ in a visible way, we do it in the following way. The subgraph of $\Gamma(\chi)$ formed by the two vertices $s^\chi_{(i_k,\dots,i_2,i_1)}$, $s^\chi_{(i_{k+1},i_k,\dots,i_2,i_1)} \in \mathcal{V}(\chi)$ and the edge $\{s^\chi_{(i_k,\dots,i_2,i_1)}, s^\chi_{(i_{k+1},i_k,\dots,i_2,i_1)}\} \in \mathcal{E}(\chi)$ is written below, where $a := \pi_\chi(\chi_{(i_k,\dots,i_2,i_1)})$ and $b := \pi_\chi(\chi_{(i_{k+1},i_k,\dots,i_2,i_1)})$. Since $s_i^{\tau_i\chi} = s_i^\chi = (s_i^\chi)^{-1}$ by Lemma 3.6 (1), $\Gamma(\chi)$ is an undirected graph, i.e., we do not need to consider a direction to any edge of $\Gamma(\chi)$. We call $\Gamma(\chi)$ the Cayley graph of $\mathcal{W}(\chi)$, where the Weyl groupoid $\mathcal{W}(\chi)$ will be defined by Definition 3.12.



$(\star)$ *In general, we do not care (resp. do care) that each vertex of the graph $\Gamma(\chi)$ exactly corresponds to an element of $\mathcal{V}(\chi)$ (resp. is assigned by an element of $\bar{\mathcal{G}}(\chi)$).*

Although a definition of a Cayley graphs of a groupoid does not seem to often appear in literatures, we may claim that it is similar to that of Definition 2.1 and that $\Gamma(\chi)$ satisfies the definition.

For $i \in I$, define the map $\tau_i : \mathcal{G}(\chi) \to \mathcal{G}(\chi)$ by $\tau_i(\chi') := \tau_i\chi'$ $(\chi' \in \mathcal{G}(\chi))$.

Definition 3.9. *Let $\chi' \in \mathcal{G}(\chi)$, $a := \pi_\chi(\chi') \in \bar{\mathcal{G}}(\chi)$, $R^+(a) := R^+(\chi')$ and $R(a) := R^+(a) \cup (-R^+(a))$. Let $s_i^a := s_i^{\chi'}$ $(i \in I)$, and define $|I| \times |I|$-matrix $C^a = (c^a_{ij})_{i,j\in I}$ over $\mathbb{Z}$ by $c^a_{ii} := -2$ and $c^a_{ij} := -N^{\chi'}_{ij}$ $(i \neq j)$. For $i \in I$, define the map $\bar{\tau}_i : \bar{\mathcal{G}}(\chi) \to \bar{\mathcal{G}}(\chi)$ by $\bar{\tau}_i(\pi_\chi(\chi')) = \pi_\chi(\tau_i\chi')$ $(\chi' \in \mathcal{G}(\chi))$. For $a \in \bar{\mathcal{G}}(\chi)$ and $k \in \mathbb{Z}_{\geq 0}$, $i_t \in I$ $(t \in J_{1,k})$, define $a_{(i_k,\dots,i_2,i_1)} \in \bar{\mathcal{G}}(\chi)$ (resp. $s^a_{(i_k,\dots,i_2,i_1)} \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)$) in the same way as those for $\chi_{(i_k,\dots,i_2,i_1)}$ (resp. $s^\chi_{(i_k,\dots,i_2,i_1)}$) with a and $\bar{\tau}_i$ in place of χ and τ_i.*

(1) *For all* $a \in \bar{\mathcal{G}}(\chi)$ *and all* $i,\, j \in I$*, we have* $\bar{\tau}_i^2 = \mathrm{id}_{\bar{\mathcal{G}}}(\chi)$ *and* $c_{ij}^{\bar{\tau}_i(a)} = c_{ij}^a$*. Namely* $\mathcal{C}(\chi) = \mathcal{C}(I, \bar{\mathcal{G}}(\chi), (\bar{\tau}_i)_{i\in I}, (C^a)_{a\in\bar{\mathcal{G}}(\chi)})$ *is a Cartan scheme, see* [8, Definition 2.1], [16, Definition 2.1] *and* [2, §2.1] *for this term.*
(2) *We have the following* (R1)-(R4)*. Namely* $\mathcal{R}(\chi) = \mathcal{R}(\mathcal{C}(\chi), (R(a))_{a\in\bar{\mathcal{G}}(\chi)})$ *is a generalized root system of type* $\mathcal{C}(\chi)$*, see* [8, Definition 2.2], [16, Definition 2.3] *and* [2, Definition 1.2, Lemma 1.5, Remark 1.6] *for this term.*

(R1) $R(a) = R^+(a) \cup (-R^+(a))$ $(a \in \bar{\mathcal{G}}(\chi))$.
(R2) $R(a) \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ $(a \in \bar{\mathcal{G}}(\chi), i \in I)$.
(R3) $s_i^a(R(a)) = R(\bar{\tau}_i(a))$ $(a \in \bar{\mathcal{G}}(\chi), i \in I)$.
(R4) *For* $a \in \bar{\mathcal{G}}(\chi)$*,* $k \in \mathbb{N}$*,* $i_t \in I$ $(t \in J_{1,k})$*, if* $s^a_{(i_k,\ldots,i_2,i_1)} = \mathrm{id}_{\mathbb{Z}\Pi}$*, then* $a_{(i_1,i_2\ldots,i_k)} = a$*. (By [2, Remark 1.6], this is equivalent to the condition that for* $a \in \bar{\mathcal{G}}(\chi)$ *and* $i, j \in I$ *with* $i \neq j$*, letting* $m_{ij}^a := |R^+(a) \cap (\mathbb{Z}_{\geq 0}\alpha_i \oplus \mathbb{Z}_{\geq 0}\alpha_j)|$*, we have*

$$a_{\underbrace{(\ldots,i,j,i)}_{m_{ij}^a}} = a_{\underbrace{(\ldots,j,i,j)}_{m_{ij}^a}} \text{ and } s^a_{\underbrace{(\ldots,i,j,i)}_{m_{ij}^a}} = s^a_{\underbrace{(\ldots,j,i,j)}_{m_{ij}^a}}.)$$

Definition 3.10. *Let* $k := |\mathcal{V}(\chi)|\,(\in \mathbb{N})$*. Let* $a := \pi_\chi(\chi) \in \bar{\mathcal{G}}(\chi)$*.*

(1) We say that a map $\theta : \mathbb{Z}/k\mathbb{Z} \to \mathcal{V}(\chi)$ *is a Hamilton circuit map of* $\Gamma(\chi)$ *if* θ *is a bijection for which* $\{(\theta\circ\hat{\pi}_k)(t), (\theta\circ\hat{\pi}_k)(t+1)\} \in \mathcal{E}(\chi)$ *for all* $t \in \mathbb{Z}$*.*
(2) Let θ *be a Hamilton circuit map of* $\Gamma(\chi)$*. Let* $a := \pi_\chi(\chi) \in \bar{\mathcal{G}}(\chi)$*, and let* $x \in J_{0,k-1}$ *be such that* $(\theta\circ\hat{\pi}_k)(x) = \mathrm{id}_{\mathbb{Z}\Pi}\,(\in \mathcal{V}(\chi))$*. Let* $j_t \in I$ $(t \in J_{1,k})$ *be such that* $s^\chi_{(j_t,\ldots,j_2,j_1)} = (\theta\circ\hat{\pi}_k)(x+t)$*. Define the map* $\varphi_\theta : \mathbb{Z}/k\mathbb{Z} \to I$ *(resp.* $\vartheta_\theta : \mathbb{Z}/k\mathbb{Z} \to \bar{\mathcal{G}}(\chi)$*) by*

$$(\varphi_\theta\circ\hat{\pi}_k)(x+t) := j_t \text{ (resp. } (\vartheta_\theta\circ\hat{\pi}_k)(x+t) := a_{(j_t,\ldots,j_2,j_1)}) \text{ for } t \in J_{1,k}.$$

(3) Let θ *be a Hamilton circuit map of* $\Gamma(\chi)$*. We call* θ *special if for every* $a \in \bar{\mathcal{G}}(\chi)$ *and every* $i \in I$*, there exists* $t \in \mathbb{Z}$ *such that* $(\vartheta_\theta\circ\hat{\pi}_k)(t) = a$ *and* $(\varphi_\theta\circ\hat{\pi}_k)(t) = i$*.*
(4) Let θ *be a Hamilton circuit map of* $\Gamma(\chi)$*. For* $i \in I$*, we say that* θ *is*

i-convenient if there exists $r \in J_{0,1}$ such that $(\varphi_\theta \circ \hat{\pi}_k)(2t+r) = i$ for all $t \in \mathbb{Z}$.

By (3.2) and the definition of $\bar{\mathcal{G}}(\chi)$, we see:

Lemma 3.11. (1) *If there exists a special Hamilton circuit map of $\Gamma(\chi)$, then for every $v \in \mathcal{V}(\chi)$ and every $i \in I$, there exists a Hamilton circuit map θ of $\Gamma(\chi)$ such that $(\theta \circ \hat{\pi}_k)(t) = v$ and $(\varphi_\theta \circ \hat{\pi}_k)(t) = i$ for some $t \in \mathbb{Z}$.*
(2) *Let $i \in I$. If there exists an i-convenient Hamilton circuit map of $\Gamma(\chi)$, then for every $v \in \mathcal{V}(\chi)$, there exists a Hamilton circuit map θ of $\Gamma(\chi)$ such that $(\theta \circ \hat{\pi}_k)(t) = v$ and $(\varphi_\theta \circ \hat{\pi}_k)(t) = i$ for some $t \in \mathbb{Z}$.*

Definition 3.12. *Let $\mathcal{W}(\chi)$ be the semigroup defined by generators*

$$0,\ e^a\,(a \in \bar{\mathcal{G}}(\chi)),\ z_i^a\,(a \in \bar{\mathcal{G}}(\chi), i \in I)$$

and relations

(z1) $0 = 00 = 0e^a = e^a 0 = 0 z_i^a = z_i^a 0,$
(z2) $e^a e^a = e^a,\ e^a e^b = 0\,(a \neq b),$
(z3) $z_i^{\bar{\tau}_i(a)} z_i^a = e^a,$
(z4) $z^a_{\underbrace{(\dots,i,j,i)}_{m_{ij}^a}} = z^a_{\underbrace{(\dots,j,i,j)}_{m_{ij}^a}} \quad (i \neq j),$

where $z^a_{(i_k,\dots,i_2,i_1)}$ is defined in the same way as that for $s^a_{(i_k,\dots,i_2,i_1)}$ with z_i^a in place of s_i^a. Let $z^a_{()} := e^a$.

Notice that by the definition [15, Section 3] of a semigroup defined by generators and relations, we see:

Lemma 3.13. *$z^a_{(i_k,\dots,i_2,i_1)} = z^a_{(j_r,\dots,j_2,j_1)}$ if and only if there exist $m \in \mathbb{N}$, $x_t \in \mathbb{Z}_{\geq 0}$, and $(i^{(t)}_{x_t},\dots,i^{(t)}_2,i^{(t)}_1) \in I^{x_t}$ $(t \in J_{0,m})$, (we let I^0 mean $\{()\}$), such that the following (r1)-(r2) are fulfilled.*

(r1) $(i^{(0)}_{x_0},\dots,i^{(0)}_2,i^{(0)}_1) = (i_k,\dots,i_2,i_1)$ *and*
$(i^{(m)}_{x_m},\dots,i^{(m)}_2,i^{(m)}_1) = (j_r,\dots,j_2,j_1)$.
(r2) *For every $t \in J_{0,m-1}$, there exist $t',t'' \in J_{0,m}$ with $(t',t'') \in \{(t,t+$

$1),(t+1,t)\}$ *such that one of* (r2-1)-(r2-2) *below is fulfilled.*
(r2-1) *We have* $x_{t''}=x_{t'}-2$ *and there exist* $y\in J_{1,x_{t'}-1}$ *such that*

$$(i^{(t'')}_{x_{t''}},\ldots,i^{(t'')}_{y+2},i^{(t')}_{y},i^{(t')}_{y},i^{(t'')}_{y-1},\ldots,i^{(t'')}_{2},i^{(t'')}_{1})=(i^{(t')}_{x_{t'}},\ldots,i^{(t')}_{2},i^{(t')}_{1}).$$

(r2-2) *Let* $a'_0:=a$ *and* $a'_u:=a_{(i^{(t')}_u,\ldots,i^{(t')}_2,i^{(t')}_1)}$ $(u\in J_{1,x_{t'}})$. *For* $u\in J_{1,x_{t'}-1}$, *if* $i^{(t')}_u\neq i^{(t')}_{u+1}$, *let* $m'_u:=m^{a'_{u-1}}_{i^{(t')}_{u+1},i^{(t')}_u}$. *We have* $x_{t''}=x_{t'}$ *and there exists* $y\in J_{1,x_{t'}-1}$ *with* $i^{(t')}_y\neq i^{(t')}_{y+1}$ *such that*

$$\begin{aligned}&(i^{(t')}_{x_{t'}},\ldots,i^{(t')}_{2},i^{(t')}_{1})\\&\quad=(i^{(t')}_{x_{t'}},\ldots,i^{(t')}_{y+m'_{y-1}},\underbrace{\ldots,i^{(t')}_{y},i^{(t')}_{y+1},i^{(t')}_{y}}_{m'_y},i^{(t')}_{y-1},\ldots,i^{(t')}_{2},i^{(t')}_{1})\end{aligned}$$

and

$$\begin{aligned}&(i^{(t'')}_{x_{t''}},\ldots,i^{(t'')}_{2},i^{(t'')}_{1})\\&\quad=(i^{(t')}_{x_{t'}},\ldots,i^{(t')}_{y+m'_{y-1}},\underbrace{\ldots,i^{(t')}_{y+1},i^{(t')}_{y},i^{(t')}_{y+1}}_{m'_y},i^{(t')}_{y-1},\ldots,i^{(t')}_{2},i^{(t')}_{1}),\end{aligned}$$

where in fact, $x_{t'}\geq m'_y$ *and* $m'_{y'}=m'_y$ $(y'\in J_{y,y+m'_y-1})$.

Theorem 3.14. ([15, Theorem 1]) *We have*

$$s^a_{(i_k,\ldots,i_2,i_1)}=s^a_{(j_r,\ldots,j_2,j_1)}\ \Leftrightarrow\ z^a_{(i_k,\ldots,i_2,i_1)}=z^a_{(j_r,\ldots,j_2,j_1)}$$

In particular, $s^a_{(i_k,\ldots,i_2,i_1)}=\mathrm{id}_{\mathbb{Z}\Pi}\Leftrightarrow s^a_{(i_k,\ldots,i_2,i_1)}=e^a$.

We have

$$|\mathcal{V}(\chi)|\in 2\mathbb{N}. \tag{3.3}$$

Proof of (3.3). Let $\mathcal{V}(\chi)':=\{s^\chi_{(i_{2k},\cdots,i_2,i_1)}\in\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)\,|\,k\in\mathbb{Z}_{\geq 0},\,i_t\in I\,(t\in J_{1,2k})\}$ and $\mathcal{V}(\chi)'':=\{s^\chi_{(i_{2k-1},\cdots,i_2,i_1)}\in\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)\,|\,k\in\mathbb{N},\,i_t\in I\,(t\in J_{1,2k-1})\}$. By Lemma 3.13 and Theorem 3.14, we have $\mathcal{V}(\chi)=\mathcal{V}(\chi)'\cup\mathcal{V}(\chi)''$ and $\mathcal{V}(\chi)'\cap\mathcal{V}(\chi)''=\emptyset$. Fix $j\in I$ and define the map $f:\mathcal{V}(\chi)'\to\mathcal{V}(\chi)''$ by $f(s^\chi_{(i_{2k},\cdots,i_2,i_1)}):=s^\chi_{(j,i_{2k},\cdots,i_2,i_1)}$. By Lemma 3.13 and Theorem 3.14, we see that f is bijective. □

Definition 3.15. *Let $\chi' \in \mathcal{X}_\Pi$, and assume $|R^+(\chi')| < \infty$. We write $\chi \cong \chi'$ if there exists a bijection $f : I \to I$ such that*

$$\forall k \in \mathbb{N}, \forall i_t \in I\,(t \in J_{1,k}),\ s^{\chi}_{(i_k,\ldots,i_2,i_1)} = s^{\chi'}_{(f(i_k),\ldots,f(i_2),f(i_1))}\,(\in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)).$$

We also write $\pi_\chi(\chi) \cong \pi_{\chi'}(\chi')$ if $\chi \cong \chi'$.

Clearly we have:

Lemma 3.16. *Assume $\chi \cong \chi'$. Then there exists a unique bijection $f : \mathcal{V}(\chi) \to \mathcal{V}(\chi')$ such that $f(\mathrm{id}_{\mathbb{Z}\Pi}) = \mathrm{id}_{\mathbb{Z}\Pi}$ and $\{\{f(x), f(y)\} | \{x, y\} \in \mathcal{E}(\chi)\} = \mathcal{E}(\chi')\,(\subset \mathfrak{p}_2(\chi'))$. In particular, if there exists a Hamilton circuit map of $\Gamma(\chi)$, then it is also the case for $\Gamma(\chi')$.*

Definition 3.17. *Let $q_{ij} := \chi(\alpha_i, \alpha_j)$ $(i, j \in I)$.*

(1) We say that χ is of (finite) Cartan-type if $q_{ii} \neq 1$ $(i \in I)$ and $q_{ii}^{N^\chi_{ij}} q_{ij} q_{ji} = 1$ $(i, j \in I, i \neq j)$. In addition, if there exists a positive definite symmetric bi-linear map $(\ ,\ ) : (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}\Pi) \times (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}\Pi) \to \mathbb{R}$ such that $[\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}]_{i,j\in I}$ is a (finite) Cartan matrix of type X and $N^\chi_{ij} = -\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ $(i, j \in I, i \neq j)$, then we say that χ is a (finite) Cartan-type bi-character of type X. (Such X really exists for any (finite) Cartan-type χ, which can be directly seen by [12].)
(2) We say that χ is of quasi-Cartan-type if there exists $\chi' \in \mathcal{X}_\Pi$ with $|R^+(\chi')| < \infty$ such that $\chi' \cong \chi$ and χ' is of Cartan-type. In addition, if χ' is a (finite) Cartan-type of type X, we say that χ' is a (finite) quasi-Cartan-type bi-character of type X.

Using Theorem 2.3, by (3.2) and the definitions of $\bar{\mathcal{G}}(\chi)$ and $\cong$, we see:

Lemma 3.18. *Assume that $|I| \geq 2$. Let $k := |\mathcal{V}(\chi)|\,(\in \mathbb{N})$ and $a := \pi_\chi(\chi) \in \bar{\mathcal{G}}(\chi)$.*

(1) *Let χ be of Cartan-type of type X. Then $|\bar{\mathcal{G}}(\chi)| = 1$, and the graph $\Gamma(\chi)$ is isomorphic to the Cayley graph $\mathcal{C}_{W,S}$ of the Coxeter system (W, S) of type X. In particular, a Hamilton circuit of $\Gamma(\chi)$ exists and it is special.*
(2) *Let χ be of quasi-Cartan-type of type X. Then the graph $\Gamma(\chi)$ is isomorphic to the Cayley graph $\mathcal{C}_{W,S}$ of the Coxeter system (W, S) of type X.*

In particular, for every $v \in \mathcal{V}(\chi)$ and every $i \in I$, there exists a Hamilton circuit map $\theta : \mathbb{Z}/k\mathbb{Z} \to \mathcal{V}(\chi)$ such that $(\vartheta_\theta \circ \hat{\pi}_k)(t) = v$ and $(\varphi_\theta \circ \hat{\pi}_k)(t) = i$ for some $t \in \mathbb{Z}$.

The following lemma seems interesting although we will not use it.

Lemma 3.19. *Assume that $|I| \geq 3$. Let χ be of quasi-Cartan-type. Let $q_{i',j'} := \chi(\alpha_{i'}, \alpha_{j'})$ $(i', j' \in I)$. Let $k := |\mathcal{V}(\chi)|$. (Notice $k \in 2\mathbb{N}$ since χ is of quasi-Cartan-type.) Let i_1, i_2, $i_3 \in I$ be such that $i_1 \neq i_2 \neq i_3 \neq i_1$. Assume that $q_{i_1,i_2}q_{i_2,i_1} \neq 1$, $q_{i_2,i_3}q_{i_3,i_2} \neq 1$ and $q_{i_1,i_3}q_{i_3,i_1} = 1$. Then there exists a Hamilton circuit map $\theta : \mathbb{Z}/k\mathbb{Z} \to \mathcal{V}(\chi)$ of $\Gamma(\chi)$ such that*

$$\forall t \in \mathbb{Z},\ (\varphi_\theta \circ \hat{\pi}_k)(2t-1) \notin \{i_1, i_3\}.$$

In particular, if $|I| = 3$, θ is an i_2-convenient Hamilton circuit map.

Proof. We use the Dynkin diagrams of the finite Coxeter systems. If $|I| = 3$, the claim follows from Fig.2 and Fig.6. If $|I| \geq 4$, the claim is inductively proved using the same argument as that of Proof of Theorem 2.3. $\square$

Definition 3.20. *Assume that $|I| \geq 2$. We call χ special if there exists a special Hamilton circuit map of $\Gamma(\chi)$.*

3.3. Main tools

In this subsection, let $\chi \in \mathcal{X}_\Pi$, and assume $|R^+(\chi)| < \infty$.

By an argument similar to that of Proof of Theorem 2.3, we have:

Proposition 3.21. *Assume $|I| \geq 3$. Fix $i \in I$. For $a \in \bar{\mathcal{G}}(\chi)$, let*

$$\begin{array}{lcl} W^a & := & \{s^a_{(u_k,\ldots,u_2,u_1)} | k \in \mathbb{Z}_{\geq 0},\ u_t \in I\,(t \in J_{1,k})\},\ (\subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)), \\ r & := & |W^a|\ (\textit{Notice that } r = |W^{\grave{a}}| \textit{ for all } \grave{a} \in \bar{\mathcal{G}}(\chi)), \\ \bar{\mathcal{G}}(\chi)^{\wedge,a} & := & \{a_{(j_k,\ldots,j_2,j_1)} | k \in \mathbb{Z}_{\geq 0},\ j_t \in I \setminus \{i\}\,(t \in J_{1,k})\}\,(\subset \bar{\mathcal{G}}(\chi)), \\ \hat{W}^a & := & \{s^a_{(j_k,\ldots,j_2,j_1)} | k \in \mathbb{Z}_{\geq 0},\ j_t \in I \setminus \{i\}\,(t \in J_{1,k})\}\,(\subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}\Pi)), \\ \hat{r}^a & := & |\hat{W}^a|. \end{array}$$

($\hat{r}^a \neq \hat{r}^{a'}$ may happen for some a, $a' \in \bar{\mathcal{G}}(\chi)$ with $a \neq a'$.) Let B be a subset of $\bar{\mathcal{G}}(\chi)$ such that $\cup_{b \in B}\bar{\mathcal{G}}(\chi)^{\wedge,b} = \bar{\mathcal{G}}(\chi)$ and $\bar{\mathcal{G}}(\chi)^{\wedge,b} \cap \bar{\mathcal{G}}(\chi)^{\wedge,b'} = \emptyset$ $(b, b' \in B,\ b \neq b')$. Assume that there exist $x^b \in \mathbb{N}$ $(b \in B)$ and surjections

$\hat{\varphi}^b_y : \mathbb{Z}/\hat{r}^b\mathbb{Z} \to I \setminus \{i\}$ $(b \in B,\, y \in J_{1,x^b})$ *satisfying the following* $(B1)$-$(B3)$. *In the following, for* $b \in B$, $y \in J_{1,x^b}$ *and* $t \in \mathbb{Z}_{\geq 0}$, *let*

$$j_{b,y;t} := \hat{\varphi}^b_y(\hat{\pi}_{\hat{r}^b}(t)),\ b_{y;t} := b_{(j_{b,y;t},\ldots,j_{b,y;2},j_{b,y;1})} \text{ and}$$
$$s_{b,y;t} := s^b_{(j_{b,y;t},\ldots,j_{b,y;2},j_{b,y;1})},$$

where $b_{y;0} := b\,(= b_{()})$ *and* $s_{b,y;0} := \mathrm{id}_{\mathbb{Z}\Pi}(= s^b_{()})$.

$(B1)$ *For* $b \in B$ *and* $y \in J_{1,x^b}$, *the map* $\hat{\theta}^b_y : \mathbb{Z}/\hat{r}^b\mathbb{Z} \to \hat{W}^b$ *defined by* $\hat{\theta}^b_y(\hat{\pi}_{\hat{r}^b}(t)) := s_{b,y;t}$ $(t \in \mathbb{Z}_{\geq 0})$ *is bijective.*
$(B2)$ *For* $b \in B$, $y \in J_{1,x^b}$ *and* $t \in J_{1,\hat{r}^b}$, *there exist* $\dot{t} \in J_{t,t+1}$, $b' \in B$, $y' \in J_{1,x^{b'}}$ *and* $t' \in J_{1,\hat{r}^{b'}}$ *such that*

$$\bar{\tau}_i(b_{y;\dot{t}-1}) = b'_{y';t'-1},\ j_{b,y;\dot{t}} = j_{b',y';t'} \text{ and } m^{b_{y;\dot{t}-1}}_{i,j_{b,y;\dot{t}-1}} = 2,$$

where notice $s^{b'_{y';t'}}_{j_{b,y';t'-1}} s^{b_{y;\dot{t}-1}}_i = s^{b_{y;\dot{t}}}_i s^{b_{y;\dot{t}-1}}_{j_{b,y;\dot{t}}}$.

