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Profit Principles and Endogenous Growth : Incorporating a Keynesian Perspective into Solow Swan Model

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Abstract: J. M. Keynes suggests the importance of animal spirits in economic activities. Animal Spirits, formally expressed by profit principle, has been well-known as a source of business fluctuations. In this paper, incorporating the profit principle into the Solow-Swan model, we construct an endogenous growth model. Our model is two-dimensional and allows a complete phase diagram analysis. We prove that the equilibrium of the model is non-oscillatory and globally asymptotically stable. We demonstrate that, in our model, growth of GDP per capita is endogenously produced as a result of the cooperation of growth of labor and various efforts for the technological innovation. We also see that the adoption of investment behavior based on profit principle is rationalized from a long-run viewpoint.

Keywords: Animal Spirits; Short-run and Long-run Profit Principles: Solow-Swan Model; Endogenous Growth.

1. Introduction

In the Solow-Swan (SS) model, technological innovations are not associated with the behaviors of economic agents. See Solow (1956) and Swan (1956). Technological innovations have been considered to be exogenously given. On the other hand, Griliches (1998) observes empirically the fact that the increasing rate of total factor productivity (abbreviated as TFP) and/or the progress rate of technology depends on R&D investment. See Griliches (1998, 2000) and Diamond (2004). This fact implies that technological innovation is closely related to the behaviors of economic agents. Romer (1986) is the first to demonstrate the fact theoretically. He constructs an endogenous growth model that the efforts and researches for technological innovations increase the growth rate of GDP per capita. See Romer (1986, 1990). In both of SS model and Romer model, technological innovations play important role. However, in SS model, technological innovations are exogenous. On the other hand, in Romer model, innovations are endogenously produced as a result of various efforts for the technological innovation. In this paper, we demonstrate that, through a mechanism different from that in Romer model, various efforts based on profit motive produces endogenous growth.

Keynes (1936, Sect. 7 of Ch. 12) emphasizes the economic important role played by animal spirits[1]. Kaldor (1940) incorporates the Keynes' perspective into the GDP dependent part of the investment function based on profit principle and constructs a endogenous business-cycle model. Adopting the Kaldor's approach to technological innovation investment and incorporating a mechanism that various efforts based on animal spirits produce endogenous growth, we construct an endogenous growth model based on profit principle. Our growth model possesses a globally stable quasi-steady state that is similar to the steady state in SS model. We also demonstrate that if animal spirits is not too intensive (in other words, if the intensity of animal spirits is sufficiently sound), the intensity of animal spirits positively correlates with the level (not growth rate) of GDP per capita at the quasi-steady state. However, the growth rate of GDP in our model is independently of such intensity and the endogenous growth in our model is produced irrespective of the intensity.

[1] For the importance of Animal Spirits, see also Akerlof and Shiller (2010).

Our model is two-dimensional tractable model that allows a complete phase diagram analysis. We demonstrate that our model possesses strong stability and gives a natural generalization of SS model.

By the way, an important question is whether the investment behavior based on profit principle is consistent. Such a behavior is adopted under the expectation that profits increases. Since our model produces endogenous growth, such a behavior becomes theoretically consistent. This implies that a self-fulfilling prophecies in the sense of Merton (1948, Sec. I) occurs. It can also be said that profit principle is rationalized from a long-run viewpoint. Thus, there is a rational link or affinity between profit principle and endogenous growth. We will discuss this point in a bit more depth.

The paper is organized as follows. In Section 2, we explain the investment behavior based on profit principle incorporating animal spirits. In Section 3, we construct an endogenous growth model with the investment behavior derived in Section 1. In Sections 4 and 5, we consider effects of parameters on growth dynamics. In Section 6, we discuss a close link between profit principle and endogenous growth. In Section 7, we provide some conclusions. In Appendix, we prove several results

2. Investment Behavior and Animal Spirits

At the beginning of a project, large technological innovators possess unpredictable possibilities to produce large innovations. In this section, we consider how will the investments supporting the innovators be carried out. Since such investments in general are economic matter, there should be some solid economic background for the investment. We consider the common grounds.

To consider it, we distinguish between the following two types of investment:

I_{stec} = in the short-run, a certain percentage surely leads to small technological innovation. We call such an investment small technological innovation investment.

I_{ltec} = the investment other than I_{stec} = the investment for large technological innovation. We call such an investment large technological innovation investment.

In the situation where the economy endogenously grows, the following process concerning two types of innovation investments will be natural. The capital accumulation for small technological innovation steadily produces small innovations. Then, expectation and impulsiveness for large technological innovation are intensified and I_{ltec} gradually increases. Thus, in the growing economy, through this process, increases in small innovation investments lead to increases in large innovation investments. We here give the following definition.

Definition 1: The process from small innovation investments to large innovation investments is called an innovation process.■

We assume that through the innovation process, the following relation holds.

Assumption 1: $I_{ltec} = \Gamma(Y) = \eta Y, \quad \eta > 0.$ ■

Through the innovation process, large innovation investments are put into the economy and an endogenous growth is naturally produced. As a result, increases in I_{stec} lead to increases in GDP. Then, since GDP is considered as a proxy variable of profit, increases in GDP lead to increases in profits. We here assume profit principle concerning investment planning:

Assumption 2: $I_{stec} = \Phi(Y) = \rho Y, \quad \rho > 0$.■

In Section 2, we discuss the affinity between profit principle and endogenous growth.

In this paper, for simplification, we assume that the $\Gamma-$function is linear:

Assumption 3: $I_{ltec} = \Gamma(Y) = \eta Y, \quad \eta > 0.$ ■

Equation (1) gives a composite of small technological and large innovation investments. Equation (1) denotes that large innovation investments also follow profit principle. We here intuitively explain the implication of (1). We assume that GDP increases. Then, following profit principle, small innovation investment increases. The increase in small innovation investment increases capital stock for small innovation. As such a capital stock increases, small innovations gradually progress. At the same time, GDP and therefore profit also gradually increase. As a result, expectation[2] and impulsiveness for large innovation are intensified. In such a situation, we consider that the animal spirits operate and large innovation investments increase. We here define as follows.

Definition 2: We call ρ the short-run animal spirits parameter and η the long-run animal spirits parameter. Moreover, we call

(1) $$I_{tec} \equiv I_{stec} + I_{ltec} = \Phi(Y) + \Gamma(Y) = (\rho + \eta)Y \equiv \zeta Y$$

the innovation investment and ζ the animal spirits parameter.■

From a Keynesian viewpoint, we modify SS model by explicitly incorporating innovation investment which follows profit principle yielded through animal spirits. Like SS model, we assume that

Assumption 4: Consumption $= C = \theta Y \quad (\theta \in (0,1))$, aggregate investment $(AI) \equiv I_{aggr} = (1-\theta)Y$, i.e. aggregate demand (AD)=aggregate supply (AS).■

Assumption 5: $1 > \theta + \zeta > 0$.■

Since ζ is considered to be fairly small, Assumption 5 is natural. It follows from Assumptions 4 and 5 that

(2) $$AD = AS \quad \Leftrightarrow \quad I_{aggr} + C = Y \quad \Leftrightarrow \quad I_{aggr} = (1-\theta)Y$$
$$\Leftrightarrow \quad [\text{investment other than } I_{tec}] = I_{aggr} - I_{tec} = (1-\theta-\zeta)Y.$$

This final equation possesses an economically important implication. We briefly explain

[2] This type of expectation is considered to be closely related to long-run expectation in Robinson (1962, Ch.2).

the implication of (2). The investment of Kaldor (1940) is determined by profit principle based on animal spirits. As demonstrated by Kaldor (1940), such an investment behavior generates instability of equilibrium. Therefore, profit principle is incompatible with $AD = AS$. Therefore, although for technological innovation investment we introduce Assumptions 2 and 3 (profits principle), we assume that the final equation of (2) is satisfied for the investment other than I_{tec}. Such an assumption is included in Assumption 4, because (2) is derived from Assumption 4. It should be noted that the final equation plays the same role of adjustment ($AD = AS$) as aggregate investment in SS model. However, if profit principle is adopted in the investment other than I_{tec}, the economy becomes unstable and $AD \neq AS$. To avoid such an unstable situation, we introduce Assumption 4 and therefore (2). If we can construct a model describing the unstable case, we will obtain a endogenous growth-cycle model with unstable equilibrium in the Keynesian tradition. Such a model will be much more complicated than KSS model. The research in this direction is ongoing and it will take time to complete. See Dohtani and Matsuyama (2022a, b).

