博士論文

Theoretical Studies on Higgs Physics in Extra Dimension Models

余剰次元モデルにおけるヒッグス物理の理論的研究

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Abstract

The Standard Model (SM) has been established as an effective theory below the electroweak scale. However, the various phenomena beyond the SM, such as the baryon number asymmetry in the universe, the existence of dark matter, inflation, and neutrino oscillation have been observed. The SM also has theoretical problems such as a hierarchy problem. These issues indicate that the SM is not a complete theory and should be improved. On the other hand, the Higgs boson with the mass of 125 GeV has been discovered and its couplings have been observed to be consistent with the SM by the Large Hadron Collider (LHC) experiment at CERN, but there are still some mysteries in the Higgs sector: existence of the guiding principle, number of the Higgs bosons, shape of the Higgs potential, dynamics behind the electroweak symmetry breaking, etc. It means that there is still a lot of room for expansion in the Higgs sector. Whereas, these mysteries are expected to be solved by the planned collider and gravitational wave experiments. Future collider experiments such as High Luminosity LHC (HL-LHC) and International Linear Collider (ILC) can measure the properties of the 125 GeV Higgs boson with good accuracy. In particular, the electron-positron colliders like the ILC and Compact Linear Collider (CLIC) measure the triple Higgs boson coupling with about 10% accuracy. Future gravitational wave experiments such as DECi-hertz Interferometer Gravitational wave Observatory (DECIGO) and Laser Interferometer Space Antenna (LISA) can observe the gravitational wave from the electroweak phase transition. Therefore, researches to approach various problems of the SM by extending the Higgs sector is actively carried out.

In this paper, we focus on extra dimensions as an expansion of the Higgs sector. Generally, we perceive a four-dimensional space-time, which consists of three spatial dimensions and a one time dimension. The extra dimension is the idea that there are hidden dimensions in addition to the four-dimensional space-time. It has been introduced in various theories such as the String Theory, Randall-Sundrum Model and Universal Extra Dimensions. The introduction of extra dimensions gives a new structure to the theory, the geometry of extra dimensions, and we can benefit from it in various ways. For example, it can control the strength of mutual constraints, resolve various hierarchies that appear in the SM, and unify the fields and forces. From a phenomenological point of view, we are interested in the TeV scale extra dimensions that exist in the experimentally accessible energy scales.

First, we discuss the gauge-Higgs unification model with a flat extra dimension in which the Higgs is embedded into the extra components of the gauge multiplets. Gauge-Higgs unification (GHU) is one of the TeV scale paradigms beyond the SM that can solve the hierarchy problem. In GHU, the Higgs potential is flat at tree level and induced by the quantum corrections due to the higher dimensional gauge symmetry. We focus on the structure difference of the Higgs potential from the SM one and investigate the triple Higgs boson coupling. As a concrete model, we consider the flat $SU(3)$ model with 5D Lorentz symmetry relaxed. In this case, the deviation of the triple Higgs boson coupling from the SM is predicted to be less than 10% when the compactification scale is larger than the experimental lower bound, which is around 5 TeV. It is also shown that the shape of the Higgs potential around the vacuum quickly approaches that of the SM potential as the compactification scale increases. This behavior is also applied to the other GHU models with a flat extra dimension.

After that, we also discuss the two-Higgs doublet model (2HDM) with extra dimensions. The 2HDM is one of the simple extensions of the SM, which only adds one additional Higgs doublet,

but it has rich phenomenology not expected in the SM. In general, when introducing more than one Higgs doublet, the Yukawa interaction matrix and the mass matrix are not diagonalized simultaneously, and an unacceptable flavor-changing neutral current (FCNC) appears. To avoid this, the 2HDM usually requires the fermions to be coupled to only one of the Higgs doublets by imposing a Z_2 symmetry. However, this Z_2 symmetry cannot be justified in the framework of the 2HDM, and its origin is left to some paradigm. Therefore, we introduce the extra dimensions as a paradigm and reproduce the viable Higgs couplings in the 2HDM without imposing the Z_2 symmetry. It is known that the fermions and scalar fields can be localized in the extra space by introducing a coupling with a kink field. Then, we localize the right-handed fermions and Higgs doublets on the extra space and construct an arrangement in which only one of the couplings to the two Higgs doublets is suppressed by the extra-dimensional integration. In this way, we can avoid the dangerous FCNC without imposing the Z_2 symmetry. This model also has a feature that the Higgs potential is not constrained by the Z_2 symmetry, unlike the usual 2HDM.

Contents

Chapter 1

Introduction

1.1 Overview

The observation of the Higgs boson by the LHC experiment at CERN [1, 2] led to the discovery of all the elementary particles predicted by the Standard Model. Consistent with most experimental results [3], including the Higgs boson coupling, the SM has been established as an effective theory below the electroweak scale $\mathcal{O}100$ GeV. The SM is composed of the following two main components. One is the gauge principle, which defines the form of interactions, and the other is the mechanism of the spontaneous symmetry breaking, which gives the mass to the elementary particles. In contrast to the gauge principle, the spontaneous symmetry breaking only assumes a minimal Higgs potential with one Higgs doublet and leaves the possibility of extensions. The Higgs potential in the SM also has a theoretical problem called hierarchy problem. In SM, the quantum corrections to the squared mass of the Higgs boson are proportional to the square of the cutoff scale of the theory. Therefore, a fine-tuning between the bare mass and the quantum correction is required to explain the observed Higgs mass. From the point of view of improving the accuracy of measurement for the Higgs properties in future collider experiments, the Higgs sector is an interesting study topic today.

1.1.1 Extention of the Higgs sector

Based on the above background, various models that extend the Higgs sector have been proposed to date. Those models can be broadly classified into two types: bottom-up and top-down. In the bottom-up approach, such as the 2HDM and the SM Effective Field Theory (SMEFT), simple extensions of the Higgs sector are applied to investigate the structure of the Higgs sector and its phenomenological features. The top-down approaches are mainly motivated to solve the hierarchical problem by assuming the TeV scale paradigms, such as supersymmetry (SUSY), compositeness, and gauge-Higgs unification. It is significant to decide which of these directions is the proper one. We should clarify the relationship between these models and investigate how to distinguish these models by using the future experiments.

1.1.2 Models in bottom-up approach

We briefly introduce the four types of the 2HDM [4], Inert doublet model (IDM) [4], and SM effective field theory (SMEFT) [5, 6] as famous bottom-up approaches. First, the 2HDM is a simple model of the SM with one additional Higgs doublet, but it has rich phenomenological features not found in the SM, such as the electroweak baryogenesis. The 2HDM usually imposes a softly broken Z_2 symmetry to avoid the flavor-changing neutral current (FCNC) at the tree level. Depending on the Z_2 parity assignment, the 2HDM are classified into four types: Type-I, II, X, and Y. In contrast, the IDM is almost identical to the 2HDM, but the exact Z_2 symmetry makes it a model in which the additional Higgs field has no vacuum expectation value (VEV) and is not coupled with the SM particles. SMEFT is a very general model that incorporates information from the BSM by including the possible dimension 6 operators to the SM Lagrangian. In particular, this model is useful experimentally because it allows us to determine physical quantities such as the Higgs couplings in a more model-independent way.

1.1.3 Models in top-down approach

The top-down approach is represented by SUSY [7], compositeness [8–11], and GHU [12–15], each of which extends the Higgs sector by different principles. In the SUSY models, the supersymmetry is imposed on the whole theory including the Higgs sector, and superpartners corresponding to the SM particles are introduced. The divergence for the square mass of the Higgs boson is canceled by the contributions of the superpartners. The composite models treat the Higgs not as an elemental particle but as a particle with an internal structure. The divergence for the square mass of the Higgs boson is relaxed by the chiral symmetry from the internal fermions. In the GHU models, the Higgs is embedded into the extra components of the gauge multiplets. In other words, the Higgs and gauge field are unified into the higher dimensional gauge field. Thus, the Higgs sector is also described by the gauge principle. As similar to the weak bosons, the Higgs mass is also protected by the gauge symmetry.

1.1.4 Future collider experiments

The extended Higgs model and future collider experiments are very closely related. The verification of the BSM by collider experiments has been carried out in two ways: direct detection and indirect detection. The direct detection is aimed at the direct production of new particles by upgrading the energy, and the hadron collider is good at this. On the other hand, indirect detection is an experiment to look for signs of BSM by precise measurement of decay rates and coupling constants, and the lepton collider is good at it because of its clear background. The planned future collider experiments, such as HL-LHC [16] and ILC [17], are also called the Higgs Factory. One of its main propositions is to produce a large amount of Higgs processes and study the properties of the Higgs in detail.

1.2 Organization

This paper is organized as follows. First, in Chapter 2, we briefly review the SM and discuss the remaining issues in the Higgs sector. In Chapter 3, we introduce the motivation for extending the Higgs sector and what kind of extended models of the Higgs sector exist. Chapter 4 gives an overview of the extra dimension and describes the behavior of particles in the extra dimension used in this study. In Chapter 5, we discuss the gauge-Higgs unification model with a flat extra dimension. We focus on the characteristic structure of the Higgs potential in gauge-Higgs unification and show the possibility of testing the model in future collider experiments by analyzing the triple Higgs boson coupling. In Chapter 6, by introducing the extra dimensions, we discuss how to reproduce the viable Higgs couplings in the 2HDM without imposing the Z_2 symmetry. Finally, we will conclude our discussion in Chapter 7.

Chapter 2

Review of the Standard model

2.1 Standard Model

The SM is currently the best description of the behavior of the elementary particles. It is composed of the following two main components. One is the gauge principle, which defines the form of interactions, and the other is the mechanism of the spontaneous symmetry breaking, which gives the mass to the elementary particles. In this section, we will explain the role of these components through a simple example. After that, we will present the Weinberg-Salam theory, which describes the electroweak interaction incorporated into the Standard Model, a significant part of this research.

2.1.1 Gauge Principle

The gauge principle is a principle that determines the form of the interaction by requiring gauge symmetry in the physical system (and the associated Lagrangian $\mathcal{L}$). We will see a simple example below. First, we consider the Lagrangian density $\mathcal{L}_D$ that describes the motion of a free Dirac particle ψ with a mass m:

$$\mathcal{L}_D = \bar{\psi}(x)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x) \tag{2.1}$$

The first and second terms represent kinetic and mass terms, respectively. This Lagrangian is invariant under the global $U(1)$ transformation defined as

$$\psi(x) \to \psi'(x) = \mathrm{e}^{-\theta}\psi(x). \tag{2.2}$$

Therefore, we can also say that this Lagrangian has a global $U(1)$ symmetry. We next extend a parameter of the global $U(1)$ symmetry θ to a function $\Lambda(x)$ that depends on the spacetime coordinates:

$$\psi(x) \to \psi'(x) = \mathrm{e}^{-iq\Lambda(x)}\psi(x) \tag{2.3}$$

This transformation is called the $U(1)$ gauge transformation (or local $U(1)$ transformation), and the gauge principle requires invariance under such local transformations. However, the Lagrangian (2.1) is not invariant under the gauge transformations at this state. Note that the extension of the parameter to a function that depends on space-time gives us an extra term through the derivative. In order to remove this extra term, we replace the derivative ∂_μ with the covariant derivative D_μ

$$\partial_\mu \to D_\mu \equiv \partial_\mu + iqA_\mu(x), \tag{2.4}$$

where A_μ is a gauge field associated with the $U(1)$ gauge symmetry and is required to be transformed as

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x). \tag{2.5}$$

Then, the covariant derivative is transformed under the $U(1)$ gauge symmetry as

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = \mathrm{e}^{-iq\Lambda(x)} D_\mu \psi(x). \tag{2.6}$$

Using the covariant derivative, the Lagrangian invariant under the $U(1)$ gauge transformation is given by

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) - q\bar{\psi}(x)\gamma^\mu A_\mu \psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},\end{aligned} \tag{2.7}$$

where $F_{\mu\nu}$ is a field strength of A_μ defined as $F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. The new terms appearing in the second and third terms of the second line mean the interaction term between ψ and A_μ and kinetic term of A_μ, respectively. Note that the $U(1)$ gauge symmetry requires the existence of the $U(1)$ gauge field A_μ and an interaction term between the $U(1)$ gauge field and Dirac particle. It is also important to note that the mass term of the gauge field $m^2 A_\mu A^\mu$ is forbidden by the gauge symmetry.