Then for $a \in \bar{\mathcal{G}}(\chi)$, *there exists a surjection* $\varphi^a : \mathbb{Z}/r\mathbb{Z} \to I$ *such that the map* $\theta^a : \mathbb{Z}/r\mathbb{Z} \to W^a$ *defined by* $\theta^a(\hat{\pi}_r(h)) := s^a_{(\varphi^a(\hat{\pi}_r(h)),\ldots,\varphi^a(\hat{\pi}_r(2)),\varphi^a(\hat{\pi}_r(1)))}$ $(h \in J_{1,r})$ *is bijective, where notice that* $\theta^a(\hat{\pi}_r(0)) = \theta^a(\hat{\pi}_r(r)) = \mathrm{id}_{\mathbb{Z}\Pi}(= s^a_{()})$. *In other words, there exists a Hamilton circuit map of* $\Gamma(\chi)$.

Proposition 3.22. (*Recall* (3.3).) *Let* $\chi \in \mathcal{X}_\Pi$, *and assume* $|R^+(\chi)| < \infty$. *Let* $N \in J_{3,\infty}$ *and assume* $I = J_{1,N}$. *Let* $i \in J_{1,N-2}$. *Let* $I_1 := J_{1,i-1}$ *and* $I_2 := J_{i+1,N}$. *Let* $k := i + 1$.

Assume that $\forall \chi' \in \mathcal{G}(\chi)$, $\forall (i_1, i_2) \in (I_1 \times I_2) \cup (\{i\} \times (I_2 \setminus \{k\}))$, $\chi'(\alpha_{i_1}, \alpha_{i_2})\chi'(\alpha_{i_2}, \alpha_{i_1}) = 1$.

For $\chi' \in \mathcal{G}(\chi)$ *and* $r \in J_{1,2}$, *if* $I_r \neq \emptyset$, *let* $\Pi_r := \{\alpha_g | g \in I_r\}$, $Y_r := \oplus_{g \in I_r}\mathbb{Z}\alpha_g$, *and* $\chi'_r := \chi'_{|Y_r \times Y_r}$, *regard* $\chi'_r \in \mathcal{X}_{\Pi_r}$ *in a natural way, and let* $h^{\chi'}_r := |\mathcal{V}(\chi'_r)|$, *where notice* $I_2 \neq \emptyset$. *For* $\chi' \in \mathcal{G}(\chi)$ *and* $r \in J_{1,2}$, *assume that if* $h^{\chi'}_r \geq 4$, *there exists a Hamilton circuit map* $\theta^{\chi'}_r : \mathbb{Z}/h^{\chi'}_r\mathbb{Z} \to \mathcal{V}(\chi'_r)$ *of* $\Gamma(\chi'_r)$, *where notice* $h^{\chi'}_2 \geq 4$.

$$\text{Assume that } \theta^{\chi'}_2 \text{ is special for every } \chi' \in \mathcal{G}(\chi). \quad \text{(See also Remark 3.23 below.)} \tag{3.4}$$

Then there exists a Hamilton circuit map of $\Gamma(\chi)$.

Proof. Let $\chi' \in \mathcal{G}(\chi)$. By Lemma 3.11 (1), we have a special Hamilton circuit map $\theta_2^{\chi'} : \mathbb{Z}/h_2^{\chi'}\mathbb{Z} \to \mathcal{V}(\chi_2')$ of $\Gamma(\chi_2')$ such that $(\theta_2^{\chi'} \circ \hat{\pi}_{h_2^{\chi'}})(0) = \mathrm{id}_{Y_2}$ and $(\varphi_{\theta_2^{\chi'}} \circ \hat{\pi}_{h_2^{\chi'}})(h_2^{\chi'}) = k$ (See also Remark 3.23 below.) Let $I_3 := I \setminus \{i\} \, (= I_1 \cup I_2)$, $\Pi_3 := \Pi \setminus \{\alpha_i\} \, (= \Pi_1 \cup \Pi_2)$, $Y_3 := Y_1 \oplus Y_2$, $\chi_3' := \chi'_{|Y_3 \times Y_3}$ and $h_3^{\chi'} := |\mathcal{V}(\chi_3')|$. Then $h_3^{\chi'} = h_1^{\chi'} \cdot h_2^{\chi'}$ if $I_1 \neq \emptyset$, and $h_3^{\chi'} = h_2^{\chi'}$ if $I_1 = \emptyset$. Define the Hamilton circuit map $\theta_3^{\chi'} : \mathbb{Z}/h_3^{\chi'}\mathbb{Z} \to \mathcal{V}(\chi_3')$ of $\Gamma(\chi_3')$ by $(\theta_3^{\chi'} \circ \hat{\pi}_{h_3^{\chi'}})(0) := \mathrm{id}_{Y_3}$ and

$$
\begin{aligned}
&(\varphi_{\theta_3^{\chi'}} \circ \hat{\pi}_{h_3^{\chi'}})(t) \\
&= \begin{cases}
(\varphi_{\theta_2^{\chi'}} \circ \hat{\pi}_{h_2^{\chi'}})(t') \\
\quad \text{for } t = t' + u \cdot h_2^{\chi'} \text{ with } t' \in J_{1,h_2^{\chi'}-1} \text{ and } u \in 2\mathbb{Z}, \\
(\varphi_{\theta_2^{\chi'}} \circ \hat{\pi}_{h_2^{\chi'}})(-t') \\
\quad \text{for } t = t' + u \cdot h_2^{\chi'} \text{ with } t' \in J_{1,h_2^{\chi'}-1} \text{ and } u \in 2\mathbb{Z}+1 \text{ if } |I_1| \geq 1, \\
k \quad \text{for } t = h_2^{\chi'} \text{ if } |I_1| = 0, \\
1 \quad \text{for } t \in \{h_2^{\chi'}, 2h_2^{\chi'}\} \text{ if } |I_1| = 1, \\
(\varphi_{\theta_1^{\chi'}} \circ \hat{\pi}_{h_1^{\chi'}})(r) \quad \text{for } t = r \cdot h_2^{\chi'} \text{ with } r \in J_{1,h_1^{\chi'}} \text{ if } |I_1| \geq 2
\end{cases}
\end{aligned}
$$

$(t \in J_{1,h_3^{\chi'}})$, where see also Fig.11. Notice that for $\chi'' \in \mathcal{G}(\chi)$ and $t \in J_{1,h_3^{\chi''}}$, there exists $r \in J_{0,1}$ such that $(\varphi_{\theta_3^{\chi''}} \circ \hat{\pi}_{h_3^{\chi''}})(t+r) \in I_2 \setminus \{k\}$. Then the claim follows from Proposition 3.21. □

Remark 3.23. *We can define the injection* $x : \mathcal{V}(\chi_r') \to \mathcal{V}(\chi')$ *by*

$$x(s^{\chi_r'}_{(i_p,\ldots,i_2,i_1)}) := s^{\chi_r'}_{(i_p,\ldots,i_2,i_1)} \; (p \in \mathbb{Z}_{\geq 0},\, i_t \in I_r \; (t \in J_{1,p})).$$

Let $a' := \pi_\chi(\chi') \, (= \pi_{\chi'}(\chi')) \in \bar{\mathcal{G}}(\chi) \, (= \bar{\mathcal{G}}(\chi'))$ *and* $a_r' := \pi_{\chi_r'}(\chi_r') \in \bar{\mathcal{G}}(\chi_r')$. *Let* $Y := \{a'_{(i_p,\ldots,i_2,i_1)} | p \in \mathbb{Z}_{\geq 0},\, i_t \in I_r \, (t \in J_{1,p})\}$. *We can define the surjection* $y : Y \to \mathcal{V}(\chi')$ *by* $y(a'_{(i_p,\ldots,i_2,i_1)}) := (a_r')_{(i_p,\ldots,i_2,i_1)}$ $(p \in \mathbb{Z}_{\geq 0}$, $i_t \in I_r \; (t \in J_{1,p}))$. *However* y *is not necessarily injective. See* Fig.78.

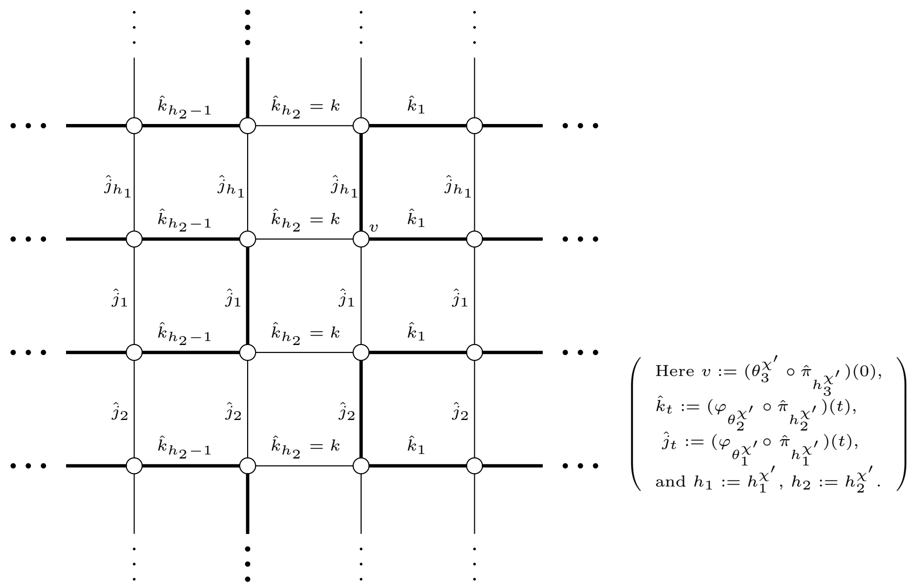


Fig.11: Idea of Hamilton circuit of $\Gamma(\chi'_3)$ in Proof of Proposition 3.22

Using an argument similar to that of Proof of Proposition 3.22, we also have:

Proposition 3.24. *Replace the assumption* (3.4) *of Proposition* 3.22 *by one of* (x) *and* (y) *below.*

(x) *For every* $\chi' \in \mathcal{G}(\chi)$*, there exists a Hamilton circuit map* $(\theta_2^{\chi'})' : \mathbb{Z}/h_2^{\chi'}\mathbb{Z} \to \mathcal{V}(\chi'_2)$ *of* $\Gamma(\chi'_2)$*. (*$(\theta_2^{\chi'})'$ *may differ from* $\theta_2^{\chi'}$*.)*
(y) *For every* $\chi' \in \mathcal{G}(\chi)$*,* $\bar{\mathcal{G}}(\chi'_2) \times (I_2 \setminus \{k\}) \subset \{((\vartheta_{\theta_2^{\chi'}} \circ \hat{\pi}_{h_2^{\chi'}})(t), (\varphi_{\theta_2^{\chi'}} \circ \hat{\pi}_{h_2^{\chi'}})(t)) | t \in \mathbb{Z}\} \cup \{((\vartheta_{(\theta_2^{\chi'})'} \circ \hat{\pi}_{h_2^{\chi'}})(t), (\varphi_{(\theta_2^{\chi'})'} \circ \hat{\pi}_{h_2^{\chi'}})(t)) | t \in \mathbb{Z}\}$.

Then there exists a Hamilton circuit map of $\Gamma(\chi)$*.*

Definition 3.25. *Assume* $|I| \geq 3$*, and let* $i \in I$ *and* $(\mathbb{Z}\Pi)' := \oplus_{j \in I \setminus \{i\}} \mathbb{Z}\alpha_j$*. Let* $\chi_{\langle i \rangle} := \chi_{|(\mathbb{Z}\Pi)' \times (\mathbb{Z}\Pi)'} \in \mathcal{X}_{\Pi \setminus \{\alpha_i\}}$*. Notice that* $\Gamma(\chi_{\langle i \rangle}) = (\mathcal{V}(\chi_{\langle i \rangle}), \mathcal{E}(\chi_{\langle i \rangle}))$ *means the one defined in the same way as that for* $\Gamma(\chi) = (\mathcal{V}(\chi), \mathcal{E}(\chi))$ *with* $\chi_{\langle i \rangle}$ *and* $\Pi \setminus \{\alpha_i\}$ *in place of* χ *and* Π *respectively. (Notice that* $R^+(\chi_{\langle i \rangle})$ *can be identified with* $R^+(\chi) \cap (\mathbb{Z}\Pi)'$*, and that* $\Gamma(\chi_{\langle i \rangle})$ *is the full subgraph of* $\Gamma(\chi)$ *with the vertices labeled by* α_j *with* $j \in I \setminus \{i\}$*.) Letting* $a := \pi_\chi(\chi) \in \bar{\mathcal{G}}(\chi)$*,* *define*

$$\bar{\mathcal{G}}_{\langle i\rangle}(\chi) := \{\bar{\tau}_{j_1}\bar{\tau}_{j_2}\cdots\bar{\tau}_{j_r}a | r\in\mathbb{N}, i_t\in I\setminus\{i\}\,(t\in J_{1,r})\}\,(\subset\bar{\mathcal{G}}(\chi)),$$

where notice $a\in\bar{\mathcal{G}}_{\langle i\rangle}(\chi)$. *For a non-empty subset* G *of* $\mathcal{G}(\chi)$, *we say that* G *(resp.* $\{\pi_\chi(\chi')|\chi'\in G\}$*) is an i-complete* $\bar{\tau}$*-representative subset of* $\mathcal{G}(\chi)$ *(resp.* $\bar{\mathcal{G}}(\chi)$*) if* $\bar{\mathcal{G}}(\chi) = \cup_{\chi'\in G}\bar{\mathcal{G}}_{\langle i\rangle}(\chi')$ *and* $\bar{\mathcal{G}}_{\langle i\rangle}(\chi')\cap\bar{\mathcal{G}}_{\langle i\rangle}(\chi'') = \emptyset$ $(\chi',\chi''\in G,\ \chi'\neq\chi'')$.

Definition 3.26. *Assume* $|I|\geq 3$.

(1) *We call* χ (i,j)*-special if there exist* $i,j\in I$ *with* $i\neq j$ *such that for every* $\chi'\in\mathcal{G}(\chi)$, *the equations* $\chi'(\alpha_i,\alpha_k)\chi'(\alpha_k,\alpha_i)=1$ $(k\in I\setminus\{i,j\})$ *hold and* $\Gamma(\chi'_{\langle i\rangle})$ *is special.*
(2) *We call* χ (i,j)*-convenient if there exist* $i,j\in I$ *with* $i\neq j$ *such that for every* $\chi'\in\mathcal{G}(\chi)$, *the equation* $\chi'(\alpha_i,\alpha_j)\chi'(\alpha_j,\alpha_i)=1$ *holds and* $\Gamma(\chi'_{\langle i\rangle})$ *is* j*-convenient.*

By Lemma 3.11 and Proposition 3.21, we have:

Proposition 3.27. *Assume* $|I|\geq 3$. *If* χ *is* (i,j)*-special or* (i,j)*-convenient for some* $i,j\in I$ *with* $i\neq j$, *then there exists a Hamilton circuit map of* $\Gamma(\chi)$.

Lemma 3.28. *Let* $\chi\in\mathcal{X}_\Pi$ *with* $|R^+(\chi)|<\infty$. *Let* I_1 *be a non-empty proper subset of* I, *and let* $I_2 := I\setminus I_1$. *Assume that* $\chi(\alpha_i,\alpha_j)\chi(\alpha_j,\alpha_i)=1$ *for all* $(i,j)\in I_1\times I_2$. *For* $r\in J_{1,2}$, *let* $\Pi_r := \{\alpha_g|g\in I_r\}$, $Y_r := \oplus_{g\in I_r}\mathbb{Z}\alpha_g$, *and* $\chi_r := \chi_{|Y_r\times Y_r}$, *regard* $\chi_r\in\mathcal{X}_{\Pi_r}$ *in a natural way, and let* $h_r := |\mathcal{V}(\chi_r)|$. *Assume that for each* $r\in J_{1,2}$, *there exists a Hamilton circuit map* $\theta^\chi_r : \mathbb{Z}/h_r\mathbb{Z}\to\mathcal{V}(\chi_r)$ *of* $\Gamma(\chi_r)$. *Let* $k := |\mathcal{V}(\chi)|\,(\in\mathbb{N})$. *Then there exists a Hamilton circuit map* $\theta : \mathbb{Z}/k\mathbb{Z}\to\mathcal{V}(\chi)$ *of* $\Gamma(\chi)$.

Proof. The map θ can be realized as a graph similar to that of Fig.11. □

3.4. Further notation, terminology and examples

Definition 3.29. *Let* $\chi\in\mathcal{X}_\Pi$. *Let* $q_{ij} := \chi(\alpha_i,\alpha_j)$ $(\in\mathbb{K}^\times)$ $(i,j\in I)$. *The generalized Dynkin diagram of* χ *is the graph composed of the* $|I|$*-vertices* *one-to-one corresponding to α_i's ($i \in I$) and the edges connected vertices corresponding to α_i and α_i ($i \neq j$) with $q_{ij}q_{ji} \neq 1$; moreover α_i and q_{ii} are attached to the vertex corresponding to α_i, and $q_{ij}q_{ji}(\neq 1)$ is attached to the edge connecting α_i and α_j. For example, the generalized Dynkin diagram $\mathcal{D}_1^{(2,9)}$ of* Fig.12 *means that $I = J_{1,3}$ and $q_{11} = q$, $q_{22} = -1$, $q_{33} = r$, $q_{12}q_{21} = q^{-1}$, $q_{23}q_{32} = r^{-1}$ and $q_{13}q_{31} = 1$.*

Let $n_{i,t}$ denote the cardinal number of the generalized Dynkin diagrams drawn in [12, Row t of Table i]. For example, $n_{1,t_1} = 1$ ($t_1 \in J_{0,1}$), $n_{1,2} = 2$, $n_{1,3} = 1$, $\ldots$, $n_{2,t_2} = 1$ ($t_2 \in J_{1,3}$), $n_{2,4} = 2$, $n_{2,3} = 3$, $\ldots$, $n_{3,t_3} = 1$ ($t_3 \in J_{1,5}$), $n_{3,6} = 3$, $n_{3,7} = 4$, $\ldots$, and $n_{4,11} = 4$, $n_{4,12} = 15$, $n_{4,13} = 2$.

Let $\mathcal{D}_k^{(i,t)}$ ($k \in J_{1,n_{i,t}}$) denote the the generalized Dynkin diagrams drawn in [12, Row t of Table i]. Let $\mathcal{D}_\bullet^{(i,t)}$ mean $\mathcal{D}_{k'}^{(i,t)}$ for any $k' \in J_{1,n_{i,t}}$. Let $l^{(i,t)}$ denote the cardinal number of the vertices of $\mathcal{D}_k^{(i,t)}$. To each vertex of $\mathcal{D}_k^{(i,t)}$, we attach the symbol α_p ($p \in J_{1,l^{(i,t)}}$). Define the order of $\mathcal{D}_k^{(i,t)}$'s and the order of vertices of $\mathcal{D}_k^{(i,t)}$ by the following (♯1)-(♯3).

(♯1) Let u and v be two different vertices of $\mathcal{D}_k^{(i,t)}$'s in the middle box of [12, Row t of Table i]. They may belong to different $\mathcal{D}_k^{(i,t)}$ and $\mathcal{D}_{k'}^{(i,t)}$. We say that u *locates at a north-east place to* v if u locates at a higher place than v, or if they locate at the same height but u locates on the left side of v. We also say that u *locates at an east-north place to* v if u locates on the left side of v, or if they locate at the same column but u locates at a higher place than v.

(♯2) If the leftmost vertex of $\mathcal{D}_{k_1}^{(i,t)}$ locates at a north-east place to the one of $\mathcal{D}_{k_2}^{(i,t)}$, then $k_1 < k_2$.

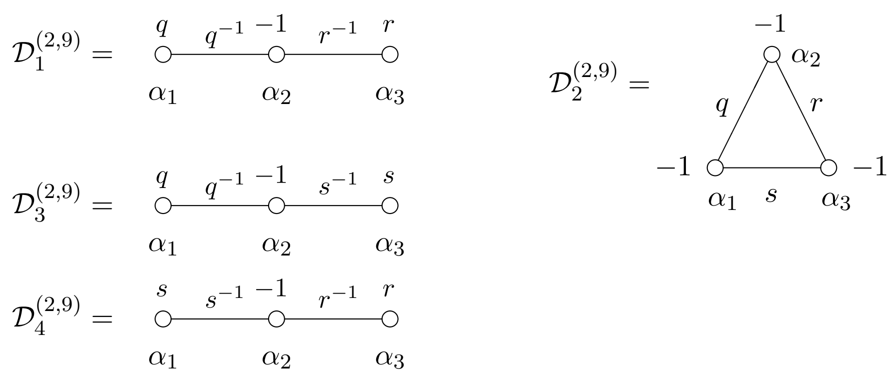
(♯3) For two different vertices of $\mathcal{D}_k^{(i,t)}$ attached by α_{p_1} and α_{p_2}, if the vertex attached by α_{p_1} locates at an east-north place to the one attached by α_{p_2}, then $p_1 < p_2$.

For example, see Fig.12 and Fig.13. Moreover for a bijection $f : J_{1,l^{(i,t)}} \to J_{1,l^{(i,t)}}$, let $\mathcal{D}_k^{(i,t)} \circ f$ mean the generalized Dynkin diagram obtained from

$\mathcal{D}_k^{(i,t)}$ by replacing α_j by $\alpha_{f^{-1}(j)}$ ($j \in J_{1,l^{(i,t)}}$), i.e., $\mathcal{D}_k^{(i,t)} \circ f$ is obtained from $\mathcal{D}_k^{(i,t)}$ by changing the order of its vertices by replacing $f(j)$ by j for $j \in J_{1,l^{(i,t)}}$. We also denote a bijection

$$f : J_{1,l^{(i,t)}} \to J_{1,l^{(i,t)}} \text{ by } \begin{bmatrix} 1 & 2 & \cdots & l^{(i,t)} \\ f(1) & f(2) & \cdots & f(l^{(i,t)}) \end{bmatrix}.$$

See Fig.14 for example.



$(q, r, s \in \mathbb{K} \setminus \{1\},\ qrs = 1,\ q \neq r \neq s \neq q)$

Fig.12: Orders of the generalized Dynkin diagrams and their vertices for [12, Row 9 of Table 2]

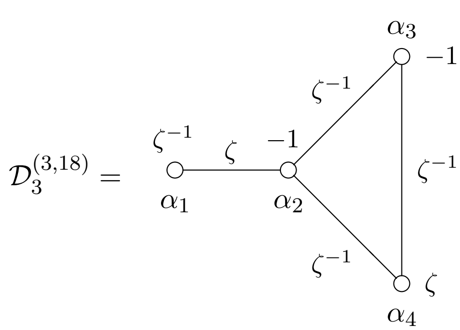


Fig.13: Order of vertices of [12, 3rd graph of Row 18 of Table 3]

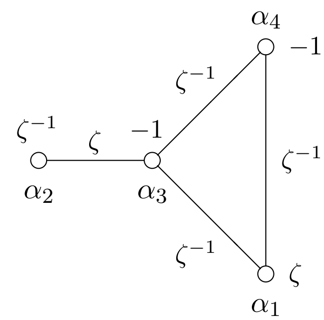
$\mathcal{D}_3^{(3,18)} \circ \begin{bmatrix} 1,2,3,4 \\ 4,1,2,3 \end{bmatrix} =$ 

Fig.14: Order changing of vertices of $\mathcal{D}_3^{(3,18)}$

Definition 3.30. *Let $\chi \in \mathcal{X}_\Pi$. We say that χ is irreducibe if the generalized Dynkin diagram of χ is connected.*

4. Hamilton Circuits of Rank-3 Cases

First of all, we comment on rank-2 cases. Let $\chi \in \mathcal{X}_\Pi$ be such that $|R^+(\chi)| < \infty$ and $|I| = 2$. Then $\Gamma(\chi)$ is a circle graph. So its Hamilton circuit map clearly exists in a similar way to that for Lemme 2.2. For example, see Fig.84 and Fig.85.

In this section, we directly draw a Hamilton circuit map of $\Gamma(\chi)$ of each irreducible $\chi \in \mathcal{X}_\Pi$ with $|R^+(\chi)| < \infty$ and $|I| = 3$.

Proposition 4.1 below tells that for $\chi \in \mathcal{X}_\Pi$ with $|R^+(\chi)| < \infty$, and $|I| = 3$, if we draw a graph inside a bounded 2-dimensional region fulfilling such an appropriate property as $(d1)$-$(d5)$ of its statement, it must be isomorphic to $\Gamma(\chi)$ as a graph.