3. Endogenous Growth Model of Solow-Swan Type

In Section 2, we consider the technological innovation investment according to profit principle based on animal spirits. In this section, incorporating such an investment behavior into model, we construct a growth model. We demonstrate the model yields an endogenous growth. In the following, we define as

K_{ltec} = capital stock for large technological innovation, K_{stec} = capital stock for small technological innovation, $K_{tec} = K_{stec} + K_{ltec}$ = capital stock for technological innovation, K_{aggr} = aggregate capital stock, $K = K_{aggr} - K_{tec}$, $I = I_{aggr} - I_{tec}$, δ = depreciation rate on K, δ_{tec} = depreciation rate on K_{tec}.

For simplification, we assume the following.

Assumption 6: $\delta_{tec} = \delta$.■

From the argument so far, we obtain the capital accumulation equation:

$$(3.1) \qquad \dot{K}_{tec} = I_{tec} - \delta K_{tec} = \zeta Y - \delta K_{tec}, \quad \delta > 0,$$

$$(3.2) \qquad \dot{K} = \dot{K}_{aggr} - \dot{K}_{tec} = I_{aggr} - \delta K_{aggr} - (I_{tec} - \delta K_{tec})$$

$$= (1-\theta)Y - \zeta Y - \delta(K_{aggr} - K_{tec}) = (1-\theta-\zeta)Y - \delta K, \quad \delta > 0.$$

The historical fact founded by Griliches (1998) is as follows. From a long-run and macro perspective, relentless research accumulations and various efforts of firms for technological innovation have gradually produced large technological innovation and therefore, effects on the growth trend. Based on this historical fact, we start to our argument. First, we define a production function as follows.

Definition 3: To incorporating the fact that in the long-run increases in K_{tec} and L_{tec} boosts productivity, we consider the production function:

$$Y = F(K_{tec}, L_{tec}, K, L) = T(K_{tec}, L_{tec})G(K, L),$$

where L_{tec} = amount of labor involved innovations (including researchers) and L = amount of labor other than L_{tec} . $T(K_{tec}, L_{tec})$ denotes the total factor productivity (TFP) which is produced by using K_{tec} and L_{tec}. We call it as TFP production function.■

Assumption 7: We assume that

$$T(K_{tec}, L_{tec}) = \beta_1 K_{tec}^a L_{tec}^b, \quad G(K, L) = \beta_2 K^c L^d, \quad Y = \beta_1 \beta_2 K_{tec}^a L_{tec}^b K^c L^d,$$
$$(a, b, c, d) \in (0,1)^4. \ ■$$

We here assume the following.

Assumption 8: $c + d = 1$ and $1 > a + c$. ■

Just like SS model, the first condition of Assumption 8 assumes that $G(K,L)$ is linear homogenous. On the other hand, since it is considered that c is sufficiently small, the second part of Assumption 8 is a natural assumption. Moreover, we assume the following.

Assumption 9: $L=\tau\exp(nt)$ and $L_{tec}=\pi L=\tau\pi\exp(nt)$, $\pi\in(0,1)$. ■

From Assumptions 7 and 9, we obtain the following.

Lemma 1: $Y=F(K_{tec},L_{tec},K,L)=\beta K_{tec}^{a}K^{c}L^{b+d}$, where $\beta=\beta_1\beta_2(\tau\pi)^{d}$. ■

Proof: Directly from Assumptions 7 and 9. ■

Under the assumptions so far, (3) and Lemma 1 yields

(4.1) $\dot{K}_{tec}=I_{tec}-\delta K_{tec}=\zeta F(K_{tec},L_{tec},K,L)-\delta K_{tec}=\zeta\beta K_{tec}^{a}K^{c}L^{b+d}-\delta K_{tec}$,

(4.2) $\dot{K}=I-\delta K=(1-\theta-\zeta)F(K_{tec},L_{tec},K,L)-\delta K=(1-\theta-\zeta)\beta K_{tec}^{a}K^{c}L^{b+d}-\delta K.$

Thus, we obtain the following system.

$$\Pi\quad\begin{cases}\dot{K}_{tec}=\zeta\beta K_{tec}^{a}K^{c}L^{b+d}-\delta K_{tec},\\ \dot{K}=(1-\theta-\zeta)\beta K_{tec}^{a}K^{c}L^{b+d}-\delta K.\end{cases}$$

Since System Π possesses an exogenous increase in labor, the system is non-autonomous. In SS model, the autonomous system is obtained by transforming capital stock into capital stock per capita. Consequently, we can obtain many economically clear results. However, this method does not hold true for System Π. However, in the following we demonstrate that, using a slightly modified method, System Π can be transformed into an autonomous differential equations system.

To transform System Π into a tractable autonomous differential equations system, we introduce the following variables.

(5) $$\tilde{k}_{tec} \equiv \frac{K_{tec}}{L^e}, \quad \tilde{k} \equiv \frac{K}{L^e}, \quad e \equiv \frac{b+d}{1-a-c} = \frac{1-c+b}{1-a-c} > \frac{1-c-a}{1-a-c} = 1.$$

It should be noted here that $\tilde{k}_{tec}$ and $\tilde{k}$ are not variables per capita. We can now derive the autonomous differential equations system concerning the variables of (5).

Lemma 2: We have

$$\Psi_{KSS} \quad \begin{cases} \dot{\tilde{k}}_{tec} = \zeta\beta\tilde{k}_{tec}^{a}\tilde{k}^{c} - (\delta + en)\tilde{k}_{tec}, \\ \dot{\tilde{k}} = (1-\theta-\zeta)\beta\tilde{k}_{tec}^{a}\tilde{k}^{c} - (\delta + en)\tilde{k}. \end{cases} \blacksquare$$

Proof: See Appendix.■

System Ψ_{KSS} corresponds to the capital accumulation equation of SS model.

Definition 4: We call System Ψ_{KSS} the Keynes-Solow-Swan (KSS) model.■

The phase diagram analysis of KSS model is as follows.

Lemma 3: We have

$$\dot{\tilde{k}}_{tec} \geq 0 \Leftrightarrow \tilde{k} \geq \left(\frac{\delta + en}{\zeta\beta}\right)^{1/c} \tilde{k}_{tec}^{(1-a)/c} \equiv \tau_1(\tilde{k}_{tec}),$$
$$\dot{\tilde{k}} \geq 0 \Leftrightarrow \tilde{k} \leq \left\{\frac{(1-\theta-\zeta)\beta}{\delta + en}\right\}^{1/(1-c)} \tilde{k}_{tec}{}^{a/(1-c)} \equiv \tau_2(\tilde{k}_{tec}). \blacksquare$$

Proof: See Appendix.■

Since $(1-a)/c > 1$ and $a/(1-c) < 1$, we obtain

$$\frac{d^2\tau_1(\tilde{k}_{tec})}{d\tilde{k}_{tec}^2} > 0, \qquad \frac{d^2\tau_2(\tilde{k}_{tec})}{d\tilde{k}_{tec}^2} < 0.$$

Noting this point, we carry out the phase diagram analysis of KSS model. See Figure 1. In Figure 1, we define

$$\text{Curve A}: \text{graph of } \dot{\tilde{k}}_{tec} = 0 \quad (\text{i.e. graph of } \tilde{k} = \tau_1(\tilde{k}_{tec})),$$

$$\text{Curve B}: \text{graph of } \dot{\tilde{k}} = 0 \quad (\text{i.e. graph of } \tilde{k} = \tau_2(\tilde{k}_{tec})),$$

$$\Sigma_1 \equiv \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} > \tau_1(\tilde{k}_{tec})\} \cap \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} > \tau_2(\tilde{k}_{tec})\},$$

$$\Sigma_2 \equiv \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} < \tau_1(\tilde{k}_{tec})\} \cap \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} > \tau_2(\tilde{k}_{tec})\},$$

$$\Sigma_3 \equiv \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} < \tau_1(\tilde{k}_{tec})\} \cap \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} < \tau_2(\tilde{k}_{tec})\},$$

$$\Sigma_4 \equiv \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} > \tau_1(\tilde{k}_{tec})\} \cap \{(\tilde{k}, \tilde{k}_{tec}) : \tilde{k} < \tau_2(\tilde{k}_{tec})\}.$$

Figure 1 about here.