The above discussion can be extended to the $SU(N)$ gauge symmetry case by a simple procedure. First, the $SU(N)$ gauge transformation can be expressed using the transformation matrix U as

$$\psi(x) \xrightarrow{SU(N)} \psi'(x) = U(x)\psi(x) \tag{2.8}$$

$$U(x) = \exp\left(-iq\Lambda(x)\right), \quad \Lambda(x) = \theta^a(x) T^a \quad \left(a = 1, \cdots, N^2 - 1\right), \tag{2.9}$$

where T^a are generators of $N^2 - 1$ transformations defined as

$$T^a T^b = \delta^{ab} \tag{2.10}$$

$$\left[T^a, T^b\right] = i f^{abc} T^c, \tag{2.11}$$

with structure constant f^{abc}. Then, the covariant derivative and gauge field for this transformation are expressed as

$$D_\mu = \partial_\mu + iqA_\mu(x) \qquad\qquad A_\mu(x) = A^a_\mu(x) T^a. \tag{2.12}$$

$SU(N)$ gauge symmetry requires the $N^2 - 1$ gauge fields $A^a_\mu(x)$ transformed as

$$A_\mu(x) \xrightarrow{SU(N)} A'_\mu(x) = U(x)\left\{A_\mu(x) + \frac{i}{q}U^{-1}(x)\left(\partial_\mu U(x)\right)\right\}U^{-1}(x). \tag{2.13}$$

Therefore, the Lagrangian invariant under the $SU(N)$ gauge transformation is given by

$$\mathcal{L} = \bar{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) - q\bar{\psi}(x)\gamma^\mu A^a_\mu T^a\psi(x) - \frac{1}{4}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right), \tag{2.14}$$

where we use the field strength of the $SU(N)$ gauge group defined as

$$F_{\mu\nu} \equiv [D_\mu, D_\nu]/ig = \left(\partial_\mu A^a_\nu - \partial\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\right)T^a. \tag{2.15}$$

2.1.2 Spontaneous Symmetry Breaking

As shown in 2.1.1, the gauge symmetry forbids the mass terms of the gauge fields. However, the W and Z bosons have been observed as massive gauge bosons. Therefore, the SM introduces a mechanism called spontaneous symmetry breaking that allows elementary particles to acquire mass. First, we consider the complex scalar field Φ with global $U(1)$ symmetry. Using the two real scalar fields ϕ_1, ϕ_2, it can be written as

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right). \tag{2.16}$$

Taking into account the renormalizability of the theory, the Lagrangian for this complex scalar with global $U(1)$ symmetry is given by

$$\mathcal{L}_\Phi = \left(\partial_\mu\Phi\right)^*\left(\partial^\mu\Phi\right) - \mu^2\Phi^*\Phi - \lambda\left(\Phi^*\Phi\right)^2. \tag{2.17}$$

When we take $\mu^2 < 0, \lambda > 0$, the potential term in the Lagrangian (2.17) $V = \mu^2\Phi^*\Phi + \lambda(\Phi^*\Phi)^2$ has a minimum at $|\Phi| = |\langle\Phi\rangle| \neq 0$:

$$\left.\frac{\partial V}{\partial|\Phi|}\right|_{\Phi=\langle\Phi\rangle} = 0 \qquad \therefore\ |\langle\Phi\rangle| = \sqrt{\frac{-\mu^2}{2\lambda}} \equiv \frac{1}{\sqrt{2}}v. \tag{2.18}$$

The potential minimums, which are called the vacuum, are degenerate on the circumference in the complex plane. When we choose one of these vacuums, it spontaneously breaks the global $U(1)$ symmetry. For example, if we choose a vacuum on the real axis

$$\Phi = \frac{1}{\sqrt{2}}\left(v + \eta(x)\right)\mathrm{e}^{i\xi(x)/v}, \tag{2.19}$$

Lagrangian (2.17) is expanded around this vacuum as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \lambda v^2\eta^2 + \text{Cubic or higher order terms for } x \text{ and } y + \text{const.} \tag{2.20}$$

In this form, η acquires mass $\sqrt{2\lambda v^2}$, while ξ is a massless particle with no mass term. This phenomenon, in which the symmetry of the Lagrangian is lost in the vacuum, is called the spontaneous symmetry breaking. The massless scalar particle that appears, as a result, is called the Nambu-Goldstone boson.

The Higgs mechanism

Now, we extend the above discussion to consider the spontaneous symmetry breaking under $U(1)$ gauge symmetry. In this case, the Lagrangian is given by

$$\mathcal{L} = |D_\mu \Phi|^2 - V - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{2.21}$$

Inserting Eq. (2.19 into the first term of this Lagrangian, we find

$$\begin{aligned} |D_\mu \Phi|^2 &= \left| (\partial_\mu + iqA_\mu(x)) \left(\frac{1}{\sqrt{2}} (v + \eta(x)) \, \mathrm{e}^{i\xi(x)/v} \right) \right|^2 \\ &= \left| \left(\partial_\mu + iq \left\{ A_\mu(x) + \frac{1}{qv} \partial_\mu \xi(x) \right\} \right) \frac{1}{\sqrt{2}} (v + \eta(x)) \right|^2 \\ &= \frac{1}{2} q^2 v^2 \left\{ A_\mu(x) + \frac{1}{qv} \partial_\mu \xi(x) \right\} \left\{ A^\mu(x) + \frac{1}{qv} \partial^\mu \xi(x) \right\} + \cdots . \end{aligned} \tag{2.22}$$

Here, if we regard $A_\prime(x) = A_\mu(x) + \partial_\mu \xi(x)/qv$, we see that the first term in the last line of Eq. (2.22) is in the form of a mass term. This substitution also implies that the massless gauge boson gets the mass by absorbing the Nambu-Goldstone boson as a longitudinally polarized component. This mechanism, in which the corresponding gauge boson absorbs the Nambu-Goldstone boson that appears upon spontaneous symmetry breaking and acquires mass, is called the Higgs mechanism.

Above discussion is also applied to the $SU(2)$ gauge symmetry. In that case, we consider a complex scalar field that behaves as a doublet under the $SU(2)$ transformation. This can be expressed using four real scalar fields as follows:

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1(x) + i\phi_2(x) \\ \phi_3(x) + i\phi_4(x) \end{pmatrix} \tag{2.23}$$

The VEV take $v/\sqrt{2}$ as same as the $U(1)$ case. If we choose the vacuum on the ϕ_3 direction, the field after symmetry breaking is represented as

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1(x) + i\xi_2(x) \\ v + \eta(x) + i\xi_3(x) \end{pmatrix} \tag{2.24}$$

This representation can be rewritten the three ξ's like a phase invariant under the $SU(2)$ gauge transformation as a $U(1)$ case:

$$\Phi(x) = \frac{1}{\sqrt{2}} \mathrm{e}^{i\tau \cdot \xi(x)/v} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix} \tag{2.25}$$

Therefore, we see that the $\eta(x)$ appears as a massive scalar, while the three $\xi_i(x)$ appear as Nambu=Goldstone bosons.

Next, we consider the kinetic term. The covariant derivative of the $SU(2)$ gauge group is given by

$$D_\mu = \partial_\mu + iq \frac{\tau^a}{2} A_\mu^a, \tag{2.26}$$

	Q_L	L_L	u_R	d_R	e_R	Φ	Φ_c
$SU(2)_L$	**2**	**2**	**1**	**1**	**1**	**2**	**2**
$U(1)_Y$	$\frac{1}{3}$	-1	$\frac{4}{3}$	$-\frac{2}{3}$	-2	1	-1

Table 2.1: Field representations and eigenvalues in Weinberg-Salam theory

where τ^a $(a = 1, 2, 3)$ are generators of the $SU(2)$ group represented by the Pauli matrixes. Therefore, the Higgs mechanism works as

$$\begin{aligned} |D_\mu \Phi|^2 &= \left| \left(\partial_\mu + iqA^a_\mu \frac{\tau^a}{2} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix} \right|^2 \\ &= \frac{1}{2} \left| \frac{i}{2} qv \begin{pmatrix} A^1_\mu - iA^2_\mu \\ -A^3_\mu \end{pmatrix} + \frac{1}{2} \begin{pmatrix} iqA^1_\mu + qA^2_\mu \\ \partial_\mu - iqA^3_\mu \end{pmatrix} \eta(x) \right|^2 \\ &= \frac{1}{8} q^2 v^2 \left({A^1_\mu}^2 + {A^2_\mu}^2 + {A^3_\mu}^2 \right) + \cdots . \end{aligned} \tag{2.27}$$

In the last line, the three $SU(2)$ gauge fields, A^i_μ, acquire the same mass by absorbing the corresponding three Nambu-Goldstone bosons.

2.1.3 Weinberg-Salam theory

The Weinberg-Salam theory is a theory that describes electromagnetism and the weak force in a unified manner among the four fundamental forces currently known: gravity, electromagnetism, the strong force, and the weak force. It consists of the $SU(2)_L \times U(1)_Y$ gauge symmetry and spontaneous symmetry breaking. The three massive gauge bosons appearing after electroweak symmetry breaking (EWSB) mediate weak interaction while the remaining massless gauge boson mediates electromagnetic interaction.

Now we consider the $SU(2) \times U(1)$ invariant Lagrangian. In this case, the covariant derivative is given by

$$D_\mu = \partial_\mu + igW^a_\mu T^a + ig' B_\mu Y \tag{2.28}$$

where $T^a (= \tau^a/2)$, Y are the generators of the $SU(2)_L$, $U(1)_Y$ gauge groups and W_μ, B_μ are corresponding gauge fields, respectively. Next, we have to decide to which representation of the gauge group the matter field expected by the theory belongs and which eigenvalue it has: the right-handed and left-handed quarks, $u_{R,L}$ $d_{R,L}$, leptons, $e_{R,L}$ ν_L, and Higgs, Φ. It is assigned as shown in Table 2.1 for consistency with experiments. In Table 2.1, the $SU(2)$ doublets are represented as

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \ L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} , \ \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \ \Phi_c = \begin{pmatrix} -\bar{\phi}^0 \\ \phi^- \end{pmatrix} , \tag{2.29}$$

where Φ_c is the charged conjugate of Φ defined by $\Phi_c = -i\tau_2 \Phi^*$. Therefore, the $SU(2) \times U(1)$

invariant Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{WS} =& i\bar{Q}_L \not{D} Q_L + i\bar{u}_R \not{D} u_R + i\bar{d}_R \not{D} d_R \\ &+ i\bar{L}_L \not{D} L_L + i\bar{e}_R \not{D} e_R + i\bar{\nu}_R \not{D} \nu_R \\ &- \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ &+ \left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right) - V\left(\Phi^*\Phi\right) \\ &- y_u \bar{Q}_L \Phi_c u_R - y_d \bar{Q}_L \Phi d_R - y_e \bar{L}_L \Phi_c e_R + h.c., \end{aligned} \tag{2.30}$$

where $W_{\mu\nu}$ and $B_{\mu\nu}$ are the tensors of the strength of the $SU(2)_L$ and $U(1)_Y$ gauge fields, respectively. In this Lagrangian, the first and second lines describe kinetic and interaction terms of the quarks and leptons, the third line describes the gauge kinetic term, and the fourth and fifth lines are Higgs sector that describes the nature of Higgs.

