Working Paper No. 356

Sequential Generalized Lorenz Dominance Criterion with Multivariate Attributes

Kazuyuki Nakamura

March, 2023

UNIVERSITY OF TOYAMA 富山大学

SCHOOL OF ECONOMICS
UNIVERSITY OF TOYAMA

Sequential Generalized Lorenz Dominance Criterion with Multivariate Attributes*

Kazuyuki Nakamura†
Faculty of Social Sciences, University of Toyama
Gofuku 3190, Toyama 930-8555, Japan

March 2023

Abstract

We extend the sequential generalized Lorenz dominance (SGL) criterion in two directions. First, we consider situations where the welfare of individuals depends on multivariate attributes. Second, we allow for demographic differences. Even under such extensions, the derived welfare ordering is basically same as in the previous analysis. We will also offer an empirical illustration.

Key words: Multivariate dominance criterion, Sequential Lorenz dominance

JEL code: D31, D13

1 Introduction

The importance of multivariate measurement in the evaluation of social welfare is undisputed. As this has long been recognized, there have been various attempts to extend Lorenz dominance criteria and inequality measures to multivariate attributes (e.g., Kolm, 1977; Marshall et al., 2011; Aaberge and Brandolini, 2015). The United Nations has established indicators to assess the welfare of countries and regions from multiple perspectives, such as the Human Development Index (HDI) and the Sustainable Development Goal (SDG) Index.

To evaluate the welfare of societies consisting of households, regions or countries, we must consider the non-transferable heterogeneity among agents. The sequential generalized Lorenz dominance (SGL) criterion of Atkinson and Bourguignon (1987) is an informative criterion for evaluating income distribution for this purpose. Using SGL, we can assess social welfare depending on households of various sizes without using equivalence scales. Furthermore, the welfare ordering

* This work was supported by JSPS KAKENHI Grant Number 19K01694.
† Email: knakamur@eco.u-toyama.ac.jp

by the SGL is consistent with a broad class of social evaluation functions. Because of these features, much literature extends the SGL criterion to various perspectives (e.g., Lambert and Ramos, 2002; Fleurbaey et al., 2003; López-Laborda and Onrubia, 2005; Ooghe and Lambert, 2006; Ooghe, 2007; Moyes, 2012; Muller and Trannoy, 2012; Fleurbaey et al., 2014).

The present paper extends the SGL criterion in two directions. First, we consider situations where the social welfare of heterogeneous agents depends on multiple attributes. We retain the basic properties of SGL while extending it to the multivariate case. That is, the concavity of the utility function for transferable attributes (income, health, education, etc.) used in Kolm (1977) and other related literature and the properties of the utility function for changes in non-transferable attributes (needs) follow the assumptions in SGL. Second, we consider demographic differences, as discussed in Jenkins and Lambert (1993) and Chambaz and Maurin (1998).

Furthermore, we show that solving a linear programming problem reveals the presence or absence of the proposed dominance relation between two distributions. We cannot test the dominance condition via a convenient graphical form such as the generalized Lorenz curve in SGL. Instead, we can obtain the optimal solution to a linear programming problem with a clear implication and providing further insights into the dominance condition. That is, even if we cannot observe a dominance relation between two distributions, we can know from the solution of a linear programming problem which economic agents and how much they should change which attributes to allow the dominance relation to hold.

As an illustration of the method presented here, we evaluate changes in global welfare based on the HDI. As is well-known, the HDI represents the geometric mean of three attributes: health, education, and income. The method presented here allows us to evaluate global well-being under a broad class of social welfare functions. Muller and Trannoy (2011) have proposed a multivariate dominance condition and analyzed worldwide well-being based on the HDI. Here, we divide countries into two categories, least developed countries (LDCs) and non-LDCs, and consider the situation where LDCs are strongly required to improve their HDI.

In the next section, we describe the analytical framework. Section 3 presents the dominance criteria proposed in this paper. Section 4 explains how to test the dominance condition using a linear programming problem. Section 5 shows an empirical application of the method described herein. Finally, we provide a summary and some remarks.

2 Analytical Framework

Consider a society consisting of n agents (households, countries, regions, etc.). Let $\mathcal{N}$ be the set of agents $\mathcal{N} := \{1, \dots, n\}$. In what follows, we compare two situations, denoted by X and Y. Agents in each situation are classified into $H(\geq 2)$ types according to their needs for the cardinal attributes. Let

$\mathcal{H} \coloneqq \{1, \dots, H\}$ be the set of needs. The group of type h agents, where $h \in H$, is the h-th mos needy among the groups.

Let $\mathcal{N}_J^h$ for $h \in \mathcal{H}$ and $J \in \{X, Y\}$ be the set of agents of type h. The number of agents of type h in situation J is denoted by $n_J^h \coloneqq \#\mathcal{N}_J^h$. Since the total number of agents is identical between X and Y, $\sum_{h \in \mathcal{H}} n_X^h = \sum_{h \in \mathcal{H}} n_Y^h$ holds. However, we allow for a situation $n_X^h \neq n_Y^h$ for some $h \in \mathcal{H}$. Without loss of generality, we define an index set such as $\mathcal{N}_J^h = \{n_J^{h-1} + 1, \dots, n_J^h\}$ with $n_J^0 = 0$ for $\forall h \in \mathcal{H}$ and $J \in \{X, Y\}$. Let $\widetilde{\mathcal{N}}_J^h$ be the set of agents whose needs are greater than or equal to h: $\widetilde{\mathcal{N}}_J^h \coloneqq \bigcup_{j=1}^h \mathcal{N}_J^j$. Let $\tilde{n}_J^h$ be the number of agents who belongs to $\widetilde{\mathcal{N}}_J^h$: $\tilde{n}_J^h = \#\widetilde{\mathcal{N}}_J^h$. From the assumption on the index for agents, we can confirm that $\widetilde{\mathcal{N}}_J^h = \{1, \dots, \tilde{n}_J^h\}$.