Figure 1 and Lemma 3 yields

Theorem 1: The directions of vector fields in the domains defined by Figure 1 are as follows.

$$(\tilde{k}, \tilde{k}_{tec}) \in \Sigma_1 \Rightarrow \dot{\tilde{k}}_{tec} > 0,\ \dot{\tilde{k}} < 0, \quad (\tilde{k}, \tilde{k}_{tec}) \in \Sigma_2 \Rightarrow \dot{\tilde{k}}_{tec} < 0,\ \dot{\tilde{k}} < 0$$

$$(\tilde{k}, \tilde{k}_{tec}) \in \Sigma_3 \Rightarrow \dot{\tilde{k}}_{tec} < 0,\ \dot{\tilde{k}} > 0, \quad (\tilde{k}, \tilde{k}_{tec}) \in \Sigma_4 \Rightarrow \dot{\tilde{k}}_{tec} > 0,\ \dot{\tilde{k}} > 0. \blacksquare$$

Proof: See Appendix.■

Directly from Theorem 1, the phase diagram of KSS model is given by Figure 2.

Figure 2 about here.

From Figure 2, we demonstrate

Theorem 2: The equilibrium point of KSS model is globally asymptotically stable.■

Proof: Directly from Figure 2.■

Figure 2 demonstrates that KSS model possesses non-oscillatory strong stability.

Example 1: We here numerically describes a typical paths of KSS model. See Figure 3. In Figure 3, we set

$$\theta = 0.85, \quad \zeta = 0.05, \quad \pi = 0.001, \quad \beta_1 = \beta_2 = 1, \quad \tau = 1, \quad a = 0.3, \quad b = 0.57,$$
$$c = 0.13, \quad d = 0.1, \quad \delta = 0.021, \quad n = 0.003.$$

Then, we have

$$\beta = \pi^d = 0.001^{0.1}, \qquad e = (b+d)/(b-c) = 67/44.$$

Then, KSS model (i.e. System Ψ_{KSS}) becomes

$$\Psi_{KSS} \quad \begin{cases} \dot{\tilde{k}}_{tec} = 0.05 \times 0.001^{0.1} \tilde{k}_{tec}^{0.3} \tilde{k}^{0.13} - (0.021 + 0.003 \times 67/44) \tilde{k}_{tec}, \\ \dot{\tilde{k}} = (1-0.9) \times 0.001^{0.1} \tilde{k}_{tec}^{0.3} \tilde{k}^{0.13} - (0.021 + 0.003 \times 67/44) \tilde{k}. \end{cases}$$

In Figure 3, blue curves describe typical paths of KSS model with different initial points and the red point describes the fixed point. Figure 3 numerically describes the result of Theorem 2.■

Figure 3 about here.

We now demonstrate the following.

Theorem 3: For the growth rates of GDP per capita $= y \equiv Y/L$ and capital stock per capita= $k_{aggr} \equiv K_{aggr}/L = (K + K_{tec})/L$, from (5) we obtain the following.

$$\lim_{t\to\infty} (\log \dot{k}_{aggr}) = n(e-1) > 0, \qquad \lim_{t\to\infty} (\log \dot{y}) = n(e-1) > 0.$$

Thus, we see from (5) that, in KSS model, the growth rates of GDP per capita at the equilibrium point is positive.■

Proof: See Appendix.■

We here remark the following.

Corollary 1: As a and b increase, e increases.■

Proof: Directly from the definition of e.■

From Corollary 1, we observe

Observation 1: The parameters of TFP production function depend on research accumulations and various efforts of firms. From Theorem 3, the efforts and accumulations increase the growth rate of GDP per capita at the equilibrium point. This implies that KSS model yields an endogenous growth.■

Concerning Observation 1, we must emphasize one important feature:

Observation: 2: The source of endogenous growth in KSS model is the Keynesian investment behavior based on animal spirits.■

By a simple calculation, we obtain the equilibrium point of KSS model:

Theorem 4: The equilibrium point of KSS model is given by

$$\tilde{k}_{tec}^{*} = \frac{\zeta^{(1-c)/(1-a-c)}\beta^{1/(1-a-c)}(1-\theta-\zeta)^{c/(1-a-c)}}{(\delta+en)^{1/(1-a-c)}},$$

$$\tilde{k}^{*} = \frac{(1-\theta-\zeta)^{(1-a)/(1-a-c)}\zeta^{a/(1-a-c)}\beta^{1/(1-a-c)}}{(\delta+en)^{1/(1-a-c)}}.■$$

Proof: See Appendix.■

The equilibrium point of Theorem 4 corresponds to the steady state of SS model. Therefore, we define the equilibrium point as

Definition 5: The equilibrium point of KSS model, $(\tilde{k}_{tec}^{*}, \tilde{k}^{*})$, is called the quasi-steady state. ■

In the next section we consider the effects of important parameters on economic growth. Figure 2 shows that the equilibrium point of KSS model possesses strong global stability. In the following, ***speaking of growth rates and levels of variables, we always consider those at the quasi-steady state*** (i.e., the equilibrium point in KSS model).

4. Effects of Parameters on Growth Dynamics of KSS Model

It follows directly from Theorem 3 that

Corollary 2: In KSS model, as parameters a and b concerning TFP production function increase the growth rates of GDP per capita and capital stock per capita also increase.■

Proof: Directly from Theorem 3.■

Corollary 3: In KSS model, as the growth rates of labor increases the growth rates of GDP per capita and capital stock per capita also increase.■

Proof: Directly from Theorem 3.■

The results of Corollaries 2 and 3 are almost the same in the related literature. However, the mechanism and the source of yielding the results are quite different from those known so far. Because, in KSS model, Corollaries 2 and 3 are derived through profit principle based on animal spirits. We here intuitively explain the result of Corollary 3.

In the following, for simplification, we assume that the number of enterprises is constant. We consider the case where the growth rate of labor increases. In the case, a chain reaction of short-run and long-run animal spirits starts. Like SS model, the growth rate of GDP also increases. GDP is considered a proxy variable for profit. Therefore, the growth rate of profit also increases. As a result, small innovation investment increases (through short-run animal spirits) and therefore, the realization rate of small innovation increases. This raises expectations for the realization of large innovation and large innovation investment start to increase (through long-run animal spirits). As a result, in the long-run, large innovation starts to increase. Thus, through short-run and long-run animal spirits (Definition 2), an increase in growth rate of labor raises the growth rate of GDP.

In KSS model, we see from Theorem 3 that the short-run and long-run animal spirits parameters (Definition 2) do not affect the **growth rate** of GDP per capita. However, we can demonstrate that the **level** or the **growth** (not growth rate) of GDP per capita is related to the animal spirits parameters:

Corollary 4: We assume that the following condition is satisfied.

(6) $$\frac{(1-\theta)a}{a+c} > \zeta.$$

Then, the level of GDP per capita increases with animal spirits parameter, ζ .■

Proof: See Appendix.■

Corollary 4 suggests that under (6), short-run and long-run animal spirits parameters are important sources of the development of capitalist economy. Condition (6) implies that, unless animal spirits parameter exceeds the threshold given by (6), it is positively correlated to the level of GDP. However, in the case where animal spirits is too intensive, animal spirits is positively correlated to the level of GDP. The reason is that in such a case, animal spirits cause overinvestment. Noting this point, we define the following.

Definition 6: Condition (6) is called a soundness condition of animal spirits. Moreover, if animal spirits parameter, ζ , satisfies the soundness condition, we say the animal spirits to be sound.■

Now, compared to Observation 2, we here make an important remark.