Then, we will review the mechanism of electroweak symmetry breaking in detail. The potential of the Higgs boson is taken as in the previous section, $V = \mu^2\Phi^*\Phi + \lambda(\Phi^*\Phi)^2$ ($\mu^2 < 0, \lambda > 0$). When we choose the VEV of the Higgs boson as

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{2.31}$$

the kinetic terms of Higgs after symmetry breaking is rewritten as

$$|D_\mu\Phi|^2 = \frac{1}{2} \left| \left(\partial_\mu + ig\frac{\tau^a}{2} W^a_{\mu\nu} + ig'\frac{1}{2} B_\mu \right) \begin{pmatrix} 0 \\ v+h \end{pmatrix} \right|^2 . \tag{2.32}$$

Extracting the part related to the mass term of the gauge field, we can summarize it as

$$\begin{aligned}\frac{1}{2} \left| \left(ig\frac{\tau^a}{2} W^a_{\mu\nu} + ig'\frac{1}{2} B_\mu \right) \langle\Phi\rangle \right|^2 &= \frac{1}{8} \left| \begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & -gW^3_\mu + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\ &= \frac{1}{8} g^2 v^2 \left(W^1_\mu - iW^2_\mu\right)\left(W^1_\mu + iW^2_\mu\right) + \frac{1}{8} v^2 (g'B_\mu - gW^3_\mu)^2 \\ &= \frac{1}{4} g^2 v^2 W^+_\mu W^{-\mu} + \frac{1}{8}(g^2 + g'^2) v^2 Z_\mu Z^\mu + 0 \cdot A_\mu A^\mu \end{aligned} \tag{2.33}$$

where $W^\pm_\mu \equiv (W^1_\mu \mp W^2_\mu)/\sqrt{2}$ are the gauge fields with electromagnetic charge ± 1 and A_μ, Z are masseigenstates defined by

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \equiv \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g & -g' \\ g' & g \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}. \tag{2.34}$$

The rotation angle θ_W of the diagonalization matrix is called the Weinberg angle, and its value is given by the mass ratio of the W and Z bosons [18]:

$$\sin^2\theta_W = 0.22343 \pm 0.00007 \tag{2.35}$$

Therefore, after EWSB, $W^\pm_\mu$ and Z acquire the mass

$$m_W = \frac{1}{2} g v \tag{2.36}$$

$$m_Z = \frac{1}{2}\sqrt{g^2 + g'^2}\, v = \frac{1}{2}\frac{g}{\cos\theta_W} v = \frac{m_W}{\cos\theta_W}, \tag{2.37}$$

respectively. On the other hand, A_μ is still massless and is the gauge field corresponding to the symmetry that remains after spontaneous symmetry breaking. For example, the vacuum state is invariant under the following transformation that combines the $SU(2)_L \times U(1)_Y$ gauge group generators T^a and Y:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix} \to \exp\left[ie\left(T^3+\frac{Y}{2}\right)\right]\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{e}^{ie} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\v\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix} \tag{2.38}$$

It is the remaining $U(1)$ symmetry, which corresponds to the electromagnetic interaction. eQ corresponds to the electromagnetic charge when the $U(1)_{EM}$ gauge group generators are defined by $Q \equiv T^2 + Y/2$.

As seen from the above Lagrangian, this theory is a non-chiral theory that distinguishes between the left-hand and right-hand of fermions. The chirality of a fermion is defined as

$$\psi_R = \frac{1+\gamma^5}{2}\psi \equiv P_R\psi \qquad \gamma^5\psi_R = +\psi_R \tag{2.39}$$

$$\psi_L = \frac{1-\gamma^5}{2}\psi \equiv P_L\psi \qquad \gamma^5\psi_L = -\psi_L. \tag{2.40}$$

The left equation expresses the projection from the Dirac spinor to the chiral spinor by the projection operator $P_{R,L}$, and the right one implies that the chiral spinor is an eigenstate of γ^5. In addition, the chiral representation has the following significant consequences:

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R \tag{2.41}$$

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \tag{2.42}$$

In other words, to write the mass term $m\bar{\psi}\psi$, we need both left-hand and right-hand chirality. The fermion mass term in the SM (Weinberg-Salam theory) is restricted to the Yukawa interaction form in Eq. (2.30) by the conditions of the gauge invariance and chirality. Then, as in the gauge fields, the fermion mass term emerges from the Yukawa sector due to the EWSB:

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} &= -\,y_u\bar{Q}_L\Phi_c u_R - y_d\bar{Q}_L\Phi d_R - y_e\bar{L}_L\Phi_c e_R + h.c.\\ &= -\,\frac{1}{\sqrt{2}}y_u v\cdot\bar{u}_L u_R - \frac{1}{\sqrt{2}}y_d v\cdot\bar{d}_L d_R - \frac{1}{\sqrt{2}}y_e v\cdot\bar{e}_L e_R + h.c. + \cdots\\ &= -\,m_u\bar{u}u - m_d\bar{d}d - m_e\bar{e}e + \cdots\end{aligned} \tag{2.43}$$

An important prediction of the SM is that the masses of the fermions are proportional to the VEV as $m_f = y_f v/\sqrt{2}$.

2.2 Open problems in the SM

The SM is a very successful theory that is consistent with most of the previous experiments. But it is not end of the story because there are phenomenological and theoretical problems that cannot be explored by the SM. In particular, the existence of dark matter, the baryon number asymmetry of the universe, inflation, and the neutrino oscillation are well known as phenomenological problems, while the gauge hierarchy problem and the unification of the gravity are known as theoretical problems.

2.2.1 Existence of the dark matter

Although the definition of the dark matter slightly varies from person to person, we will consider it to be any matter with the mass that has no electromagnetic interaction. Indirect evidence for the existence of dark matter has been presented in several ways. One of them is the measurement of the rotational velocity of galaxies by V. Rubin and K. Ford in 1970 [19]. This observation showed that the mass density calculated from that was much greater than the mass density of the optically observable matter. It indirectly proves that there is a large amount of matter in the universe that does not emit the photons but only has the mass. Candidates for the dark matter can be classified into two major categories: those derived from the astrophysics and those derived from the particle physics. Astrophysical candidates include the black hole that swallow the light and the white dwarf and neutron star that do not emit the enough light to be observable. These candidates are formed by the baryons, whose abundance is predicted by the Big Bang cosmology. According to this prediction, these sources of the dark matter cannot satisfy the required amount of the dark matter. On the other hand, the SM does not contain any candidate particles for the dark matter. Therefore, to solve this problem from a particle physical perspective, it is necessary to extend the SM to include the dark matter candidates. The Weakly Interacting Massive Particle has been recognized as the most famous dark matter candidate.

2.2.2 Baryon number asymmetry of the universe

In the SM, the particles and antiparticles are equivalent. However, we have known from the observations that the ratio of baryon number to photon number in the universe is given as follows [20]:

$$\eta = \frac{n_B}{n_\gamma} = \frac{n_q - n_{\bar{q}}}{3n_\gamma} \sim 10^{-9} \tag{2.44}$$

This result indicates that the present universe has matter dominance. Therefore, from an early universe where matter and antimatter existed symmetrically, some mechanism would create an asymmetry in which the matter would be dominant. The conditions for generating a baryon number from the symmetric early universe are proposed by Sakharov as follows [21]:

- Baryon number violation
- C and CP symmetry violation
- Interactions out of thermal equilibrium

In the SM framework, the possibility of an electroweak baryogenesis scenario has been investigated. In this scenario, above three conditions are translated as follows:

- Baryon number violation $\rightarrow$ sphaleron process
- C and CP symmetry violation $\rightarrow$ CKM matrix
- Interactions out of thermal equilibrium $\rightarrow$ strongly 1st order phase transition of the Higgs potential

However, in the SM, the CKM phase is not sufficient to produce the CP violation [22] and the 125 GeV Higgs cannot realize the 1st order phase transition [23]. Therefore, to solve the baryon asymmetry in the EW scale, we need an additional source of the CP violation and a suitable structure of the Higgs potential, which realize the strongly 1st order phase transition.

2.2.3 Neutrino oscillation

In 1998, an experiment at the Super-Kamiokande confirmed the neutrino oscillation [24]. It is the phenomenon in which three generations of the neutrinos change from one generation to another during their transportation. When neutrinos have the masses, there are mass eigenstates $\nu_i^{(m)}$ separate from the gauge eigenstates ν_i. Then, these eigenstates are related by some unitary transformation:

$$\nu_i = U_{ij}\nu_j^{(m)} \tag{2.45}$$

Since the time evolution of $|\nu_i\rangle$ can be written as

$$|\nu_i, t\rangle = \sum_j \mathrm{e}^{-iE_j t} U_{ij} |\nu_j^{(m)}\rangle \ , \tag{2.46}$$

using the energy E_i of $\nu_i^{(m)}$, the transition probability from ν_i to ν_j after the time has passed by t is given by

$$P_{\nu_i \to \nu_j} = |\langle \nu_j | \nu_i, t\rangle|^2 = \sum_{k,l} U_{jk}^* U_{ki} U_{jl} U_{li}^{ast} \mathrm{e}^{-i(E_k - E_l)t} \ . \tag{2.47}$$

Assuming for simplicity that the only 1st and 2nd generations are mixed, the mixing matrix becomes

$$U = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{2.48}$$

Then, we find

$$\begin{aligned} P_{\nu_i \to \nu_j} &= \sin^2\theta \cos^2\theta \left(2 + \mathrm{e}^{-i(E_1 - E_2)t} + \mathrm{e}^{-i(E_2 - E_1)}\right) \\ &\sim \sin^2 2\theta \sin^2 \frac{m_2^2 - m_1^2}{4E} L \ , \end{aligned} \tag{2.49}$$

where L denotes the neutrino moving distance and we expand the energy as $E \sim |\vec{p}| + m_i^2/2|\vec{p}|$ by using the fact that the neutrino masses are very small. This result indicates that the mixing between the masseigenstates and a non-zero mass difference are required for the neutrino oscillations. Since neutrinos are the massless particles in the SM, we need some mechanism to give them small masses.

2.2.4 Inflation

Inflation was introduced to solve the flatness problem and the horizon problem, which were cosmological problems that could not be solved by the Big Bang cosmology. This idea makes it possible to solve the above problem by causing an exponential expansion in the early universe. However, the source of this exponential expansion is not yet understood. Since the early universe is considered as a very high energy, the particle physics valid on such an energy scale. Therefore, the source of inflation should be included in the general relativity or SM. However, the SM does not contain the scalar field that causes inflation, and thus we need to extend the SM.

2.2.5 Hierarchy problem

Then, we will discuss the cut-off scale Λ of the SM. Assuming that the SM will be switched to another theory by at least the Planck scale, to what energy scale can the SM be applied? The quadratic divergence of the squared mass for the Higgs boson provides critical insight into this question. The observed mass is given by the bare mass in the Lagrangian and the quantum corrections: $m^2 = m_0^2 + \Delta m^2$. The quantum corrections to the masses of the gauge bosons and fermions are protected from drastic divergences by their symmetry (gauge symmetry and chiral symmetry). On the other hand, the Higgs field is the only particle in the SM that is not protected by any symmetry, and its quantum corrections are proportional to the square of the cut-off scale. Focusing on the top quark that mainly contributes to the Higgs potential, we obtain

$$\begin{aligned}\Delta m_h^2 &= i\left(-i\frac{y_t}{\sqrt{2}}\right)^2(-1)\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{i(\not{p}+m_t)}{p^2-m_t^2}\cdot\frac{i(\not{p}+\not{k}+m_t)}{(p+k)^2-m_t^2}\right]\\ &= -i\int\frac{d^4p}{(2\pi)^4}\int_0^1 dx\frac{2y_t^2\left(p\cdot(p+k)+m_t^2\right)}{\left(p^2+2xp\cdot k+xk^2-m_t^2\right)^2}. \qquad (2.50)\end{aligned}$$

In the second line, we introduce the Feynman parameter x. By replacing $l = p + xk$ and $\Delta = m_t^2 - x(1-x)k^2$, Eq. (2.50) becomes

$$\begin{aligned}\Delta m_h^2 &= -2i\int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\frac{l^2+\Delta}{\left(l^2-\Delta\right)^2}\\ &= -2y_t^2\int\frac{d\Omega_4}{(2\pi)^4}\int_0^1 dx\int_0^\Lambda dl_E\frac{l_E^2-\Delta}{\left(l_E^2+\Delta\right)^2}\\ &= -\frac{3y_t^2}{8\pi^2}\left(\Lambda^2-3m_t^2\ln\left(\frac{\Lambda^2}{m_t^2}\right)+2m_t^2\right) \qquad (2.51)\end{aligned}$$

where we perform the Wick rotation in the second line: $l^0 = il_E^0$. Therefore, if the SM is applicable up to the Planck scale $\mathcal{O}(10^{19})$ GeV, a delicate fine-tuning between the bare mass and the quantum corrections $\mathcal{O}(10^{34})$ GeV^2 is needed to explain the observed Higgs mass $m_h = 125$ GeV. It indicates the existence of the new physics beyond the SM around the TeV scale.