Proposition 4.1. *Let $\chi \in \mathcal{X}_\Pi$ be such that $|I| = 3$ and $|R^+(\chi)| < \infty$. Let X be a non-empty finite subset of $\mathbb{R}^2$ fulfilling the following conditions $(d1)$-$(d6)$.*

$(d1)$ For $i \in I$, we have $\sigma_i^X \in \mathfrak{S}_X$ with $(\sigma_i^X)^2 = \mathrm{id}_X$ and $\sigma_i^X(x) \neq x$ $(x \in X)$, where $\mathfrak{S}_X$ is the group formed by all bijections from X to X.
$(d2)$ We have a surjection $\eta : X \to \bar{\mathcal{G}}(\chi)$ with $\eta \circ \sigma_i^X = \bar{\tau}_i \circ \eta$ for all $i \in I$.
$(d3)$ For $x \in X$ and $i \in I$, we have a piecewise smooth curve $\gamma_{x,i} : [0,1] \to \mathbb{R}^2$ such that $\gamma_{x,i}(0) = x$, $\gamma_{x,i}(1) = \sigma_i^X(x)$, $\gamma_{\sigma_i^X(x),i}(v) = \gamma_{x,i}(1-v)$

($v \in [0,1]$) and $\gamma_{x,i}$ is injective. Moreover, for $x \in X$ and $i, j \in I$ with $i \neq j$, we have $\gamma_{x,i}([0,1]) \cap \gamma_{x,j}([0,1]) = \{x\}$.

($d4$) Let $x \in X$, and let i, $j \in I$ with $i \neq j$. Let $m := m_{i,j}^{\eta(x)}$. Then $(\sigma_j^X \sigma_i^X)^m(x) = x$. Let $i_{2t-1} := i$ and $i_{2t} := j$ for $t \in J_{1,m}$. Let $x_0 := x$ and $x_u := \sigma_u^X(x_{u-1})$ for $u \in J_{1,2m}$ (whence $x_{2m} = x$). Define the piece-wise smooth closed curve $\hat{\gamma}_{x,i,j} : [0,2m] \to \mathbb{R}^2$ by $\hat{\gamma}_{x,i,j}(t) = \gamma_{x_u,i_u}(t+u)$ ($u \in J_{0,m-1}$, $t \in [u,u+1]$). Then $\hat{\gamma}_{x,i,j}$ is injective.

($d5$) For $k \in J_{1,2}$, let $x_k \in X$, i_k, $j_k \in I$ with $i_k \neq j_k$ and let $\hat{m}_k := m_{i_k,j_k}^{\eta(x_k)}$ and $H_{x_1,i_1,j_1} := \hat{\gamma}_{x,i,j}([0,2\hat{m}_k])$. Then exactly one of the following $(d5)_1$-$(d5)_4$ holds.

$(d5)_1$ $H_{x_1,i_1,j_1} \cap H_{x_2,i_2,j_2} = \emptyset$.

$(d5)_2$ $H_{x_1,i_1,j_1} = H_{x_2,i_2,j_2}$, $\{i_1,j_1\} = \{i_2,j_2\}$ and $\hat{\gamma}_{x_1,i_1,j_1}(u) = x_2$ for some $u \in J_{0,\hat{m}_1-1}$.

$(d5)_3$ $\exists u_1 \in J_{0,\hat{m}_1-1}$, $\exists u_2 \in J_{0,\hat{m}_2-1}$, $H_{x_1,i_1,j_1} \cap H_{x_2,i_2,j_2} = \{\hat{\gamma}_{x_1,i_1,j_1}(u_1)\} = \{\hat{\gamma}_{x_2,i_2,j_2}(u_2)\}$.

$(d5)_4$ $\exists u_1 \in J_{0,\hat{m}_1-1}$, $\exists u_2 \in J_{0,\hat{m}_2-1}$, $H_{x_1,i_1,j_1} \cap H_{x_2,i_2,j_2} = \hat{\gamma}_{x_1,i_1,j_1}([u_1,u_1+1]) = \hat{\gamma}_{x_2,i_2,j_2}([u_2,u_2+1])$.

($d6$) For $x \in X$ and i, $j \in I$ with $i \neq j$, let $\widehat{H}_{x,i,j}$ be the bounded simply connected closed subset of $\mathbb{R}^2$ whose boundary is $\hat{\gamma}_{x,i,j}([0,2m])$. Then $\cup_{x\in X} \cup_{i,j\in I, i\neq j} \widehat{H}_{x,i,j}$ is simply connected.

($d7$) Let $\hat{x} \in X$ be such that $\pi_{\chi}(\chi) = \eta(\hat{x})$. For $l \in \mathbb{Z}_{\geq 0}$ and $i_{t'} \in I$ ($t' \in J_{1,l}$), let $\hat{x}_{(i_l,\ldots,i_2,i_1)} := \sigma_{i_l}^X \cdots \sigma_{i_2}^X \sigma_{i_1}^X(\hat{x})$, where $\hat{x}_{(i_l,\ldots,i_2,i_1)} := \hat{x}_{()} := x$ if $l = 0$. Then $X = \{\hat{x}_{(i_t,\ldots,i_2,i_1)} | l \in \mathbb{Z}_{\geq 0}, i_{t'} \in I\,(t' \in J_{1,l})\}$.

Then there exists a bijection $\varphi : \mathcal{V}(\chi) \to X$ such that $\varphi(s^{\chi}_{(i_l,\ldots,i_2,i_1)}) = \hat{x}_{(i_l,\ldots,i_2,i_1)}$ for all $l \in \mathbb{Z}_{\geq 0}$ and $i_{t'} \in I$ ($t' \in J_{1,l}$), where notice $\varphi(\mathrm{id}_{\mathbb{Z}\Pi}) = \hat{x}$.

Proof. Let $\mathbb{R}X$ be an $|X|$-dimensional $\mathbb{R}$-linear space whose basis is X. For $b \in \bar{\mathcal{G}}(\chi)$ and $i \in I$, define $f_i^b \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}X)$ by $f_i^b(x) := \delta_{b,\eta(x)} \sigma_i^X(x)$ ($x \in X$). By ($d1$) and ($d4$), we have the semigroup homomorphism $\zeta : \mathcal{W}(\chi) \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}X)$ with $\zeta(z_i^b) := f_i^b$ ($b \in \bar{\mathcal{G}}(\chi), i \in I$). By Theorem 3.14, there exists a surjection $\varphi : \mathcal{V}(\chi) \to X$ of the statement.

Let $l \in \mathbb{N}$ and $i_t \in I$ ($t \in J_{1,l}$) be such that $\hat{x}_{(i_l,\ldots,i_2,i_1)} = \hat{x}$. Let $\hat{x}_0 := \hat{x}$

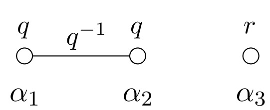
and $\hat{x}_t := \hat{x}_{(i_t,\ldots,i_2,i_1)}$ $(t \in J_{1,l})$. Then $\hat{x}_l = \hat{x}$. We have $l \geq 2$ by $(d1)$. Let h_1, $h_2 \in J_{0,l-1}$ be such that $h_1 < h_2$, $\hat{x}_{h_1} = \hat{x}_{h_2}$ and $\hat{x}_{h'} \neq \hat{x}_{h_1}$ for $h' \in J_{h_1+1,h_2-1}$. We have $h_1 \leq h_2 - 2$ by $(d1)$. If $h_1 = h_2 - 2$, we have $i_{h_1+1} = i_{h_2}$ by $(d1)$ and $(d3)$, whence $s^{\chi}_{(i_l,\ldots,i_2,i_1)} = s^{\chi}_{(i_l,\ldots,i_{h_1+3},i_{h_1},\ldots,i_2,i_1)}$, where if $h_2 = l$, let $s^{\chi}_{(i_l,\ldots,i_{h_1+3},i_{h_1},\ldots,i_2,i_1)}$ mean $s^{\chi}_{(i_{l-2},\ldots,i_2,i_1)}$. Assume $h_2 \geq h_1 + 3$. Let Z be the bounded simply connected closed subset of $\mathbb{R}^2$ whose boundary is composed of $\gamma_{x_{t-1},i_t}([0,1])$ $(t \in J_{h_1,h_2-1})$. Let $||Z||$ mean the area of Z. Then $0 < ||Z|| < \infty$. By $(d4)$, $(d5)$ and $(d6)$, using an induction on $||Z||$, we see that $s^{\chi}_{(i_l,\ldots,i_2,i_1)} = s^{\chi}_{(i_l,\ldots,i_{h_2+1},i_{h_1},\ldots,i_2,i_1)}$. Thus we prove $s^{\chi}_{(i_l,\ldots,i_2,i_1)} = \mathrm{id}_{\mathbb{Z}\Pi}$. Hence by $(d7)$, the inverse map φ^{-1} of φ can be defined. In particular, φ is bijective. $\square$

Using Proposition 4.1, we directly have:

Theorem 4.2. *Assume that* $|I| = 3$. *Then for every* χ *of* [12, Table 2], *there exists a Hamilton circuit map of* $\Gamma(\chi)$. *In fact,* (*up to elements of* $\bar{\mathcal{G}}(\chi)$) *it is drawn by* Fig.20 ($\mathcal{D}^{(2,1)}_{\bullet}$, $\mathcal{D}^{(2,4)}_{\bullet}$, $\mathcal{D}^{(2,8)}_{\bullet}$), Fig.25 ($\mathcal{D}^{(2,2)}_{\bullet}$, $\mathcal{D}^{(2,3)}_{\bullet}$, $\mathcal{D}^{(2,5)}_{\bullet}$, $\mathcal{D}^{(2,12)}_{\bullet}$, $\mathcal{D}^{(2,14)}_{\bullet}$), Fig.27 ($\mathcal{D}^{(2,6)}_{\bullet}$), Fig.29 ($\mathcal{D}^{(2,7)}_{\bullet}$), Fig.32 ($\mathcal{D}^{(2,9)}_{\bullet}$, $\mathcal{D}^{(2,10)}_{\bullet}$, $\mathcal{D}^{(2,11)}_{\bullet}$), Fig.35 ($\mathcal{D}^{(2,13)}_{\bullet}$, $\mathcal{D}^{(2,18)}_{\bullet}$), Fig.37 ($\mathcal{D}^{(2,15)}_{\bullet}$), Fig.39 ($\mathcal{D}^{(2,16)}_{\bullet}$) *and* Fig.41 ($\mathcal{D}^{(2,17)}_{\bullet}$).

Rank-3-Case-0: We shall also need the Hamilton circuit map of Fig.16 for χ of $A_2 \times A_1$-case.

$a := [\mathcal{D}^{(1,1)}_1 \times \mathcal{D}^{(1,0)}_1]$



$(q, r \in \mathbb{K}^{\times}, q \neq 1)$

Fig.15: the generalized Dynkin diagram for $A_2 \times A_1$

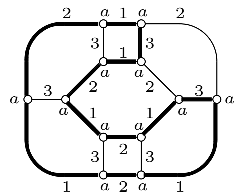


Fig.16: Special and 1-convenient Hamilton circuit for $A_2 \times A_1$

Rank-3-Case-1: Let χ be of $\mathcal{D}_1^{(2,8)}$. We draw a Hamilton circuit map of $\Gamma(\chi)$ by Fig.20. Notice that χ is of [12, Table 2, Row 8] and that $\chi_1 \cong \chi_2$ for any two χ_1 and χ_2 of [12, Table 2, Row 1,4,8].

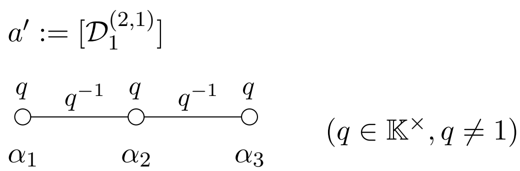


Fig.17: The generalized Dynkin diagram for [12, Table 2, Row 1]

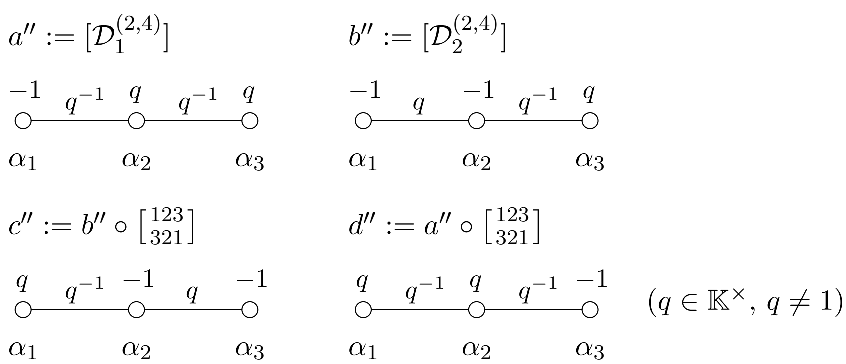


Fig.18: Generalized Dynkin diagrams for [12, Table 2, Row 4]

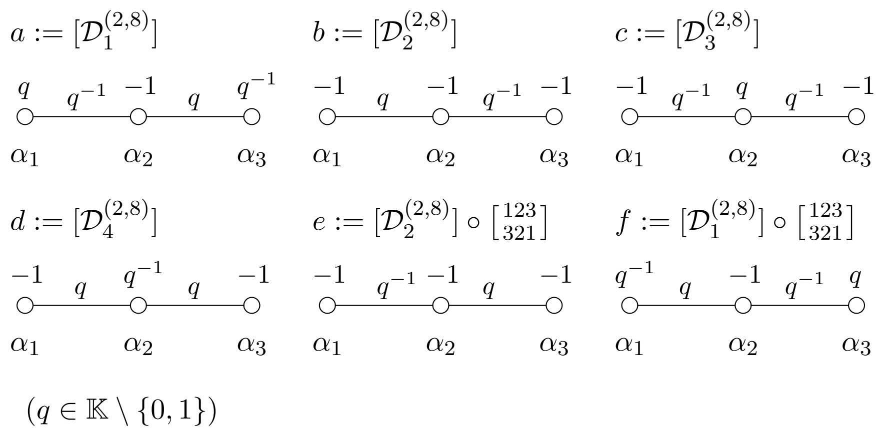
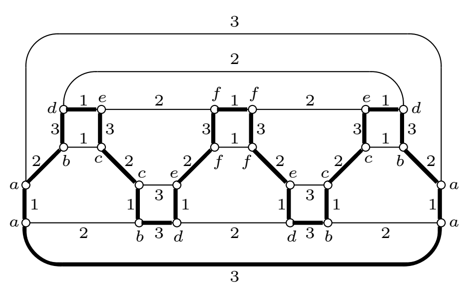
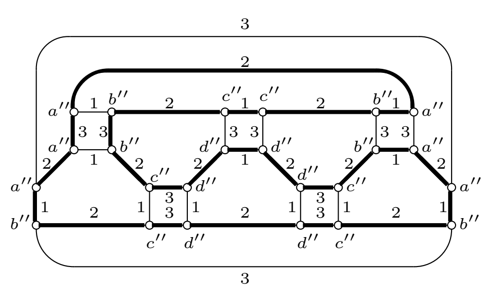


Fig.19: The generalized Dynkin diagrams for [12, Table 2, Row 8]



$(a' \cong a'' \cong a)$

Fig.20: Hamilton circuit for [12, Table 2, Row 8 (≅ Rows 1,4)]



$(a' \cong a'' \cong a)$

Fig.21: 2-convenient and special Hamilton circuit
for [12, Table 2, Row 4 (≅ Rows 1,8)]

Rank-3-Case-2: Let χ be of $\mathcal{D}_1^{(2,5)}$ or $\mathcal{D}_1^{(2,14)}$. We draw Hamilton circuit maps of $\Gamma(\chi)$ by Fig.25. Notice that χ is of [12, Table 2, Rows 5,14] and that $\chi_1 \cong \chi_2$ for any two χ_1 and χ_2 of [12, Table 2, Row 2,3,5,12,14].

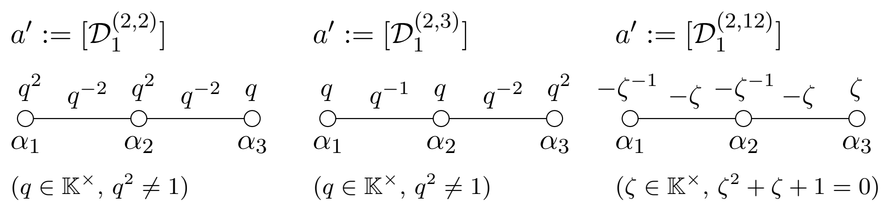


Fig.22: Generalized Dynkin diagrams of [12, Table 2, Rows 2,3,12]

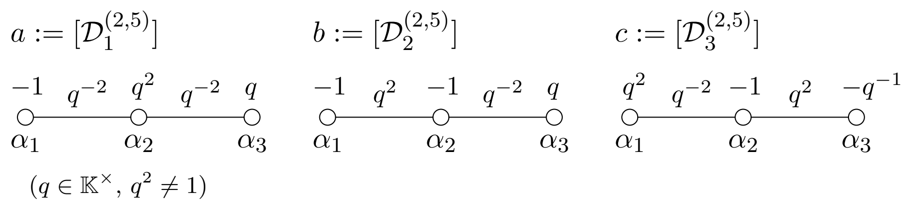


Fig.23: Generalized Dynkin diagrams for [12, Table 2, Row 5]

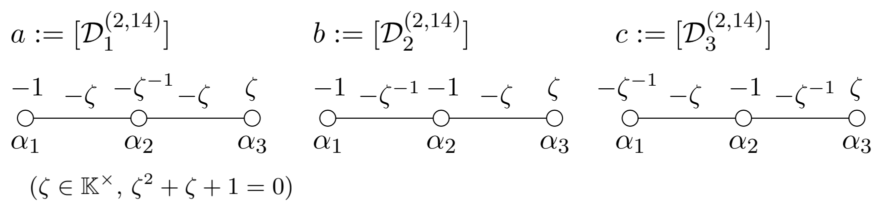
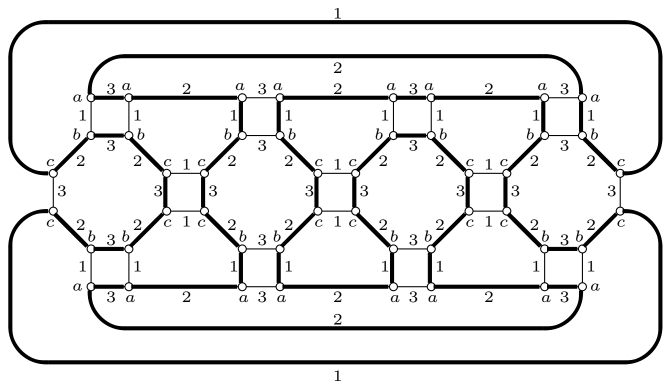


Fig.24: The generalized Dynkin diagrams for [12, Table 2, Row 14]



$(a' \cong a)$

Fig.25: Special Hamilton circuit of
[12, Table 2, Rows 5,14 (and Rows 2,3,12)]

The Cayley graphs of Fig.3 and Fig.25 are essentially the same because we have $R(\chi_1) = R(\chi_2)$ for χ_1 of Fig.3 and χ_2 of Fig.25. The same is true for Fig.6 and Fig.20. Recall the notation $\cong$ from Definition 3.15.

Rank-3-Case-3: Let χ be of $\mathcal{D}_1^{(2,6)}$. We draw a Hamilton circuit map of $\Gamma(\chi)$ by Fig.27. Notice that χ is of [12, Table 2, Row 6].

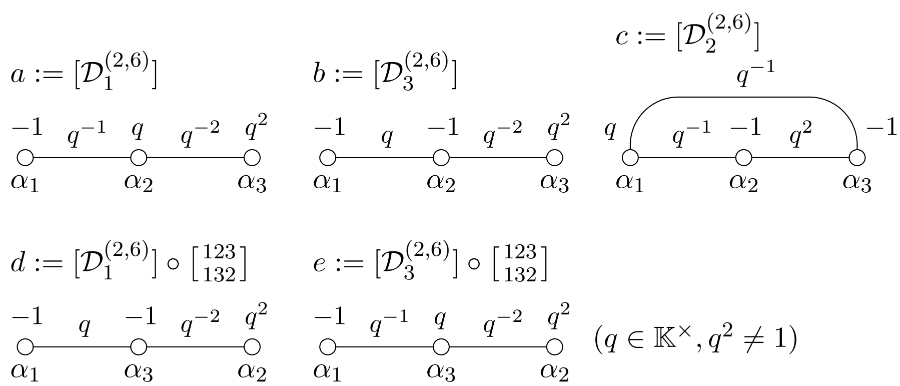


Fig.26: Generalized Dynkin diagrams for [12, Table 2, Row 6]

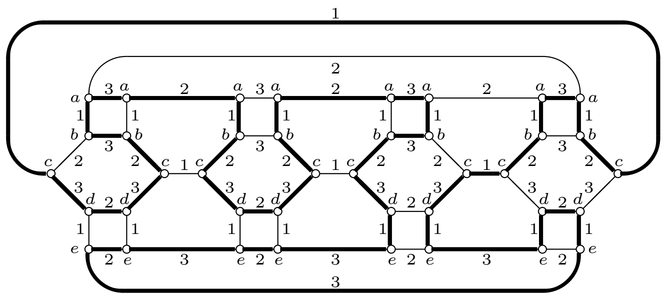


Fig.27: Special Hamilton circuit of [12, Table 2, Row 6]

Rank-3-Case-4: Let χ be of $\mathcal{D}_1^{(2,7)}$. We draw Hamilton circuit maps of $\Gamma(\chi)$ by Fig.29 and Fig.30. Notice that χ is of [12, Table 2, Rows 7].

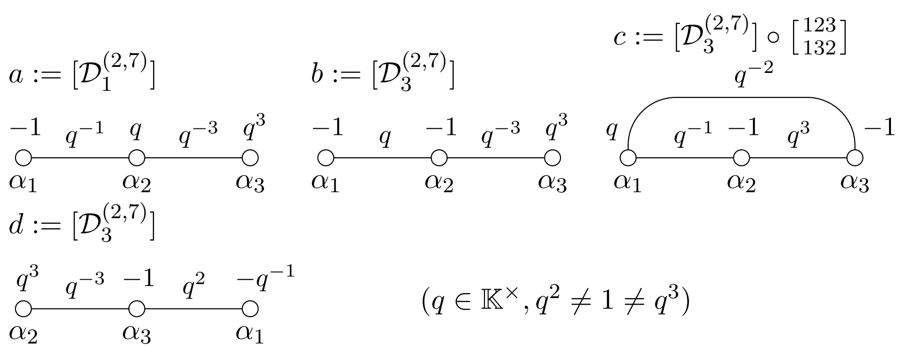


Fig.28: Generalized Dynkin diagrams of [12, Table 2, Row 7]

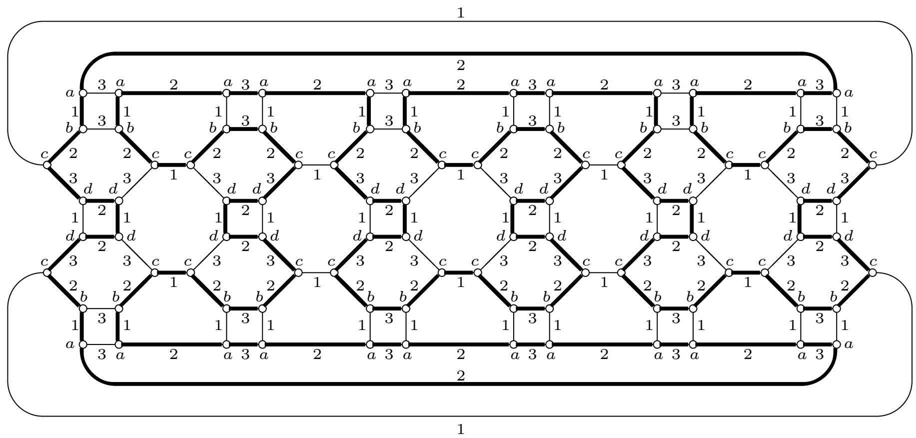


Fig.29: 2-convenient Hamilton circuit of [12, Table 2, Row 7]

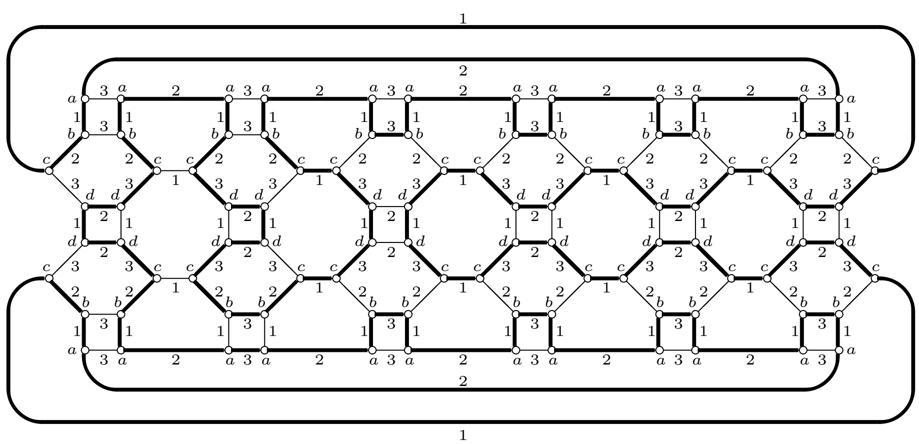


Fig.30: Special Hamilton circuit of [12, Table 2, Row 7]

Rank-3-Case-5: Let χ be of $\mathcal{D}_1^{(2,9)}$, $\mathcal{D}_1^{(2,10)}$ or $\mathcal{D}_1^{(2,11)}$. We draw a Hamilton circuit map of $\Gamma(\chi)$ by Fig.32. Notice that χ is of [12, Table 2, Rows 9,10,11].