Observation 3: In Observation 2, we see that the source of endogenous growth in KSS model is in profit principle yielded through animal spirits and, therefore, in research accumulations and the efforts of firms for innovation. Moreover, Corollary 4 states that, as far as animal spirits is sound, the intensity of animal spirits increases the level of GDP per capita.■

Concerning Observation 3, we make one important remark:

Remark: From Theorem 3, we see that the growth rate of GDP per capita is independent of animal spirits parameter, ζ. Thus, although the endogenous growth of KSS model is yielded by the profit principle based on animal spirits, the growth rate is determined irrespective of the intensity of animal spirits.■

5. Remarks on KSS Model

In the following, we examine KSS model with well-known endogenous growth models. The remarkable feature of KSS model is the mechanism of generating endogenous growth. We find it in the short-run and long-run animal spirits (Assumptions 1 and 2). Moreover, as far as animal spirits is sound, an increase in the intensity of animal spirits leads to an increase in the level of GDP per capita. Thus, in KSS model, the intensity of animal spirits is the source of abundance.

Taking a hint from Griliches (1998) and Diamond (2004), we assume that, in the long-run, increase in large innovation investment leads to increase in technological innovations concerning TFP. By the way, the empirical fact found by Griliches (1998) and Diamond (2004) is that R&D investment positively correlates the growth rate of TFP. We now demonstrate that the empirical fact is found in KSS model:

Corollary 5: If the growth rate of labor and/or the parameters concerning TFP (a and b) increase, the growth rate of TFP and R&D investment increase at the same time.■

Proof: See Appendix.■

Noting Observation 1, we see from Corollary 5 that the empirical result in Griliches (1998) and Diamond (2004) holds true in the case where a and b increase.

The Romer model (Romer (1990)) is the epochal starting point of endogenous growth theory. In the Romer model, the growth rate of an economy with a large labor is larger than that that with small labor. From a theoretical viewpoint, this is considered a significant drawback. Assuming that the growth rate of TFP depends on the growth rate of labor and the parameters concerning technical innovation, Jones (1995) tries to correct this drawback. We demonstrate that such an assumption of Jones (1995) are derived as a result of KSS model.

Corollary 6: Growth rate of TFP $= \frac{(1-c)(a+b)}{1-a-c} n$.■

Proof: See Appendix.■

From Corollary 6, we observe that the growth rate of TFP positively correlates with the growth rate of labor and the technical parameters concerning TFP, a and b, which depend on research accumulations and the efforts of firms for large technological innovation. For the parameter b, our results are similar to those of Jones (1995). Therefore, enhancing education from various perspectives leads to an increase in growth rate. This result has been well-known.

However, in this paper we emphasize that the growth rate of TFP positively correlates with the parameter a. It is the technological parameter concerning K_{tec}.

> KSS model possesses the accumulation equation of K_{tec}, into which the profit principle is incorporated. Thus, through the profit principle based on the animal spirits becomes a source of the TFP growth.

The parameter a represents the intensity of its reflection. In this sense, the parameter plays an important role in our endogenous growth model.

Moreover, we obtain the following.

Corollary 7: An increase in $\pi = L_{tec} / L$ arises the level of GDP per capita.■

Proof: See Appendix.■

Corollary 7 shows that although an increase in $\pi = L_{tec} / L$ may not rise the growth rate of GDP per capita, the increase enriches people's lives.

KSS model is constructed by incorporating profit principle into SS model. The investment based on profit principle depends on GDP. In the case where the factor increasing GDP operates, profit principle operates in the economy and GDP per capita start to grow. In KSS model, such a factor is the growth of labor. If labor does not grow, KSS model does not produce the growth of GDP per capita and becomes SS model. We here briefly explain it. We assume that the growth rate of labor as the factor is zero. Then, defining $L = \overline{L}$ (constant), it follows from (A.3.3) in Appendix that

$$y = Y / L = \beta \tilde{k}_{tec}^{a} \tilde{k}^{c} \overline{L}^{b+d-1+e(a+c)}.$$

From Theorem 2, we see that $\tilde{k}$ and $\tilde{k}_{tec}$ converge to quasi-stationary state. Therefore, we obtain that if labor does not grow, GDP per capita does not grow. Therefore, GDP also does not grow. Thus, we obtain

Observation 4: In KSS model, the cooperation of profit principle (i.e. animal spirits) and growth of labor produces the growth of GDP per capita.■

6. Rationality and Consistency of Economic Behaviors Based on Profit Principles

The investment behavior we adopt is classical profit principle in which technological innovation investment depends on GDP (i.e., profit). If this investment behavior is not consistent, such a behavior will be discarded sooner or later. In this section, we consider this problem from a longer-run perspective than the business cycle.

What does the consistency of profit principle mean? We first consider profit principle based on short-run animal spirits. We call such a profit principle short-run profit principle. Firms adopting the investment behavior of short-run profit principle expect larger future profits than present profits. Therefore, in the situation that profits grow, it can be said that short-run profit principle is consistent. This also implies that expectations for future profits based on short-run profit principle becomes a self-fulfilling prophesy. Thus, short-run profit principle becomes consistent in a growing economy.

Secondly, we considered profit principle based on long-run animal spirits. We call such a profit principle long-run profit principle. In long-run profits principle, firms expect growth at a higher level than in short-run profit principle. In KSS model, GDP per capita grows. Then, the economy gradually enjoys a higher level of prosperity. Therefore, profits of firms gradually grow. This implies that long-run profit principle also becomes consistent. Thus, expectations for the future profits based on long-run profit principle also becomes a self-fulfilling prophesy. Thus, in KSS model, long-run profit principle becomes consistent in the economy where GDP per capita grows and both profits principles become consistent. In other words, although both profits principles may be not ex ante rational, they are rationalized ex post. Thus, in a roughly sense, there is a rational link or affinity between profit principle and endogenous growth.

7. Conclusions and Final remarks

In this paper, we considered Keynesian profit principle based on animal spirits and incorporated profit principle into the investment function concerning technological innovations. Moreover, incorporating the innovation investment function and modifying SS model, we constructed a growth model from a Keynesian perspective. The model is non-autonomous. Modifying the way used in Solow (1956) and Swan (1956), we derived an autonomous system that allows a complete phase diagram analysis. We named it KSS model. We proved that KSS model produces growth of GDP per capita and that the equilibrium of KSS model is globally asymptotically stable. The equilibrium is similar to the steady state in SS model. We call the equilibrium the quasi-steady state.

The mechanism producing it is intuitively as follows. We assumed that labor grows at a constant rate. Like SS model, GDP also grows. Then, through the operation of (short-run) profit principle, small technological innovation investment increases. Small technological innovation investments will gradually lead to the realizations of small innovations. As a result, expectation and impulsiveness for large technological innovation are gradually intensified. Then, through the operation of (long-run) profit principle, large technological innovation investments (containing R&D investments) increase. Here, based on the empirical fact by Griliches (1986) that the increasing rate of TFP and/or the progress rate of technology depend on R&D investment, we assumed that large technological innovation investments increase TFP. In assuming it, we considered that the investments produce TFP. As the TFP production function, we assumed the Cobb-Douglas production function. As a result, in the long-run, TFP gradually increases and KSS model produces growth of GDP per capita. We demonstrated that if research accumulations increase and various efforts of firms for technological innovation are intensified, TFP increases through the TFP production function and, as a result, growth rate of GDP per capita also increases. Moreover, we demonstrated that growth of labor leads to growth of GDP per capita.

We proved that if labor does not grow, GDP cannot even grow. This implies that growth in KSS model is yielded by the operation of growth of labor and various efforts for technological innovation. More precisely, we can say that growth of labor produces growth of GDP (in the same way as of SS model) and the operation of them leads to the endogenous growth.

We also proved that, as far as animal spirits are sound, the intensity of animal spirits correlates with the level or the growth of GDP per capita at the quasi-steady state. However, we proved that the growth rate of GDP in KSS model is independently of such intensity and the endogenous growth of KSS model yielded by animal spirits is produced irrespective of the intensity.