2.3 Effective Potential in the SM

As shown in Sec. 2.1.3, the Higgs potential has a negative mass term required for spontaneous symmetry breaking. Note that there are no odd power terms because the Higgs field of the SM is a $SU(2)_L$ doublet, and therefore the Higgs potential is given by

$$V_0 = -\mu^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2 = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \qquad (2.52)$$

where μ^2 and λ are the positive model parameters. These model parameters can be rewritten in terms of the vacuum expectation value and the Higgs mass from the first and second derivatives of the Higgs potential:

$$\left.\frac{\partial V_0}{\partial \phi}\right|_{\phi=v} = 0 \qquad \therefore \mu^2 = \lambda v^2$$
$$\left.\frac{\partial^2 V_0}{\partial \phi^2}\right|_{\phi=v} = m_h^2 \quad \therefore m_h^2 = 2\mu^2 \tag{2.53}$$

Therefore, at the tree level, the Higgs self-couplings are uniquely predicted as

$$\lambda^0_{hhh} = \left.\frac{\partial^3 V_0}{\partial \phi^3}\right|_{\phi=v} = 3\frac{m_h^2}{v},$$
$$\lambda^0_{4h} = \left.\frac{\partial^4 V_0}{\partial \phi^4}\right|_{\phi=v} = 3\frac{m_h^2}{v^2}. \tag{2.54}$$

In the following, we will see how these results are modified by the quantum corrections. The one-loop contributions to the Higgs potential is given by

$$V = V_0 + V_1 + \cdots, \quad V_1 = \sum_I \frac{\sigma_I}{2} \int \frac{d^4 p_E}{(2\pi)^4} \ln\left(p_E^2 + m_I^2\right), \tag{2.55}$$

where $\sigma_I = 1$ for bosons and $\sigma_I = -1$ for fermions, and the sum runs over all particles whose masses depend on the VEV. Similar to the Higgs mass calculation, focusing on the top quark that gives the main contribution, the one-loop contribution can be calculated as follows:

$$\begin{aligned} -\int \frac{d^4 p_E}{(2\pi)^4} \ln\left(p_E^2 + m_t^2\right) &= -\frac{\partial}{\partial \alpha} \int \frac{d^4 p_E}{(2\pi)^4} \left.\frac{1}{(k_E^2 + m_t^2)^\alpha}\right|_{\alpha=0} \\ &= -\frac{\partial}{\partial \alpha} \left.\left(\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \frac{1}{(m_t^2)^{\alpha - d/2}} \right)\right|_{\alpha=0} \\ &= -\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{(m_t^2)^{-d/2}} \end{aligned} \tag{2.56}$$

Evaluating this result with the $\overline{MS}$ scheme, we obtain

$$\begin{aligned} -\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{(m_t^2)^{-d/2}} &= -\frac{1}{\frac{d}{2}\left(\frac{d}{2} - 1\right)} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{d/2}} (m_t^2)^{d/2} \\ &= -\frac{m_t^4}{2(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) - \ln(m_t^2) + \frac{3}{2} \right) \\ &\to -\frac{m_t^4}{2(4\pi)^2} \left(-\ln \frac{m_t^2}{Q^2} + \frac{3}{2} \right), \end{aligned} \tag{2.57}$$

where we use $\epsilon = d - 4$ and Q is an arbitrary mass scale parameter. Thus, the one-loop effective potential is given by

$$V_{\text{eff}} = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - \frac{3}{64\pi^2} \frac{y_t^4 \phi^4}{4} \left(-\ln \frac{y_t^2 \phi^2}{2Q^2} + \frac{3}{2} \right), \tag{2.58}$$

where we use $m_t(\phi) = y_t\phi/\sqrt{2}$. Then, the tadpole and Higgs mass conditions are modified as

$$\left.\frac{\partial V_{\text{eff}}}{\partial \phi}\right|_{\phi=v} = 0 \qquad \therefore \mu^2 = \lambda v^2 + \frac{3}{4v^2} m_t^4 \left(1 - \ln \frac{m_t^2}{Q^2}\right),$$
$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial \phi^2}\right|_{\phi=v} = m_h^2 \quad \therefore m_h^2 = 2\mu^2 + \frac{3}{2v^2} m_t^4. \tag{2.59}$$

Note that the arbitrary mass parameter Q is renormalized by these conditions. Therefore, by using these relations, the SM triple Higgs boson coupling is predicted at the one-loop level as

$$\lambda_{hhh} = \left.\frac{\partial^3 V_{\text{eff}}}{\partial \phi^3}\right|_{\phi=v} = \frac{3m_h^2}{v}\left(1 - \frac{1}{\pi^2}\frac{m_t^4}{v^2 m_h^2}\right). \tag{2.60}$$

When the mass is proportional to the vacuum expectation value, as in the case of SM particles, the one-loop contribution is proportional to the 4th power of its mass spectrum. Therefore, the contribution of the top quark is the largest, about 10% of the tree level.

Chapter 3

Review of the extended Higgs sector

In this chapter, we introduce the extended Higgs model that we will study in the following chapters. we see the model structures for the bottom-up approaches and how to solve the hierarchy problems in the SM for the top-down approaches.

3.1 Two-Higgs-Doublet Model

The 2HDM has almost the same structure as the SM. The most significant difference is the existence of one additional Higgs doublet. In other words, there are two Higgs doublets with different VEV:

$$\Phi_1 = \begin{pmatrix} w_1^+ \\ \frac{1}{\sqrt{2}}\left(h_1 + iz_1 + v_1\right) \end{pmatrix} \ , \quad \Phi_2 = \begin{pmatrix} w_2^+ \\ \frac{1}{\sqrt{2}}\left(h_2 + iz_2 + v_2\right) \end{pmatrix} \tag{3.1}$$

In general, there exist two Yukawa interactions, and thus the mass matrix and Higgs coupling matrixes are not diagonalized simultaneously [25]. For example, we consider the down sector. Using the Yukawa couplings y_{1d}^{ij}, y_{2d}^{ij}, the mass matrix m, and the Higgs couplings g^{ij}, the gauge and mass eigenstates can be related as

$$y_{1d}^{ij}\bar{q}_L^i \Phi_1 d_R^j + y_{2d}^{ij}\bar{q}_L^i \Phi_2 d_R^j + (\text{h.c.}) = \bar{d}^i m_d^{ij} d^j + \bar{d}^i \left(h g_h^{ij} + H g_H^{ij} + H' g_{H'}^{ij}\right) d^j, \tag{3.2}$$

where h, H, H' are the neutral masseigenstates determined by the linear combination of h_1, h_2, z_1, z_2. Then, the unacceptable FCNC are appear in the off-diagonal components of the Higgs couplings. If $y_{2d}^{ij} = 0$ is proportional to y_{1d}^{ij}, then the mass matrix and Higgs coupling matrixes are diagonalized simultaneously. It is called the Yukawa alignment [26] or Alined 2HDM [27]. To avoid it more naturally, we usually only couple fermions with either Higgs doublets, y_{1d}^{ij} or $y_{2d}^{ij} = 0$, by imposing the Z_2 symmetry. There are four types of the Z_2 parity assignments, Type-I,II,Y,X, that achieve the above, as shown in the table 3.1. The Higgs potential is also modified as

$$\begin{aligned} V(\Phi_1, \Phi_2) &= m_1^2 \Phi_1^\dagger \Phi_1 + m_2^2 \Phi_2^\dagger \Phi_2 - \left(m_3^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}\right) + \frac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \frac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 \\ &+ \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \left(\frac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \text{h.c.}\right). \end{aligned} \tag{3.3}$$

Of the terms that break the Z_2 symmetry, fourth order terms such as $(\Phi_1^\dagger \Phi_1)(\Phi_1^\dagger \Phi_2)$, $(\Phi_2^\dagger \Phi_2)(\Phi_1^\dagger \Phi_2)$ are forbidden, while second order terms such as $\Phi_1^\dagger \Phi_2$ are kept to make the additional Higgs

	Φ_1	Φ_2	q_L	l_L	u_R	d_R	e_R
Type-I	$+$	$-$	$+$	$+$	$-$	$-$	$-$
Type-II	$+$	$-$	$+$	$+$	$-$	$+$	$+$
Type-X	$+$	$-$	$+$	$+$	$-$	$-$	$+$
Type-Y	$+$	$-$	$+$	$+$	$-$	$+$	$-$

Table 3.1: Z_2 parity assignment in the 2HDM

bosons heavier. Therefore, what we impose on the 2HDM is not exact Z_2 symmetry, but so-called softly broken Z_2 symmetry. The two VEVs v_1, v_2 are determined by the minimum of the Higgs potential as

$$\begin{aligned} \left.\frac{\partial V}{\partial \Phi_1}\right|_{\Phi_1=v_1} &= 0 \quad \therefore -m_1^2 = -\frac{v_2}{v_1}m_3^2 + \frac{1}{2}v_1^2\lambda_1 + \frac{1}{2}v_2^2\left(\lambda_3+\lambda_4+\lambda_5\right), \\ \left.\frac{\partial V}{\partial \Phi_2}\right|_{\Phi_2=v_2} &= 0 \quad \therefore -m_2^2 = -\frac{v_1}{v_2}m_3^2 + \frac{1}{2}v_2^2\lambda_2 + \frac{1}{2}v_1^2\left(\lambda_3+\lambda_4+\lambda_5\right). \end{aligned} \tag{3.4}$$

From the mass representations of the gauge bosons, we get the relations

$$v_1^2 + v_2^2 = v^2 = 246 \text{ GeV} \ , \quad \tan\beta = \frac{v_2}{v_1}, \tag{3.5}$$

where rotation angle β is defined as

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{3.6}$$

We can get the mass matrixes of the Higgs doublets from the second derivatives of the Higgs potential around the VEV. For the imaginary parts of the neutral components in Φ_1, Φ_2, we find the massless Goldstone boson z and CP odd Higgs A with mass M_A

$$\begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} M_A^2 c_\beta^2 & -M_A^2 c_\beta s_\beta \\ -M_A^2 c_\beta s_\beta & M_A^2 s_\beta^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & M_A^2 \end{pmatrix} \begin{pmatrix} z \\ A \end{pmatrix}, \tag{3.7}$$

where the mass M_A and relation between the two basis are given by

$$M_A^2 = \frac{m_3^2}{c_\beta s_\beta} - \lambda_5 v^2 = M^2 - \lambda_5 v^2 \ , \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_\beta & -s_\beta \\ s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} z \\ A \end{pmatrix}. \tag{3.8}$$

For the charged components in Φ_1, Φ_2, we find the two massless Goldstone bosons $w^\pm$ and charged Higgs bosons $H^\pm$ with mass $M_{H^\pm}$

$$\begin{pmatrix} w_1^- & w_2^- \end{pmatrix} \begin{pmatrix} M_{H^\pm}^2 c_\beta^2 & -M_{H^\pm}^2 c_\beta s_\beta \\ -M_{H^\pm}^2 c_\beta s_\beta & M_{H^\pm}^2 s_\beta^2 \end{pmatrix} \begin{pmatrix} w_1^+ \\ w_2^+ \end{pmatrix} = \begin{pmatrix} w^- & H^- \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & M_{H^\pm}^2 \end{pmatrix} \begin{pmatrix} w^+ \\ H^+ \end{pmatrix}, \tag{3.9}$$

where the mass $M_{H^\pm}$ and relation between the two basis are given by

$$M_{H^\pm}^2 = M^2 - \frac{1}{2}\left(\lambda_4+\lambda_5\right)v^2 \ , \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_\beta & -s_\beta \\ s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} z \\ A \end{pmatrix}. \tag{3.10}$$

Finally, for the real parts of the neutral components in Φ_1, Φ_2, we find the two CP even Higgs bosons H and h with two different mass m_H and m_h

$$\begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} M^2 + \lambda_1 v^2 c_\beta^2 & -M^2 + \lambda_{345} v^2 c_\beta s_\beta \\ -M^2 + \lambda_{345} v^2 c_\beta s_\beta & M^2 + \lambda_2 v^2 s_\beta^2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} H & h \end{pmatrix} \begin{pmatrix} M_H^2 & 0 \\ 0 & M_h^2 \end{pmatrix} \begin{pmatrix} H \\ h \end{pmatrix}, \tag{3.11}$$

where we take $\lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5$ and relation between the two basis is given by

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{pmatrix} \begin{pmatrix} H \\ h \end{pmatrix}. \tag{3.12}$$

We identify h as the 125 GeV Higgs boson and H as an additional heavier Higgs boson.