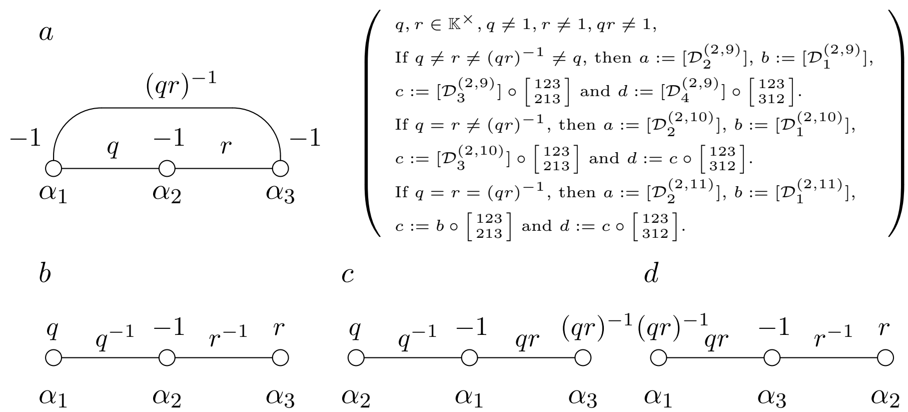


Fig.31: Generalized Dynkin diagrams for [12, Table 2, Rows 9,10,11]

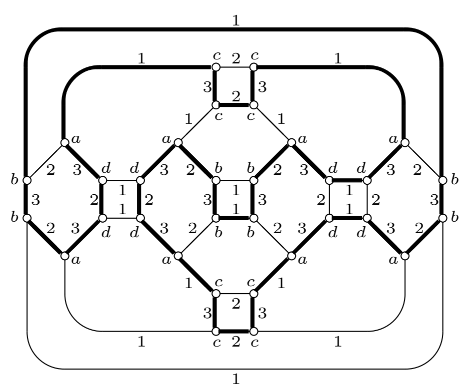


Fig.32: Special and 3-convenient Hamilton circuit for [12, Table 2, Rows 9, 10, 11]

Rank-3-Case-6: Let χ be of $\mathcal{D}_1^{(2,13)}$ or $\mathcal{D}_1^{(2,18)}$. We draw a Hamilton circuit map of $\Gamma(\chi)$ by Fig.35. Notice that χ is of [12, Table 2, Rows 13,18].

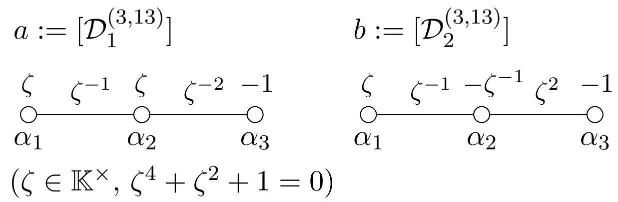


Fig.33: Generalized Dynkin diagrams for [12, Table 2, Rows 13]

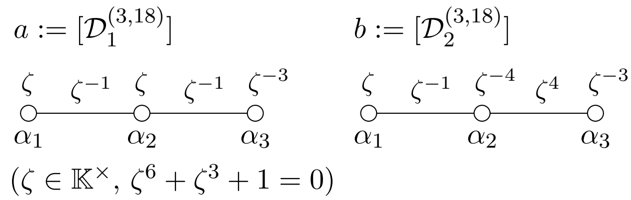


Fig.34: Generalized Dynkin diagrams for [12, Table 2, Rows 18]

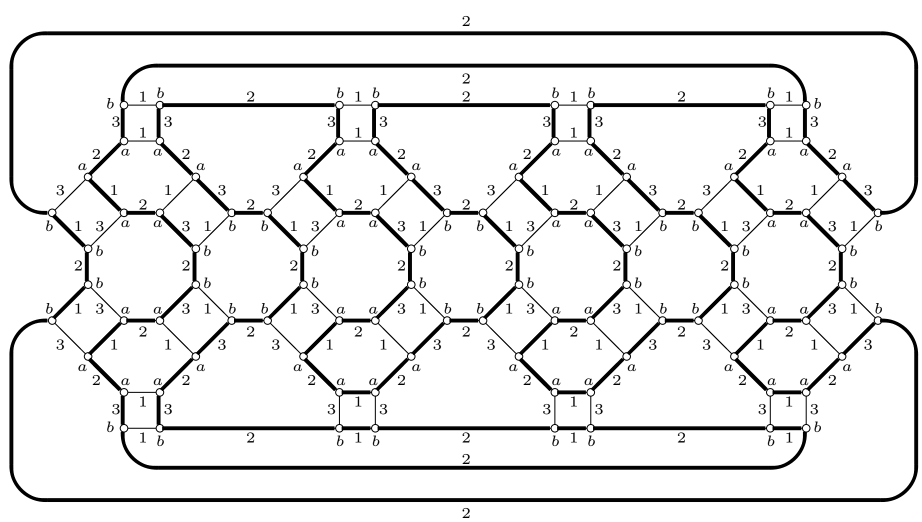


Fig.35: Special and 2-convenient Hamilton circuit of
[12, Table 3, Rows 13,18]

Rank-3-Case-7: Let χ be of $\mathcal{D}_1^{(2,15)}$, where let $\zeta \in \mathbb{K}^\times$ be such that $\zeta^2+\zeta+1=0$. We draw a Hamilton circuit map of $\Gamma(\chi)$ by Fig.37. Notice that χ is of [12, Table 2, Rows 15].

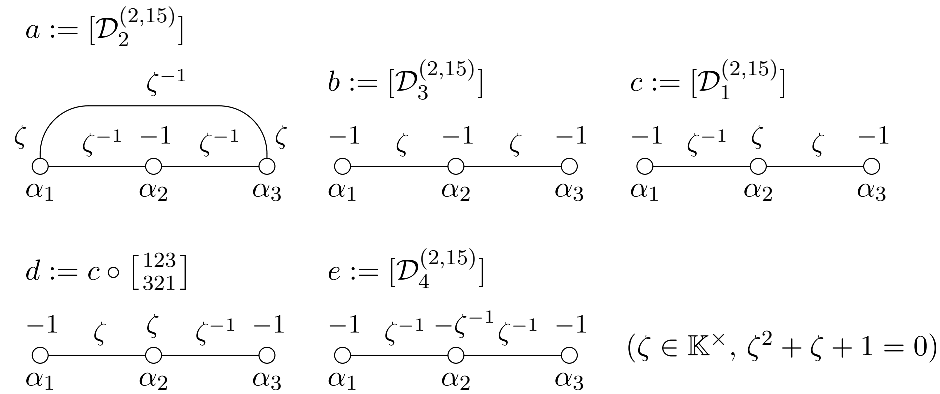


Fig.36: The generalized Dynkin diagrams for [12, Table 2, Row15]

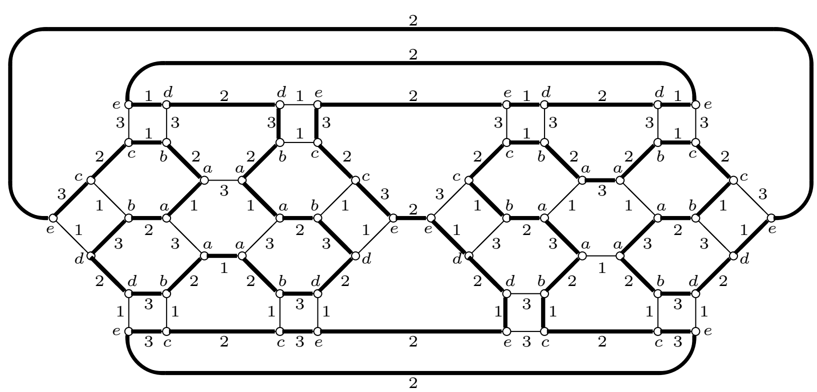


Fig.37: Special and 2-convenient Hamilton circuit of [12, Table 2, Row 15]

Rank-3-Case-8: Let χ be of $\mathcal{D}_1^{(2,16)}$, where let $\zeta \in \mathbb{K}^\times$ be such that $\zeta^2+\zeta+1=0$. We draw a Hamilton circuit map of $\Gamma(\chi)$ by Fig.38. Notice that χ is of [12, Table 2, Rows 16].

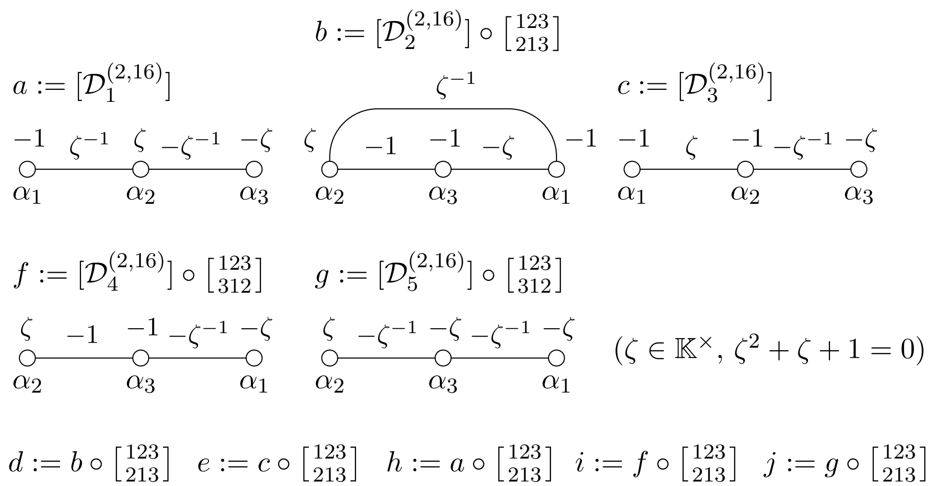


Fig.38: The generalized Dynkin diagrams for [12, Table 2, Row16]

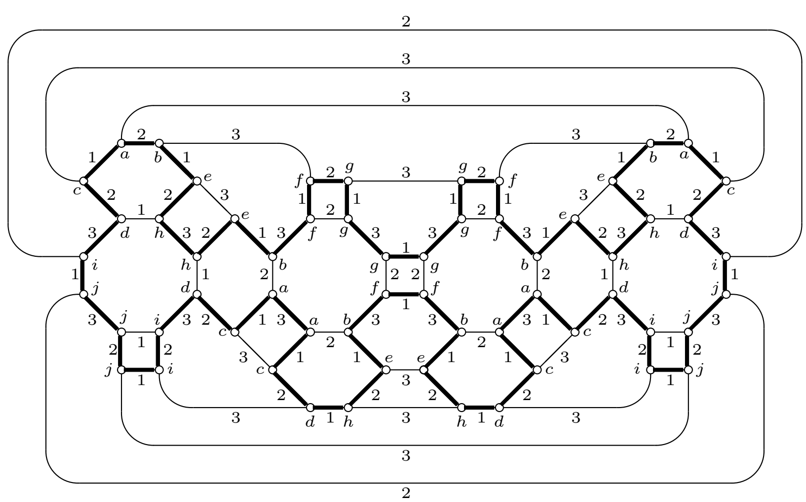
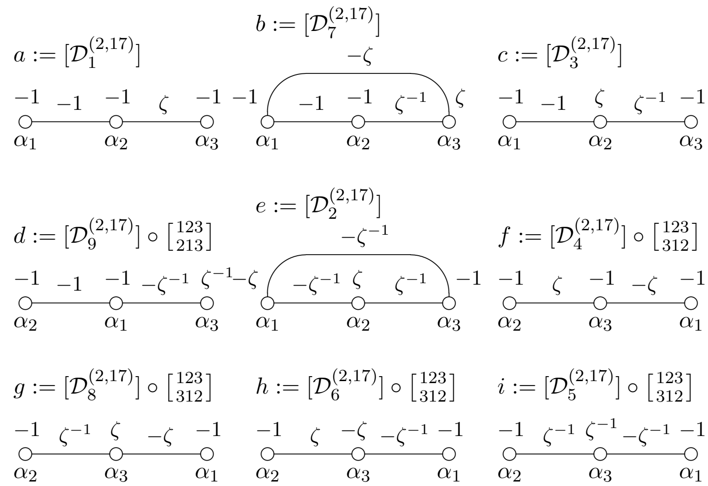


Fig.39: Hamilton circuit for [12, Table 2, Row 16]

Rank-3-Case-9: Let χ be of $\mathcal{D}_1^{(2,17)}$, where let $\zeta \in \mathbb{K}^{\times}$ be such that $\zeta^2+\zeta+1=0$. We draw a Hamilton circuit map of $\Gamma(\chi)$ by Fig.41. Notice that χ is of [12, Table 2, Rows 17].



$(\zeta \in \mathbb{K}^{\times},\ \zeta^2+\zeta+1=0)$

Fig.40: Generalized Dynkin diagrams in [12, Table 2, Row 17]

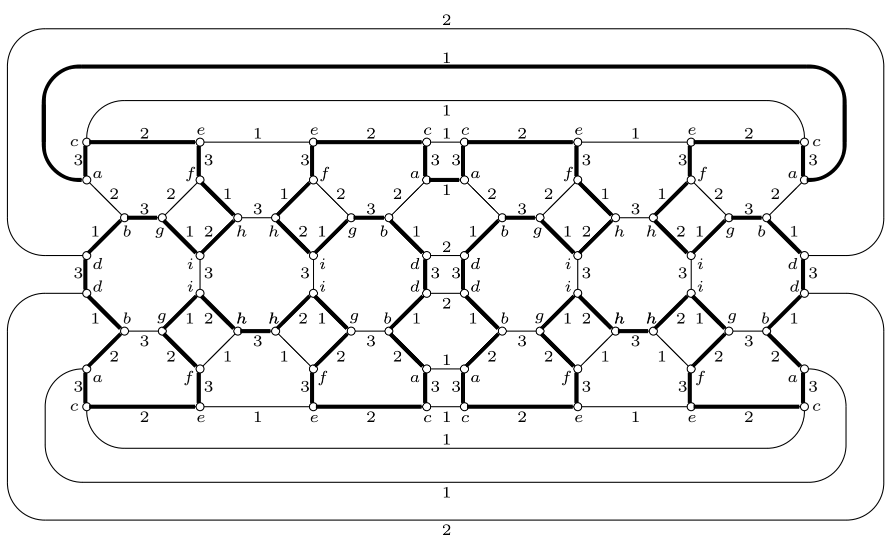


Fig.41: Hamilton Circuit for [12, Table 2, Row 17]

5. Hamilton Circuits of Rank-4 Cases

In this section, we directly draw a Hamilton circuit map of $\Gamma(\chi)$ of each irreducible $\chi \in \mathcal{X}_\Pi$ with $|R^+(\chi)| < \infty$ and $|I| = 4$.

In this section, analyzing carefully in each case, we shall see:

Theorem 5.1. *Assume that* $|I| = 4$. *Then for every* χ *of* [12, Table 3], *a Hamilton circuit map of* $\Gamma(\chi)$ *exists. In fact,* (up *to elements of* $\bar{\mathcal{G}}(\chi)$) *it is drawn by* Fig.8 ($\mathcal{D}_\bullet^{(3,1)}$, $\mathcal{D}_\bullet^{(3,6)}$, $\mathcal{D}_\bullet^{(3,10)}$), Fig.4 ($\mathcal{D}_\bullet^{(3,2)}$, $\mathcal{D}_\bullet^{(3,3)}$, $\mathcal{D}_\bullet^{(3,7)}$, $\mathcal{D}_\bullet^{(3,11)}$, $\mathcal{D}_\bullet^{(3,15)}$, $\mathcal{D}_\bullet^{(3,16)}$, $\mathcal{D}_\bullet^{(3,19)}$), Fig.10 ($\mathcal{D}_\bullet^{(3,4)}$), Fig.9 ($\mathcal{D}_\bullet^{(3,5)}$), Fig.45 ($\mathcal{D}_\bullet^{(3,8)}$), Fig.48 ($\mathcal{D}_\bullet^{(3,12)}$), Fig.52 ($\mathcal{D}_\bullet^{(3,13)}$), Fig.55 ($\mathcal{D}_\bullet^{(3,20)}$), Fig.58 ($\mathcal{D}_\bullet^{(3,21)}$), Fig.61 ($\mathcal{D}_\bullet^{(3,9)}$), Fig.64 ($\mathcal{D}_\bullet^{(3,14)}$), Fig.67 ($\mathcal{D}_\bullet^{(3,17)}$), Fig.74 ($\mathcal{D}_\bullet^{(3,18)}$) *and* Fig.81 ($\mathcal{D}_\bullet^{(3,22)}$).

Rank-4-Case-1: Let χ be of [12, Table 3, Rows 1-7,10,11,15,16,19]. Then it is of Cartan-type or of quasi-Cartan-type. See Fig.42. If χ is of $\mathcal{D}_1^{(3,1)}$ (resp. $\mathcal{D}_1^{(3,2)}$, $\mathcal{D}_1^{(3,3)}$, $\mathcal{D}_1^{(3,4)}$, $\mathcal{D}_1^{(3,5)}$), then it is of Cartan-type of type A_4 (resp. B_4, resp. C_4, resp. F_4, resp. D_4), a Hamilton circuit of $\Gamma(\chi)$ is given by Fig.8 (resp. Fig.4, resp. Fig.4, resp. Fig.10, resp. Fig.9). If χ is of $\mathcal{D}_1^{(3,1)}$ (resp. $\mathcal{D}_1^{(3,2)}$), we have $\chi \cong \chi'$ for χ' of $\mathcal{D}_1^{(3,6)}$ and $\mathcal{D}_1^{(3,10)}$ (resp. $\mathcal{D}_1^{(3,3)}$, $\mathcal{D}_1^{(3,7)}$, $\mathcal{D}_1^{(3,11)}$, $\mathcal{D}_1^{(3,15)}$, $\mathcal{D}_1^{(3,16)}$ and $\mathcal{D}_1^{(3,19)}$), where χ' is of quasi-Cartan-type of type A_4 (resp. B_4).

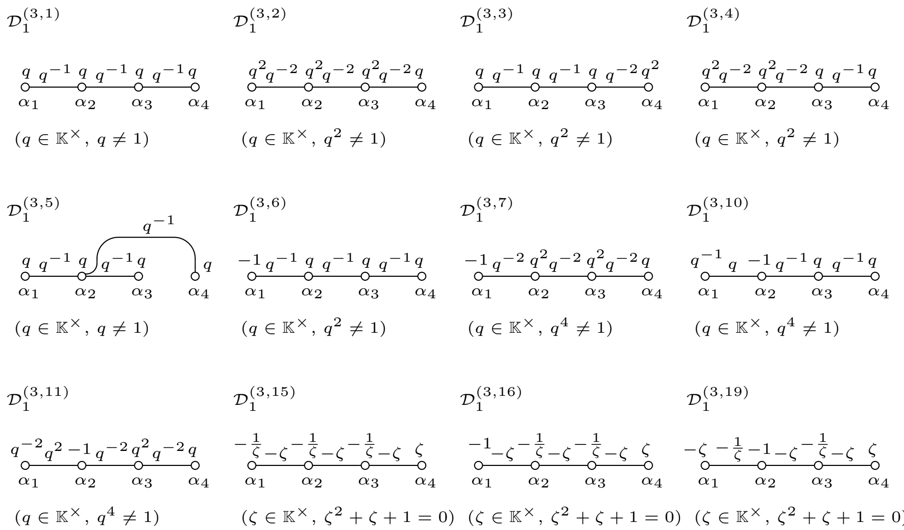
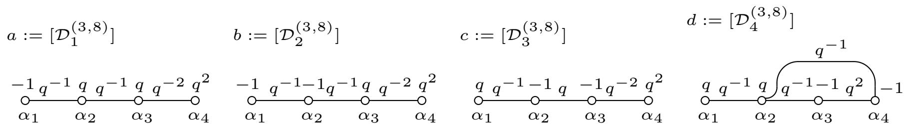


Fig.42: First diagrams of [12, Table 3, Rows 1-7,10,11,15,16,19]

Rank-4-Case-2: Let χ be of [12, Table 3, Rows 8, 12, 13, 20, 21]. As mentioned by Table 1, it is of $(1,2)$-special, whence a Hamilton circuit map of $\Gamma(\chi)$ exists by Proposition 3.27.

Table 1: Rank-4 $(1,2)$-special bicharacters

χ	$\mathcal{D}_1^{(3,8)}$	$\mathcal{D}_1^{(3,12)}$	$\mathcal{D}_1^{(3,13)}$	$\mathcal{D}_1^{(3,20)}$	$\mathcal{D}_1^{(3,21)}$
Fig.s	43, 44	46, 47	49, 50,	53, 54	56, 57
Extracted α_i	α_1	α_1	α_1	α_1	α_1
Rep.s	$\{a,b,g\}$	$\{a,c\}$	$\{a,e\}$	$\{c,\bar{d}\}$	$\{\bar{a},c\}$
Fig.s for $\chi_{\langle i\rangle}$	25, 27, 25,	27, 32	32, 21	37, 27	35, 37
Special	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)



$(q \in \mathbb{K}^{\times},\ q \neq -1)$

$e := c \circ \begin{bmatrix} 1234 \\ 1243 \end{bmatrix}$ $f := b \circ \begin{bmatrix} 1234 \\ 1243 \end{bmatrix}$ $g := a \circ \begin{bmatrix} 1234 \\ 1243 \end{bmatrix}$

Fig.43: Generalized Dynkin diagram of [12, Table 3, Row 8]

$$a \overset{\bar{\tau}_1}{\longleftrightarrow} b \overset{\bar{\tau}_2}{\longleftrightarrow} c \overset{\bar{\tau}_3}{\longleftrightarrow} d \overset{\bar{\tau}_4}{\longleftrightarrow} e \overset{\bar{\tau}_2}{\longleftrightarrow} f \overset{\bar{\tau}_1}{\longleftrightarrow} g$$

Fig.44: Changing of diagrams of Fig.43

$(s_3^a)(s_2^a)s_3^a s_2^a s_3^a (s_4^a) s_3^a s_2^a s_3^a s_4^a s_3^a s_2^a s_3^a s_4^a s_3^a s_4^a s_3^a s_2^a s_3^a s_4^a s_3^a s_4^a s_3^a s_4^a s_3^a s_2^a s_3^a s_2^a s_3^a s_4^a s_3^a s_2^a s_3^a$
$\cdot s_4^a s_3^a s_2^a s_3^a s_4^a s_3^a s_4^a$
$\cdot s_3^a s_2^a s_3^a s_4^a s_3^a s_4^a s_3^a (s_1^b)(s_2^c)(s_3^d)(s_2^d)(s_4^e)(s_3^e)(s_4^d)(s_3^c)(s_4^c)(s_2^b)(s_4^b)(s_3^b) s_2^c s_3^d s_4^e s_3^e s_4^d s_3^c s_2^b$
$\cdot s_3^b s_4^b s_2^c s_4^c s_3^d s_4^e (s_2^f)(s_4^f)(s_3^f) s_4^f s_3^f s_4^f (s_2^e) s_3^e$
$\cdot s_2^f s_4^f s_2^e s_4^d s_2^d (s_1^d) s_2^d s_4^e s_3^e s_4^d s_3^c s_4^c s_2^b s_4^b s_3^b s_2^c s_3^d s_4^e s_3^e s_4^d s_3^c s_2^b s_3^b s_4^b s_2^c s_4^c s_3^d s_4^e s_2^f s_4^f s_3^f s_4^f s_3^f s_4^f s_2^e$
$\cdot s_3^e s_2^f s_4^f s_2^e (s_1^e)$
$\cdot s_3^e s_4^d s_3^c s_4^c s_2^b s_4^b s_3^b s_2^c s_3^d s_4^e s_3^e s_4^d s_3^c s_2^b s_3^b s_4^b s_2^c s_4^c s_3^d s_4^e s_2^f s_4^f s_3^f s_4^f s_3^f s_4^f s_2^e s_3^e s_2^f s_4^f s_2^e s_1^e s_3^e s_4^d s_3^c s_4^c$
$\cdot s_2^b s_4^b s_3^b s_2^c$
$\cdot s_3^d s_4^e s_3^e s_4^d s_3^c s_2^b s_3^b s_4^b s_2^c s_4^c s_3^d s_4^e s_2^f s_4^f s_3^f s_4^f s_3^f s_4^f s_2^e s_3^e s_2^f s_4^f s_2^e s_1^e s_3^e s_4^d s_3^c s_4^c s_2^b s_4^b s_3^b s_2^c s_3^d s_4^e s_3^e s_4^d$
$\cdot s_3^c s_2^b s_3^b s_4^b$
$\cdot s_2^c s_4^c s_3^d s_4^e s_2^f s_4^f s_3^f s_4^f s_3^f s_4^f s_2^e s_3^e s_2^f (s_1^g)(s_3^g)(s_4^g)(s_2^g) s_4^g s_2^g s_4^g s_3^g s_4^g s_2^g s_4^g s_3^g s_4^g s_2^g s_4^g s_3^g s_4^g$
$\cdot s_3^g s_4^g s_2^g s_4^g s_3^g s_4^g s_3^g s_4^g s_3^g s_4^g$
$\cdot s_2^g (s_1^f) s_2^e s_4^d s_2^d s_3^c s_2^b s_4^b s_2^c s_3^d s_2^d s_4^e s_3^e s_4^d s_3^c s_4^c s_2^b s_4^b s_3^b s_2^c s_3^d s_4^e s_3^e s_4^d s_3^c s_2^b s_3^b s_4^b s_2^c s_4^c s_3^d s_4^e s_2^f s_4^f$
$\cdot s_3^f s_4^f s_3^f s_4^f s_2^e s_3^e$
$\cdot s_2^f s_1^g s_2^g s_4^g s_3^g s_4^g s_2^g s_4^g s_3^g s_4^g s_2^g s_4^g s_3^g s_4^g s_3^g s_4^g s_2^g s_4^g s_3^g s_4^g s_3^g s_1^f s_2^e s_4^d s_2^d s_3^c s_2^b s_4^b s_2^c s_3^d s_2^d s_1^d s_2^d s_3^c s_2^b s_4^b$
$\cdot s_2^c s_3^d s_2^d s_1^d$
$\cdot s_2^d s_3^c s_2^b s_4^b s_2^c s_3^d s_2^d s_1^d s_2^d s_3^c s_2^b s_4^b s_2^c (s_1^c) s_2^b (s_1^a)$