Thus, KSS model demonstrates that sound animal spirits are an important pulling force of the capitalist economy. However, it has been well-known that the animal spirits are a source of market instability and therefore business fluctuations. These results are derived from qualitatively different models. The result on pulling force (resp. instability) is derived from equilibrium (resp. disequilibrium) model. To derive seemingly contradictory two results simultaneously, we need construct an endogenous growth-cycle model. Research in this direction is currently in progress. See Dohtani and Matsuyama (2022a, b).

Finally, we demonstrated that the investment behavior of profit principle, in which technological innovation investment depends on GDP (i.e., profit), is consistent (i.e. rational) in KSS model. Firms adopting profit principle based on animal spirits expect larger future profits than present profits. Therefore, in the situation that profits grow, it can be said that profit principle is consistent. This also implies that expectations for future profits based on profit principle becomes a self-fulfilling prophesy. Thus, profit principle becomes consistent in a growing economy. In (long-run) profits principle concerning large technological innovation, firms will expect growth at a higher quality level. In KSS model, GDP per capita grows. Then, the economy enjoys a higher level of prosperity and profits of firms step up to a higher level. This implies that long-run profit principle also becomes consistent. In any case, profit principle becomes consistent. In other words, although both profits principle seems ex ante irrational, they are rationalized as a result. Thus, there is a rational link or affinity between profit principle and endogenous growth.

We could consider slightly modified profit principle in which innovation investments depends on expected income. If we incorporate such a profit principle into KSS model, KSS model becomes three dimensional. It will be not easy to analyze dynamic behavior of the three-dimensional version. However, since our original two-dimensional KSS model possesses non-oscillatory strong stability, it is much expected that almost the same results will hold true for the three-dimensional version. However, we must leave its confirmation for future research.

Appendix

In the following, we prove several Lemmas, Theorems and Corollaries.

Proof of Lemma 1: Assumptions 7 and 8 yields

$$F(K_{tec}, L_{tec}, K, L) = \beta_1\beta_2 K_{tec}^a L_{tec}^b K^c L^d = \beta_1\beta_2 K_{tec}^a \pi^b L^b K^c L^d = \beta K_{tec}^a K^c L^{b+d}.$$

This proves Lemma 1.■

Proof of Lemma 2: It follows from definitions and (4) that

(A.1.1) $$\dot{\tilde{k}}_{tec} = \dot{\left(\frac{K_{tec}}{L^e}\right)} = \frac{\dot{K}_{tec} L^e - eL^{e-1}\dot{L} K_{tec}}{L^{2e}} = \frac{\dot{K}_{tec}}{L^e} - e\frac{\dot{L}}{L} \bullet \frac{K_{tec}}{L^e}$$

$$= \frac{\zeta\beta K_{tec}^a K^c L^{b+d} - \delta K_{tec}}{L^e} - en\tilde{k}_{tec} = \frac{\zeta\beta K_{tec}^a K^c L^{b+d}}{L^e} - (\delta + en)\tilde{k}_{tec}$$

$$= \zeta\beta\left(\frac{K_{tec}}{L^e}\right)^a \left(\frac{K}{L^e}\right)^c \frac{1}{L^{e-ea-ec-(b+d)}} - (\delta + en)\tilde{k}_{tec}$$

$$= \zeta\beta\tilde{k}_{tec}^a \tilde{k}^c \frac{1}{L^{e(1-a-c)-(b+d)}} - (\delta + en)\tilde{k}_{tec} = \zeta\beta\tilde{k}_{tec}^a \tilde{k}^c - (\delta + en)\tilde{k}_{tec},$$

(A.1.2) $$\dot{\tilde{k}} = \dot{\left(\frac{K}{L^e}\right)} = \frac{\dot{K} L^e - eL^{e-1}\dot{L} K}{L^{2e}} = \frac{\dot{K}}{L^e} - e\frac{\dot{L}}{L} \bullet \frac{K}{L^e}$$

$$= \frac{(1-\theta-\zeta)\beta K_{tec}^a K^c L^{b+d} - \delta K}{L^e} - en\tilde{k}$$

$$= \frac{(1-\theta-\zeta)\beta K_{tec}^a K^c L^{b+d}}{L^e} - (\delta + en)\tilde{k}$$

$$= (1-\theta-\zeta)\beta\left(\frac{K_{tec}}{L^e}\right)^a \left(\frac{K}{L^e}\right)^c \frac{1}{L^{e-ea-ec-(b+d)}} - (\delta + en)\tilde{k}$$

$$= (1-\theta-\zeta)\beta\tilde{k}_{tec}^a \tilde{k}^c \frac{1}{L^{e(1-a-c)-(b+d)}} - (\delta + en)\tilde{k}$$

$$= (1-\theta-\zeta)\beta\tilde{k}_{tec}^a \tilde{k}^c - (\delta + en)\tilde{k}.$$

This proves Lemma 2.■

Proof of Lemma 3: It follows from Lemma 2 that

(A.2.1) $$\dot{\tilde{k}}_{ltec} \geq 0 \Leftrightarrow \zeta\beta\tilde{k}_{tec}^a \tilde{k}^c \geq (\delta + en)\tilde{k}_{tec} \Leftrightarrow \tilde{k}^c \geq \frac{\delta + en}{\zeta\beta}\tilde{k}_{tec}^{1-a}$$

$$\Leftrightarrow \tilde{k} \geq \left(\frac{\delta + en}{\zeta\beta}\right)^{1/c} \tilde{k}_{tec}^{(1-a)/c} \equiv \tau_1(\tilde{k}_{tec}),$$

(A.2.2) $\dot{\tilde{k}} \geq 0 \Leftrightarrow (1-\theta-\zeta)\beta\tilde{k}_{tec}^{a}\tilde{k}^{c} \geq (\delta+en)\tilde{k} \Leftrightarrow \left(\frac{1-\theta-\zeta}{\delta+en}\right)\beta\tilde{k}_{tec}^{a} \geq \tilde{k}^{1-c}$

$$\Leftrightarrow \tilde{k} \leq \left\{\frac{(1-\theta-\zeta)\beta}{\delta+en}\right\}^{1/(1-c)} \tilde{k}_{tec}{}^{a/(1-c)} \equiv \tau_2(\tilde{k}_{tec}).$$

This proves Lemma 3.■

Proof of Theorem 1: From (A.2) and the definitions of Σ_j ($j \in \{1,2,3,4\}$), we obtain

$$(\tilde{k},\tilde{k}_{tec}) \in \Sigma_1 \Rightarrow \tilde{k} > \tau_1(\tilde{k}_{tec}),\ \tilde{k} > \tau_2(\tilde{k}_{tec}) \Rightarrow \dot{\tilde{k}}_{tec} > 0,\ \dot{\tilde{k}} < 0,$$
$$(\tilde{k},\tilde{k}_{tec}) \in \Sigma_2 \Rightarrow \tilde{k} < \tau_1(\tilde{k}_{tec}),\ \tilde{k} > \tau_2(k_{tec}) \Rightarrow \dot{\tilde{k}}_{tec} < 0,\ \dot{\tilde{k}} < 0,$$
$$(\tilde{k},\tilde{k}_{tec}) \in \Sigma_3 \Rightarrow \tilde{k} < \tau_1(\tilde{k}_{tec}),\ \tilde{k} < \tau_2(\tilde{k}_{tec}) \Rightarrow \dot{\tilde{k}}_{tec} < 0,\ \dot{\tilde{k}} > 0,$$
$$(\tilde{k},\tilde{k}_{tec}) \in \Sigma_4 \Rightarrow \tilde{k} > \tau_1(\tilde{k}_{tec}),\ \tilde{k} < \tau_2(\tilde{k}_{tec}) \Rightarrow \dot{\tilde{k}}_{tec} > 0,\ \dot{\tilde{k}} > 0.$$

This completes the proof.■

Proof of Theorem 3: It follows from the definitions that

(A.3.1) $k_{tec} \equiv \frac{K_{tec}}{L} = \frac{\tilde{k}_{tec}L^{e}}{L} = \tilde{k}_{tec}L^{e-1},$