3.2 Inert Doublet Model

The Inert Doublet Model also has two Higgs doublets. However, in contrast to the 2HDM, which has a soft broken Z_2 symmetry, the IDM has an unbroken (exact) Z_2 symmetry. Under this Z_2 symmetry only the additional Higgs doublet η has a negative Z_2 parity: $\eta \to -\eta$. Therefore, η does not appear in the Yukawa sector. The Higgs potential with exact Z_2 symmetry is given by

$$\begin{aligned} V(\Phi, \eta) = {} & m_1^2 \Phi^\dagger \Phi + m_2^2 \eta^\dagger \eta + \frac{1}{2}\lambda_1 (\Phi^\dagger \Phi)^2 + \frac{1}{2}\lambda_2 (\eta^\dagger \eta)^2 \\ & + \lambda_3 (\Phi^\dagger \Phi)(\eta^\dagger \eta) + \lambda_4 (\Phi^\dagger \eta)(\eta^\dagger \Phi) + \left(\frac{1}{2}\lambda_5 (\Phi^\dagger \eta)^2 + \text{h.c.} \right). \end{aligned} \tag{3.13}$$

where the two Higgs doublets are written by

$$\Phi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(h + iG^0 + v_1\right) \end{pmatrix}, \quad \eta = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(H + iA\right) \end{pmatrix}. \tag{3.14}$$

η does not have a VEV due to the Z_2 symmetry, and thus the VEV of Φ becomes the electroweak vacuum $v \simeq 246$ GeV. There are three Goldstone bosons, G^+, G^-, G^0, SM like Higgs, h, and additional Higgs bosons, H^+, H^-, H, A. From the second derivatives of the Higgs potential, these masses are given by

$$\begin{aligned} m_h^2 &= \lambda_1 v^2, \\ m_{H^\pm} &= m_2^2 + \frac{1}{2}\lambda_3 v^2, \\ m_H &= m_2^2 + \frac{1}{2}\left(\lambda_3 + \lambda_4 + \lambda_5\right) v^2, \\ m_A &= m_2^2 + \frac{1}{2}\left(\lambda_3 + \lambda_4 - \lambda_5\right) v^2, \end{aligned} \tag{3.15}$$

where m_1 is eliminated by the stationary condition of the Higgs potential. Note that the lightest additional Higgs boson can be candidate of the dark matter because it does not decay into the SM particles thanks to the Z_2 symmetry. We note that the lightest Z_2 odd field, H or A, can be a dark matter candidate since it does not decay into the SM particles [28, 29].

3.3 TeV scale paradigms

As seen in Sec. 2.2, the Higgs mass in the SM is sensitive to the cut-off scale of the theory. It is because the SM Higgs is an elementary scalar with no symmetry. Therefore, we naturally consider that the SM is replaced by some new physics models around the TeV scale. The candidates of such a new physics model are supersymmetry, compositeness, and gauge-Higgs unification. They protect the mass of the Higgs boson from the second divergence to introduce some symmetries in the Higgs sector in a different way. In the following, we will discuss SUSY as a famous and clean example of canceling divergence. In brief, the SUSY model introduces partner particles (Sparticles) that differ only in spin from the SM particles and imposes symmetry (called supersymmetry) between them. For the introduction of Sparticles, the Yukawa sector is modified as

$$\mathcal{L}_{\text{Yukawa}} = -\frac{y_t}{\sqrt{2}}\phi\bar{t}t - f_L\phi|\tilde{t}_L|^2 - f_R\phi|\tilde{t}_R|^2 - \frac{\lambda_L}{2}\phi^2|\tilde{t}_L|^2 - \frac{\lambda_R}{2}\phi^2|\tilde{t}_R|^2 + \cdots , \tag{3.16}$$

where h, t, and $\tilde{t}$ denote the Higgs boson, top quark, and top superpartner (stop), respectively. Focusing on the top quark again, there are new contributions from the stop:

$$\Delta m_h^2 = \Delta m_h^2|_{(a)} + \Delta m_h^2|_{(b)} + \Delta m_h^2|_{(c)} \tag{3.17}$$

$\Delta m_h^2|_{(a)}$ is a contribution from the top quark given by

$$\Delta m_h^2|_{(a)} = -\frac{3y_t^2}{8\pi^2}\left(\Lambda^2 - 3m_t^2\ln\left(\frac{\Lambda^2}{m_t^2}\right) + 2m_t^2\right). \tag{3.18}$$

$\Delta m_h^2|_{(b)}$ denotes the contribution from the four point coupling with the stop. Noting that stop is a boson, it is calculated as

$$\begin{aligned}\Delta m_h^2|_{(b)} &= 3i\sum_{i=L,R}\left(-i\frac{\lambda_i}{2}\right)\int\frac{d^4p}{(2\pi)^4}\text{tr}\left[\frac{i}{p^2-m_{\tilde{t}}^2}\right]\\ &= \sum_{i=L,R}\frac{3\lambda_i}{16\pi^2}\left(\Lambda^2 - m_{\tilde{t}_i}^2\ln\left(\frac{\Lambda^2}{m_{\tilde{t}_i}^2}\right)\right).\end{aligned} \tag{3.19}$$

Similarly, $\Delta m_h^2|_{(c)}$ denotes the contribution from the three point coupling with the stop. It also calculated as

$$\begin{aligned}\Delta m_h^2|_{(b)} &= 3i\sum_{i=L,R}(-if_i)^2\int\frac{d^4p}{(2\pi)^4}\text{tr}\left[\frac{i}{p^2-m_{\tilde{t}}^2}\frac{i}{(p+k)^2-m_{\tilde{t}}^2}\right]\\ &= \sum_{i=L,R}\frac{3f_i^2}{16\pi^2}\left(-\ln\left(\frac{\Lambda^2}{m_{\tilde{t}_i}^2}\right)+1\right).\end{aligned} \tag{3.20}$$

Since supersymmetry generally leads to the relationship between y_t and λ_i,

$$\lambda_L = \lambda_R = y_t^2, \tag{3.21}$$

we can see that the divergence proportional to the second-order of the cut-off cancels out. If supersymmetry is not broken at any scale,

$$\begin{aligned} m_{\tilde{t}_L} &= m_{\tilde{t}_R} = m_t, \\ f_L &= f_R = \sqrt{2} y_t m_t, \end{aligned} \tag{3.22}$$

even further log divergence can be elegantly canceled out. The possibility of canceling out the whole cutoff scale is a unique property of SUSY and one of the reasons why SUSY has been favored. It is generally known that the quantum divergence of a particle with some symmetry can be suppressed to a log divergence by that symmetry. In GHU, the Higgs can be regarded as part of a higher-dimensional gauge field and thus has a higher dimensional gauge symmetry. Therefore, although GHU is a high-dimensional theory, the Higgs mass is believed to be protected by the high-dimensional gauge symmetry. Actually, the finiteness of the Higgs potential has been confirmed up to the two-loop level in several GHU models. The Higgs potential at the higher loop level has also been investigated. Finally, in the composite model, the Higgs is not regarded as an elementary particle, but as a particle with an internal structure. Therefore, the symmetry (e.g., chiral symmetry) of the internal particles that make up the Higgs protects the Higgs mass.

Chapter 4

Overview of the extra dimensions

We usually perceive only three-dimensional space (length, width, and height) and one-dimensional time. The theory of relativity treats them as equals and describes them as a unified four-dimensional space-time. However, the possibility of other extra dimensions has not been eliminated as long as they satisfy the experiment. In other words, as long as it is satisfied, there are many variations of extra dimensions: the number of dimensions, structure of the space. For example, the Kaluza-Klein theory attempted to unify gravity and electromagnetism by expanding spacetime to five dimensions in the 1920s after the publication of general relativity [30]. It also introduced the compactification of space, which is still frequently used today [31]. In the String Theory [32], six extra dimensions are introduced for consistency. The Randall-Sundrum model [33] introduces a "warped" extra dimension defined by the metric

$$ds^2 = \mathrm{e}^{-2kx_5}\eta^{\mu\nu}x_\mu x_\nu + dx_5^2 \tag{4.1}$$

where x_5 is an extra-dimensional coordinate. In the following, we first see how the signs of extra dimensions would appear if there were one. Then we discuss the compactification of space in a flat extra dimension and its benefits. Finally, we introduce Orbifolding of space to create a chiral theory as a 4D effective theory.

4.1 Power laws in the higher dimensions

If the elementary particles can move in the extra-dimensional direction, then the effect of the extra dimensions will be reflected in the power-law and so on. For example, we consider Gauss's law of electromagnetic field: In classical electromagnetism, the electric field emitted by a point charge Q_1 is given by

$$\int_S \boldsymbol{E}\cdot\boldsymbol{n}\,dS = \frac{Q_1}{\epsilon_0} \quad \therefore\ \boldsymbol{E}(\boldsymbol{r}) = \frac{Q_1}{4\pi\epsilon_0}\frac{1}{r^2}\,\boldsymbol{e}_r. \tag{4.2}$$

It indicates that the electric field originating from the point charge is inversely proportional to the surface area of the 3D sphere $4\pi r^2$ because it is isotropically emitted into 3D space. Similarly, the electrostatic potential created from this electric field is given by

$$V(r) = \int \boldsymbol{E}(\boldsymbol{r})\cdot d\boldsymbol{r} = \frac{1}{4\pi\epsilon_0 r}. \tag{4.3}$$

The Coulomb force acting between the point charges can be expressed as $\boldsymbol{F}(\boldsymbol{r}) = Q_2\boldsymbol{E}(\boldsymbol{r})$ using the above electric field $\boldsymbol{E}$, which is still inversely proportional to the square of the distance r. Here, introducing the extra space dimensions of d-dimension, the Coulomb force is modified as

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{Q_1Q_2}{\epsilon_0}\frac{1}{r^{2+d}}\left(\int d\Omega_{3+d}\right), \tag{4.4}$$

since the magnitude of the electric field is inversely proportional to the surface area of the $3+d$-dimensional sphere. As you can see, any extra spatial dimensions that would be equal to our three-spacial dimensions visibly change the power law and must be eliminated. Then, what kind of extra-space dimensions are left as possibilities? In the following, as an example, we introduce the concept of compactification, which has been adopted in a lot of the extra-dimensional models.

4.2 Compactification of the space

The extra spatial dimension, which seems to expand infinitely just like three-dimensional space, was not a candidate. Then, what about extra dimensions that can move only slightly in its direction? Compactification is one of the methods to make it possible. In this idea, we think of the d-dimensional extra space as a small circle in its direction. In the following, we consider the five-dimensional space-time where the coordinates are denoted by $x_M = (x_\mu, x_5)$ $(M = 0,1,2,3,5)$. The metric is given by $\eta_{MN} = (+1,-1,-1,-1,-1)$ from

$$ds^2 = \eta_{MN}dx^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu - dx^5 dx^5. \tag{4.5}$$

We identify x^5 with $x^5 + 2\pi R$ for the fifth dimension coordinate. Then we can represent the fifth dimension as a circumference of a circle S_1 with radius R. Considering the Lagrangian density, this identification can be expressed as invariance to the Lagrangian as

$$\mathcal{L}(x_\mu, x_5) = \mathcal{L}(x_\mu, x_5 + 2\pi R). \tag{4.6}$$

In the flat space-time, the same invariance is required for each field in the Lagrangian

$$\Phi(x_\mu, x_5) = \mathrm{e}^{i\theta}\Phi(x_\mu, x_5 + 2\pi R), \tag{4.7}$$

where, θ is the free phase parameter, and this expression implies that the fields are periodic for the fifth dimension. Therefore, the field Φ propagating in this 5D space-time can be expanded in the Fourier series as

$$\Phi(x_\mu, x_5) = \frac{1}{\sqrt{2\pi R}}\phi^0(x_\mu) + \frac{1}{\sqrt{\pi R}}\sum_{n=1}^{\infty}\left(\phi^{+n}(x_\mu)\cos\frac{nx_5}{R} + \phi^{-n}(x_\mu)\sin\frac{nx_5}{R}\right). \tag{4.8}$$

After expansion, each mode satisfies the normalization and orthogonal conditions

$$\int_0^{2\pi R} dx^5\ \eta_i\eta_j f^i(x^5) f^j(x^5) = \delta^{ij}, \tag{4.9}$$

with

$$f^n(x_5) \equiv \frac{1}{\sqrt{\pi R}} \begin{cases} \cos\frac{|n|x_5}{R} & (n>0) \\ 1 & (n=0) \\ \sin\frac{|n|x_5}{R} & (n<0) \end{cases} \quad , \quad \eta_n = \begin{cases} 1 & (n>0) \\ \frac{1}{\sqrt{2}} & (n=0) \\ 1 & (n<0) \end{cases} \quad . \tag{4.10}$$

Now we consider the theory of free scalar fields in five dimensions. The action for the scalar Φ with 5D mass M is given by

$$\mathcal{S} = \int d^4x \int_0^{2\pi R} dx^5 \mathcal{L}_{5D}, \quad \mathcal{L}_{5D} = \frac{1}{2}\left(\partial_\mu \Phi(x_M)\right)^2 - \frac{1}{2}M^2\left(\Phi(x_M)\right)^2. \tag{4.11}$$

Integrating into the fifth dimension, we can see how this theory looks from our four-dimensional space-time

$$\mathcal{S}_{4D} = \sum_{i=-\infty}^{\infty} \int d^4x \left[\frac{1}{2}\left(\partial_\mu \phi^i(x_\mu)\right)^2 - \frac{1}{2}m_i^2\left(\phi^i(x_\mu)\right)^2\right]. \tag{4.12}$$

We can see that this action include the infinite number of scalar fields ϕ^i with the mass $m_i = M^2 + |i|^2/R^2$. In this case, the mode ϕ^0 with constant configuration in the fifth dimension is called a zero mode, and the oscillating modes $\phi^{\pm n}$ are called the Kaluza-Klein (KK) mode. Note that the oscillation energy in the extra-dimensional direction is observed as mass from a 4D point of view, and its magnitude is inversely proportional to the radius R of the extra dimension. The above discussion can be applied to other fields.