The utility of an agent depends on m attributes, which have cardinal properties. Let $\mathcal{M} = \{1, \dots, m\}$ be the set of attributes. We denote by $z_{ij} \in \mathcal{D}_i = [\underline{z}_i, \bar{z}_i]$ for $i \in \mathcal{M}$ and $j \in \mathcal{N}$ the i-th attribute with which the j-th agent is endowed, where $\mathcal{D}_i$ is the conceivable range for the i-th attribute. For the later discussion, we denote by $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_m)$ the vector consisting of the maximum conceivable amounts of all attributes. Thus, the cardinal attributes with which the j-th agent is endowed can be represented by a column vector such as $\mathbf{z}_j = (z_{1j}, \dots, z_{mj})^t \in \mathcal{D} \subset \mathbb{R}^m$, where $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_m$.[1]

Let $\mathcal{R}(m, n)$ be a set of m-by-n matrices with real entries. The distribution of attributes in the h-th needy group in society X is represented as follows:

$$\mathbf{X}^h = \left[\mathbf{x}_{n_X^{h-1}+1}, \dots, \mathbf{x}_{n_X^h}\right] \in \mathcal{R}(m, n_X^h) \; for \; h \in \mathcal{H}, x_{ij} \in \mathcal{D}. \tag{1}$$

Thus, the full distribution of attributes in situation X can be written as follows:

$$\mathbf{X} = [\mathbf{X}^1, \dots, \mathbf{X}^H] \in \mathcal{R}(m, n). \tag{2}$$

Let $\widetilde{\mathbf{X}}^h$ be a sub-matrix consisting of the distributions of needy groups from 1 to h. That is,

$$\widetilde{\mathbf{X}}^h = [\mathbf{X}^1, \dots, \mathbf{X}^h] \in \mathcal{R}(m, \tilde{n}_X^h), \qquad h \in \mathcal{H}. \tag{3}$$

For situation Y, we similarly define $\mathbf{y}_j$, $\mathbf{Y}^h$, $\mathbf{Y}$, and $\widetilde{\mathbf{Y}}^h$.

We denote by $U(\mathbf{z}, h)$ the utility of agents having attributes $\mathbf{z} \in (-\infty, \infty) \times, \dots, \times (-\infty, \infty)$, where this utility function is continuous in $\mathbf{z}$ and we denote by $\mathcal{U}$ the set of all such functions. Furthermore, we consider a class of utility functions based on the following properties.

U1 For a given $h \in \mathcal{H}$, $U(\mathbf{z}, h)$ is non-decreasing and concave in $\mathbf{z}$;

[1] Superscript t denotes transpose of a vector.

U2 For a given $h \in \mathcal{H} \setminus \{H\}$, $G(\mathbf{z}, h) = U(\mathbf{z}, h) - U(\mathbf{z}, h+1)$ is non-decreasing and concave in $\mathbf{z}$;

U3 $G(\bar{\mathbf{z}}, h) = 0$ holds $\forall h \in \mathcal{H} \setminus \{H\}$.

We introduce the following class of utility function:

$$\mathcal{U}_{\bar{\mathbf{z}}} = \{U \in \mathcal{U}\text{: U1, U2 and U3 are satisfied}\}.$$

Assumptions U1–U3 can be interpreted as an extended version of Jenkins and Lambert (1993). Indeed, when we consider the case of $\#\mathcal{M} = 1$, $\mathcal{U}_{\bar{\mathbf{z}}}$ coincides with the class considered in Jenkins and Lambert (1993). In the present analysis, we do not impose differentiability with respect to $\mathbf{z}$ on U, as this property is not essential. From U1, U2, and U3, we can verify that $G(\mathbf{z}, h) \leq 0$ $\forall h \in \mathcal{H} \setminus \{H\}$.

For example, we can construct a utility function which belongs to $\mathcal{U}_{\bar{\mathbf{z}}}$ as follows.

Example 1 Let $\mathcal{V}^k = \{[c_1^k, \mathbf{v}_1^k], \dots, [c_r^k, \mathbf{v}_r^k]\}$ be the set of $(m+1)$-dimensional real vector, where $c_s^k \in \mathbb{R}$ and $\mathbf{v}_s^k \in \mathbb{R}_+^m$ for $s = 1,2,\dots,r$. The following utility function $\widehat{U}$ belong to $\mathcal{U}_{\bar{\mathbf{z}}}$:

$$\widehat{U}(\mathbf{z}, h) = \alpha_h + \sum_{k=h}^{H} \varphi^k(\mathbf{z}), h \in \mathcal{H}, \mathbf{z} \in \mathcal{D}, \tag{4}$$

where

$$\varphi^k(\mathbf{z}) = \min_{[c_s^k, v_s^k] \in \mathcal{V}^k} \left\{ [c_s^k, v_s^k] \begin{bmatrix} 1 \\ \mathbf{z} \end{bmatrix} \right\},$$

$$\alpha_{h+1} = \alpha_h + \varphi^h(\bar{\mathbf{z}}), h \in \mathcal{H} \setminus \{H\},$$

and α_1 is given.[2]

Social welfare is utilitarian. In society X, we denote by W_X social welfare as follows:

[2] Since $\varphi^h(\mathbf{z})$ is non-decreasing and concave in $\mathbf{z}$, $\widehat{U}$ is also non-decreasing and concave. Furthermore, $\widehat{G}(\mathbf{z}, h) = \alpha_h - \alpha_{h+1} + \varphi^h(\mathbf{z})$ is also non-decreasing and concave. Finally, $\widehat{U}(\bar{\mathbf{z}}, h) = \alpha_h + \varphi^h(\bar{\mathbf{z}}) + \sum_{k=h+1}^{H} \varphi^h(\bar{\mathbf{z}}) = \widehat{U}(\bar{\mathbf{z}}, h+1)$ holds.

$$W_X = \frac{1}{n}\sum_{h\in\mathcal{H}}\sum_{j\in\mathcal{N}_X^h} U(\mathbf{x}_j, h) = \frac{1}{n}\left[\sum_{h\in\mathcal{H}\setminus\{H\}}\sum_{j\in\tilde{\mathcal{N}}_X^h} G(\mathbf{x}_j, h) + \sum_{j\in\tilde{\mathcal{N}}} U(\mathbf{x}_j, H)\right]. \tag{5}$$

We define W_Y in a similar way.

3 Multivariate Dominance Criteria

First, we consider a dominance criterion which is an extended version of uniform majorization introduced by Kolm (1977) in the economics literature.

Definition 1 For two matrices, $\mathbf{A}$ and $\mathbf{B} \in \mathcal{R}(m, n)$, $\mathbf{A} \prec^w \mathbf{B}$ implies that there exists some doubly stochastic matrix, $\mathbf{P} \in \mathcal{R}(n, n)$ such that $\mathbf{BP} \leq \mathbf{A}$ holds.

The above definition is known as increasing concave ordering (ICV) in the literature of the theory of stochastic order (e.g., Muller and Stoyan, 2002). In the theory of majorization, if $\mathbf{BP} = \mathbf{A}$ holds for some doubly stochastic matrix, $\mathbf{P}$, it is said that $\mathbf{B}$ uniformly majorizes $\mathbf{A}$ (e.g., Kolm, 1977). The notion of uniform majorization can be regarded as an extension of the familiar Lorenz dominance criterion. In this analogy, Definition 1 can be regarded as an extension of the generalized Lorenz dominance criterion of Shorrocks (1983) to multivariate attributes.

Indeed, the following property, which is interpreted as a multivariate version of the Shorrocks theorem, can be obtained.