Fig.45: Special Hamilton circuit of [12, Table 3, Row 8], where (s_i^x)'s mean that it is really special (Length = 336)

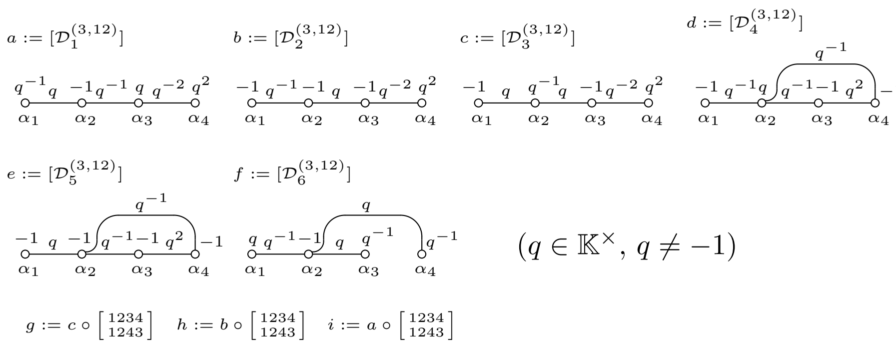


Fig.46: Generalized Dynkin diagram of [12, Table 3, Row 12]

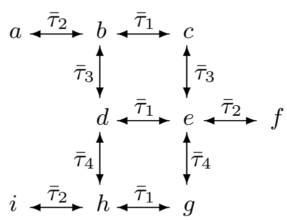
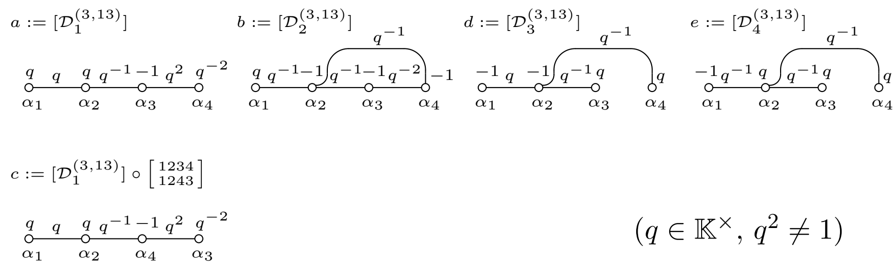


Fig.47: Changing of diagrams of Fig.46

$$
\begin{aligned}
&(s_2^f)(s_4^f)(s_3^f)s_4^f(s_2^e)(s_4^g)(s_2^g)(s_4^e)(s_3^c)(s_4^c)(s_2^c)s_4^c(s_3^e)s_4^g(s_3^g)s_4^e s_2^f s_4^f s_3^f s_4^f s_2^e \\
&\cdot s_4^g s_2^g s_4^e s_3^c s_4^c s_2^c s_4^c s_3^e s_4^g s_3^g (s_1^h)(s_3^h)(s_4^d)(s_3^b)(s_2^a)(s_3^a)(s_4^a)(s_2^b)(s_4^b) \\
&\cdot (s_3^d)(s_4^h)s_3^h s_4^d(s_2^d)s_3^b s_2^a s_4^a s_2^b s_3^d s_2^d s_4^h(s_2^i)(s_4^i)(s_2^h)s_3^h s_2^i s_4^i(s_3^i)s_4^i s_3^i s_4^i s_2^h s_4^d s_3^b s_4^b s_2^a s_4^a \\
&\cdot (s_1^a)s_4^a s_2^b s_4^b s_3^d s_4^h s_3^h s_4^d s_2^d s_3^b s_2^a s_4^a \\
&\cdot s_2^b s_3^d s_2^d s_4^h s_2^i s_4^i s_2^h s_3^h s_2^i s_4^i s_3^i s_4^i s_3^i s_4^i s_2^h s_4^d s_3^b(s_1^c)s_3^e s_4^g s_3^g s_4^e s_2^f s_4^f s_3^f s_4^f s_2^e s_4^g s_2^g s_4^e \\
&\cdot s_3^c s_4^c s_2^c s_4^c s_3^e s_4^g s_3^g s_4^e s_2^f s_4^f \\
&\cdot s_3^f s_4^f s_2^e(s_1^d)s_3^b s_2^a s_3^a s_4^a s_2^b s_4^b s_3^d s_4^h s_3^h s_4^d s_2^d s_3^b s_2^a s_4^a s_2^b s_3^d s_2^d s_4^h s_2^i s_4^i s_2^h s_3^h s_2^i s_4^i s_3^i s_4^i \\
&\cdot s_3^i s_4^i s_2^h s_4^d s_3^b s_4^b s_2^a s_4^a s_1^a s_4^a \\
&\cdot s_2^b s_4^b s_3^d s_4^h s_3^h s_4^d s_2^d s_3^b s_2^a s_4^a s_2^b s_3^d s_2^d s_4^h s_2^i s_4^i s_2^h(s_1^g)s_4^e s_2^f s_4^f s_3^f s_4^f s_2^e s_4^g s_2^g s_4^e s_3^c s_4^c s_2^c \\
&\cdot s_4^c s_3^e s_4^g s_3^g s_4^e s_2^f s_4^f s_3^f(s_1^f)s_2^e \\
&\cdot s_4^g s_2^g s_4^e s_3^c s_4^c s_2^c s_4^c s_3^e s_4^g s_3^g s_4^e s_2^f s_4^f s_3^f s_4^f s_2^e s_4^g s_2^g s_4^e s_3^c s_4^c s_2^c s_4^c s_3^e s_4^g s_3^g s_4^e s_2^f s_4^f s_3^f \\
&\cdot s_1^f s_2^e s_4^g s_2^g s_4^e s_3^c s_4^c s_2^c s_4^c s_3^e \\
&\cdot s_4^g s_1^h s_2^i s_4^i s_3^i s_4^i s_3^i s_4^i s_2^h s_4^d s_3^b s_4^b s_2^a s_4^a s_3^a s_2^b s_3^d s_4^h s_3^h s_4^d s_3^b s_2^a s_1^a s_2^b s_3^d s_4^h s_3^h s_1^g s_2^g s_4^e \\
&\cdot s_3^c s_4^c s_2^c(s_1^b)s_2^a s_4^a s_3^a s_2^b s_3^d s_4^h \\
&\cdot s_3^h s_4^d s_3^b s_2^a s_1^a s_2^b s_3^d(s_1^e)
\end{aligned}
$$

Fig.48: Hamilton circuit of [12, Table 3, Row 12] missing s_1^i, where (s_i^x)'s mean that s_v^u with $(u,v) \neq (i,1)$ exist (Length = 288)



$(q \in \mathbb{K}^\times,\ q^2 \neq 1)$

Fig.49: Generalized Dynkin diagram of [12, Table 3, Row 13]

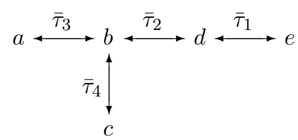


Fig.50: Changing of diagrams of Fig.49

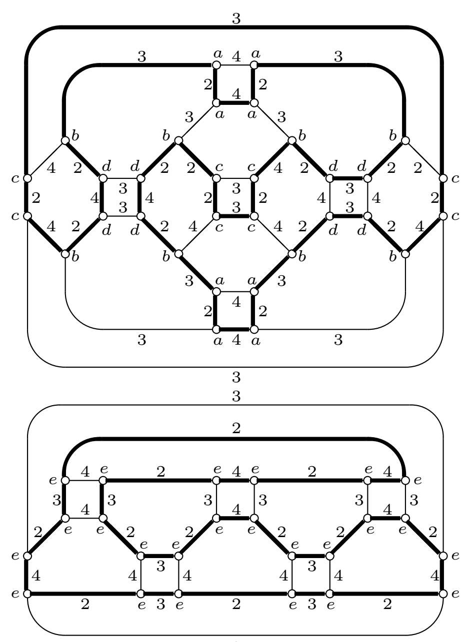


Fig.51: Parts of Hamilton circuit of [12, Table 3, Row 13]
being 2-convenient and special

$$(s_2^a)(s_4^a)(s_3^b)(s_4^c)(s_2^c)(s_4^b)(s_2^d)(s_4^d)(s_3^d)s_4^d(s_2^b)s_4^c(s_3^c)s_4^b(s_3^a)s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d s_4^d s_3^d$$
$$\cdot s_4^d s_2^b s_4^c s_3^c s_4^b s_3^a (s_1^a) s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d s_4^d$$
$$\cdot s_3^d s_4^d s_2^b s_4^c s_3^c s_4^b s_3^a s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d s_4^d s_3^d (s_1^e)(s_2^e)(s_3^e) s_2^e (s_4^e) s_2^e s_4^e s_2^e s_3^e s_2^e s_3^e s_2^e s_4^e$$
$$\cdot s_2^e s_3^e s_2^e s_4^e s_2^e s_4^e s_2^e (s_1^d) s_4^d s_2^b$$
$$\cdot s_4^c s_3^c s_4^b s_3^a s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d s_4^d s_3^d s_4^d s_2^b s_4^c s_3^c s_4^b s_3^a s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c (s_1^c) s_3^c s_4^b s_3^a$$
$$\cdot s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d s_4^d s_3^d$$
$$\cdot s_4^d s_2^b s_4^c s_3^c s_4^b s_3^a s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c s_1^c s_3^c s_4^b s_3^a s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d s_4^d s_3^d s_4^d s_2^b s_4^c s_3^c$$
$$\cdot s_4^b s_3^a s_1^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d$$
$$\cdot s_4^d s_3^d s_1^e s_2^e s_3^e s_2^e s_4^e s_2^e s_4^e s_2^e s_3^e s_2^e s_3^e s_2^e s_4^e s_2^e s_3^e s_2^e s_4^e s_2^e s_4^e s_2^e s_3^e s_2^e s_3^e s_2^e s_1^d s_2^b s_4^c s_3^c$$
$$\cdot s_4^b s_3^a s_4^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d$$
$$\cdot s_4^d s_3^d s_4^d s_2^b s_4^c s_3^c s_4^b s_3^a s_1^a s_2^a s_4^a s_3^b s_4^c s_2^c s_4^b s_2^d s_4^d s_3^d s_4^d s_2^b (s_1^b) s_2^d s_4^d s_3^d s_4^d s_2^b s_1^b s_2^d s_4^d s_1^e$$
$$\cdot s_2^e s_3^e s_2^e s_1^d s_2^b s_4^c s_3^c s_4^b s_3^a s_1^a$$

Fig.52: Special Hamilton circuit of [12, Table 3, Row 13] by Fig.51, where (s_i^x)'s mean that it is really special (Length $= 240$)

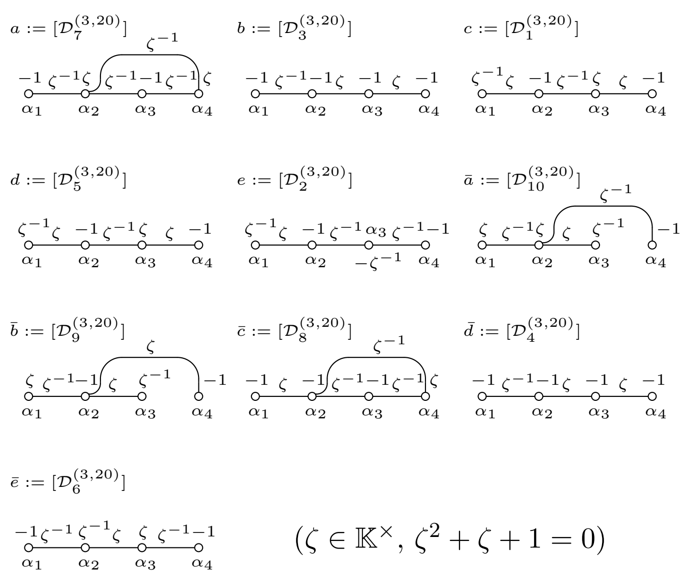


Fig.53: Generalized Dynkin diagrams of [12, Table 3, Row 20]

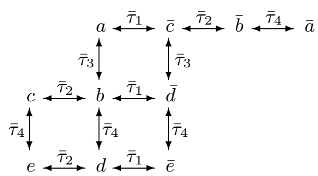


Fig.54: Changing of diagrams of Fig.53

$$S' := (s_3^e)(s_2^d)(s_3^d)(s_2^e)s_3^e(s_4^c)(s_3^c)(s_4^e)s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_4^c s_3^c (s_2^b)(s_3^a)(s_4^a)(s_3^b)$$
$$\cdot(s_2^c)s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b (s_4^d) s_3^d s_2^e s_3^e s_4^c s_3^c s_2^b s_3^a (s_2^a) s_3^b s_4^d s_3^d (s_4^b) s_3^a s_2^a$$
$$\cdot s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d (s_1^{\bar e})(s_4^{\bar d})(s_3^{\bar c})(s_2^{\bar b})(s_3^{\bar b})(s_4^{\bar a})(s_3^{\bar a})(s_2^{\bar a})(s_4^{\bar b})(s_2^{\bar c})(s_3^{\bar d})(s_2^{\bar d}) s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_3^{\bar a}$$
$$\cdot s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} (s_4^{\bar c}) s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} (s_4^{\bar e})(s_3^{\bar e}) s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} (s_2^{\bar e})(s_1^d) s_2^e s_3^e s_2^d$$
$$\cdot s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_4^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b$$
$$\cdot s_3^a s_4^a s_3^b s_4^d s_3^d s_2^e s_3^e s_4^c s_3^c s_2^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d$$
$$\cdot s_3^d s_4^b (s_1^{\bar d}) s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_3^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e}$$
$$\cdot s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} (s_1^{\bar a}) s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b}$$
$$\cdot s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_3^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} (s_1^a) s_2^a s_3^b s_4^d s_3^d s_2^e s_3^e s_2^d$$
$$\cdot s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_4^c s_3^c s_2^b s_3^a s_4^a$$
$$\cdot s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_4^d s_3^d s_2^e s_3^e s_4^c s_3^c s_2^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b}$$
$$\cdot s_4^{\bar a} s_2^{\bar a} s_3^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d}$$
$$\cdot s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_1^d s_2^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_4^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c$$
$$\cdot s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_4^d s_3^d s_2^e s_3^e s_4^c s_3^c s_2^b s_3^a s_2^a$$
$$\cdot s_3^b s_4^d s_3^d s_4^b s_1^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_3^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e}$$
$$\cdot s_3^{\bar e} s_2^{\bar e} s_1^d s_2^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e$$
$$\cdot s_3^e s_4^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_4^d s_3^d s_2^e s_3^e s_4^c (s_1^c) s_2^b s_3^a s_2^a s_3^b s_4^d$$
$$\cdot s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_2^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c$$
$$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_4^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_4^d$$
$$\cdot s_1^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_3^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_3^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c}$$

Fig.55′: Definition of S' of Fig.55

$S' \cdot s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_1^{d} s_3^{d} s_2^{e} s_3^{e} s_2^{d} s_3^{d} s_2^{e} s_3^{e} s_4^{c} s_3^{c} s_4^{e} s_3^{e} s_2^{d} s_3^{d} s_2^{e} s_3^{e} s_4^{c}$
$\cdot s_3^{c} s_4^{e} s_3^{e} s_4^{c} s_3^{c} s_2^{b} s_3^{a} s_4^{a} s_3^{b} s_2^{c} s_3^{c} s_2^{b} s_3^{a} s_4^{a} s_3^{b} s_2^{c} s_3^{c} s_2^{b} s_3^{a} s_4^{a}$
$\cdot s_3^{b} s_4^{d} s_3^{d} s_2^{e} s_3^{e} s_4^{c} s_3^{c} s_2^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_1^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_3^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c}$
$\cdot s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_2^{\bar e}$
$\cdot s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_1^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_2^{\bar e}$
$\cdot s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_1^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c}$
$\cdot s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_1^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_1^{\bar a} s_2^{\bar a} s_3^{\bar a}$
$\cdot s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_1^{d}$
$\cdot s_2^{e} s_3^{e} s_2^{d} s_3^{d} s_2^{e} s_3^{e} s_4^{c} s_3^{c} s_4^{e} s_3^{e} s_2^{d} s_3^{d} s_2^{e} s_3^{e} s_4^{c} s_3^{c} s_4^{e} s_3^{e} s_4^{c} s_3^{c} s_2^{b} s_3^{a} s_4^{a} s_3^{b} s_2^{c} s_3^{c} s_2^{b} s_3^{a} s_4^{a} s_3^{b}$
$\cdot s_2^{c} s_3^{c} s_2^{b} s_3^{a} s_4^{a} s_3^{b} s_4^{d} s_1^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_3^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c}$
$\cdot s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_3^{\bar a} s_4^{\bar b} s_3^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_4^{\bar c} s_2^{\bar b} s_4^{\bar a} s_3^{\bar a} s_4^{\bar b} s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_2^{\bar e} s_1^{d} s_2^{e} s_3^{e}$
$\cdot s_4^{c} s_3^{c} s_2^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_1^{\bar e}$
$\cdot s_2^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_3^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} (s_1^{\bar b}) s_2^{\bar c} s_4^{\bar c} s_3^{\bar d} s_4^{\bar e} s_3^{\bar e} s_4^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_2^{\bar e}$
$\cdot s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_3^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_1^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c}$
$\cdot s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_1^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_3^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_1^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_1^{a} s_2^{a} s_3^{b} s_1^{\bar d} s_2^{\bar d} s_4^{\bar e} s_3^{\bar e} s_2^{\bar e} s_3^{\bar e} s_2^{\bar e}$
$\cdot s_1^{d} s_2^{e} s_3^{e} s_4^{c} s_1^{c} s_2^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_3^{a} s_2^{a}$
$\cdot s_3^{b} s_4^{d} s_1^{\bar e} s_2^{\bar e} s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b} s_3^{\bar b} s_4^{\bar a} s_3^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_1^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_3^{a} s_2^{a} s_3^{b} s_4^{d} s_1^{\bar e} s_2^{\bar e} s_3^{\bar e} s_4^{\bar d} s_3^{\bar c} s_2^{\bar b}$
$\cdot s_3^{\bar b} s_4^{\bar a} s_3^{\bar a} s_2^{\bar a} s_4^{\bar b} s_2^{\bar c} s_1^{a} s_2^{a} s_3^{b} s_4^{d} s_3^{d} s_4^{b} s_1^{\bar d} s_2^{\bar d} s_3^{\bar c} s_2^{\bar b} s_4^{\bar a} s_2^{\bar a} s_1^{\bar a} s_2^{\bar a}$
$\cdot s_4^{b} s_2^{\bar c} s_1^{a} s_2^{a} s_3^{b} s_4^{d} s_1^{\bar e} s_2^{\bar e} s_1^{d} s_2^{e}$

Fig.55: Hamilton circuit of [12, Table 3, Row 20] missing s_1^b, s_1^e and $s_1^{\bar c}$
(In this circuit, there exists s_i^x for every $(x, i) \in \{a, \dots, \bar e\} \times J_{2,4}$.)
(Length = 960)

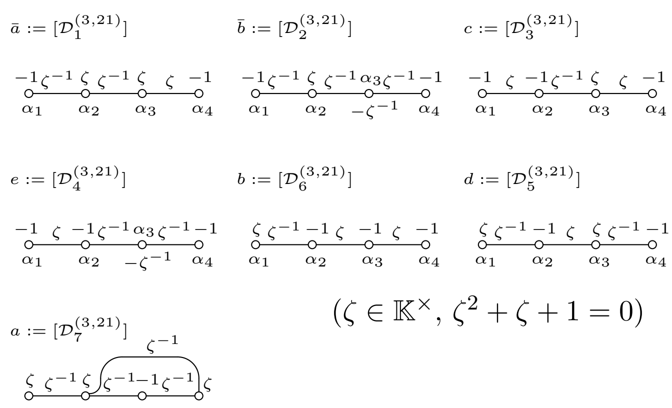


Fig.56: Generalized Dynkin diagrams of [12, Table 3, Row 21]

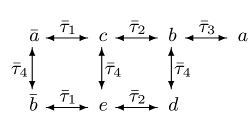


Fig.57: Changing of diagrams of Fig.56

$S'' := (s_3^y)(s_4^x)(s_3^x)(s_4^y)s_3^y(s_2^y)s_3^y s_2^y s_3^y s_4^x s_3^x(s_2^x)s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y$

$\cdot s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y$

$\cdot s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y$

$\cdot s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x(s_1^c)(s_3^c)(s_4^e)(s_3^e)(s_2^d)$

$\cdot(s_3^d)(s_2^e)s_3^e s_2^d s_3^d(s_4^b)(s_3^a)(s_2^a)(s_3^b)(s_4^d)s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b(s_2^c)s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a(s_4^a)$

$\cdot s_3^b s_2^c s_3^c(s_2^b)s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e(s_4^c)s_3^c s_4^e$

$\cdot s_3^e s_2^d(s_1^d)s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c$

$\cdot s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d$

$\cdot s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b(s_1^b)s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e$

$\cdot s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b$

$\cdot s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_2^e(s_1^y)s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y$

$\cdot s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x$

$\cdot s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y$

$\cdot s_3^y s_2^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y(s_1^e)s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b$

$\cdot s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_2^e s_1^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x$

$\cdot s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x$

$\cdot s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x$

$\cdot s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a$

$\cdot s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b$

$\cdot s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b$

$\cdot s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c$

$\cdot s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_1^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x$

$\cdot s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x$

$\cdot s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x$

$\cdot s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_1^e s_2^d s_3^d s_4^b$

$\cdot s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a$

$\cdot s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e$

$\cdot s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e$

$\cdot s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e$

$\cdot s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b$

$\cdot s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e$

$\cdot s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c$

$\cdot s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e$

$\cdot s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b$

Fig.58$''$: Definition of S'' for Fig.58

$S' := s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e$

$\cdot s_3^e s_2^d s_3^d s_2^e s_1^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x$

$\cdot s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x$

$\cdot s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y$

$\cdot s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a$

$\cdot s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b$

$\cdot s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_1^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y$

$\cdot s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y$

$\cdot s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x$

$\cdot s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a$

$\cdot s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c$

$\cdot s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_1^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y$

$\cdot s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y$

$\cdot s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x$

$\cdot s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b$

$\cdot s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c$

$\cdot s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_1^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y$

$\cdot s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x$

$\cdot s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x$

$\cdot s_3^x s_2^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y$

$\cdot s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_1^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a$

$\cdot s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b$

$\cdot s_3^a s_4^a (s_1^a) s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e$

$\cdot s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_1^b s_2^c s_3^c s_4^e s_3^e s_2^d$

$\cdot s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e$

$\cdot s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_1^d s_4^b s_3^a s_2^a s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_1^a s_2^a$

$\cdot s_3^b s_2^c s_3^c s_4^e s_3^e s_2^d s_3^d s_4^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b$

$\cdot s_3^a s_4^a s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d$

$\cdot s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c s_3^c s_2^b s_3^a s_4^a$

$\cdot s_3^b s_2^c s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b$

$\cdot s_3^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d$

$\cdot s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d$

$\cdot s_1^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b$

$\cdot s_3^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_1^b s_2^c s_3^c s_2^b s_3^a s_4^a s_3^b s_2^c$

$\cdot s_3^c s_4^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_3^e s_4^c (s_1^x) s_2^x s_3^x s_4^y s_1^e s_2^d s_3^d s_2^e$

$\cdot s_1^y s_2^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a$

$\cdot s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_4^c$

$\cdot s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_1^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_1^e s_2^d s_3^d s_2^e$

$\cdot s_1^y s_2^y s_3^y s_4^x s_3^x s_4^y s_3^y s_2^y s_3^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_1^e s_2^d$

$\cdot s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a s_3^b s_4^d s_1^d s_2^e$

$\cdot s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a$

Fig.58′: Definition of S' for Fig.58

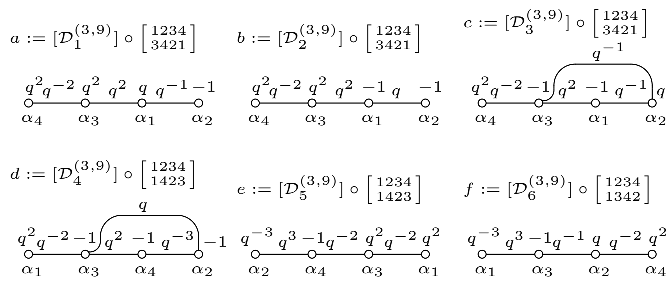
$S'' \cdot S' \cdot s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a$
$\cdot s_1^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e$
$\cdot s_1^y s_2^y s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a$
$\cdot s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_1^y s_2^y$
$\cdot s_3^y s_2^y s_3^y s_4^x s_3^x s_2^x s_3^x s_4^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a s_3^b s_4^d$
$\cdot s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_4^c s_3^c s_4^e s_3^e s_2^d s_3^d s_2^e s_1^y s_2^y s_3^y s_4^x$
$\cdot s_3^x s_2^x s_3^x s_4^y s_1^e s_2^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a s_1^a s_2^a s_3^b s_4^d s_3^d s_4^b s_3^a s_2^a$
$\cdot s_1^a s_2^a s_3^b s_4^d s_1^d s_2^e s_3^e s_1^y$

Fig.58: Special Hamilton circuit of [12, Table 3, Row 21], where S'' and S' are those of Fig.58$''$ and Fig.58$'$ respectively, and let $x := \bar{a}$ and $y := \bar{b}$ and let (s_i^z)'s mean that this is exactly special (Length = 2688)

Rank-4-Case-3: Let χ be of [12, Table 3, Rows 9, 14, 17]. As mentioned by Table 2, it is of (i, j)-convenient for some i and j, whence a Hamilton circuit map of $\Gamma(\chi)$ exists by Proposition 3.27.