Now that we have formulated the 5D theory from the 4D point of view, we will see how the power law is modified in this case. As an analogy to the previous electromagnetic interaction, let us assume a $U(1)$ interaction in 5 dimensions where the coupling constant is g_5. If we look at this interaction from the fourth dimension, the $U(1)$ gauge field, which was originally one in the fifth dimension, appears as an infinite number of gauge fields with mass m_n, and these will have the same coupling constant g_4 in the fourth dimension. In such a situation, the electrostatic potential can be calculated by adding up the Green's functions representing the propagation of each mode as

$$V(r) = g_4^2 G(r, m_0) + 2\sum_{n=1}^{\infty} g_4^2 G(r, m_n) = \frac{1}{4\pi R}\left[1 + \frac{2\mathrm{e}^{-r/R}}{1-\mathrm{e}^{-r/R}}\right]. \tag{4.13}$$

The first and second terms represent the contribution from the zero mode and KK modes, respectively. When the radius R of the extra dimension is sufficiently small, the second term can be neglected. Therefore, we see that the four-dimensional power law (4.3) is successfully reproduced as

$$V(r) \sim \frac{g_4^2}{4\pi r}\ , \quad \text{for } R \ll r. \tag{4.14}$$

On the other hand, if R is large enough, the potential is inversely proportional to r^2:

$$\begin{aligned} V(r) &\sim \frac{g_4^2}{4\pi r}\left[1 + \frac{2R}{r}\right] = \frac{g_4^2}{4\pi^2 r^2}(2\pi R)\left[\frac{r}{2R} + 1\right] \\ &\sim 2\pi R \times \frac{g_4^2}{4\pi^2 r^2} \quad \text{for } R \gg r \end{aligned} \tag{4.15}$$

In other words, we can see that the power-law changes around the radius R.

Finally, we will finish by introducing some significant features of the compactification. As a simple example, we consider the action of the $U(1)$ gauge field $A_M(x_\mu, x_5)$ in five dimensions given by

$$\mathcal{S} = \int d^4x \int dx^5 \mathcal{L}_{5D} \tag{4.16}$$

$$\mathcal{L}_{5D} = -\frac{1}{4}F_{MN}F^{MN} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{\mu 5}F^{\mu 5} \tag{4.17}$$

where we use the field strength defined as $F_{MN} = \partial_M A_N - \partial_N A_M$. We will discuss what happens when we look at this theory from the 4D perspective. As in the case of the scalar field, if we expand the gauge fields and integrate them in the fifth dimension, we obtain

$$\mathcal{S} = \int d^4x \ \mathcal{L}^0_{4D} + \sum_{n=1}^{\infty} \int d^4x \ \mathcal{L}^{\pm n}_{4D}. \tag{4.18}$$

where the Lagrangian densities of the zero-mode and KK modes are given by

$$\mathcal{L}^0_{4D} = -\frac{1}{4}\left(F^0_{\mu\nu}\right)^2 - \frac{1}{2}\left(\partial_\mu A^0_5\right)^2, \tag{4.19}$$

$$\mathcal{L}^{\pm n}_{4D} = -\frac{1}{4}\left(F^{\pm n}_{\mu\nu}\right) - \frac{m_n^2}{2}\left(A^{\pm n}_\mu - \frac{1}{m_n}\partial_\mu A^{\mp n}_5\right)^2. \tag{4.20}$$

The zero-mode for the extra component of the 5D gauge field appears as a scalar in the zero-mode Lagrangian. On the other hand, the KK modes for the extra component of the 5D gauge field are absorbed by the corresponding KK modes for the 4D gauge field as these longitudinal components. Note that when we consider the 5D gauge field from the four-dimensional perspective, there appear not only the gauge fields but also the scalar field.

4.3 Orbifolding of the space

Compactification is not enough to construct the extra dimensions for the chiral fermions. Then, we discuss it considering the 5D Dirac equation

$$\left(i\Gamma^M\partial_M - W\right)\Psi = \left(i\gamma^\mu\partial_\mu - \gamma^5\partial_5 - W\right)\Psi = 0, \tag{4.21}$$

where we use the 5D Gamma matrix $\Gamma^M = (\gamma^\mu, i\gamma^5)$ satisfying the anti-commutation relation $\{\Gamma^M, \Gamma^N\} = \eta^{MN}$. Note that γ^5 is appears as equal to the others gamma matrixes in the Dirac equation. In other words, we can not consider a chiral theory as long as keeping the proper 5D Lorentz invariance. It implies that under the 5D Lorentz transformation, the right-hand and left-hand fermions are mixed. For the compactified extra dimension, it still remains.

Then, we use the method called orbifolding to get the chiral theory. By using the γ^5 matrix, 5D fermions are transformed under the Z_2 symmetry in the x_5 direction as

$$\Psi(x_\mu, x_5) \to \Psi^P(x_\mu, x_5) \equiv \gamma^5\Psi(x_\mu, -x_5), \tag{4.22}$$

because $\Psi(x_\mu, x_5)$ and $\Psi(x_\mu, -x_5)$ are obey the same 5D Dirac equation. Using the anti-commutation relation $\{\Gamma^5, \Gamma^\mu\} = 0$, we find

$$\left[i\gamma^\mu \frac{\partial}{\partial x_\mu} - \gamma^5 \frac{\partial}{\partial x_5}\right] \Psi^P(x_\mu, x_5) = -\gamma^5 \left[i\gamma^\mu \frac{\partial}{\partial x_\mu} - \gamma^5 \frac{\partial}{\partial(-x_5)}\right] \Psi(x_\mu, -x_5). \tag{4.23}$$

Since the mass term $m\Psi$ is not invariant to this transformation, this discussion is only valid for the massless 5D fermions. Eq. (4.22) is rewritten by

$$\begin{aligned} \Psi_R(x^\mu, x^5) &\to \Psi_R^P(x^\mu, x^5) = +\Psi_R(x^\mu, -x^5), \\ \Psi_L(x^\mu, x^5) &\to \Psi_L^P(x^\mu, x^5) = -\Psi_L(x^\mu, -x^5), \end{aligned} \tag{4.24}$$

where the 4D chiral eigenstates are defined by

$$\begin{aligned} \Psi_R &\equiv \frac{1}{\sqrt{2\pi R}}\psi_R^0(x_\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left(\psi_R^{+n}(x_\mu) \cos\frac{nx_5}{R} + \psi_R^{-n}(x_\mu) \sin\frac{nx_5}{R}\right), \\ \Psi_L &\equiv \frac{1}{\sqrt{2\pi R}}\psi_L^0(x_\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left(\psi_L^{+n}(x_\mu) \cos\frac{nx_5}{R} + \psi_L^{-n}(x_\mu) \sin\frac{nx_5}{R}\right). \end{aligned} \tag{4.25}$$

Focusing on the Z_2 invariant states, Ψ_R and Ψ_L are the even and odd functions for the x_5 direction, respectively:

$$\Psi_R(x_\mu, x_5) = +\Psi_R(x_\mu, -x_5), \quad \Psi_L(x_\mu, x_5) = -\Psi_L(x_\mu, -x_5) \tag{4.26}$$

Therefore, if we impose the Z_2 symmetry discussed above on this theory, only the right-hand fermion remains for the zero modes: ψ_R^0. It allows us to create a chiral asymmetric theory as a low energy effective theory in four dimensions. In the 5D case, it is possible to create a chiral asymmetric 4D effective theory by identifying x_5 with $2\pi R - x_5$, which is called the orbifolding. It also changes the compactified space of $[0, 2\pi R]$ without edges into a space of $[0, \pi R]$ with two edges (boundaries). The identification of this space can be read as a boundary condition on the field as

$$\begin{aligned} \Phi(x_\mu, x_5) &= P_0\Phi(x_\mu, x_5) \equiv \eta_{P_0}\Phi(x_\mu, -x_5) \\ \Phi(x_\mu, \pi R + x_5) &= P_1\Phi(x_\mu, \pi R + x_5) \equiv \eta_{P_1}\Phi(x_\mu, \pi R - x_5) \end{aligned} \tag{4.27}$$

where P_0, P_1 are parity transformation operators at two boundaries and η_{P_0}, η_{P_1} are the eigenvalues of them, which take ± 1. In addition, these boundary conditions are related to that of the compactification as

$$P_0 P_1 = U, \tag{4.28}$$

where the operator U is defined by

$$\Phi(x_\mu, x_5) = U\Phi(x_\mu, x_5) = \eta_U \Phi(x_\mu, 2\pi R + x_5). \tag{4.29}$$

Chapter 5

Gauge-Higgs unification

Gauge-Higgs Unification is one of the TeV scale paradigms beyond the SM that solve the hierarchy problem. In GHU, the Higgs is embedded into the extra components of the gauge multiplets. Namely, the 4D Higgs and gauge field are unified into the higher dimensional gauge field. The Higgs potential in GHU is flat at tree level and induced by the quantum correction due to the higher dimensional gauge symmetry. The realistic GHU models have been constructed in a flat or warped S^1/Z_2 extra dimension [34–38]. In these models, the vacuum expectation value of the Higgs field corresponds to the Aharonov-Bohm phase for the S^1 extra space, which has a non-zero value through the quantum corrections, causing the gauge symmetry breaking [13, 14]. This mechanism is called the Hosotani mechanism. The Higgs and top quark masses have been a bottleneck in constructing the realistic GHU models, but various models have been proposed to overcome this problem. Now, we are in the verification phase of these models. We focus on the structure difference of the Higgs potential from the SM and test the models by analyzing the triple Higgs boson coupling. The triple Higgs boson coupling has been analyzed by the previous studies in the warped models [39, 40] and the flat $SU(3)$ model with a large representation [41, 42]. Therefore, we analyze it on the $SU(3)$ model with 5D Lorentz symmetry relaxed [34]. We then discuss the structure of the Higgs potential that is common to GHU models with a flat extra dimension.