Lemma 1 For two matrices consisting of m-dimensional column vector, $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ and $\mathbf{B} = [\mathbf{b}_1, \dots, \mathbf{b}_n] \in \mathcal{R}(m, n)$, the following two conditions are equivalent:

(*i*) $\mathbf{A} \prec^w \mathbf{B}$,

(*ii*) $\sum_{i=1}^{n} f(\mathbf{a}_i) \geq \sum_{i=1}^{n} f(\mathbf{b}_i)$ holds for all non-decreasing concave function f.

Proof: See Nakamura (2012).

We turn to the dominance criterion that we propose herein. Following Atkinson and Bourguignon (1987), we could compare $\widetilde{\mathbf{X}}^h$ and $\widetilde{\mathbf{Y}}^h$, sequentially. However, it should be noted that $\tilde{n}_X^h = \tilde{n}_Y^h$ $\forall h \in \mathcal{H}$ does not necessarily hold. Thus, we begin by adjusting the sizes of matrices. Let $\tilde{n}_M^h$ for $h \in \mathcal{H}$ be $\max\{\tilde{n}_X^h, \tilde{n}_Y^h\}$. Based on $\widetilde{\mathbf{X}}^h$ and $\widetilde{\mathbf{Y}}^h$, we construct the following matrices.

$$\widehat{\mathbf{X}}^h = \begin{cases} [\widetilde{\mathbf{X}}^h, \mathbf{Z}^h] & if\ \tilde{n}_X^h < \tilde{n}_Y^h, \\ \widetilde{\mathbf{X}}^h & if\ \tilde{n}_X^h = \tilde{n}_M^h, \end{cases} \tag{6a}$$

and

$$\widehat{\mathbf{Y}}^h = \begin{cases} [\widetilde{\mathbf{Y}}^h, \mathbf{Z}^h] & if\ \tilde{n}_X^h > \tilde{n}_Y^h, \\ \widetilde{\mathbf{Y}}^h & if\ \tilde{n}_Y^h = \tilde{n}_M^h, \end{cases} \tag{6b}$$

where $\mathbf{Z}^h \in \mathcal{R}(m, |\tilde{n}_X^h - \tilde{n}_Y^h|)$ is a matrix consisting of identical column vectors, $\bar{\mathbf{z}} \in \mathbb{R}^m$:

$$\mathbf{Z}^h = [\bar{\mathbf{z}}, \dots, \bar{\mathbf{z}}]. \tag{7}$$

Next, we define the following dominance criterion.

Definition 2 For $\widehat{\mathbf{X}}^h$ and $\widehat{\mathbf{Y}}^h \in \mathcal{R}(m, \tilde{n}_M^h)$ and a given vector for $\bar{\mathbf{z}}$, if $\widehat{\mathbf{X}}^h \prec^w \widehat{\mathbf{Y}}^h\ \forall h \in \mathcal{H}$ holds, then **X** dominates **Y** in the sense of an extended form of Jenkins and Lambert's dominance criterion (in symbols $\mathbf{Y}(\bar{\mathbf{z}}) \prec^{JL} \mathbf{X}(\bar{\mathbf{z}})$).

If $m = 1$, then Definition 2 is basically the same as the welfare dominance criterion proposed by Jenkins and Lambert (1993) and the dominance criterion of Chambaz and Maurin (1998). In addition to $m = 1$, if we concentrate on the case of $\tilde{n}_X^h = \tilde{n}_Y^h\ \forall h \in \mathcal{H}$, Definition 2 reduces to Atkinson and Bourguignon's SGL.

The following Lemma characterizes the dominance condition described above using the Kronecker product, ⊗, and the column stacking operator, vec.

Lemma 2 The following two conditions are equivalent.

(*i*) $\mathbf{Y}(\bar{\mathbf{z}}) \prec^{JL} \mathbf{X}(\bar{\mathbf{z}})$

(*ii*) The following system of inequalities has a non-negative solution $\mathbf{q}^h \in \mathbb{R}_+^{(\tilde{n}_M^h)^2}\ \forall h \in \mathcal{H}$:

$$\begin{bmatrix} \mathbf{A}^1 & & \boldsymbol{O} \\ & \ddots & \\ \boldsymbol{O} & & \mathbf{A}^H \end{bmatrix} \begin{bmatrix} \mathbf{q}^1 \\ \vdots \\ \mathbf{q}^H \end{bmatrix} \leq \begin{bmatrix} \mathbf{b}^1 \\ \vdots \\ \mathbf{b}^H \end{bmatrix}, \tag{8}$$

where,

$$\mathbf{A}^h=\begin{bmatrix}\mathbf{I}_{\tilde{n}_M^h}\otimes\widehat{\mathbf{Y}}^h\\ \mathbf{e}_{\tilde{n}_M^h}^t\otimes\mathbf{I}_{\tilde{n}_M^h}\\ -\mathbf{e}_{\tilde{n}_M^h}^t\otimes\mathbf{I}_{\tilde{n}_M^h}\\ \mathbf{I}_{\tilde{n}_M^h}\otimes\mathbf{e}_{\tilde{n}_M^h}^t\\ -\mathbf{I}_{\tilde{n}_M^h}\otimes\mathbf{e}_{\tilde{n}_M^h}^t\end{bmatrix},\quad \mathbf{b}^h=\begin{bmatrix}\mathrm{vec}\widehat{\mathbf{X}}^h\\ \mathbf{e}_{\tilde{n}_M^h}\\ -\mathbf{e}_{\tilde{n}_M^h}\\ \mathbf{e}_{\tilde{n}_M^h}\\ -\mathbf{e}_{\tilde{n}_M^h}\end{bmatrix},\quad \text{for } h\in\mathcal{H}$$

In the above definition, $\mathbf{I}_{\tilde{n}_M^h}\in\mathcal{R}(\tilde{n}_M^h,\tilde{n}_M^h)$ is an identity matrix and $\mathbf{e}_{\tilde{n}_M^h}\in\mathbb{R}^{\tilde{n}_M^h}$ is a column vector whose entries are all equal to 1.