Table 2: Rank-4 (i, j)-convenient bicharacters

χ	$\mathcal{D}_1^{(3,9)}$	$\mathcal{D}_1^{(3,14)}$	$\mathcal{D}_1^{(3,17)}$
Fig.s	59, 60	62, 63,	64, 66
Extracted α_i	α_3	α_1	α_3
Rep.s	$\{a, c\}$	$\{a, h, j\}$	$\{g, b, i\}$
Fig.s for $\chi_{\langle i \rangle}$	16, 32	32, 16, 16	17, 32, 17
Convenient	(3,1)	(1,3)	(3,1)



$(q \in \mathbb{K}^{\times},\ q^2 \neq 1 \neq q^3)$

Fig.59: Generalized Dynkin diagrams of [12, Table 3, Row 9]

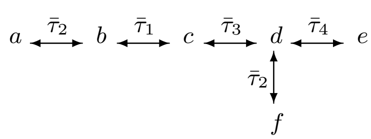
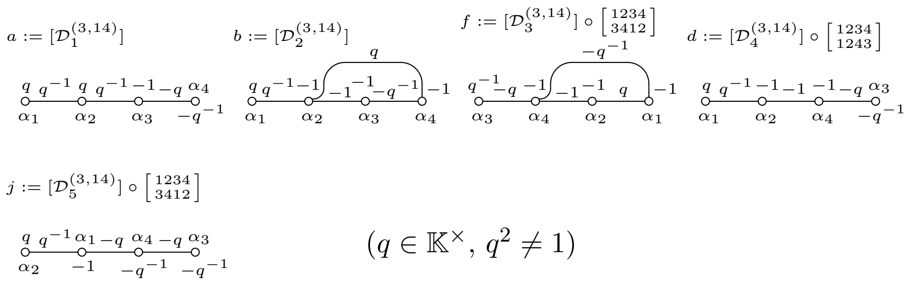


Fig.60: Changing of diagrams of Fig.59

$(s_2^f)(s_4^f)(s_3^f)s_4^f(s_2^d)(s_4^e)(s_2^e)(s_4^d)(s_3^c)(s_4^c)(s_2^c)s_4^c(s_3^d)s_4^e(s_3^e)s_4^d s_2^f s_4^f s_3^f s_4^f$
$\cdot s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e (s_1^e) s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e$
$\cdot s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c$
$\cdot s_2^c (s_1^b)(s_2^a)(s_4^a)(s_3^a) s_4^a s_3^a s_4^a (s_2^b)(s_1^c) s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e$
$\cdot s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a s_4^a s_2^b (s_4^b)(s_3^b) s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e$
$\cdot s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a s_4^a s_2^b s_4^b s_3^b s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e$
$\cdot s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a s_4^a s_2^b s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d$
$\cdot s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a s_4^a s_2^b s_4^b s_3^b s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d$
$\cdot s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a s_4^a s_2^b s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c$
$\cdot s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a s_4^a s_2^b s_4^b s_3^b s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c$
$\cdot s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f (s_1^f) s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_1^f s_2^d s_4^e$
$\cdot s_2^e s_4^d s_3^c s_4^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c s_2^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a (s_1^a) s_2^b s_4^b s_3^b s_4^b s_3^b s_4^b s_2^a s_4^a s_3^a s_1^a s_2^b s_4^b$
$\cdot s_3^b s_4^b s_3^b s_4^b s_2^a s_4^a s_3^a s_1^a s_2^b s_4^b s_3^b s_4^b s_3^b s_4^b s_2^a s_1^a s_2^b s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_1^b s_2^a s_4^a s_3^a s_4^a s_3^a s_4^a s_2^b s_4^b$
$\cdot s_3^b s_4^b s_3^b s_1^c s_2^c s_4^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c s_2^c s_1^b s_3^b s_4^b s_3^b s_4^b s_2^a s_4^a s_3^a s_1^a s_2^b s_4^b s_3^b s_4^b s_3^b s_4^b s_2^a s_1^a s_2^b s_4^b$
$\cdot s_3^b s_4^b s_3^b s_4^b s_2^a s_1^a s_2^b s_4^b s_3^b s_4^b s_3^b s_4^b s_2^a s_4^a s_3^a s_4^a s_3^a s_1^a s_3^a s_4^a s_3^a s_1^a s_3^a s_4^a s_3^a s_1^a s_3^a s_1^a s_3^a s_4^a s_3^a s_1^a s_3^a s_1^a s_3^a s_1^a s_2^b s_4^b s_3^b s_4^b$
$\cdot s_3^b s_1^c s_3^d s_4^e s_3^e s_4^d s_2^f s_4^f s_3^f s_1^f s_3^f s_1^f s_3^f s_4^f s_2^d s_4^e s_2^e s_4^d s_3^c s_4^c s_2^c s_1^b s_3^b s_1^c s_2^c s_1^b s_3^b s_4^b s_3^b s_1^c s_2^c s_1^b s_3^b s_1^c s_2^c s_1^b s_3^b s_4^b s_3^b s_1^c$
$\cdot s_2^c s_1^b s_3^b s_1^c s_2^c s_1^b s_3^b s_1^c s_2^c s_1^b s_3^b s_4^b s_3^b s_1^c s_3^d (s_1^d)$

Fig.61: Special Hamilton circuit of [12, Table 3, Row 9] (Length = 576)



$c := b \circ \left[\begin{smallmatrix}1234\\3214\end{smallmatrix}\right]$ $e := d \circ \left[\begin{smallmatrix}1234\\3214\end{smallmatrix}\right]$ $h := a \circ \left[\begin{smallmatrix}1234\\3214\end{smallmatrix}\right]$ $g := f \circ \left[\begin{smallmatrix}1234\\3214\end{smallmatrix}\right]$ $i := j \circ \left[\begin{smallmatrix}1234\\3214\end{smallmatrix}\right]$

Fig.62: Generalized Dynkin diagrams of [12, Table 3, Row 14]

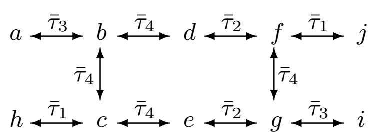


Fig.63: Changing of diagrams of Fig.62

$$(s^b_3)(s^d_4)(s^d_3)(s^f_2)(s^f_3)(s^g_4)(s^i_3)(s^i_2)(s^g_3)(s^e_2)(s^e_3)(s^c_4)(s^c_3)(s^b_2)(s^a_3)(s^a_4)$$
$$\cdot s^b_3 s^d_4 s^d_3 s^f_2 s^f_3 s^g_4 s^i_3 s^i_2 s^g_3 s^e_2 s^e_3 s^c_4 s^c_3 s^b_2 (s^b_1)(s^c_2) s^c_3 (s^e_4) s^e_3 (s^g_2) s^i_3 s^i_2 s^g_3 (s^f_4)$$
$$\cdot s^f_3 (s^d_2) s^d_3 (s^b_4) s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 (s^j_1)(s^j_4)(s^j_3) s^j_4 s^j_3 (s^j_2) s^j_3 s^j_4 (s^f_1)$$
$$\cdot s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4$$
$$\cdot s^f_3 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 (s^h_1)(s^h_2)(s^h_3) s^h_2 s^h_3 (s^h_4) s^h_3 s^h_2 (s^c_1)$$
$$\cdot s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 s^f_3 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4$$
$$\cdot s^j_1 s^j_4 s^j_3 s^j_4 s^j_3 s^j_2 s^j_3 s^j_4 s^f_1 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 s^f_3 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^h_1 s^h_2 s^h_3 s^h_2 s^c_1 s^e_4 s^e_3 s^g_2$$
$$\cdot s^i_3 s^i_2 s^g_3 s^f_4 s^f_3 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 (s^i_1) s^i_2 s^g_3 s^f_4 s^f_3 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 s^f_3 s^d_2$$
$$\cdot s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^h_1 s^h_2 s^h_3 s^h_2 s^h_3 s^h_4 s^c_1 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 s^j_1 s^j_4 s^f_1 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 s^f_3 s^d_2$$
$$\cdot s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 s^j_1 s^j_4 s^j_3 s^j_2 s^j_3 s^j_4 s^j_3 s^j_4 s^j_3 s^j_2 s^f_1 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^c_3 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4$$
$$\cdot s^f_3 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^h_1 s^h_2 s^h_3 s^h_2 s^h_3 s^h_4 s^c_1 s^e_4 s^e_3 s^g_2 (s^g_1) s^f_4 s^f_3 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^h_1 s^h_4 s^h_3 s^h_2 s^h_3 s^h_2 s^h_3 s^h_4 s^c_1 s^e_4 s^e_3 s^g_2$$
$$\cdot s^i_3 s^i_2 s^g_3 s^f_4 s^j_1 s^j_4 s^j_3 s^j_2 s^f_1 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 s^b_3 s^c_2 s^h_1 s^h_2 s^h_3 s^h_4 s^c_1 s^e_4 s^e_3 s^g_2 s^i_3 s^i_2 s^g_3 s^f_4 s^j_1 s^j_4 s^j_3 s^j_2 s^f_1 s^d_2 s^d_3 s^b_4 s^a_3 s^a_4 (s^a_1) s^a_4$$

Fig.64: Hamilton circuit of [12, Table 3, Row 14] missing s^a_2, s^d_1, s^e_1 and s^i_4 (From this Hamilton circuit, by Fig.62, we obtain another Hamilton circuit of [12, Table 3, Row 14] missing s^h_2, s^e_3, s^d_3 and s^j_4. Indeed it is obtained by replacing 1, 3 and $x \in \{a, \ldots, j\}$ by 3, 1 and $x \circ \left[\begin{smallmatrix}1234\\3214\end{smallmatrix}\right]$ respectively.) (Length = 360)

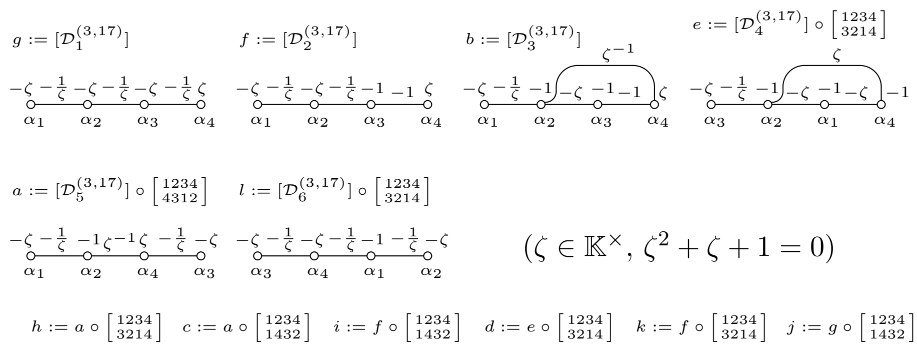


Fig.65: Generalized Dynkin diagrams of [12, Table 3, Row 17]

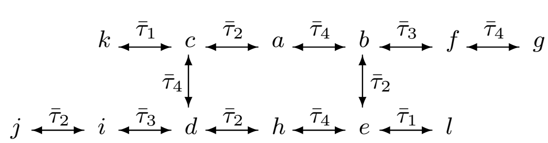


Fig.66: Changing of diagrams of Fig.65

$S' := (s_2^b)(s_1^b)(s_4^a)(s_1^a)(s_2^c)(s_1^k)(s_2^k)(s_1^c)(s_4^d)(s_1^d)(s_2^h)(s_1^h)(s_4^e)(s_1^l)(s_4^l)(s_1^e)s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l$
$\cdot s_4^l(s_3^l)s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$
$\cdot s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d(s_3^i)(s_2^j)(s_1^j)(s_4^j)s_1^j s_4^j s_1^j(s_2^i)(s_1^i)(s_4^i)(s_3^d)s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a$
$\cdot s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b(s_3^f)(s_4^g)(s_1^g)(s_2^g)s_1^g s_2^g s_1^g$
$\cdot(s_4^f)(s_1^f)(s_2^f)(s_3^b)s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j$
$\cdot s_4^j s_1^j s_4^j s_1^j s_2^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k$
$\cdot s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g s_2^g s_1^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d$
$\cdot s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j$
$\cdot s_4^j s_1^j s_2^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g$
$\cdot s_2^g s_1^g s_2^g s_1^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l$
$\cdot s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_1^j s_2^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b$
$\cdot s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g$
$\cdot s_2^g s_1^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g s_2^g s_1^g s_4^f s_3^b$
$\cdot s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k$
$\cdot s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_1^j s_2^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d$
$\cdot s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g s_2^g s_1^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a$
$\cdot s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_1^j s_2^i s_1^i$
$\cdot s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h$
$\cdot s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g s_2^g s_1^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$
$\cdot s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j(s_3^j)s_2^i s_1^i s_4^i s_1^i$
$\cdot s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k$
$\cdot s_2^k(s_3^k)s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_3^k$
$\cdot s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g$
$\cdot s_2^g s_1^g s_2^g s_1^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l$
$\cdot s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_1^j s_2^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b$
$\cdot s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g$
$\cdot s_2^g(s_3^g)s_4^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j$
$\cdot s_4^j s_1^j s_4^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_1^i s_2^j s_1^j s_4^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_1^i$
$\cdot s_2^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$
$\cdot s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g s_2^g s_1^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d$

Fig.67′: Definition of S' for Fig.67

$S' \cdot s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_3^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g$

$\cdot s_2^g s_1^g s_4^f s_1^f s_2^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_3^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h$

$\cdot s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g s_2^g s_3^g s_4^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$

$\cdot s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_3^j s_2^i s_1^i s_4^i s_1^i$

$\cdot s_4^i s_1^i s_2^j s_1^j s_4^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_1^i s_2^j s_1^j s_4^j s_1^j s_4^j s_3^j s_4^j s_3^j s_2^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$

$\cdot s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_1^j s_2^i s_3^d s_2^h s_1^h$

$\cdot s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_3^g s_4^f s_1^f s_2^f s_1^f s_2^f s_1^f s_4^g s_1^g s_2^g s_1^g s_2^g s_3^g s_2^g s_1^g s_2^g s_1^g s_4^f s_1^f s_2^f s_1^f s_2^f s_3^b$

$\cdot s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_3^g s_4^f s_1^f$

$\cdot s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_1^i s_2^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_1^i$

$\cdot s_2^j s_1^j s_4^j s_1^j s_4^j s_3^j s_4^j s_1^j s_4^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l$

$\cdot s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_1^f s_4^g s_1^g s_2^g s_1^g s_2^g s_3^g s_2^g s_1^g$

$\cdot s_2^g s_1^g s_4^f s_1^f s_2^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_4^i s_1^i s_4^i s_3^d$

$\cdot s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_1^f s_4^g s_3^g$

$\cdot s_4^f s_1^f s_2^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$

$\cdot s_2^b s_1^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_4^g s_1^g s_2^g s_1^g s_2^g s_3^g s_2^g s_1^g s_2^g s_3^g s_4^f s_1^f$

$\cdot s_2^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b$

$\cdot s_4^a s_1^a s_2^c (s_3^c) s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_3^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_3^b$

$\cdot s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_3^j s_4^j s_1^j s_4^j s_3^j s_4^j s_3^j$

$\cdot s_2^i s_1^i s_4^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_1^f s_2^f s_1^f s_4^g s_3^g s_4^f s_1^f s_2^f s_1^f s_2^f s_1^f s_4^g s_1^g s_2^g s_1^g$

$\cdot s_2^g s_3^g s_2^g s_1^g s_2^g s_3^g s_4^f s_1^f s_2^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i$

$\cdot s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_3^b s_4^a s_1^a s_2^c s_3^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_1^b s_4^a s_1^a s_2^c s_3^c$

$\cdot s_4^d s_3^i s_2^j s_1^j s_4^j s_1^j s_4^j s_3^j s_2^i s_1^i s_4^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$

$\cdot s_2^b s_3^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c$

$\cdot s_4^d s_3^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_1^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k$

$\cdot s_2^k s_1^c s_4^d s_1^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e$

$\cdot s_2^b s_3^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_4^i s_1^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l$

$\cdot s_4^l s_1^e s_2^b s_3^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k s_2^k s_1^c s_4^d s_3^i s_4^i s_3^d s_2^h s_1^h s_4^e s_1^l s_4^l s_1^e s_2^b s_3^f s_2^f s_3^b s_4^a s_1^a s_2^c s_1^k$

$\cdot s_2^k s_1^c s_4^d s_3^i s_4^i s_3^d s_2^h s_1^h s_4^e (s_3^e)$

Fig.67: Hamilton circuit of [12, Table 3, Row 17]
missing s_4^b, s_2^e, s_2^a, s_3^a, s_3^h, s_4^h, s_4^c, s_2^d, s_4^k and s_2^l (Length = 1440)

Rank-4-Case-3: Let χ be of $\mathcal{D}_1^{(3,18)}$. Use the symbols in Fig.68. For $t \in \{a, b, c, d\}$ of Fig.68, let t' be the one obtained from t by changing α_1 and α_4. Let $\chi_1 := \chi \in \mathcal{X}_\Pi$. Then $\pi_\chi(\chi_1) = a$. Let $\chi_2 := \tau_1\tau_2\tau_3\tau_4\chi \in \mathcal{G}(\chi)$ and $\chi_3 := \tau_1\tau_2\tau_3\tau_4\chi_2 \in \mathcal{G}(\chi)$. Then $\pi_\chi(\chi_2) = e$ and $\pi_\chi(\chi_3) = a'$. By Fig.69, $\{\chi_1, \chi_2, \chi_3\}$ is a 1-complete $\bar{\tau}$-representative subset of $\mathcal{G}(\chi)$. Moreover Hamilton circuit maps of $\Gamma((\chi_1)_{\langle 1\rangle})$, $\Gamma((\chi_2)_{\langle 1\rangle})$, $\Gamma((\chi_3)_{\langle 1\rangle})$ are drawn in by Fig.70, Fig.71, Fig.72 respectively. Then by Proposition 3.21 and Fig.73, we see an existence of a Hamilton circuit map for χ. In fact, it is written in Fig.74.

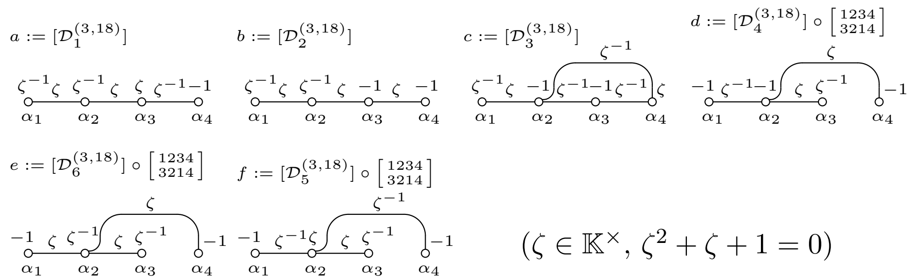


Fig.68: Generalized Dynkin diagrams of [12, Table 3, Row 18]

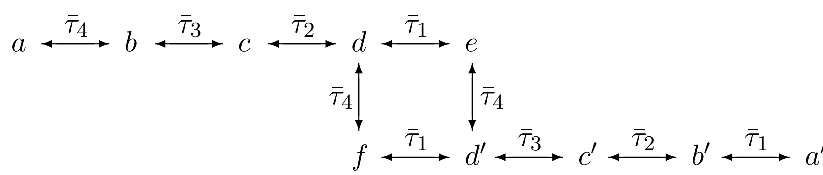


Fig.69: Changing of diagrams of Fig.68

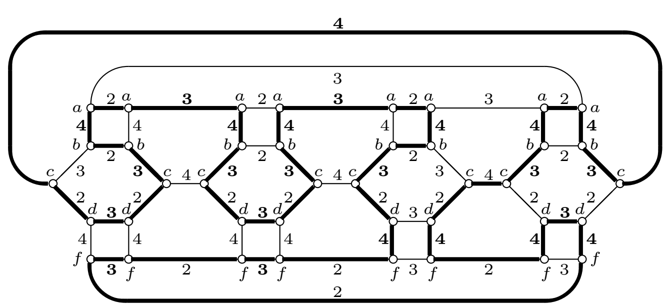


Fig.70: First part of Hamilton circuit for [12, Table 3, Row 18]

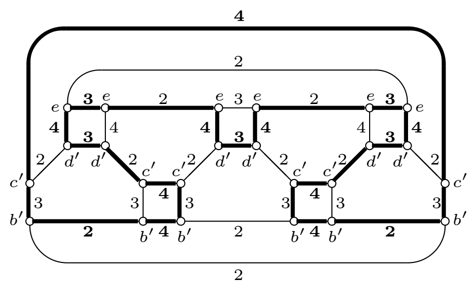


Fig.71: Second part of Hamilton circuit for [12, Table 3, Row 18]

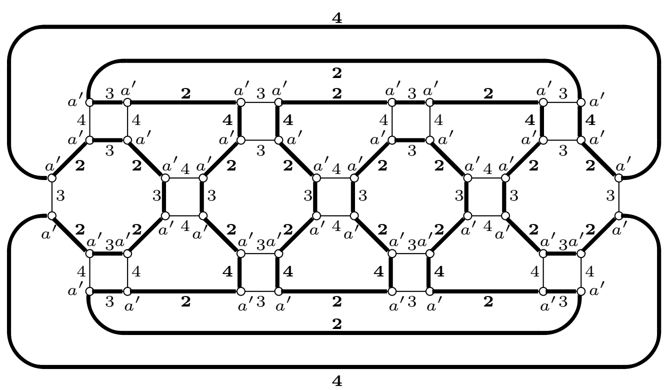
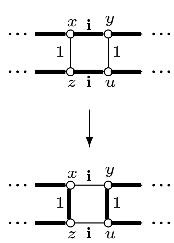


Fig.72: Third part of Hamilton circuit for [12, Table 3, Row 18]



$$\left(\begin{array}{l} \{(\mathbf{i},x,y,z,u),(\mathbf{i},z,u,x,y),(\mathbf{i},y,x,u,z),(\mathbf{i},u,z,y,x)\} \\ \cap\{(\mathbf{3},a,a,a,a),(\mathbf{4},a,b,a,b),(\mathbf{3},b,c,b,c),(\mathbf{4},c,c,c,c), \\ (\mathbf{3},d,d,e,e),(\mathbf{4},d,f,e,d'),(\mathbf{3},f,f,d',d'),(\mathbf{4},c',c',c',c'), \\ (\mathbf{2},b',b',a',a'),(\mathbf{4},b',b',a',a')\}\neq\emptyset \end{array}\right)$$