5.1 Toy $SU(3)$ Model

We first consider the toy $SU(3)$ model, which consists of the flat $M^4 \times S^1/Z_2$ spacetime and $SU(3)_c \times SU(3)_w$ gauge symmetry. The extended electroweak gauge group $SU(3)_w$ is the minimal group with the $SU(2) \times U(1)$ group as a subgroup. The gauge field $A_M = A_M^a T^a$ of the $SU(3)_w$ gauge group has boundary conditions on the S^1/Z_2 orbifold,

$$\begin{aligned} S^1 :& A_M(x_5 + 2\pi R) = \eta A_M(x_5) \\ Z_2 :& A_\mu(-x^5) = P^\dagger A_\mu(x^5) P, \quad A_5(-x^5) = -P^\dagger A_5(x^5) P \end{aligned} \tag{5.1}$$

where η stands for the periodicity of the field, periodic (1) and antiperiodic (-1), P is the eigenvalue matrix of the Z_2 parity. By choosing the boundary conditions as

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \quad \eta = 1, \tag{5.2}$$

the $SU(3_c) \times SU(3)_w$ gauge group breaks down to the $SU(3)_c \times SU(2)_L \times U(1)_w$ gauge group. Since the zero-modes are the even functions for the x_5 direction, we get

$$A_\mu^0 = \frac{1}{2}\begin{pmatrix} A_\mu^{3,0} + \frac{1}{\sqrt{3}}A_\mu^{8,0} & A_\mu^{1,0} - iA_\mu^{2,0} & 0 \\ A_\mu^{1,0} + iA_\mu^{2,0} & -A_\mu^{3,0} + \frac{1}{\sqrt{3}}A_\mu^{8,0} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}}A_\mu^{8,0} \end{pmatrix}, \tag{5.3}$$

$$A_5^0 = \frac{1}{2}\begin{pmatrix} 0 & 0 & A_5^{4,0} - iA_5^{5,0} \\ 0 & 0 & A_5^{6,0} - iA_5^{7,0} \\ A_5^{4,0} + iA_5^{5,0} & A_5^{6,0} + iA_5^{7,0} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \Phi \\ \Phi^\dagger & 0 \end{pmatrix}. \tag{5.4}$$

A_μ^0 is a gauge field of the $SU(2)_L \times U(1)_w$ gauge group, while A_5^0 is consist of the broken generators and Φ is regarded as a $SU(2)_w$ doublet scalar. Thus we can identify Φ as the SM Higgs because it can obtain the non-zero VEV from the Hosotani mechanism. The KK modes of A_5 are absorbed into the KK modes of A_μ as seen in Sec. 4.2. For the following calculations, we define the VEV of A_5 as

$$\langle A_5^a \rangle = \frac{2\alpha}{g_5 R}\delta^{a7}. \tag{5.5}$$

So far, we have seen how the gauge field and the Higgs field can be unified. Now let us turn our attention to the matter field. It is tempting to construct a theory based only on the bulk field, which, like the gauge field, propagates freely in the extra-dimensional direction. However, if we consider only the bulk field, the zero modes of the fermion are also naively determined by the gauge coupling. Then these masses are degenerate to the EW scale, which is inconsistent with the observation. To solve this problem, we consider two types of fermions in this model: massive bulk fermions and localized massless brane fermions. The brane fermions are localized at either end of the S^1/Z_2 orbifold. By mixing the bulk and brane fermion and through the heavy bulk fermion, we can get the brane fermion to have a mass lighter than the EW scale, like a seesaw mechanism. For simplicity, we will ignore flavor structure and consider the quark Lagrangian for one generation. This model contains the five types of matter fields: bulk fermion pairs $\{\Psi_j, \tilde{\Psi}_j\}$, left-handed quark doublet $Q_L = (t_L, b_L)^{\mathrm{T}}$, right-handed up-type quark u_R, right-handed down-type quark d_R. These representations and quantum numbers are summarized in Table 5.1. The bulk fermion pair is required to have a mass sufficiently larger than the EW scale to form a mixing term with a 5D mass parameter M_j. We introduce e_1 and e_2 as parameters to represent the degree of mixing between bulk and brane fermions with a mass dimension. From the remarks above, the 5D matter Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{matter}} = \sum_{j=u,d} &\left\{ \bar{\Psi}_j \left(i\not{D}_4 - D_5\gamma^5 \right) \Psi_j + \bar{\tilde{\Psi}}_j \left(i\not{D}_4 - D_5\gamma^5 \right) \tilde{\Psi}_j + \left(\bar{\Psi}_j M_j \tilde{\Psi}_j + \text{h.c.} \right) \right\} \\ &+ \delta(x_5 - l_1) \left\{ \bar{Q}_L i\not{D}_4 Q_L + \left(e_1^d \bar{Q}_L \psi_d + e_1^u \bar{Q}_R^c \psi_u + \text{h.c.} \right) \right\} \\ &+ \delta(x_5 - l_2) \left\{ \bar{u}_R i\not{D}_4 u_R + \bar{d}_R i\not{D}_4 d_R + \left(e_2^d \bar{d}_R \chi_d + e_2^u \bar{u}_L^c \chi_u + \text{h.c.} \right) \right\},\end{aligned} \tag{5.6}$$

where $\not{D}_4 \equiv \gamma^\mu D_\mu$ with D_μ and D_5 being covariant derivatives, and $\psi_{u,d}$ and $\chi_{u,d}$ are the $SU(2)_L$ doublet and singlet components of the bulk fermions $\Psi_{u,d}$. In place of M_j and e_i^a, we introduce dimensionless parameters $\lambda^j = \pi R M_j$ and $\epsilon_i^a = \sqrt{\pi R/2}\, e_i^a$ ($a = u, d$; $i = 1, 2$), respectively. There are totally 6 model parameters: $\lambda^u, \lambda^d, \epsilon_1^u, \epsilon_2^u, \epsilon_1^d$ and ϵ_2^d.

Table 5.1: Matter contents and the quantum numbers. The color factor is denoted by C_F.

Fields	$SU(3)_c \times SU(3)_w$	periodicity (η)	$SU(3)_c \times SU(2)_L \times U(1)_Y$	C_F
$(\Psi_t, \tilde{\Psi}_t)$	$(\mathbf{3}, \bar{\mathbf{6}})$	periodic (0)	$(\mathbf{3},\mathbf{1})_{2/3} + (\mathbf{3},\mathbf{2})_{1/6} + (\mathbf{3},\mathbf{3})_{-1/3}$	3
$(\Psi_b, \tilde{\Psi}_b)$	$(\mathbf{3}, \mathbf{3})$	periodic (0)	$(\mathbf{3},\mathbf{1})_{-1/3} + (\mathbf{3},\mathbf{2})_{1/6}$	3
Q_L			$(\mathbf{3},\mathbf{2})_{1/6}$	3
t_R			$(\mathbf{3},\mathbf{1})_{2/3}$	3
b_R			$(\mathbf{3},\mathbf{1})_{-1/3}$	3

5.1.1 Mass spectrum of a bulk gauge field

We will now discuss the mass spectrums of the bulk and brane fields. As a simple example, we first consider the bulk gauge fields. The action for the $SU(3)_w$ gauge field, which contains the SM Higgs-like field, is given by

$$\begin{aligned}
\mathcal{S}_{\text{gauge}} &= \int d^4x \int_0^{2\pi R} dx_5 \; \mathcal{L}_{5D}, \\
\mathcal{L}_{5D} &= -\frac{1}{2}\mathrm{Tr}\left(F_{MN}F^{MN}\right) = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \mathrm{Tr}\left(F_{\mu 5}F^{\mu 5}\right),
\end{aligned} \tag{5.7}$$

where the strength of the 5D gauge field is defined by $F_{MN} = \partial_M A_N - \partial_N A_M + ig_5[A_M, A_N]$ with its coupling constant g_5. By integrating out the fifth dimension, effective 4D Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{4D} = &-\frac{1}{2}\mathrm{Tr}\left(\partial_\mu A_\nu^0 - \partial_\nu A_\mu^0\right)^2 - \mathrm{Tr}\left(\partial_\mu A_5^0\right)^2 \\
&- \sum_{n=1}^{\infty}\left(\frac{1}{2}\mathrm{Tr}\left(\partial_\mu A_\nu^{\pm n} - \partial_\nu A_\mu^{\pm n}\right)^2 + m_n^2 \mathrm{Tr}\left(A_\mu^{\pm n} \pm \frac{1}{m_n}\partial_\mu A_5^{\mp n}\right)^2\right) \\
&- \sum_{l,m,n=-\infty}^{\infty} ig_4 \lambda_{lmn} \mathrm{Tr}\left(\frac{1}{2}\left\{\partial_\mu A_\nu^l - \partial_\nu A_\mu^l, \left[A_\mu^m, A_\nu^n\right]\right\} + \left\{\partial_\mu A_5^l - \frac{i}{R}A_\mu^{-l}, \left[A_\mu^m, A_5^n\right]\right\}\right) \\
&- \sum_{k,l,m,n=-\infty}^{\infty} (ig_4)^2 \lambda_{klmn} \mathrm{Tr}\left(\frac{1}{2}\left[A_\mu^k, A_\nu^l\right]\left[A_\mu^m, A_\nu^n\right] + \left[A_\mu^k, A_5^l\right]\left[A_\mu^m, A_5^n\right]\right)
\end{aligned} \tag{5.8}$$

where we define the 4D gauge coupling as $g_4 = g_5/\sqrt{2\pi R}$. The first and second lines denote the kinetic terms of the zero-mode and KK modes gauge fields, respectively. The third and fourth lines also denote the three-point and four-point interaction terms between all modes of gauge field. The three and four point coupling constants are written by

$$\begin{aligned}
(ig_4)\lambda_{lmn} &= \int_0^{2\pi R} dx_5 \,(ig_5)\, \eta_l \eta_m \eta_n f^l f^m f^n, \\
(ig_4)^2 \lambda_{klmn} &= \int_0^{2\pi R} dx_5 \,(ig_5)^2\, \eta_k \eta_l \eta_m \eta_n f^k f^l f^m f^n,
\end{aligned} \tag{5.9}$$

with

$$\eta_n = \begin{cases} \frac{1}{\sqrt{2}} & (n=0) \\ 1 & (n \neq 0) \end{cases}, \tag{5.10}$$

$$f^0 = \frac{1}{\sqrt{2\pi R}}, \ f^{+n}(x^5) = \frac{1}{\sqrt{\pi R}} \cos\frac{nx^5}{R}, \ f^{-n}(x^5) = \frac{1}{\sqrt{\pi R}} \sin\frac{nx^5}{R}. \tag{5.11}$$

We start from the mass of the zero-mode gauge field. The part of the 4D Lagrangian in Eq. (5.8) that is relevant to the zero-mode mass is

$$\begin{aligned} \mathcal{L}^0_{4D} &= -\frac{1}{2}\mathrm{Tr}\left(F^0_{\mu\nu}\right)^2 - \mathrm{Tr}\left(F^0_{\mu 5}\right)^2 \\ &= -\frac{1}{2}\mathrm{Tr}\left(F^0_{\mu\nu}\right)^2 - 2\mathrm{Tr}|D_\mu H|^2, \end{aligned} \tag{5.12}$$

where each strength of the zero-mode gauge field are given by

$$\begin{aligned} F^0_{\mu\nu} =& \partial_\mu A^0_\nu - \partial_\nu A^0_\mu + ig_4[A^0_\mu, A^0_\nu], \\ F^0_{\mu 5} =& \partial_\mu A^0_5 - \partial_5 A^0_\mu + ig_4[A^0_\mu, A^0_5] = \partial_\mu A^0_5 + ig_4[A^0_\mu, A^0_5]. \end{aligned} \tag{5.13}$$

The first and second terms of this Lagrangian represent the kinetic term of $SU(2)_L \times U(1)_w$ gauge field and kinetic and interaction terms of the scalar field belonging to the $SU(2)_L$ doublet, respectively. This scalar field corresponds to the SM Higgs field, and its VEV causes spontaneous symmetry breaking. It can be clarified by rewriting the second term of this Lagrangian as follows:

$$\begin{aligned} 2\mathrm{Tr}\,|D_\mu H|^2 &= 2\mathrm{Tr}|\partial_\mu H + ig_4[A^0_\mu, H]|^2 \\ &= \frac{1}{2}\left|\left(\partial_\mu + ig_4 {W_\mu}^{a,0}\frac{\tau^a}{2} + ig_4\frac{\sqrt{3}}{2}B^0_\mu\right)(v + h(x^\mu))\right|^2 \\ &= \frac{1}{4}g_4^2 v^2 W^+_\mu W^{-\mu} + \frac{1}{2}g_4^2 v^2 Z_\mu Z^\mu + 0\, A_\mu A^\mu + \cdots \end{aligned} \tag{5.14}$$