Proof First, consider a matrix inequality $\widehat{\mathbf{Y}}^h\mathbf{Q}^h\leq\widehat{\mathbf{X}}^h$ for $h\in\mathcal{H}$. Vectorizing this matrix inequality, we obtain

$$\mathbf{I}_{\tilde{n}_M^h}\otimes\widehat{\mathbf{Y}}^h\mathrm{vec}\mathbf{Q}^h\leq\mathrm{vec}\widehat{\mathbf{X}}^h \tag{9}$$

Next, if $\mathbf{Q}^h$ is doubly stochastic, then $\mathbf{Q}^h\mathbf{e}_{\tilde{n}_M^h}=\mathbf{e}_{\tilde{n}_M^h}$ and $\mathbf{e}_{\tilde{n}_M^h}^t\mathbf{Q}^h=\mathbf{e}_{\tilde{n}_M^h}^t$ must hold. Vectorizing $\mathbf{Q}^h\mathbf{e}_{\tilde{n}_M^h}=\mathbf{e}_{\tilde{n}_M^h}$ and $\mathbf{e}_{\tilde{n}_M^h}^t\mathbf{Q}^h=\mathbf{e}_{\tilde{n}_M^h}^t$, we get $\left(\mathbf{e}_{\tilde{n}_M^h}^t\otimes\mathbf{I}_{\tilde{n}_M^h}\right)\mathrm{vec}\mathbf{Q}^h=\mathbf{e}_{\tilde{n}_M^h}$ and $\left(\mathbf{I}_{\tilde{n}_M^h}\otimes\mathbf{e}_{\tilde{n}_M^h}^t\right)\mathrm{vec}\mathbf{Q}^h=\mathbf{e}_{\tilde{n}_M^h}$, respectively.[3] Thus, if $\mathbf{Y}(\bar{\mathbf{z}})\prec^{JL}\mathbf{X}(\bar{\mathbf{z}})$, then (8) has a non-negative solution, $\mathbf{q}^h=\mathrm{vec}\mathbf{Q}^h$, $\forall h\in\mathcal{H}$. Conversly, if (8) has a non-negative solution, then $\widehat{\mathbf{X}}^h\prec^w\widehat{\mathbf{Y}}^h$ $\forall h\in\mathcal{H}$ holds, which implies $\mathbf{Y}(\bar{\mathbf{z}})\prec^{JL}\mathbf{X}(\bar{\mathbf{z}})$.

Lemma 2 implies that the dominance condition defined above is characterized by sequential dominance relations for matrix inequalities, $\widehat{\mathbf{A}}^h\mathbf{q}^h\leq\mathbf{b}^h$ for $h\in\mathcal{H}$. Now, we can state the main result.

Proposition 1 For a given vector, $\bar{\mathbf{z}}\in\mathbb{R}^m$, the following two conditions are equivalent.

(*i*). $\mathbf{Y}(\bar{\mathbf{z}})\prec^{JL}\mathbf{X}(\bar{\mathbf{z}})$,

(*ii*). $W_X\geq W_Y\ \forall U\in\mathcal{U}_{\bar{\mathbf{z}}}$.

Proof See Appendix.

[3] For the vectrizing procedure, see Rao and Mitra (1971).

In the next section, we consider how to verify the validity of this extended SGL.

4 Equivalent Linear Programming Problem

We can easily transform the dominance condition that we presented into a linear programming model. Based on (8), consider the following problem.

Problem 1 For $h \in \mathcal{H}$,

$$\min_{\mathbf{q}^h} \mathbf{v}^h \mathbf{q}^h \tag{P1}$$

subject to

$$\hat{\mathbf{A}}^h \mathbf{q}^h = \hat{\mathbf{b}}^h \tag{P2}$$

and

$$\mathbf{q}^h \geq \mathbf{0}, \tag{P3}$$

where $\mathbf{v}^h = \left[\mathbf{0}_{(\tilde{n}_M^h)^2 + m\tilde{n}_M^h}, \hat{\mathbf{v}}^h\right] \in \mathbb{R}^{(\tilde{n}_M^h)^2 + 2m\tilde{n}_M^h}$,

$$\hat{\mathbf{A}}^h = \begin{bmatrix} \mathbf{I}_{\tilde{n}_M^h} \otimes \hat{\mathbf{Y}}^h & \mathbf{I}_{m\tilde{n}_M^h} & -\mathbf{I}_{m\tilde{n}_M^h} \\ \mathbf{e}_{\tilde{n}_M^h}^t \otimes \mathbf{I}_{\tilde{n}_M^h} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{\tilde{n}_M^h} \otimes \mathbf{e}_{\tilde{n}_M^h}^t & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{b}^h = \begin{bmatrix} \mathrm{vec}\hat{\mathbf{X}}^h \\ \mathbf{e}_{\tilde{n}_M^h} \\ \mathbf{e}_{\tilde{n}_M^h} \end{bmatrix},$$

and $\hat{\mathbf{v}}^h \in \mathbb{R}^{m\tilde{n}_M^h}$ is a vector whose entries are all constant and strictly positive: $\hat{\mathbf{v}}^h > \mathbf{0}$.

Problem 1 is a standard linear programming model. We can easily show that the dual of Problem 1 has a bounded optimal solution. Thus, by the duality theorem, Problem 1 also has a bounded optimal solution.[4] In (P1), $\hat{\mathbf{v}}^h$ can be interpreted as the shadow prices of the (negative) slack vector. Thus, we have the following results.

---

[4] Consider the dual problem of Problem 1: $\max_{\boldsymbol{s}^h} \boldsymbol{s}^h \hat{\mathbf{b}}^h$ subject to $\boldsymbol{s}^h \hat{\mathbf{A}}^h \leq \mathbf{v}^p$. It is obvious that $\boldsymbol{s}^h = \mathbf{0}$ is a feasible solution. We decompose $\boldsymbol{s}^h$ as $\boldsymbol{s}^h = [\boldsymbol{s}_1^h, \boldsymbol{s}_2^h]$, where $\boldsymbol{s}_1^h \in \mathbb{R}^{m\tilde{n}_M^p}$ and $\boldsymbol{s}_2^h \in \mathbb{R}^{2\tilde{n}_M^p}$. Since $\boldsymbol{s}_1^h$ is restricted to $-\mathbf{v}^p \leq \boldsymbol{s}_1^h \leq \mathbf{0}$, the optimal solution is also bounded.

Proposition 2. The following two conditions are equivalent:

(i) $\mathbf{Y}(\bar{\mathbf{z}}) \prec^{JL} \mathbf{X}(\bar{\mathbf{z}})$;

(ii) The optimal solution of Problem 1 is zero $\forall h \in \mathcal{H}$.

Proof The claim is clear from Lemma 2 and Proposition 1.

Proposition 2 states that we can confirm $\prec^{JL}$ by solving Problem 1 sequentially. We can easily solve the problem if the number of agents is not too large.