Fig.73: Joints of Fig.70, Fig.71, Fig.72

$$(s^a_2)(s^a_3)(s^b_4)(s^c_3)(s^d_2)(s^d_3)(s^c_2)(s^b_3)(s^a_4)s^a_3 s^a_2 s^b_4(s^b_2)s^c_3 s^d_2(s^e_4)(s^e_2)(s^e_3)s^e_2 s^e_3 s^e_2(s^d_4)s^d_3 s^e_4 s^e_2 s^d_4 s^c_2(s^c_4)$$
$$\cdot s^b_3 s^a_4 s^a_2 s^b_4 s^c_3 s^c_4 s^d_2 s^d_3 s^c_2 s^b_3 s^b_2(s^b_1)$$
$$\cdot s^c_3 s^c_4 s^d_2 s^d_3 s^c_2 s^b_3 s^b_2 s^a_4 s^a_2 s^a_3 s^b_4 s^c_3 s^d_2 s^d_3 s^c_2 s^b_3 s^a_4 s^a_3 s^a_2 s^b_4 s^b_2 s^c_3 s^d_2 s^e_4 s^e_2 s^e_3 s^e_2 s^e_3 s^e_2 s^d_4 s^d_3 s^e_4 s^e_2 s^d_4 s^c_2 s^c_4 s^b_3 s^b_1 s^c_3 s^c_4$$
$$\cdot s^d_2 s^d_3 s^c_2 s^b_3 s^b_2 s^a_4 s^a_2 s^a_3 s^b_4 s^c_3 s^d_2 s^d_3 s^c_2 s^b_3 s^a_4 s^a_3 s^a_2 s^b_4 s^b_2 s^c_3 s^d_2 s^e_4 s^e_2 s^e_3 s^e_2 s^e_3 s^e_2 s^d_4 s^d_3 s^e_4 s^e_2$$
$$\cdot(s^g_1)(s^g_3)(s^h_2)(s^h_4)(s^i_3)(s^i_4)(s^i_2)(s^h_3)s^h_4$$
$$\cdot s^i_3 s^i_2 s^i_4 s^h_3 s^h_4(s^g_2)s^g_3(s^f_4)(s^f_3)(s^f_2)(s^g_4)s^g_3(s^e_1)s^e_2 s^d_4 s^c_2 s^c_4 s^b_3 s^a_4 s^a_2 s^b_4 s^c_3 s^c_4 s^d_2 s^d_3 s^c_2 s^b_3 s^b_2 s^a_4 s^a_2 s^a_3 s^b_4 s^c_3$$
$$\cdot s^d_2 s^d_3 s^c_2 s^b_3 s^a_4 s^a_3 s^a_2$$
$$\cdot s^b_4 s^b_2 s^c_3 s^d_2 s^e_4 s^e_2 s^e_3 s^e_2 s^g_1 s^f_4 s^f_3 s^f_2 s^g_4 s^g_3 s^f_4 s^f_2 s^f_3 s^g_4 s^g_3 s^h_2 s^h_4 s^i_3 s^i_4 s^i_2 s^h_3 s^h_4 s^i_3 s^i_2$$
$$\cdot(s^j_1)(s^j_2)(s^j_3)s^j_2(s^j_4)s^j_2 s^j_4 s^j_2 s^j_3 s^j_2 s^j_3 s^j_2$$
$$\cdot s^j_3 s^j_2 s^j_4 s^j_2 s^j_3 s^j_2 s^j_3 s^j_2 s^j_4 s^j_2 s^j_3 s^j_2 s^j_4 s^j_2 s^j_3 s^j_2 s^j_4 s^j_2 s^j_4 s^j_2 s^j_3 s^j_2 s^j_3 s^j_2 s^j_3 s^j_2 s^j_4 s^j_2 s^j_3 s^j_2 s^j_3 s^j_2 s^j_4(s^i_1)s^i_4 s^h_3 s^h_4 s^g_2 s^g_3 s^f_4$$
$$\cdot s^f_3 s^f_2 s^g_4 s^g_3 s^f_4 s^f_2 s^f_3 s^g_4 s^g_3 s^h_2(s^h_1)s^g_2 s^g_3 s^f_4 s^f_3 s^f_2 s^g_4 s^g_3 s^f_4 s^f_2 s^f_3 s^g_4 s^g_3 s^h_2 s^h_4 s^i_3 s^i_4 s^i_2 s^h_3 s^h_1 s^g_2 s^g_3 s^f_4 s^f_3 s^f_2 s^g_4 s^g_3 s^f_4 s^f_2 s^f_3$$
$$\cdot s^g_4 s^g_3 s^h_2 s^h_4 s^i_3 s^i_4 s^i_2 s^h_3 s^h_1 s^g_2 s^g_3 s^f_4 s^f_3 s^f_2 s^g_4 s^g_3 s^f_4 s^f_2 s^f_3 s^g_4 s^g_3 s^h_2 s^h_1 s^g_2 s^g_3 s^f_4 s^f_3 s^f_2 s^g_4 s^g_3 s^e_1 s^e_2 s^d_4 s^c_2 s^c_4 s^b_3 s^a_4 s^a_2 s^b_4 s^c_3$$
$$\cdot s^c_4 s^d_2 s^d_3 s^c_2 s^b_3 s^b_2 s^a_4 s^a_2 s^a_3 s^b_4 s^c_3 s^d_2 s^d_3 s^c_2 s^b_3 s^a_4 s^a_3 s^a_2 s^b_4 s^b_2 s^b_1 s^b_2 s^a_4 s^a_2 s^a_3 s^b_4 s^c_3 s^d_2 s^d_3 s^c_2 s^b_3 s^a_4 s^a_3 s^a_2 s^b_4 s^b_2 s^c_3 s^d_2 s^e_4 s^e_2$$
$$\cdot s^e_3 s^e_2 s^e_3 s^e_2 s^d_4 s^d_3 s^e_4 s^e_2 s^d_4 s^c_2 s^c_4 s^b_3 s^a_4 s^a_2 s^b_4 s^c_3 s^c_4 s^d_2 s^d_3 s^c_2(s^c_1)s^d_2 s^e_4 s^e_2 s^e_3 s^e_2 s^e_3 s^e_2 s^d_4 s^d_3(s^f_1)s^f_2 s^f_3 s^g_4 s^g_3 s^h_2 s^h_4 s^i_3 s^i_4 s^i_2$$
$$\cdot s^h_3 s^h_1 s^g_2 s^g_3 s^f_4 s^f_3 s^f_2 s^g_4 s^g_3 s^f_4 s^f_2 s^f_3 s^g_4 s^g_3 s^h_2 s^h_4 s^i_3 s^i_4 s^i_2 s^h_3 s^h_4 s^i_3 s^i_2 s^i_4 s^h_3 s^h_1 s^i_3 s^i_2 s^i_4 s^h_3 s^h_1 s^i_3 s^i_4 s^i_2 s^h_3 s^h_4 s^i_3 s^i_2 s^i_4 s^h_3$$
$$\cdot s^h_1 s^i_3 s^i_2 s^i_4 s^h_3 s^h_1 s^i_3 s^i_2 s^i_4 s^h_3 s^h_1 s^i_3 s^i_4 s^i_2 s^h_3 s^h_4 s^i_3 s^j_1 s^j_3 s^j_2 s^i_1 s^h_3 s^h_4 s^g_2 s^e_1 s^e_2 s^d_4 s^d_3 s^f_1 s^f_2 s^f_3(s^d_1)s^c_2 s^c_4 s^b_3 s^a_4 s^a_2(s^a_1)s^a_2 s^a_1$$

Fig.74: Special Hamilton circuit of [12, Table 3, Row 18] obtained from Fig.70, Fig.71, and Fig.72 by joints mentioned by Fig.68, where $g := d'$, $h := c'$, $i := b'$ and $j := a'$ (Length = 480)

Rank-4-Case-4: Let $\chi \in \mathcal{X}_\Pi$ be that of $\mathcal{D}_1^{(3,22)}$. Use the symbols in Fig.75 and Fig.76. As shown in Fig.75, for $t \in \{a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ of Fig.75, let t' mean the one obtained from t by changing α_2 and α_4. Let $\chi_1 := \chi$. Then $\pi_\chi(\chi) = a$. By Fig.76, for χ', $\chi'' \in \mathcal{G}(\chi)$ with $\tau_1(\chi') = \chi''$, we have

$$(\pi_\chi(\chi'), \pi_\chi(\chi'')) \in \{(\bar{a}, \bar{a}'), (\bar{c}, \bar{d}'), (\bar{b}, \bar{b}'), (\bar{d}, \bar{c}')\}. \tag{5.1}$$

Let $\chi_1 := \chi$. Then $\pi_\chi(\chi_1) = a$. Let χ_2, (resp. χ_3) $\in \bar{\mathcal{G}}(\chi)$ be such that $\pi_\chi(\chi_2) = \bar{a}$ (resp. $\pi_\chi(\chi_3) = a'$). By Fig.76, $\{\chi_1, \chi_2, \chi_3\}$ is a 1-complete $\bar{\tau}$-representative subset of $\mathcal{G}(\chi)$. Notice that the generalized Dynkin diagram of $\pi_{(\chi_1)_{\langle 1\rangle}}((\chi_1)_{\langle 1\rangle})$ (resp. $\pi_{(\chi_3)_{\langle 1\rangle}}((\chi_3)_{\langle 1\rangle})$ is the same as a of Fig.28 with ζ, 2, 3, 4 (resp. ζ, 2, 4, 3) in place of q, 1, 2, 3 respectively, and that the generalized Dynkin diagram of $\pi_{(\chi_2)_{\langle 1\rangle}}((\chi_2)_{\langle 1\rangle})$ is the same as a of Fig.31. with ζ, 2, 3, 4 in place of $-q = -r$, 1, 2, 3 respectively, By Fig.30 and Fig.32, we see that for $t \in J_{1,3}$, there exists a special Hamilton circuit map of $\Gamma(\pi_{(\chi_t)_{\langle 1\rangle}}((\chi_t)_{\langle 1\rangle}))$. By an argument similar to that for $\mathcal{D}_\bullet^{(3,18)}$, we see the existence of a Hamilton circuit map for χ. In fact, it is written in Fig.81.

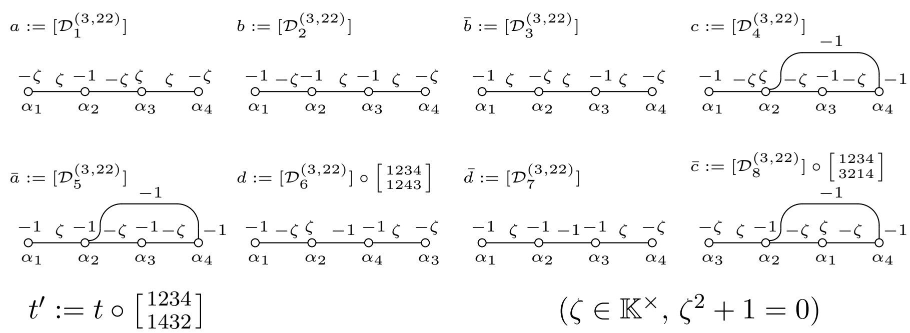


Fig.75: Generalized Dynkin diagrams of [12, Table 3, Row 22]

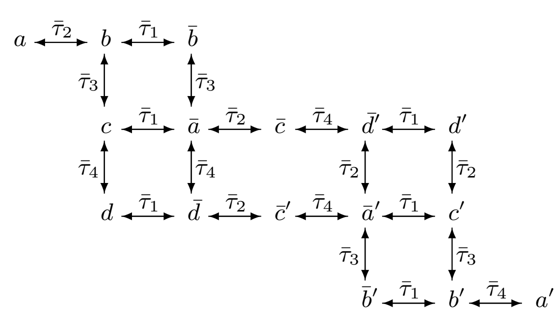


Fig.76: Changing of diagrams of Fig.75

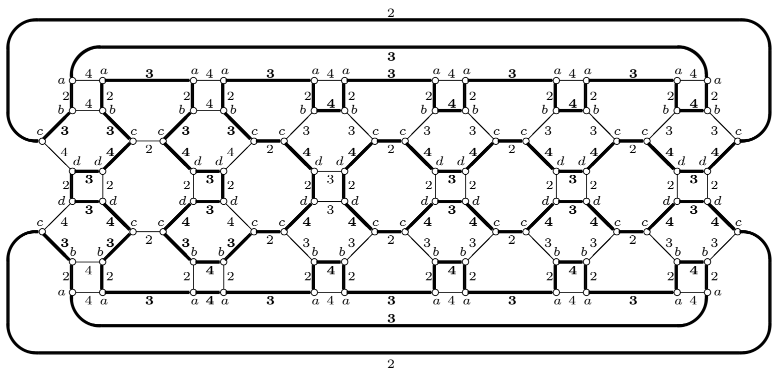


Fig.77: First part for Hamilton circuit of [12, Table 3, Row 22]

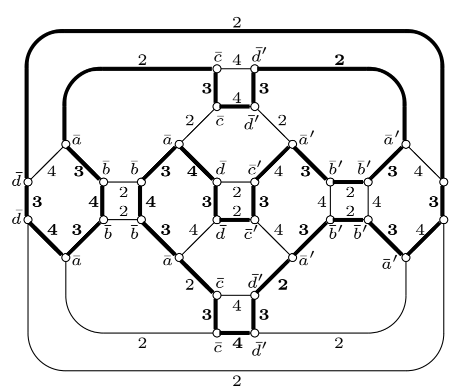


Fig.78′: Second part for Hamilton circuit of
[12, Table 3, Row 22]

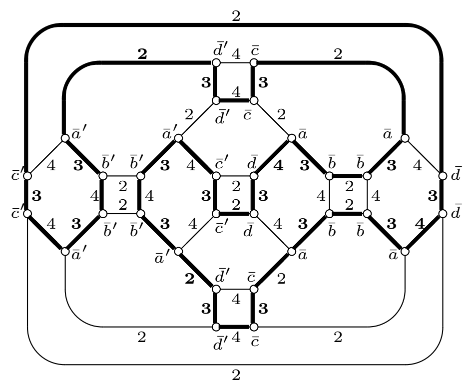


Fig.78: Third part for Hamilton circuit of
[12, Table 3, Row 22]

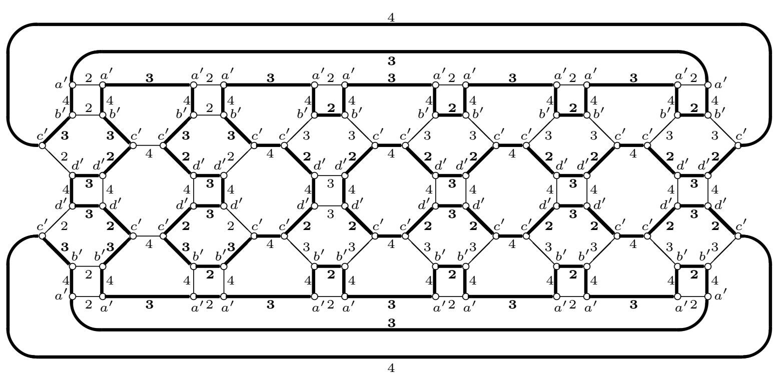
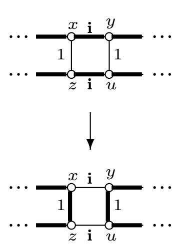


Fig.79: Fourth part for Hamilton circuit of [12, Table 3, Row 22]



$$\left(\begin{array}{l} \{(\mathbf{i},x,y,z,u),(\mathbf{i},z,u,x,y),(\mathbf{i},y,x,u,z),(\mathbf{i},u,z,y,x)\} \\ \cap\{(\mathbf{3},a,a,a,a),(\mathbf{4},a,a,a,a),(\mathbf{3},b,c,\bar{b},\bar{a}),(\mathbf{4},b,b,\bar{b},\bar{b}), \\ (\mathbf{4},c,d,\bar{a},\bar{d}),(\mathbf{3},d,d,\bar{d},\bar{d}),(\mathbf{3},\bar{c},\bar{c},\bar{c},\bar{c}),(\mathbf{3},\bar{c}',\bar{c}',\bar{c}',\bar{c}'), \\ (\mathbf{2},\bar{d}',\bar{a}',d',c'),(\mathbf{3},\bar{d}',\bar{d}',d',d'),(\mathbf{3},\bar{a}',\bar{b}',c',b'),(\mathbf{2},\bar{b}',\bar{b}',b',b'), \\ (\mathbf{2},a',a',a',a'),(\mathbf{3},a',a',a',a')\} \neq \emptyset \end{array}\right)$$

Fig.80: Joints of Fig.77, Fig.78, Fig.79

$$S' := (s_3^a)(s_2^b)(s_4^b)(s_2^a)s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b (s_3^c)(s_2^c)(s_4^d)(s_2^d)(s_4^c)s_2^c(s_3^b)s_4^b s_3^c s_4^d(s_3^d)s_2^d$$
$$\cdot s_3^d s_4^c s_3^b s_2^a s_3^a s_2^b s_3^c s_4^d s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a (s_4^a) s_3^a s_2^b s_4^b s_2^a$$
$$\cdot s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_3^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_2^d$$
$$\cdot s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c (s_1^f)(s_2^h)(s_3^h)(s_4^j)(s_3^j)(s_2^k)$$
$$\cdot (s_3^l)(s_2^l)(s_3^k)(s_4^i)(s_3^i)(s_2^g)(s_3^g)(s_4^f)(s_3^e)(s_4^e)(s_3^f) s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i (s_1^i) s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j$$
$$\cdot s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l$$
$$\cdot (s_1^o)(s_4^p)(s_3^p)(s_4^o)(s_2^o) s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o (s_3^n)(s_4^n)(s_2^m)(s_4^m)(s_2^n) s_4^n (s_3^o) s_2^o s_3^n s_2^m$$
$$\cdot (s_3^m) s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_4^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p (s_2^p) s_3^p s_4^o$$
$$\cdot s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n$$
$$\cdot s_2^m s_4^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m (s_1^j) s_3^j (s_4^h) s_3^h (s_2^f) s_3^e (s_2^e) s_3^f (s_4^g) s_3^g$$
$$\cdot (s_2^i) s_3^i (s_4^k) s_3^l (s_4^l) s_3^k (s_2^j) s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_4^g s_3^g s_2^i s_3^i s_4^k (s_1^n) s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m$$
$$\cdot s_3^m s_2^n s_4^n s_2^m s_4^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_3^o$$
$$\cdot s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_4^m s_2^n s_4^n s_3^o s_2^o s_3^n s_2^m s_3^m$$
$$\cdot s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_4^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_2^p s_3^p s_4^o s_2^o$$
$$\cdot s_4^p s_3^p s_4^o (s_1^l) s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h$$
$$\cdot (s_1^h) s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k$$
$$\cdot s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e (s_1^b) s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b$$
$$\cdot s_3^c s_2^c s_4^d s_2^d s_4^c s_2^c s_3^b s_4^b s_3^c s_4^d s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_2^b s_3^c s_4^d$$
$$\cdot s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_4^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_3^c s_2^c s_4^d s_3^d$$
$$\cdot s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_2^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d$$
$$\cdot (s_1^g) s_3^g s_2^i s_3^i s_4^k s_3^l s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_4^g s_3^g s_2^i s_3^i s_4^k s_3^l s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h s_2^f (s_1^c)$$
$$\cdot s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_2^d s_4^c s_2^c s_4^d s_3^d s_4^c$$
$$\cdot s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_3^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_3^c$$
$$\cdot s_2^c s_4^d s_2^d s_4^c s_2^c s_3^b s_4^b s_3^c s_4^d s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_2^b s_3^c s_4^d s_3^d$$
$$\cdot s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_4^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b (s_1^e) s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e$$
$$\cdot s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j$$
$$\cdot s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_1^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p$$
$$\cdot s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_4^m s_2^n s_4^n s_3^o s_2^o s_3^n s_2^m$$
$$\cdot s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_4^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_2^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p$$
$$\cdot s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n$$
$$\cdot s_2^m s_4^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_1^j s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_4^g s_3^g s_2^i s_3^i s_4^k s_3^l s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h$$
$$\cdot s_2^f s_3^e s_2^e s_3^f s_4^g s_3^g s_2^i s_3^i s_4^k s_1^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m$$
$$\cdot s_3^m s_2^n s_4^n s_2^m s_4^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_3^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p$$
$$\cdot s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_4^m s_2^n s_4^n s_3^o s_2^o s_3^n s_2^m s_3^m$$
$$\cdot s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_4^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_2^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_1^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f$$
$$\cdot s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h$$
$$\cdot s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e$$
$$\cdot s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j$$
$$\cdot s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_1^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a$$
$$\cdot s_3^a s_2^b s_3^c s_2^c s_4^d s_2^d s_4^c s_2^c s_3^b s_4^b s_3^c s_4^d s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_2^b$$
$$\cdot s_3^c s_4^d s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_4^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_3^c s_2^c$$
$$\cdot s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_2^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c$$
$$\cdot s_4^d s_3^d s_1^g s_3^g s_2^i s_3^i s_4^k s_3^l s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_4^g s_3^g s_2^i s_3^i s_4^k s_1^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m$$
$$\cdot s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_4^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m$$
$$\cdot s_3^m s_2^n s_4^n s_3^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_4^m s_2^n s_4^n s_3^o$$
$$\cdot s_2^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_4^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p$$
$$\cdot s_2^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_1^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k$$
$$\cdot s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i$$
$$\cdot s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l$$
$$\cdot s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g$$

Fig.81′: Definition of S' for Fig.81

$S' \cdot s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_1^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a$
$\cdot s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_3^c s_2^c s_4^d s_2^d s_4^c s_2^c s_3^b s_4^b s_3^c s_4^d$
$\cdot s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_2^b s_3^c s_4^d s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_4^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a$
$\cdot s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_3^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c$
$\cdot s_4^d s_2^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_1^g s_3^g s_2^i s_3^i s_4^k s_3^l s_4^l s_3^k s_2^j s_3^j s_4^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i$
$\cdot s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e$
$\cdot s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g$
$\cdot s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f$
$\cdot s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e$
$\cdot s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h$
$\cdot s_4^j s_3^j s_2^k s_3^l s_2^l s_1^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_4^m s_2^n$
$\cdot s_4^n s_3^o s_2^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o s_4^p s_3^p s_4^o s_3^n s_2^m s_3^m s_4^m s_3^m s_2^n s_3^o$
$\cdot s_4^p s_3^p s_2^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_3^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n$
$\cdot s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_4^m s_2^n s_4^n s_2^m s_3^m s_1^j s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_4^g$
$\cdot s_3^g s_2^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k$
$\cdot s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_1^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a$
$\cdot s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_3^c s_2^c s_4^d s_2^d s_4^c s_2^c s_3^b s_4^b s_3^c s_4^d s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_2^b s_3^c s_4^d$
$\cdot s_3^d s_2^d s_3^d s_4^c s_3^b s_2^a s_3^a s_4^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_4^b s_2^a$
$\cdot s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_1^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k$
$\cdot s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i$
$\cdot s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_1^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c$
$\cdot s_4^d s_3^d s_4^c s_2^c s_4^d s_2^d s_4^c s_2^c s_4^d s_3^d s_4^c s_2^c s_4^d s_3^d s_1^g s_3^g s_2^i s_3^i s_4^k s_3^l$
$\cdot s_4^l s_3^k s_2^j s_3^j s_4^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i$
$\cdot s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e$
$\cdot s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_2^f s_3^e s_2^e s_3^f s_4^g s_3^g s_2^i s_3^i s_4^k s_3^l$
$\cdot s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_1^c s_2^c s_4^d s_3^d s_4^c s_2^c s_1^f s_2^h s_3^h s_4^j$
$\cdot s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h$
$\cdot s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k$
$\cdot s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j$
$\cdot s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_4^k s_3^l s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_4^g$
$\cdot s_3^g s_2^i s_3^i s_4^k s_3^l s_4^l s_3^k s_1^n s_4^n s_2^m s_3^m s_2^n s_4^n s_2^m s_3^m s_2^n s_4^n (s_1^k) s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_3^h s_4^j s_3^j$
$\cdot s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k$
$\cdot s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l$
$\cdot s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i$
$\cdot s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_2^f s_3^e s_2^e s_3^f s_4^g s_3^g s_2^i s_3^i s_4^k s_3^l s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h s_2^f s_3^e s_2^e s_3^f s_1^c s_2^c$
$\cdot s_4^d s_3^d s_4^c s_2^c s_1^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_4^e$
$\cdot s_3^f s_2^h s_3^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_3^i s_2^g s_3^g s_4^f s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_2^l s_3^k s_4^i s_1^i s_2^g s_3^g s_4^f$
$\cdot s_3^e s_4^e s_3^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_1^o s_4^p s_3^p s_4^o s_1^l s_4^l s_3^k s_2^j s_3^j s_4^h s_3^h$
$\cdot s_2^f s_3^e s_2^e s_3^f s_1^c s_2^c s_4^d s_3^d s_4^c s_2^c s_1^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_1^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p s_4^o s_1^l$
$\cdot s_4^l s_3^k s_1^n s_4^n s_2^m s_3^m s_2^n s_4^n s_1^k s_4^i s_1^i s_2^g s_3^g s_4^f s_3^e s_1^b s_2^a s_3^a s_2^b s_4^b$
$\cdot s_2^a s_3^a s_2^b s_4^b s_2^a s_3^a s_2^b s_1^e s_2^e s_3^f s_1^c s_2^c s_4^d s_3^d s_4^c s_2^c s_1^f s_2^h s_1^h s_4^j s_3^j s_2^k s_3^l s_1^o s_4^p s_3^p s_4^o s_2^o s_4^p s_3^p$
$\cdot s_4^o s_2^o s_4^p s_3^p s_4^o s_1^l s_4^l s_3^k s_1^n s_4^n s_2^m s_3^m s_2^n s_4^n s_1^k s_4^i s_1^i s_2^g s_3^g s_4^f$
$\cdot s_3^e s_4^e s_1^b s_2^a$

Fig.81: Hamilton circuit of [12, Table 3, Row 22] missing s_1^a, s_1^d, s_1^m and s_1^p, where S' is that of Fig.81$'$, and let $e := \bar{b}$, $f := \bar{a}$, $g := \bar{d}$, $h := \bar{c}$, $i := \bar{c}'$, $j := \bar{d}'$, $k := \bar{a}'$, $l := \bar{b}'$, $m := d'$, $n := c'$, $o := b'$, $p := a'$ (Length $= 2304$)

6. Existence of a Hamilton circuit of $\Gamma(\chi)$ with $|I| \geq 5$

In this section, analyzing carefully in each case, we shall see:

Theorem 6.1. *Assume $|I| \geq 5$. For every χ of* [12, Table 4], *there exists a Hamilton circuit map of $\Gamma(\chi)$.*

Rank $\geq$ 5-Case-1: Let χ be one of [12, Table 4, Rows 1-8, 16, 20, 22]. Then we can easily see that χ is of Cartan-type or quasi-Cartan-type. By Theorem 2.3, a Hamilton circuit of $\Gamma(\chi)$ exists.