(A.3.2) $k \equiv \frac{K}{L} = \frac{\tilde{k}L^{e}}{L} = \tilde{k}L^{e-1},$

(A.3.3) $y = \frac{Y}{L} = \frac{\beta K_{tec}^{a}K^{c}L^{b+d}}{L} = \beta\frac{K_{tec}^{a}}{L^{ea}}\frac{K^{c}}{L^{ec}}L^{b+d+ea+ec-1}$

$$= \beta\tilde{k}_{tec}^{a}\tilde{k}^{c}L^{b+d-1+e(a+c)}.$$

Since $\tilde{k}_{tec}$ and $\tilde{k}$ are solutions of KSS model, they converge. Therefore,

$$\lim_{t\to\infty}\dot{\tilde{k}} = 0, \qquad \lim_{t\to\infty}\dot{\tilde{k}}_{tec} = 0.$$

From this, we obtain

(A.4.1) $\lim_{t\to\infty}\left(\log\dot{\tilde{k}}\right) = \lim_{t\to\infty}\dot{\tilde{k}}/\tilde{k} = 0, \quad \lim_{t\to\infty}\left(\log\dot{\tilde{k}}_{tec}\right) = \lim_{t\to\infty}\dot{\tilde{k}}_{tec}/\tilde{k}_{tec} = 0,$

(A.4.2) $\lim_{t\to\infty}\left\{\log(\dot{\tilde{k}}+\tilde{k}_{tec})\right\} = \lim_{t\to\infty}(\dot{\tilde{k}}+\dot{\tilde{k}}_{tec})/(\tilde{k}+\tilde{k}_{tec}) = 0.$

Moreover, we have

$$k_{aggr} \equiv \frac{K_{aggr}}{L} = \frac{K + K_{tec}}{L} = k + k_{tec} = (\tilde{k} + \tilde{k}_{tec})L^{e-1}.$$

Therefore, (A.3) and (A.4) yield

$$\begin{aligned}\lim_{t\to\infty}(\log \overset{\bullet}{k}_{aggr}) &= \lim_{t\to\infty}\left[\overset{\bullet}{\log}\{(\tilde{k} + \tilde{k}_{tec})L^{e-1}\}\right] \\ &= \lim_{t\to\infty}\left\{\overset{\bullet}{\log}(\tilde{k} + \tilde{k}_{tec})\right\} + (e-1)\lim_{t\to\infty}(\log \overset{\bullet}{L}) \\ &= n(e-1) > 0,\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty}(\log \overset{\bullet}{y}) &= a\lim_{t\to\infty}(\log \overset{\bullet}{\tilde{k}}_{tec}) + c\lim_{t\to\infty}(\log \overset{\bullet}{\tilde{k}}) \\ &\quad + \{b + d - 1 + e(a + c)\}\lim_{t\to\infty}(\log \overset{\bullet}{L}) \\ &= n\{b + d - 1 + e(a + c)\} \\ &= n\{b + d - e(1 - a - c) + e - 1\} = n(e-1) > 0.\end{aligned}$$

Thus, we complete the proof.■

Proof of Theorem 4: The equilibrium point of KSS model is given by the solution of the equations:

$$\tilde{k} = \left(\frac{\delta + en}{\zeta\beta}\right)^{1/c} \tilde{k}_{tec}^{(1-a)/c}, \qquad \tilde{k} = \left\{\frac{(1-\theta-\zeta)\beta}{\delta + en}\right\}^{1/(1-c)} \left(\tilde{k}_{tec}\right)^{a/(1-c)}.$$

Therefore, we have

$$\left(\frac{\delta + en}{\zeta\beta}\right)^{1/c} \tilde{k}_{tec}^{(1-a)/c} = \left\{\frac{(1-\theta-\zeta)\beta}{\delta + en}\right\}^{1/(1-c)} \left(\tilde{k}_{tec}\right)^{a/(1-c)}.$$

We see from this equation that

$$\tilde{k}_{tec}^{(1-a)/c - a/(1-c)} = \left(\frac{\zeta\beta}{\delta + en}\right)^{1/c} \left\{\frac{(1-\theta-\zeta)\beta}{\delta + en}\right\}^{1/(1-c)} \quad \Rightarrow$$

$$\tilde{k}_{tec}^{(1-a-c)/c(1-c)} = \left(\frac{\zeta\beta}{\delta + en}\right)^{1/c} \left\{\frac{(1-\theta-\zeta)\beta}{\delta + en}\right\}^{1/(1-c)}.$$

It follows from this fact that

(A.5.1) $$\tilde{k}_{tec}^{*} = \left(\frac{\zeta\beta}{\delta + en}\right)^{(1-c)/(1-a-c)} \left\{\frac{(1-\theta-\zeta)\beta}{\delta + en}\right\}^{c/(1-a-c)}$$

$$= \frac{\zeta^{(1-c)/(1-a-c)}\beta^{1/(1-a-c)}(1-\theta-\zeta)^{c/(1-a-c)}}{(\delta + en)^{1/(1-a-c)}},$$

(A.5.2) $$\tilde{k}^* = \left\{\frac{(1-\theta-\zeta)\beta}{\delta+en}\right\}^{1/(1-c)} \tilde{k}_{tec}^{*a/(1-c)}$$

$$= \left\{\frac{(1-\theta-\zeta)\beta}{\delta+en}\right\}^{1/(1-c)} \left\{\frac{\zeta^{(1-c)/(1-a-c)}\beta^{1/(1-a-c)}(1-\theta-\zeta)^{c/(1-a-c)}}{(\delta+en)^{1/(1-a-c)}}\right\}^{a/(1-c)}$$

$$= \frac{\zeta^{a/(1-a-c)}(1-\theta-\zeta)^{(1-a)/(1-a-c)}\beta^{1/(1-a-c)}}{(\delta+en)^{1/(1-a-c)}}.$$

This proves Theorem 4.■

Proof of Corollary 4: We see from Theorem 2 that any solution of KSS model converges to the quasi-steady state:

$$\lim_{t\to\infty} \tilde{k}_{tec} = \tilde{k}_{tec}^*, \quad \lim_{t\to\infty} \tilde{k} = \tilde{k}^*.$$

We define

$$M \equiv \frac{\beta^{1/(1-a-c)}}{(\delta+en)^{1/(1-a-c)}}, \qquad N \equiv \beta L^{b+d-1+e(a+c)}.$$

It follows from (A.3.3) that

$$y = \beta L^{b+d-1+e(a+c)} \tilde{k}_{tec}^a \tilde{k}^c = N \tilde{k}_{tec}^a \tilde{k}^c.$$

Therefore, from (A.5), we see that GDP per capita is given by

$$y^* \equiv \lim_{t\to\infty} y = N \tilde{k}_{tec}^{*\,a} \tilde{k}^{*c}$$

$$= NM^a \zeta^{\frac{a(1-c)}{1-a-c}} (1-\theta-\zeta)^{\frac{ac}{1-a-c}} \bullet M^c \zeta^{\frac{ac}{1-a-c}} (1-\theta-\zeta)^{\frac{c(1-a)}{1-a-c}}$$

$$= NM^{a+c} \zeta^{\frac{a}{1-a-c}} (1-\theta-\zeta)^{\frac{c}{1-a-c}}.$$

Therefore, we see that

$$\frac{d\tilde{y}^*}{d\zeta} = \frac{NM^{a+c}}{1-a-c}\left\{a\zeta^{\frac{a}{1-a-c}-1}(1-\theta-\zeta)^{\frac{c}{1-a-c}} - c\zeta^{\frac{a}{1-a-c}}(1-\theta-\zeta)^{\frac{c}{1-a-c}-1}\right\}.$$

Therefore, we have

$$\frac{d\tilde{y}^*}{d\zeta} > 0 \ \text{ if and only if } \ a\zeta^{\frac{a}{1-a-c}-1}(1-\theta-\zeta)^{\frac{c}{1-a-c}} > c\zeta^{\frac{a}{1-a-c}}(1-\theta-\zeta)^{\frac{c}{1-a-c}-1}.$$

This implies that

$$\frac{d\tilde{y}^*}{d\zeta} > 0 \quad \text{if and only if} \quad a(1-\theta-\zeta) > c\zeta.$$

Therefore, we obtain

$$\frac{d\tilde{y}^*}{d\zeta} > 0 \quad \text{if and only if} \quad \frac{(1-\theta)a}{a+c} > \zeta.$$