5 Empirical Illustration

In this section, we offer a simple application of the procedure proposed in the previous section. We compare the global distribution of welfare between different years using data from the United Nations Human Development Index (HDI). As mentioned previously, the HDI consists of three sub-indices: health, education, and income. The health index (HI) comes from the life expectancy at birth (LE); the education index (EI) consists of the mean years of schooling (MYS) for adults aged 25 years and the expected years of schooling (EYS) for children of school entering age; the income index (II) comes from the gross national income per capita ($GNIpc$). We use normalized indices such as follows:

$$HI_i = \min\left\{\frac{LE_i - 20}{85 - 20}, 1\right\},$$

$$EI_i = \min\left\{\frac{1}{2}\left(\frac{MYS_i - 0}{15 - 0} + \frac{EYS_i - 0}{18 - 0}\right), 1\right\},$$

$$II_i = \min\left\{\frac{\ln(GNIpc_i) - \ln(100)}{\ln(75000) - \ln(100)}, 1\right\}.$$

We assume that the needs for the attributes differ between LDCs and non-LDCs. Specificaly, a one-unit improvement in an index of LDC brings about a more prominent effect on social welfare than in non-LDCs, even if the attribute value is the same. This might because the HDI does not fully capture the factors affecting well-being, or improvements in the HDI in LDCs may have a positive externality to the world.

We use data published by the United Nations Development Programme (UNDP) for 2000, 2005, 2010, 2015, and 2020, and select 174 countries with complete data for the survey years. The definition of LDCs follows the United Nations Department of Economic and Social Affairs. During the period, some countries included among the LDCs and later graduate[5].

Table 1 presents summary statistics. The average achievement of each sub-index has increased year by year in both LDCs and non-LDCs. At the same time, standard deviations are decreasing year by year, except for income from 2015 to 2020. These facts suggest an improvement in social welfare. On the other hand, the minimum value of the health index from 2000 to 2005 decreased in both LDCs and non-LDCs. This implies that the distribution of attributes in 2005 does not dominate the distribution in 2000. The average achievement in non-LDCs is higher than in LDCs in all sub-indices for all years. However, the maximum value of each index in LDCs is greater than the minimum value in non-LDCs. That is, the sub-indices are distributed with overlap.

Table 1 Summary statistics

		LDCs				Non-LDCs			
year		mean	max	min	std	mean	max	min	std
2000	HI	0.547	0.783	0.377	0.111	0.783	0.941	0.380	0.111
	EI	0.316	0.695	0.117	0.116	0.628	0.911	0.253	0.139
	II	0.446	0.738	0.273	0.101	0.733	1.000	0.419	0.146
	# of countries	44				130			
2005	HI	0.588	0.842	0.357	0.106	0.801	0.954	0.340	0.116
	EI	0.359	0.710	0.147	0.112	0.668	0.942	0.313	0.137
	II	0.466	0.795	0.304	0.103	0.754	1.000	0.451	0.141
	# of countries	44				130			
2010	HI	0.619	0.887	0.394	0.098	0.824	0.969	0.410	0.105
	EI	0.400	0.726	0.182	0.105	0.700	0.946	0.342	0.131
	II	0.484	0.798	0.320	0.098	0.769	1.000	0.498	0.132
	# of countries	43				131			
2015	HI	0.649	0.777	0.478	0.069	0.842	0.989	0.539	0.095
	EI	0.425	0.654	0.225	0.090	0.730	0.941	0.383	0.127
	II	0.489	0.810	0.319	0.094	0.782	1.000	0.536	0.127
	# of countries	41				133			
2020	HI	0.668	0.800	0.504	0.065	0.841	1.000	0.600	0.092
	EI	0.445	0.614	0.264	0.088	0.749	0.959	0.392	0.126
	II	0.486	0.655	0.301	0.083	0.782	1.000	0.512	0.128
	# of countries	40				134			

Source: Author's calculation from the UNDP data.

Based on these data, we consider the dominance relationship by applying Problem 1. Following Muller and Trannoy (2011), we do not weight each country's attributes by its population. We set the

[5] See Table A1 in Appendix for the list of LDCs.

maximum conceivable amount for all attributes to 1. That is, $\bar{\mathbf{z}} \equiv (1,1,1)$. We also set the shadow price vector in Problem 1 to $\hat{\mathbf{v}}^1 = (500, \ldots, 500)$. In what follows, **Y** is referred to as the distribution for the *base year* and **X** as the distribution for the *comparison year*.

Table 2 shows the results of solving Problem 1 for LDCs only. We can confirm that the distributions for 2010, 2015, and 2020 dominate the distribution for the 2000. Furthermore, the 2010 and 2015 distributions dominate the 2005 distribution. In other pair-wise comparisons, we do not observe a dominance relationship.

Table 3 shows the optimal values obtained by solving Problem 1 for all countries including non-LDCs. Here we observe the same dominance relationship as in Table 2. From this, we can find the welfare orderings shown in Figure 1. In summary, global well-being has improved over the 20-year period.

Table 2 Dominance relationships for LDCs

		Base year				
		2000	2005	2010	2015	2020
Comparison year	2000	-	2300.39	5023.74	7753.89	9187.33
	2005	9.89	-	2723.35	5453.50	6886.94
	2010	0.00	0.00	-	2730.15	4163.59
	2015	0.00	0.00	13.26	-	1433.44
	2020	0.00	1.30	9.62	38.51	-

Source: Author's own calculation.

Table 3 Dominance relationships for all countries

		Base year				
		2000	2005	2010	2015	2020
Comparison year	2000	-	7500.53	14423.15	20517.94	22732.33
	2005	31.76	-	6955.61	13030.06	15247.68
	2010	0.00	0.00	-	6099.17	8315.06
	2015	0.00	0.00	13.26	-	2477.09
	2020	0.00	1.30	9.62	44.28	-

Source: Author's own calculation.

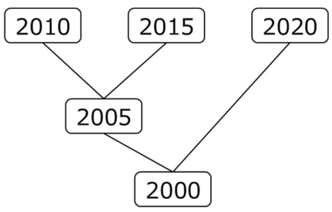

Figure 1 Welfare ordering

From the solution of Problem 1, we can obtain more detailed information. For example, Table 3 shows that the optimal solution for 2020 (comparison year) compared to 2005 (base year) is 1.30, and there is no dominance relationship. The solution vector of Problem 1 indicates that a dominance relationship is established if the income index (II) in 2020 in Burundi, Sub-Saharan Africa, increases by 0.0026, which corresponds to a 1.02 (in 2017 PPP $) increase in GNI per capita.

6 Remarks

In this paper, we extended the sequential generalized Lorenz dominance (SGL) criterion in two directions. First, we considered a situation where the welfare of individuals depends on multivariate attributes in addition to the needs for these their attributes. Second, as considered in Chambaz and Maurin (1998) and Jenkins and Lambert (1993), we allowed for demographic differences. Even under these extensions, the welfare ordering derived was basically the same as in the analysis with the original SGL criterion. We also offered an empirical illustration.

We can consider further extensions based on the framework in this paper. For example, the maximum conceivable value can be adjustable attribute-wise. We can also apply the present procedure to multivariate poverty analysis by setting $\bar{\mathbf{z}}$ as a poverty line vector. In the empirical illustration, we give an equal weight to each country. We could instead use the population size of each country as a weight. In the latter case, a modified matrix based on the doubly stochastic matrix would characterize the dominance relation.