(1) (2) (3) (4) (5) (6) (7) (8)

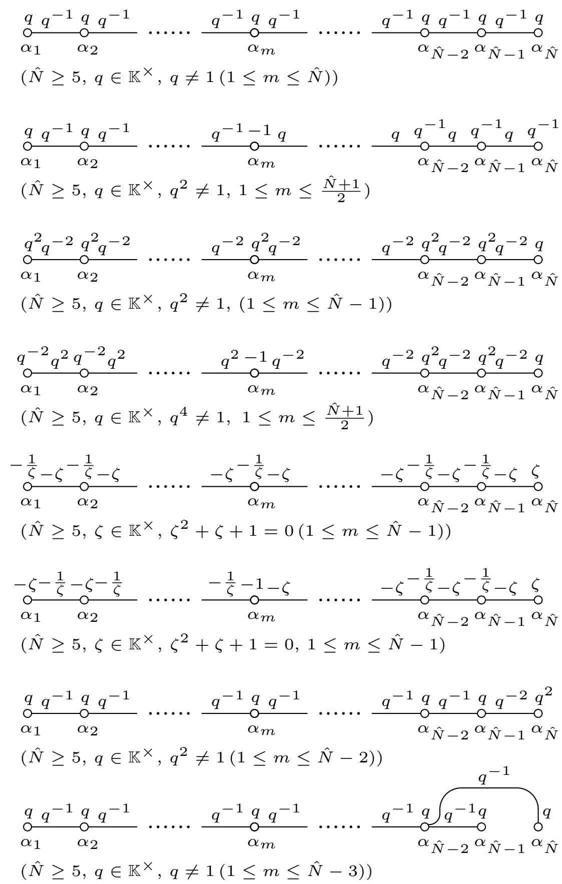


Fig.82: One of diagrams of each row of [12, Table 4, Rows 1-8]

Rank $\geq$ 5-Case-2: Let $q \in \mathbb{K}^\times \setminus \{1,-1\}$, $\hat{N} \in J_{5,\infty}$, $I := J_{1,\hat{N}}$ and $m \in J_{1,\hat{N}-1}$.

Let $\mathcal{H}$ be a rank-$\hat{N}$ free $\mathbb{Z}$-module with a basis $\{\epsilon_i|i \in I\}$. Let $\hat{I} := J_{1,\hat{N}+1}$. Let $\hat{P}(\hat{N},m)$ be the set formed by all maps $\hat{p} : \hat{I} \to J_{0,1}$ with $(-1)^{(\hat{p}(\hat{N})+1)\hat{p}(\hat{N}+1)} = 1$ and $|\{i \in I|\hat{p}(i) = 0\}| = m$. For $\hat{p} \in \hat{P}(\hat{N},m)$, define the $\mathbb{Z}$-bihomomophism $\lambda_{\hat{p}} : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}$ by $\lambda_{\hat{p}}(\epsilon_i, \epsilon_j) := \delta_{ij}(-1)^{\hat{p}(i)}$ $(i, j \in I)$. For $\hat{p} \in \hat{P}(\hat{N},m)$ and $i \in I$, define $\alpha_i^{\hat{p}} \in \mathcal{H}$ by

$$\alpha_i^{\hat{p}} := \begin{cases} \epsilon_i - \epsilon_{i+1} & \text{if } i \in J_{1,\hat{N}-2}, \\ \epsilon_{\hat{N}-1} - \epsilon_{\hat{N}} & \text{if } \hat{p}(\hat{N}+1) = 0 \text{ and } i = \hat{N}-1, \\ 2\epsilon_{\hat{N}} & \text{if } \hat{p}(\hat{N}) = \hat{p}(\hat{N}+1) = 1 \text{ and } i = \hat{N}-1, \\ \epsilon_{\hat{N}-1} + \epsilon_{\hat{N}} & \text{if } \hat{p}(\hat{N}) = \hat{p}(\hat{N}+1) = 0 \text{ and } i = \hat{N}, \\ 2\epsilon_{\hat{N}} & \text{if } \hat{p}(\hat{N}) - 1 = \hat{p}(\hat{N}+1) = 0 \text{ and } i = \hat{N}, \\ \epsilon_{\hat{N}-1} - \epsilon_{\hat{N}} & \text{if } \hat{p}(\hat{N}) = \hat{p}(\hat{N}+1) = 1 \text{ and } i = \hat{N}. \end{cases}$$

For $\hat{p} \in \hat{P}(\hat{N},m)$, define the map $\mu_{\hat{p}} : I \to J_{0,1}$ by $\mu_{\hat{p}}(i) := \delta_{0,t}$, where $t := \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) = 0$. For $\hat{p} \in \hat{P}(\hat{N},m)$, define $\chi^{\hat{p}} \in \mathcal{X}_\Pi$ by

$$\chi^{\hat{p}}(\alpha_i, \alpha_j) := \begin{cases} (-1)^{\mu_{\hat{p}}(i)\mu_{\hat{p}}(j)} q^{\lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}})} & \text{if } i \leq j, \\ 1 & \text{if } i > j, \end{cases}$$

for $i, j \in I$. Then we see $|R^+(\chi^{\hat{p}})| < \infty$ by [12, Table 4, Rows 9,10]. For $\dot{\chi}_1$, $\dot{\chi}_2 \in \mathcal{X}_\Pi$, we write $\dot{\chi}_1 \doteqdot \dot{\chi}_2$ if $\dot{\chi}_1(\alpha_i, \alpha_i) = \dot{\chi}_2(\alpha_i, \alpha_i)$ and $\dot{\chi}_1(\alpha_i, \alpha_j)\dot{\chi}_1(\alpha_j, \alpha_i) = \dot{\chi}_2(\alpha_i, \alpha_j)\dot{\chi}_2(\alpha_j, \alpha_i)$ for all $i, j \in I$. It follows from [2, Lemma 4.22] that if $\dot{\chi}_1 \doteqdot \dot{\chi}_2$ and $|R^+(\dot{\chi}_1)| < \infty$, then $R^+(\dot{\chi}_2) = R^+(\dot{\chi}_1)$. By [12, Table 4, Rows 9,10], we have

$$\begin{aligned} \forall \hat{p} \in \hat{P}(\hat{N},m),\ &\{\dot{\chi}_1 \in \mathcal{X}_\Pi \,|\, \exists \dot{\chi}_2 \in \mathcal{G}(\chi^{\hat{p}}),\ \dot{\chi}_1 \doteqdot \dot{\chi}_2\} \\ &= \{\dot{\chi}_3 \in \mathcal{X}_\Pi \,|\, \exists \hat{p}' \in \hat{P}(\hat{N},m),\ \dot{\chi}_3 \doteqdot \chi^{\hat{p}'}\}. \end{aligned}$$

(For every $\tilde{\chi} \in \mathcal{X}_\Pi$ of [12, Table 4, Rows 9,10], we have $\tilde{\chi} \doteqdot \chi^{\hat{p}}$ for some $q \in \mathbb{K}^\times \setminus \{1, -1\}$, $\hat{N} \in J_{5,\infty}$, $m \in J_{1,\hat{N}-1}$ and $\hat{p} \in \hat{P}(\hat{N},m)$. See Fig.83.)

For $\hat{p} \in \hat{P}(\hat{N},m)$ and $i, j \in I$ with $i \neq j$, we have

$$N_{i,j}^{\chi^{\hat{p}}} = \begin{cases} 0 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) = 0, \\ 1 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) = -2\lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) \neq 0, \\ 1 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) = 0 \text{ and } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) \neq 0, \\ 2 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) = -\lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) \neq 0. \end{cases} \tag{6.1}$$

Define $\hat{p}_0 \in \hat{P}(\hat{N}, m)$ by $\hat{p}_0^{-1}(\{0\}) = J_{1,m} \cup \{\hat{N}+1\}$ and $\hat{p}_0^{-1}(\{1\}) = J_{m+1,\hat{N}}$. Let $\chi := \chi^{\hat{p}_0}$. We can define the bijection $\Omega : \hat{P}(\hat{N}, m) \to \bar{\mathcal{G}}(\chi)$ in a way that for every $\hat{p} \in \hat{P}(\hat{N}, m)$, there exists a unique $\Omega(\hat{p}) \in \bar{\mathcal{G}}(\chi)$ with $\chi' \doteqdot \chi^{\hat{p}}$ for $\chi' \in \mathcal{G}(\chi)$ satisfying $\pi_\chi(\chi') = \Omega(\hat{p})$. For $i \in I$, define the bijection $\hat{\tau}_i : \hat{P}(\hat{N}, m) \to \hat{P}(\hat{N}, m)$ by $\Omega(\hat{\tau}_i(\hat{p})) = \bar{\tau}_i(\Omega(\hat{p}))$ ($\hat{p} \in \hat{P}(\hat{N}, m)$).

Let $\mathfrak{S}(\hat{N}+1)$ be the symmetric group of degree $\hat{N}+1$, and let $\sigma_i := (i, i+1) \in \mathfrak{S}(\hat{N}+1)$ ($i \in J_{1,\hat{N}}$). Regarding $\hat{\tau}_i(\hat{p})$ for $\hat{p} \in \hat{P}(\hat{N}, m)$ and $i \in I$, by (6.1), we have the following.

- If $\lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) \neq 0$, then $\hat{\tau}_i(\hat{p}) = \hat{p}$.
- If $i \in J_{1,\hat{N}-2}$, then $\hat{\tau}_i(\hat{p}) = \hat{p} \circ \sigma_i$.
- If $\lambda_{\hat{p}}(\alpha_{\hat{N}-1}^{\hat{p}}, \alpha_{\hat{N}-1}^{\hat{p}}) = 0$ (whence $\hat{p}(\hat{N}+1) = 0$), then $\hat{\tau}_i(\hat{p}) = \hat{p} \circ \sigma_{\hat{N}-1}$.
- If $\lambda_{\hat{p}}(\alpha_{\hat{N}}^{\hat{p}}, \alpha_{\hat{N}}^{\hat{p}}) = 0$, then $\hat{\tau}_i(\hat{p})_{|I} = (\hat{p})_{|I} \circ \sigma_{\hat{N}-1}$ and $(\hat{\tau}_i(\hat{p}))(\hat{N}+1) = 1 - \hat{p}(\hat{N}+1)$. Hence by (6.1), for $\hat{p} \in \hat{P}(\hat{N}, m)$ and $i, j \in I$ with $i \neq j$, letting $a := \Omega(\hat{p})$, we have

$$m_{ij}^a = \begin{cases} 0 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) = 0, \\ 3 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) = -2\lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) \neq 0, \\ 3 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) = 0 \text{ and } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) \neq 0, \\ 4 & \text{if } \lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_i^{\hat{p}}) = -\lambda_{\hat{p}}(\alpha_i^{\hat{p}}, \alpha_j^{\hat{p}}) \neq 0. \end{cases}$$

Notice $\hat{N} \geq 5$. By Fig.83, Fig.84 and Fig.85, using the claim obtained from that of Proposition 3.22 by letting $N = \hat{N}$, $i = N - 2$ and then changing $1, 2, \ldots, \hat{N}$ by $\hat{N}, \hat{N}-1, \ldots, 1$, we see that a Hamilton circuit map of $\Gamma(\chi^{\hat{p}})$ exists. Similarly, by Proposition 3.22 for $N = \hat{N}$ and $i = N - 3$, using Fig.3, Fig.6, Fig.27 and Fig.32, we also see that fact.

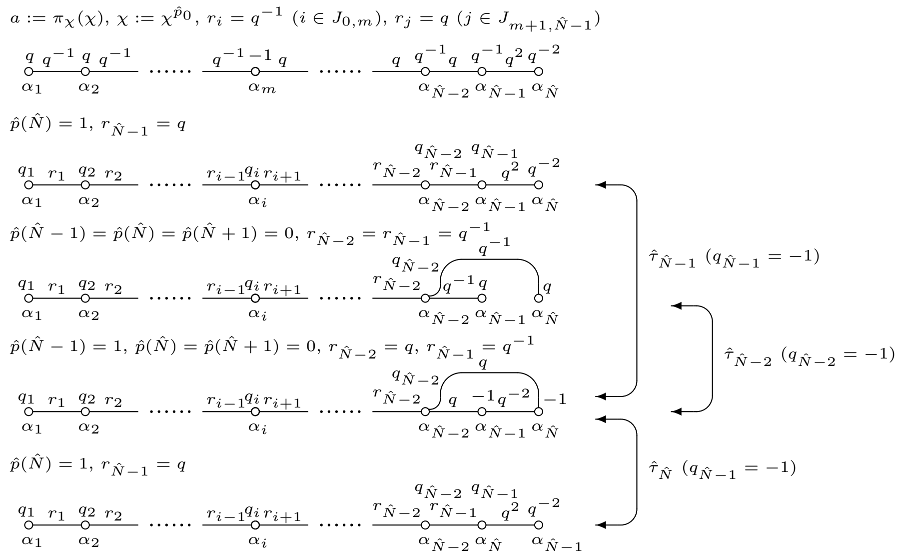


Fig.83: Diagrams of [12, Table 4, Rows 9,10], where $i \in J_{1,\hat{N}-2}$, $q \in \mathbb{K}^\times$, $q^2 \neq 1$, $q_j \in \{-1, q^{\pm 1}\}$ $(j \in J_{1,\hat{N}-1})$, $r_k \in \{q^{\pm 1}\}$ $(k \in J_{0,\hat{N}-1})$, $r_{i-1}q_i^2 r_i = 1$ and $m = |\{i \in J_{0,\hat{N}} | r_i = q^{-1}\}|$.

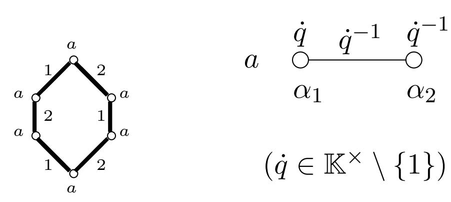


Fig.84: Special Hamilton circuit of [12, Table 1, Row 1]

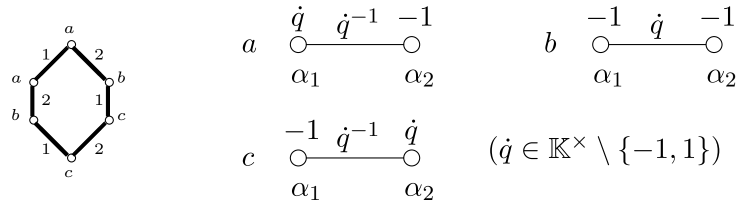


Fig.85: Special Hamilton circuit of [12, Table 1, Row 2]

We emphasize:

Theorem 6.2. *For every* χ *of* [12, Table 4, Rows 9,10]*, there exists a Hamilton circuit map of* $\Gamma(\chi)$*.*

Rank $\geq$ **5-Case-3:** Let χ be of [12, Table 4, Rows 11-15, 17-19, 21]. Let $N := |\Pi|$. Here $N \in J_{5,7}$ and $I = J_{1,N}$. Notice Tables 3 and 4, where we show a Hamilton circuit of $\Gamma(\hat{\chi}_{|\mathbb{Z}\alpha_{N-3}\oplus\mathbb{Z}\alpha_{N-2}\oplus\mathbb{Z}\alpha_{N-1}\oplus\mathbb{Z}\alpha_N})$ of $\hat{\chi} \in \mathcal{G}(\chi)$. Recall that we have a special Hamilton circuit map of $\Gamma(\chi')$ for χ' of [12, Table 3, Rows 11,13,18] by Fig.61, Fig.52 and Fig.74. Hence by Proposition 3.22, we have a Hamilton circuit map of $\Gamma(\chi)$ for χ of [12, Table 4, Rows 11, 12, 15, 17, 18, 21]. Although the Hamilton circuit maps Fig.64 and Fig.55 of of $\Gamma(\chi'')$ for χ'' of [12, Table 3, Rows 14, 20] are not special, it follows from Proposition 3.24 that we have a Hamilton circuit map of $\Gamma(\chi)$ for χ of [12, Table 4, Rows 13, 14, 19]. Incidentally, if $\hat{N} \geq 6$, then by Fig.84 and Fig.85, using the claim obtained from that of Proposition 3.22 by letting $N = \hat{N}$, $i = N-2$ and then changing $1, 2, \ldots, \hat{N}$ by $\hat{N}, \hat{N}-1, \ldots, 1$, we see that a Hamilton circuit map of $\Gamma(\chi)$ exists.

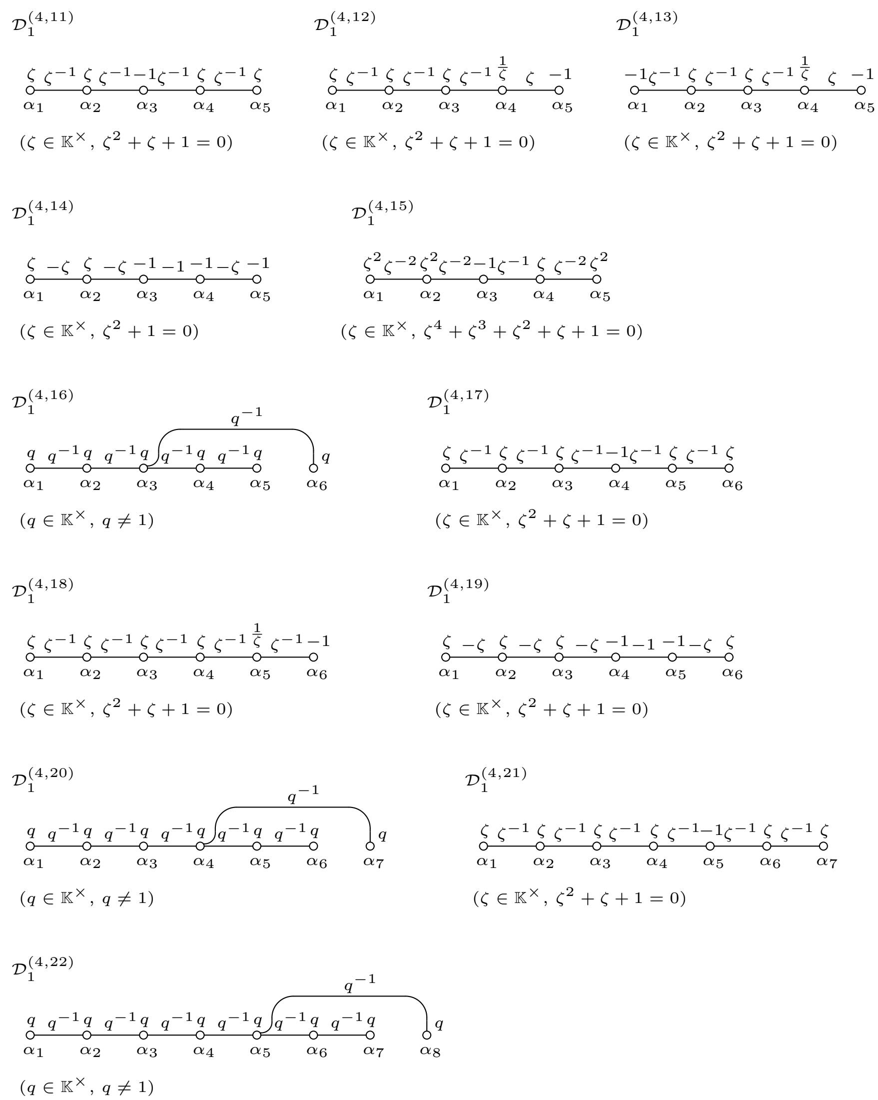


Fig.86: First diagrams of [12, Table 4, Rows 11-22]

Table 3: Rank-5's rank-4 edge sub-Hamilton circuits

χ	$\mathcal{D}_\bullet^{(4,11)}$	$\mathcal{D}_\bullet^{(4,12)}$	$\mathcal{D}_\bullet^{(4,13)}$	$\mathcal{D}_\bullet^{(4,14)}$	$\mathcal{D}_\bullet^{(4,15)}$
Fig.s	8, 52	52, 74	8, 74, 55	8, 64	8, 61

Table 4: Rank-6, 7 and 8's rank-4 edge sub-Hamilton circuits

χ	$\mathcal{D}_\bullet^{(4,16)}$	$\mathcal{D}_\bullet^{(4,17)}$	$\mathcal{D}_\bullet^{(4,18)}$	$\mathcal{D}_\bullet^{(4,19)}$	$\mathcal{D}_\bullet^{(4,20)}$	$\mathcal{D}_\bullet^{(4,21)}$	$\mathcal{D}_\bullet^{(4,22)}$
Fig.s	8	8, 52	8, 52, 74	8, 64	8	8, 52	8

To obtain the above Tables 3 and 4, we can use some diagrams in [1] together with Section 6 below.

After all, by Theorems 4.2, 5.1 and 6.1, we prove our main theorem:

Theorem 6.3. *For every $\chi \in \mathcal{X}_\Pi$ with $|R^+(\chi)| < \infty$, there exists a Hamilton circuit map of $\Gamma(\chi)$.*

Appendix: Names by [1]

As mentioned in Definition 3.29, $\mathcal{D}_\bullet^{(x,y)}$ means the family of the generalized Dynkin diagrams belonging to Row-y of Table x of [12]. In [1], a suitable name of each $\mathcal{D}_\bullet^{(x,y)}$ has been introduced with consideration of not only simple Lie superalgerbas but also modular simple Lie superalgerbas. The names are RHSs of the following.

(2) See [1, Section 7].

$\mathcal{D}_\bullet^{(3,14)} = \mathtt{wk}(\mathbf{4})$, $\mathcal{D}_\bullet^{(1,5)} = \mathtt{br}(\mathbf{2})$, $\mathcal{D}_\bullet^{(2,18)} = \mathtt{br}(\mathbf{3})$.

(2) See [1, Section 8].

$\mathcal{D}_\bullet^{(1,9)} = \mathtt{brj}(\mathbf{2};\mathbf{3})$, $\mathcal{D}_\bullet^{(2,13)} = \mathbf{g}(\mathbf{1},\mathbf{6})$, $\mathcal{D}_\bullet^{(2,15)} = \mathbf{g}(\mathbf{2},\mathbf{3})$, $\mathcal{D}_\bullet^{(3,18)} = \mathbf{g}(\mathbf{3},\mathbf{3})$, $\mathcal{D}_\bullet^{(3,20)} = \mathbf{g}(\mathbf{4},\mathbf{3})$, $\mathcal{D}_\bullet^{(3,21)} = \mathbf{g}(\mathbf{3},\mathbf{6})$, $\mathcal{D}_\bullet^{(4,11)} = \mathbf{g}(\mathbf{2},\mathbf{6})$, $\mathcal{D}_\bullet^{(4,12)} = \mathtt{el}(\mathbf{5};\mathbf{3})$, $\mathcal{D}_\bullet^{(4,13)} = \mathbf{g}(\mathbf{8},\mathbf{3})$, $\mathcal{D}_\bullet^{(4,17)} = \mathbf{g}(\mathbf{4},\mathbf{6})$, $\mathcal{D}_\bullet^{(4,18)} = \mathbf{g}(\mathbf{6},\mathbf{6})$, $\mathcal{D}_\bullet^{(4,21)} = \mathbf{g}(\mathbf{8},\mathbf{6})$.

(3) See [1, Section 9].

$\mathcal{D}_\bullet^{(1,13)} = \mathtt{brj}(\mathbf{2};\mathbf{5})$, $\mathcal{D}_\bullet^{(4,15)} = \mathtt{el}(\mathbf{5};\mathbf{5})$.

(3) See [1, Section 10].

$\mathcal{D}_\bullet^{(4,14)} = \mathtt{ufo}(\mathbf{1})$, $\mathcal{D}_\bullet^{(4,19)} = \mathtt{ufo}(\mathbf{2})$, $\mathcal{D}_\bullet^{(2,16)} = \mathtt{ufo}(\mathbf{3})$, $\mathcal{D}_\bullet^{(2,17)} = \mathtt{ufo}(\mathbf{4})$, $\mathcal{D}_\bullet^{(3,17)} = \mathtt{ufo}(\mathbf{5})$, $\mathcal{D}_\bullet^{(3,22)} = \mathtt{ufo}(\mathbf{6})$, $\mathcal{D}_\bullet^{(1,7)} = \mathtt{ufo}(\mathbf{7})$, $\mathcal{D}_\bullet^{(1,8)} = \mathtt{ufo}(\mathbf{8})$, $\mathcal{D}_\bullet^{(1,12)} = \mathtt{ufo}(\mathbf{9})$, $\mathcal{D}_\bullet^{(1,14)} = \mathtt{ufo}(\mathbf{10})$, $\mathcal{D}_\bullet^{(1,15)} = \mathtt{ufo}(\mathbf{11})$, $\mathcal{D}_\bullet^{(1,16)} = \mathtt{ufo}(\mathbf{12})$.

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