Thus, under Condition (6), we see that as animal spirits parameters ζ increase, the growth of GDP per capita also increases.■

Proof of Corollary 5: We define:

$$\text{Increase rate of } K_{tec} = \overset{\bullet}{\left(\log K_{tec}\right)} \equiv GRK_{tec}, \quad \text{Increase rate of TFP} \equiv GRT_{tec}.$$

Then, we have

(A.6.1) $$GRT_{tec} = \overset{\bullet}{\{\log T(K_{tec}, L_{tec})\}} = \overset{\bullet}{\log\left(\lambda K_{tec}^{a} L_{tec}^{b}\right)} = aGRK_{tec} + bn,$$

(A.6.2) $$\frac{I_{tec}}{K_{tec}} = \frac{\overset{\bullet}{K}_{tec} + \delta K_{tec}}{K_{tec}} = GRK_{tec} + \delta.$$

The definition (5) yields $K_{tec} = \tilde{k}_{tec} L^{e}$ and $en = (b+d)n/(1-a-c)$. Moreover, as stated in the proof of Theorem 3, we have $\lim_{t\to\infty} \overset{\bullet}{\tilde{k}}_{tec} = 0$. Therefore, we see

(A.7) $$\lim_{t\to\infty} GRK_{tec} = \lim_{t\to\infty} \overset{\bullet}{\{\log(K_{tec})\}} = \lim_{t\to\infty} \overset{\bullet}{\left\{\log(\tilde{k}_{tec} L^{e})\right\}}$$

$$= \lim_{t\to\infty} \overset{\bullet}{\left\{\log(\tilde{k}_{tec})\right\}} + en = \frac{b+d}{1-a-c} n.$$

The capital stock for technological innovation at the quasi-steady state is given by $K_{tec}^{*} \equiv \tilde{k}_{tec}^{*} L^{e}$. Therefore, from (A.6.2) and (A.7), we see that R&D investment at the quasi-steady state ($\equiv I_{tec}^{*}$) is given by

(A.8) $$I_{tec}^{*} = K_{tec}^{*} \bullet (\lim_{t\to\infty} GRK_{tec} + \delta) = \tilde{k}_{tec}^{*} L^{e} \bullet \left(\frac{b+d}{1-a-c} n + \delta\right).$$

From (A.7) and (A.8), we see that as the parameters a and b concerning TFP function increase under the condition $1 > a+c$ of Assumption 8, $\lim_{t\to\infty} GRK_{tec}$ and

I_{tec}^* also increase. Therefore, we see from (A.6.1) that as the parameters a and b increase, $\lim_{t\to\infty} GRT_{tec}$ also increases. These results imply that the growth rate of TFP and the R&D investment increase at the same time. This completes the proof.■

Proof of Corollary 6: From Assumption 8, the definition and (A.6), we have

$$\text{Growth rate of TFP } = aGRK_{tec} + bn = a\frac{b+d}{1-a-c}n + bn$$
$$= \left\{\frac{a(b+1-c)}{1-a-c} + \frac{b(1-a-c)}{1-a-c}\right\}n$$
$$= \left(\frac{a-ac+b-bc}{1-a-c}\right)n = \frac{(1-c)(a+b)}{1-a-c}n.$$

This completes the proof.■

Proof of Corollary 7: From Theorem 4, K_{tec} and K at the steady state, we have

$$K_{tec}^* \equiv \tilde{k}_{tec}^* L^e = \frac{\zeta^{(1-c)/(1-a-c)}\beta^{1/(1-a-c)}(1-\theta-\zeta)^{c/(1-a-c)}}{(\delta+en)^{1/(1-a-c)}}L^e,$$
$$K^* \equiv \tilde{k}^* L^e = \frac{(1-\theta-\zeta)^{(1-a)/(1-a-c)}\zeta^{a/(1-a-c)}\beta^{1/(1-a-c)}}{(\delta+en)^{1/(1-a-c)}}L^e.$$

Therefore, noting Assumption 8, K_{tec}^* and K^* are monotonously increasing functions with respect to β. Moreover, we see from Lemma 1 that GDP per capita at the quasi-steady state is given by

$$y^* \equiv \frac{Y^*}{L} \equiv \frac{\beta K_{tec}^{*\,a} K^{*c} L^{b+d}}{L} = \beta K_{tec}^{*\,a} K^{*c} L^{b+d-1}.$$

Therefore, GDP per capita is monotonously increasing functions with respect to β. Since $\beta = \beta_1\beta_2(\tau\pi)^d$, GDP at per capita is monotonously increasing functions with respect to π. This completes the proof.■

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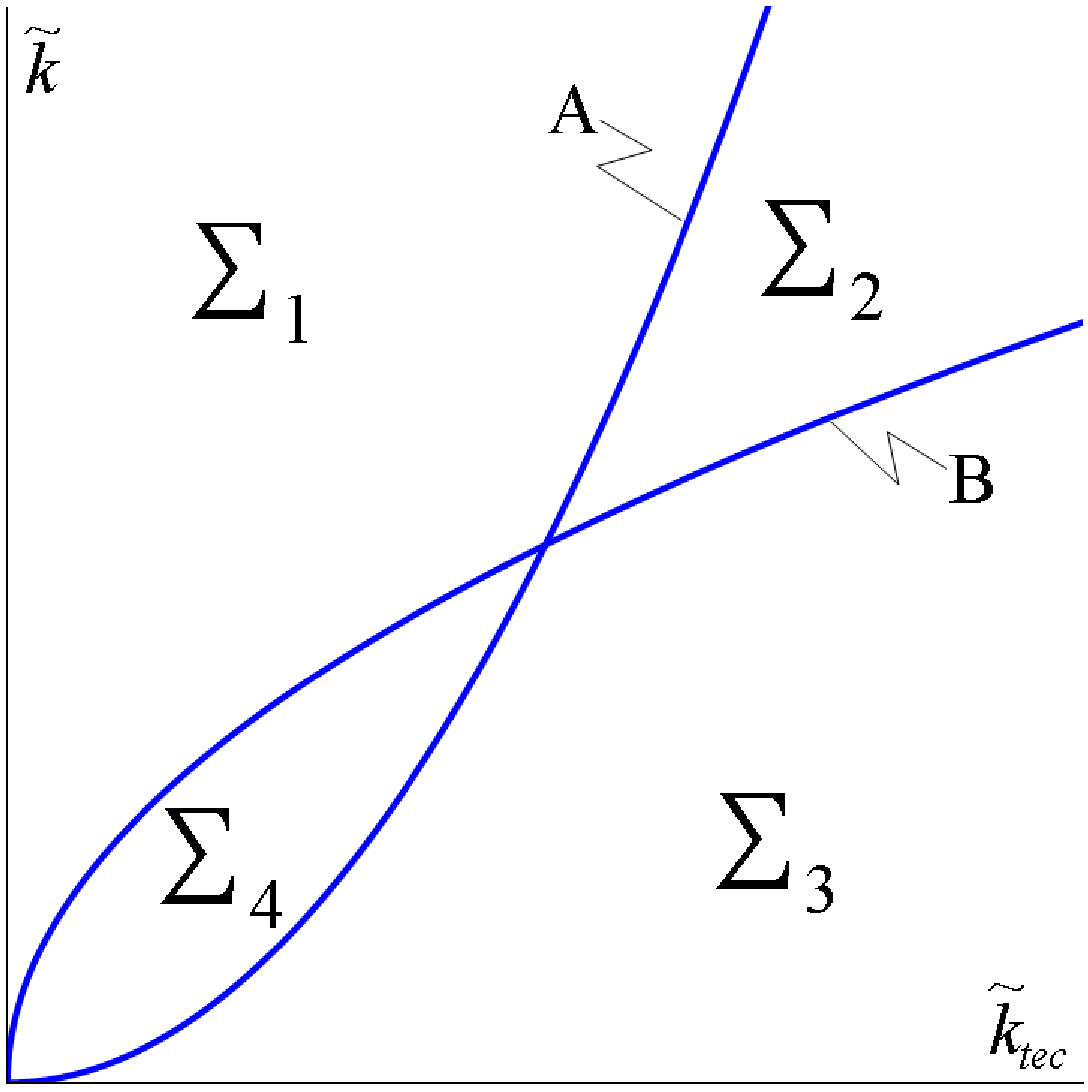

Figure 1 : The description of Σ_j $(j = 1,2,3,4)$.