Appendix

Proof of Proposition 1

First, suppose that $\mathbf{Y}(\bar{\mathbf{z}}) \prec^{JL} \mathbf{X}(\bar{\mathbf{z}})$. From Definition 2, $\widehat{\mathbf{X}}^h \prec^w \widehat{\mathbf{Y}}^h\ \forall h \in \mathcal{H}$ holds. Noting that $U \in \mathcal{U}_{\bar{\mathbf{z}}}$ and G are non-decreasing concave, $\widehat{\mathbf{X}}^h \prec^w \widehat{\mathbf{Y}}^h$ implies that

$$\sum_{j \in \tilde{\mathcal{N}}_X^h} G(\mathbf{x}_j, h) - \sum_{j \in \tilde{\mathcal{N}}_Y^h} G(\mathbf{y}_j, h) + (\tilde{n}_Y^h - \tilde{n}_X^h) G(\bar{\mathbf{z}}, h) \geq 0, h \in \mathcal{H} \setminus \{H\}, \tag{A1}$$

$$\sum_{j \in \tilde{\mathcal{N}}} U(\mathbf{x}_j, H) - \sum_{j \in \tilde{\mathcal{N}}} U(\mathbf{y}_j, H) \geq 0. \tag{A2}$$

Furthermore, since $G(\bar{\mathbf{z}}, h) = 0$ holds from U3, we can eliminate the third term of the LHS of (A1). Inserting (A1) and (A2) into the difference in the social welfare, we can confirm $W_X \geq W_Y\ \forall U \in \mathcal{U}_{\bar{\mathbf{z}}}$.

Next, suppose that $\mathbf{Y}(\bar{\mathbf{z}}) \nprec^{JL} \mathbf{X}(\bar{\mathbf{z}})$. In such a situation, from Lemma 2, Inequality (8) does not have a non-negative solution. Thus, from the theorem of alternative (e.g., Gale, 1960, Theorem 2.8), there exists a non-negative vector, $\mathbf{w}^h \in \mathbb{R}_+^{m+4\tilde{n}_M^h}$ for $h \in \mathcal{H}$ such that

$$\mathbf{w}^h \mathbf{A}^h \geq \mathbf{0}\ \forall h \in \mathcal{H}, \tag{A3}$$

and

$$[\mathbf{w}^1, \ldots, \mathbf{w}^H]\begin{bmatrix}\mathbf{b}^1\\ \vdots\\ \mathbf{b}^H\end{bmatrix} < 0, \tag{A4}$$

hold. Let us decompose $\mathbf{w}^h$ as $\mathbf{w}^h = [\mathbf{w}_1^h, \ldots, \mathbf{w}_{\tilde{n}_M^h}^h, \check{\mathbf{d}}^h, \hat{\mathbf{d}}^h, \check{\mathbf{c}}^h, \hat{\mathbf{c}}^h]$, where $\mathbf{w}_j^h \in \mathbb{R}_m$ and $\check{\mathbf{d}}^h, \hat{\mathbf{d}}^h, \check{\mathbf{c}}^h, \hat{\mathbf{c}}^h \in \mathbb{R}_{\tilde{n}_M^h}$. Let $\mathbf{d}^h = \check{\mathbf{d}}^h - \hat{\mathbf{d}}^h$ and $\mathbf{c}^h = \check{\mathbf{c}}^h - \hat{\mathbf{c}}^h$. Thus, an entry-wise representation of (A3) is as follows:

$$\mathbf{w}_j^h \mathbf{y}_i^h + d_i^h + c_j^h \geq 0 \ \text{ for } i \in \widetilde{\mathcal{N}}_Y^h \text{ and } j = 1, \ldots, \tilde{n}_M^h. \tag{A5}$$

Furthermore, when $\tilde{n}_Y^h < \tilde{n}_M^h$, we have

$$\mathbf{w}_j^h \bar{\mathbf{z}} + d_i^h + c_j^h \geq 0 \ \text{ for } i = \tilde{n}_Y^h + 1, \ldots, \tilde{n}_M^h \text{ and } \ j = 1, \ldots, \tilde{n}_M^h. \tag{A6}$$

Let $\mathcal{V}^h$ be a set of $(m+1)$-dimensional row vector defined as $\mathcal{V}^h = \{[\mathbf{w}_1^h, c_1^h], \ldots, [\mathbf{w}_{\tilde{n}_M^h}^h, c_{\tilde{n}_M^h}^h]\}$. As in Example 1, consider the following function:

$$\varphi^h(\mathbf{z}) = \min_{[\mathbf{w}_j^h, c_j^h] \in \mathcal{V}^h} \left\{[\mathbf{w}_j^h, c_j^h]\begin{bmatrix}\mathbf{z}\\ 1\end{bmatrix}\right\}, \tag{A7}$$

From (A6) and the definition of φ^h, we have

$$\varphi^h(\mathbf{y}_i) + d_i^h \geq 0 \ \forall i \in \widetilde{\mathcal{N}}_Y^h. \tag{A8}$$

It should be noted that φ^h is non-decreasing concave in $\mathbf{z}$. Furthermore, if $\tilde{n}_Y^h < \tilde{n}_M^h$, then

$$\varphi^h(\bar{\mathbf{z}}) + d_i^h \geq 0 \ \forall i = \tilde{n}_Y^h + 1, \ldots, \tilde{n}_M^h. \tag{A9}$$

Summing (A8) and (A9) over $i \in \widetilde{\mathcal{N}}_Y^h$ and $h \in \mathcal{H}$, we have

$$\sum_{h \in \mathcal{H}} \sum_{i \in \widetilde{\mathcal{N}}_Y^h} \varphi^h(\mathbf{y}_i) + \sum_{h \in \mathcal{H}} \sum_{i=1}^{\tilde{n}_M^h} d_i^h + \sum_{h \in \mathcal{H}} \left(\tilde{n}_M^h - \tilde{n}_Y^h\right)\varphi^h(\bar{\mathbf{z}}) \geq 0 \tag{A10}$$

Next, the LHS of (A4) can be written as follows:

$$\mathbf{w}^h\mathbf{b}^h = \begin{cases} \sum_{i\in\widetilde{\mathcal{N}}_X^h} \left([\mathbf{w}_i^h, c_i^h]\begin{bmatrix}\mathbf{x}_i\\1\end{bmatrix}\right) + \sum_{i\in\widetilde{\mathcal{N}}_X^h} d_i^h, & if\ \tilde{n}_X^h = \tilde{n}_M^h, \\ \sum_{i\in\widetilde{\mathcal{N}}_X^h} \left([\mathbf{w}_i^h, c_i^h]\begin{bmatrix}\mathbf{x}_i\\1\end{bmatrix}\right) + \sum_{i=1}^{\tilde{n}_M^h} d_i^h + \sum_{i=\tilde{n}_X^h+1}^{\tilde{n}_M^h} \left([\mathbf{w}_i^h, c_i^h]\begin{bmatrix}\bar{\mathbf{z}}\\1\end{bmatrix}\right), & if\ \tilde{n}_X^h < \tilde{n}_M^h, \end{cases} \tag{A11}$$

for $h \in \mathcal{H}$. Since $\mathbf{w}_j^h\mathbf{x}_i + c_j^h \geq \varphi^h(\mathbf{x}_i)$ $\forall i \in \widetilde{\mathcal{N}}_X^h$, $\forall j \in \{1, \dots, \tilde{n}_M^h\}$, and $\forall h \in \mathcal{H}$ holds from the definition of φ^h, we can confirm

$$\mathbf{w}^h\mathbf{b}^h \geq \sum_{i\in\widetilde{\mathcal{N}}_X^h} \varphi^h(\mathbf{x}_i) + \sum_{i=1}^{\tilde{n}_M^h} d_i^h + (\tilde{n}_M^h - \tilde{n}_X^h)\varphi^h(\bar{\mathbf{z}}), \qquad \forall h \in \mathcal{H}. \tag{A12}$$

Thus, summing (A12) over $h \in \mathcal{H}$, we obtain

$$\sum_{h\in\mathcal{H}} \mathbf{w}^h\mathbf{b}^h \geq \sum_{h\in\mathcal{H}} \sum_{i\in\widetilde{\mathcal{N}}_X^h} \varphi^h(\mathbf{x}_i) + \sum_{h\in\mathcal{H}} \sum_{i=1}^{\tilde{n}_M^h} d_i^h + \sum_{h\in\mathcal{H}} (\tilde{n}_M^h - \tilde{n}_X^h)\varphi^h(\bar{\mathbf{z}}). \tag{A13}$$

Because $\sum_{h\in\mathcal{H}} \mathbf{w}^h\mathbf{b}^h < 0$, together with (A10) and (A13), we obtain

$$\sum_{h\in\mathcal{H}} \sum_{i\in\widetilde{\mathcal{N}}_Y^h} \varphi^h(\mathbf{y}_i) > \sum_{h\in\mathcal{H}} \sum_{i\in\widetilde{\mathcal{N}}_X^h} \varphi^h(\mathbf{x}_i) + \sum_{h\in\mathcal{H}} (\tilde{n}_Y^h - \tilde{n}_X^h)\varphi^h(\bar{\mathbf{z}}). \tag{A14}$$

In (A14), for a given α_1, let α_{h+1} be given by $\alpha_{h+1} = \alpha_h + \varphi^h(\bar{\mathbf{z}})$ for $h \in \mathcal{H} \setminus \{H\}$. Noting that $\tilde{n}_X^H = \tilde{n}_Y^H$, we have

$$\sum_{h\in\mathcal{H}} (\tilde{n}_Y^h - \tilde{n}_X^h)\varphi^h(\bar{\mathbf{z}}) = \sum_{h\in\mathcal{H}\setminus\{H\}} (\tilde{n}_Y^h - \tilde{n}_X^h)\varphi^h(\bar{\mathbf{z}}) \tag{A15}$$

The above setting is equivalent to setting $\varphi^h(\bar{\mathbf{z}}) = \alpha_{h+1} - \alpha_h$ for $h \in \mathcal{H} \setminus \{H\}$. Thus, applying summation by parts to the RHS in (A15), we obtain,

$$\begin{aligned} \sum_{h=1}^{H-1} (\tilde{n}_Y^h - \tilde{n}_X^h)\varphi^h(\bar{\mathbf{z}}) &= \sum_{h=1}^{H-1} (\tilde{n}_Y^h - \tilde{n}_X^h)(\alpha_{h+1} - \alpha_h) \\ &= -\alpha_1(\tilde{n}_Y^1 - \tilde{n}_X^1) - \sum_{h=1}^{H-1} \alpha_{h+1}(n_Y^{h+1} - n_X^{h+1}) \\ &= \sum_{h=1}^{H} \alpha_h(n_X^h - n_Y^h). \end{aligned} \tag{A16}$$

The result in (A15) can be written as follows:

$$\sum_{h\in\mathcal{H}}\left(\tilde{n}_Y^h-\tilde{n}_X^h\right)\varphi^h(\bar{\mathbf{z}})=\sum_{h\in\mathcal{H}}\left(n_X^h-n_Y^h\right)\alpha_h. \tag{A17}$$

Furthermore, we can rewrite the LHS of (A14) as follows:

$$\begin{aligned}\sum_{h\in\mathcal{H}}\sum_{i\in\tilde{\mathcal{N}}_Y^h}\varphi^h(\boldsymbol{y}_i)&=\sum_{i\in\tilde{\mathcal{N}}_Y^1}\varphi^1(\boldsymbol{y}_i)+\sum_{i\in\tilde{\mathcal{N}}_Y^2}\varphi^2(\boldsymbol{y}_i)+,\dots,+\sum_{i\in\tilde{\mathcal{N}}_Y^2}\varphi^H(\boldsymbol{y}_i)\\&=\sum_{i\in\mathcal{N}_Y^1}\varphi^1(\boldsymbol{y}_i)+\sum_{k=1}^{2}\sum_{i\in\mathcal{N}_Y^k}\varphi^k(\boldsymbol{y}_i)+,\dots,+\sum_{k=1}^{H}\sum_{i\in\mathcal{N}_Y^k}\varphi^k(\boldsymbol{y}_i)\\&=\sum_{i\in\mathcal{N}_Y^1}\sum_{k=1}^{H}\varphi^k(\boldsymbol{y}_i)+\sum_{i\in\mathcal{N}_Y^2}\sum_{k=2}^{H}\varphi^k(\boldsymbol{y}_i)+,\dots,+\sum_{i\in\mathcal{N}_Y^H}\varphi^H(\boldsymbol{y}_i)\\&=\sum_{h\in\mathcal{H}}\sum_{i\in\mathcal{N}_Y^h}\sum_{k=h}^{H}\varphi^k(\boldsymbol{y}_i).\end{aligned} \tag{A18}$$