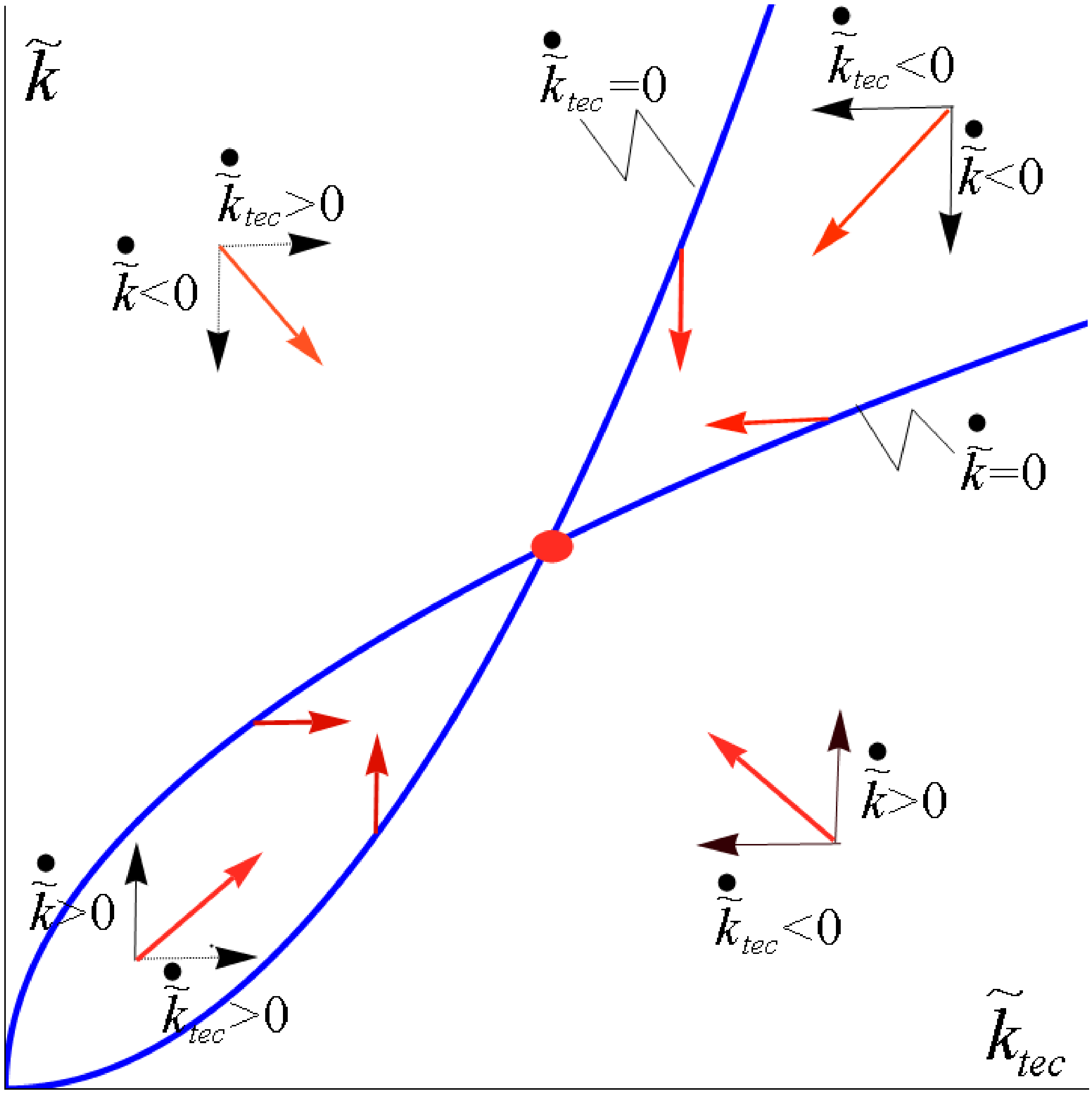

Figure 2 : Vector field demonstrates global asymptotical stability

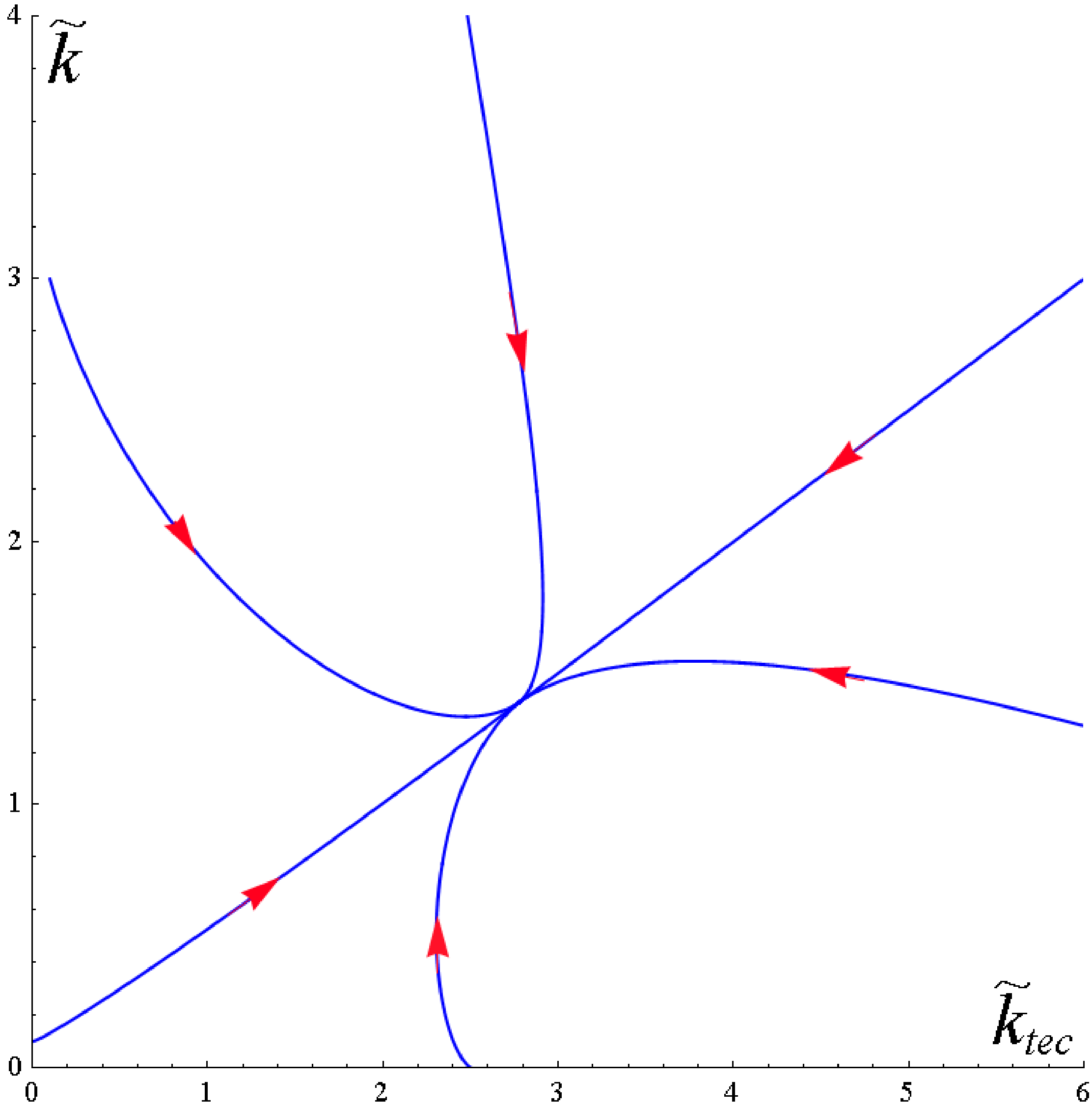

Figure 3 : Typical paths of KSS model