Similarly, we can confirm that $\sum_{h\in\mathcal{H}}\sum_{i\in\tilde{\mathcal{N}}_X^h}\varphi^h(\mathbf{x}_i)=\sum_{h\in\mathcal{H}}\sum_{i\in\mathcal{N}_X^h}\sum_{k=h}^{H}\varphi^k(\mathbf{x}_i)$ holds. Substituting (A17) and (A18) into (A14), we obtain

$$\sum_{h\in\mathcal{H}}\sum_{i\in\mathcal{N}_Y^h}\left[\alpha_h+\sum_{k=h}^{H}\varphi^k(\mathbf{y}_i)\right]>\sum_{h\in\mathcal{H}}\sum_{i\in\mathcal{N}_X^h}\left[\alpha_h+\sum_{k=h}^{H}\varphi^k(\mathbf{x}_i)\right]. \tag{A19}$$

Now consider the following utility function:

$$\widehat{U}(\mathbf{z},h)=\alpha_h+\sum_{k=h}^{H}\varphi^k(\mathbf{z}),\qquad h\in\mathcal{H},\mathbf{z}\in\mathcal{D}.$$

Then as shown in Example 1, $\widehat{U}\in\mathcal{U}_{\bar{\mathbf{z}}}$. Therefore, $\mathbf{Y}(\bar{\mathbf{z}})\nprec^{JL}\mathbf{X}(\bar{\mathbf{z}})$ implies that there exists $\widehat{U}\in\mathcal{U}_{\bar{\mathbf{z}}}$ such that $W_X<W_Y$.

List of LDCs

Table A1 List of LDCs

	2000	2005	2010	2015	2020
Afghanistan	●	●	●	●	●
Angola	●	●	●	●	●
Burundi	●	●	●	●	●
Benin	●	●	●	●	●
Burkina Faso	●	●	●	●	●
Bangladesh	●	●	●	●	●
Bhutan*	●	●	●	●	●
Central African Republic	●	●	●	●	●
Congo (Democratic Republic of the)	●	●	●	●	●
Comoros	●	●	●	●	●
Cabo Verde	●	●			
Djibouti	●	●	●	●	●
Eritrea*	●	●	●	●	●
Ethiopia	●	●	●	●	●
Guinea	●	●	●	●	●
Gambia	●	●	●	●	●
Guinea-Bissau*	●	●	●	●	●
Equatorial Guinea	●	●	●	●	
Haiti	●	●	●	●	●
Cambodia	●	●	●	●	●
Kiribati	●	●	●	●	●
Lao People's Democratic Republic	●	●	●	●	●
Liberia	●	●	●	●	●
Lesotho	●	●	●	●	●
Madagascar	●	●	●	●	●
Maldives	●	●	●		
Mali	●	●	●	●	●
Myanmar	●	●	●	●	●
Mozambique	●	●	●	●	●
Mauritania	●	●	●	●	●
Malawi	●	●	●	●	●
Niger	●	●	●	●	●
Nepal	●	●	●	●	●
Rwanda	●	●	●	●	●
Sudan	●	●	●	●	●
Senegal	●	●	●	●	●
Solomon Islands	●	●	●	●	●
Sierra Leone	●	●	●	●	●
Somalia*	●	●	●	●	●
South Sudan*				●	●
Sao Tome and Principe	●	●	●	●	●
Chad	●	●	●	●	●
Togo	●	●	●	●	●
Timor-Leste*		●	●	●	●
Tuvalu	●	●	●	●	●
Tanzania (United Republic of)	●	●	●	●	●
Uganda	●	●	●	●	●
Vanuatu*	●	●	●	●	
Samoa	●	●	●		
Yemen	●	●	●	●	●
Zambia	●	●	●	●	●

Note: * excluded from the analysis due to missing data.
Source: Author's own compilation based on Committee for Development Policy and United Nations Department of Economic and Social Affairs (2021).

References

Aaberge, R., and Brandolini, A. (2015) “Multidimensional poverty and inequality,” in *Handbook of Income Distribution* (Vol. 2, pp. 141-216). Elsevier.

Atkinson, A.B., and Bourguignon, F. (1987). "Income Distribution and Differences in Needs," in G.R. Feiwel, ed., *Arrow and the Foundations of the Theory of Economic Policy*, 350-370, Macmillan, London.

Chambaz, C., and Maurin, E. (1998) "Atkinson and Bourguignon's Dominance Criteria: Extended and Applied to the Measurement of Poverty in France," *Review of Income and Wealth*, 44, 497-513.

Committee for Development Policy and United Nations Department of Economic and Social Affairs (2021). *Handbook on the Least Developed Country Category: Inclusion, Graduation and Special Support Measures*. United Nations publication.

Fleurbaey, M., C. Hagneré, and A. Trannoy, A. (2003) “Welfare comparisons with bounded equivalence scales,” *Journal of Economic Theory*, 110, 309–336.

Fleurbaey, M., Hagneré, C., and Trannoy, A. (2014) “Welfare comparisons of income distributions and family size: An individualistic approach,” *Journal of Mathematical Economics*, 51, 12-27.

Jenkins, S., and Lambert, P. (1993) "Ranking income distributions when needs differ," *Review of Income and Wealth*, 39, 337-356.

Kolm, S. C. (1977). Multidimensional egalitarianisms. *Quarterly Journal of Economics*, 91(1), 1-13.

Lambert, P.J., and Ramos, X. (2002) “Welfare Comparisons: Sequential Procedures for Heterogeneous Populations,” *Economica*, 69, 549-562.

López-Laborda, J., and Onrubia, J. (2005) "Personal Income Tax Decentralization, Inequality, and Social Welfare," *Public Finance Review*, 33, 213-235, 2005.

Marshall, A.W., Olkin, I., and Arnold, B.C. (2011) *Inequalities: Theory of Majorization and its Applications*, 2nd. ed., Springer, New York.

Moyes, P., (2012) “Comparisons of heterogeneous distributions and dominance criteria,” *Journal of Economic Theory,* 147, 1351–1383.

Müller, A., and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks* (Vol. 389). Wiley.

Müller, C., and Trannoy, A. (2012) “Multidimensional inequality comparisons: A compensation perspective,” *Journal of Economic Theory*, 147(4), 1427-1449.

Müller, C., and Trannoy, A. (2011) “A dominance approach to the appraisal of the distribution of well-being across countries,” *Journal of Public Economics*, 95(3-4), 239-246.

Nakamura, K. (2012) “Extension of generalized Lorenz dominance criterion to multivariate attributes,” Faculty of Economics, University of Toyama, Working Paper, 266.

Ooghe, E., and Lambert, P. (2006) “On bounded dominance criteria,” *Mathematical Social Sciences*, 52(1), 15-30.

Ooghe, E. (2007) “Sequential Dominance and Weighted Utilitarianism,” *Economics Letters*, 94, 208-212.

Rao, C.R., and Mitra, S.K. (1971) *Generalized Inverse of Matrices and Its Applications*. John Wiley & Sons

Shorrocks, A. F., (1983) "Ranking Income Distributions," *Economica*, 50, 1-17, 1983.