We identify $A^{a,0}_\mu \equiv W^{a,0}_\mu$ $(i = 1, 2, 3)$ and $A^{8,0}_\mu \equiv B^0_\mu$. The mass eigenstates A_μ and Z_μ are given by

$$A_\mu = \frac{\sqrt{3}A^{3,0}_\mu + A^{8,0}_\mu}{2}, \tag{5.15}$$

$$Z_\mu = \frac{A^{3,0}_\mu - \sqrt{3}A^{8,0}_\mu}{2}. \tag{5.16}$$

In this case, the Weinberg angle is predicted as $\sin\theta_W = \sqrt{3}/2$, or $\theta_W = \pi/3$. The masses corresponding to W and Z bosons are given by

$$m^0_W = \frac{1}{2}g_4 v = \frac{\alpha}{R}, \tag{5.17}$$

$$m^0_Z = g_4 v = \frac{2\alpha}{R}. \tag{5.18}$$

Then we consider the masses of the KK modes. The corresponding Lagrangian part is extracted as

$$\begin{aligned}\mathcal{L}_{4D}^{n} = &-\frac{1}{2}\mathrm{Tr}\left(\partial_\mu A_\nu^{+n} - \partial_\nu A_\mu^{+n}\right)^2 - m_n^2 \mathrm{Tr}\left(A_\mu^{+n} + \frac{1}{m_n^2}\partial_\mu A_5^{-n}\right)^2 \\ &-\frac{1}{2}\mathrm{Tr}\left(\partial_\mu A_\nu^{-n} - \partial_\nu A_\mu^{-n}\right)^2 - m_n^2 \mathrm{Tr}\left(A_\mu^{-n} - \frac{1}{m_n^2}\partial_\mu A_5^{+n}\right)^2 \\ &- i g_4 \mathrm{Tr}\left(\left\{\frac{n}{R}A_\mu^{+n}, \left[A_\mu^{-n}, \langle A_5^0\rangle\right]\right\} - \left\{\frac{n}{R}A_\mu^{-n}, \left[A_\mu^{+n}, \langle A_5^0\rangle\right]\right\}\right) \\ &- (ig_4)^2 \mathrm{Tr}\left(\left[A_\mu^{+n}, \langle A_5^0\rangle\right]\left[A_\mu^{+n}, \langle A_5^0\rangle\right] + \left[A_\mu^{-n}, \langle A_5^0\rangle\right]\left[A_\mu^{-n}, \langle A_5^0\rangle\right]\right),\end{aligned} \tag{5.19}$$

where each KK mode of gauge field $A_\mu^{(\pm n)}(n = 1, 2, \cdots)$ are represented by boundary conditions as following form:

$$A_\mu^{+n} = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{3}}A_\mu^{8,+n} + A_\mu^{3,+n} & A_\mu^{1,+n} + iA_\mu^{2,+n} & 0 \\ A_\mu^{1,+n} - iA_\mu^{2,+n} & \frac{1}{\sqrt{3}}A_\mu^{8,+n} - A_\mu^{3,+n} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}}A_\mu^{8,+n} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A_\mu^{+n} & \sqrt{2}W_\mu^{+,+n} & 0 \\ \sqrt{2}W_\mu^{-,+n} & -2Z_\mu^{1,+n} & 0 \\ 0 & 0 & -2Z_\mu^{2,+n} \end{pmatrix}$$

$$A_\mu^{-n} = \frac{1}{2}\begin{pmatrix} 0 & 0 & A_\mu^{4,-n} + iA_\mu^{5,-n} \\ 0 & 0 & A_\mu^{6,-n} + iA_\mu^{7,-n} \\ A_\mu^{4,-n} - iA_\mu^{5,-n} & A_\mu^{6,-n} - iA_\mu^{7,-n} & 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & W_\mu^{\prime +,-n} \\ 0 & 0 & Z_\mu^{\prime +,-n} \\ W_\mu^{\prime -,-n} & Z_\mu^{\prime -,-n} & 0 \end{pmatrix}$$

By defining $Z \equiv Z^1 - Z^2 = (A^3 - \sqrt{3}A^8)/2$ and $W^\pm = (A^1 \pm A^2)/\sqrt{2}$, we can rewrite the third and fourth lines of Eq. (5.19) as

$$\text{3rd line} = -\frac{n}{R}\frac{\alpha}{R}\left\{2i\left(W_\mu^{-,+n}W_\mu^{\prime +,-n} - W_\mu^{+,+n}W_\mu^{\prime -,-n}\right) + 4A_\mu^{7,-n}\cdot Z_\mu^{+n}\right\} \tag{5.20}$$

$$\text{4th line} = -\frac{1}{2}\left(\frac{\alpha}{R}\right)^2\left\{2\left(W_\mu^{-,+n}W_\mu^{+,+n} + W_\mu^{\prime +,-n}W_\mu^{\prime -,-n}\right) + 4\left(A_\mu^{7,-n}\right)^2 + 4\left(Z_\mu^{+n}\right)^2\right\}. \tag{5.21}$$

where we use following relations:

$$\left[A_\mu^{+n}, \langle A_5^0\rangle\right] = \frac{v}{4}\begin{pmatrix} 0 & 0 & \sqrt{2}W_\mu^{+,+n} \\ 0 & 0 & -2Z_\mu^{+n} \\ -\sqrt{2}W_\mu^{-,+n} & 2Z_\mu^{+n} & 0 \end{pmatrix} \tag{5.22}$$

$$\left[A_\mu^{-n}, \langle A_5^0\rangle\right] = \frac{\sqrt{2}v}{4}\begin{pmatrix} 0 & W_\mu^{\prime +,-n} & 0 \\ -W_\mu^{\prime -,-n} & \sqrt{2}\,i\,A_\mu^{7,-n} & 0 \\ 0 & 0 & -\sqrt{2}\,i\,A_\mu^{7,-n} \end{pmatrix} \tag{5.23}$$

Therefore, for the basis of $(W_\mu^{+,+n}, W_\mu^{\prime +,-n})$ and $(Z_\mu^{+n}, Z_\mu^{\prime +,-n})$, we can write the mass matrixes of the KK modes of W and Z bosons as

$$\frac{1}{R^2}\left(W_\mu^{-,+n}, W_\mu^{\prime -,-n}\right)\begin{pmatrix} n^2+\alpha^2 & 2in\alpha \\ -2in\alpha & n^2+\alpha^2 \end{pmatrix}\begin{pmatrix} W_\mu^{+,+n} \\ W_\mu^{\prime +,-n} \end{pmatrix} \tag{5.24}$$

$$\frac{1}{2R^2}\left(Z_\mu^{+n}, A_\mu^{7,-n}\right)\begin{pmatrix} n^2+4\alpha^2 & 4n\alpha \\ 4n\alpha & n^2+4\alpha^2 \end{pmatrix}\begin{pmatrix} Z_\mu^{+n} \\ A_\mu^{7,-n} \end{pmatrix} \tag{5.25}$$

These mass matrixes can be diagonalized by the following unitary matrixes:

$$U_W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} \quad , \quad U_Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{5.26}$$

Then, the mass eigenstates and mass spectrums of the gauge fields are defined as

$$\begin{aligned}
\hat{A}_\mu^{+n} &= \left(\frac{\sqrt{3}A_\mu^{3,+n} + A_\mu^{8,+n}}{2} \right) & m_{A^+} &= \frac{n}{R}, \\
\hat{A}_\mu^{-n} &= \sqrt{2}\,\mathrm{Re}\left[Z_\mu'^{+,-n} \right] = A_\mu^{6,-n} & m_{A^-} &= \frac{n}{R}, \\
\hat{W}_\mu^{+,+n} &= \frac{1}{\sqrt{2}} \left(W_\mu^{+,+n} + iW_\mu'^{+,-n} \right) & m_{W^+} &= \frac{n+\alpha}{R}, \\
\hat{W}_\mu^{+,-n} &= \frac{1}{\sqrt{2}} \left(W_\mu^{+,+n} - iW_\mu'^{+,-n} \right) & m_{W^-} &= \frac{n-\alpha}{R}, \\
\hat{Z}_\mu^{+n} &= \frac{1}{\sqrt{2}} \left(\frac{A_\mu^{3,+n} - \sqrt{3}A_\mu^{8,+n}}{2} + A_\mu^{7,-n} \right) & m_{Z^+} &= \frac{n+2\alpha}{R}, \\
\hat{Z}_\mu^{+n} &= \frac{1}{\sqrt{2}} \left(\frac{A_\mu^{3,+n} - \sqrt{3}A_\mu^{8,+n}}{2} - A_\mu^{7,-n} \right) & m_{Z^-} &= \frac{n-2\alpha}{R}.
\end{aligned} \tag{5.27}$$

If we consider only the SU(3) gauge field, the value of the Weinberg angle will deviate significantly from the experimental value. Two methods have been devised to correct it: introducing a gauge kinetic term in the brane or additional U(1) gauge symmetry [43, 44].

5.1.2 Mass spectrum of a bulk fermion

Next, we will see how the bulk fermion appears in the 4D effective theory. The boundary conditions imposed on the bulk fermion are given by

$$\begin{aligned}
\Psi(x_\mu, x_5) &= \hat{U}\Psi(x_\mu, x_5) = \eta_U \Psi(x_\mu, x_5 + 2\pi R) \\
\Psi(x_\mu, x_5) &= \hat{P}\Psi(x_\mu, x_5) = \eta_P \gamma_5 \Psi(x_\mu, -x_5).
\end{aligned} \tag{5.28}$$

Since bulk fermions also have periodicity for S^1, they can be Fourier expanded by trigonometric functions. From Eq. (5.28), each chiral eigenstate is restricted to an even or odd function depending on the eigenvalue η_P. For $\eta_U = 1$ and $\eta_P = +1(-1)$, we find $\Psi_R = \text{even}\,(\text{odd})$ and $\Psi_L = \text{odd}\,(\text{even})$. For the sake of convenience, we will define chiral fermions as

$$\Psi = \begin{pmatrix} \xi \\ \bar{\zeta}^T \end{pmatrix}. \tag{5.29}$$

Then, the action for a free bulk fermion is given by

$$\begin{aligned}
\mathcal{S} &= \int d^4x \int_0^{2\pi R} dx^5 \, i\bar{\Psi} \left(\Gamma^M \partial_M \right) \Psi \\
&= \int d^4x \int_0^{2\pi R} dx^5 \left\{ i \left(\zeta^T, \bar{\xi} \right) \begin{pmatrix} -i\partial_5 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & i\partial_5 \end{pmatrix} \begin{pmatrix} \xi \\ \bar{\zeta}^T \end{pmatrix} \right\},
\end{aligned} \tag{5.30}$$

where the Z_2 invariance forbid the mass term $\bar{\Psi}M\Psi$. In the case of $\eta_U = 1$ and $\eta_P = +1$, the 4D effective Lagrangian for the zero mode and KK modes are calculated as

$$\begin{aligned}
\mathcal{L}^0_{4D} &= \frac{1}{2\pi R}\int_0^{2\pi R} dx^5\, i\left(\zeta^{0^T}, 0\right)\begin{pmatrix} 0 & \sigma^\mu\partial_\mu \\ \bar{\sigma}^\mu\partial_\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \bar{\zeta}^{0^T}\end{pmatrix} \\
&= i\,\zeta^{0^T}\sigma^\mu\partial_\mu\bar{\zeta}^{0^T}, \qquad (5.31) \\
\mathcal{L}^{\pm n}_{4D} &= \sum_{n=1}^{\infty}\int_0^{2\pi R} dx^5\, i\eta_n^2\left(f^{+n}\zeta^{+n^T}, -f^{-n}\bar{\xi}^{-n}\right)\begin{pmatrix} -i\partial_5 & \sigma^\mu\partial_\mu \\ \bar{\sigma}^\mu\partial_\mu & i\partial_5 \end{pmatrix}\begin{pmatrix} -f^{-n}\xi^{-n} \\ f^{+n}\bar{\zeta}^{+n^T}\end{pmatrix} \\
&= \sum_{n=1}^{\infty}\left(i\bar{\xi}^{-n}\bar{\sigma}^\mu\partial_\mu\xi^{-n} + i\zeta^{+n^T}\sigma^\mu\partial_\mu\bar{\zeta}^{+n^T} - \frac{n}{R}\left(\zeta^{+n^T}\xi^{-n} + \bar{\xi}^{-n}\bar{\zeta}^{+n^T}\right)\right) \\
&= \sum_{n=1}^{\infty}\bar{\psi}^n\left(i\gamma^\mu\partial_\mu - \frac{n}{R}\right)\psi^n, \quad : \psi^n = \begin{pmatrix}\xi^{-n} \\ \bar{\zeta}^{+n^T}\end{pmatrix} \qquad (5.32)
\end{aligned}$$

with

$$\Psi_R = \sum_{n=0}^{\infty}\eta_n f^{+n}(x^5)\psi_R^{+n}(x^\mu), \quad \Psi_L = -\sum_{n=0}^{\infty}\eta_n f^{-n}(x^5)\psi_L^{-n}(x^\mu). \qquad (5.33)$$

We can see that the effective 4D theory includes massless left-handed fermion and both chirality fermions with mass spectrums $m_n = n/R$. While in the case of $\eta_U = 1$ and $\eta_P = -1$, the 4D effective Lagrangian for the zero mode and KK modes are also calculated as

$$\begin{aligned}
\mathcal{L}^0_{4D} &= \frac{1}{2\pi R}\int_0^{2\pi R} dx^5\, i\left(0, \bar{\xi}^0\right)\begin{pmatrix} 0 & \sigma^\mu\partial_\mu \\ \bar{\sigma}^\mu\partial_\mu & 0 \end{pmatrix}\begin{pmatrix} \xi^0 \\ 0\end{pmatrix} \\
&= i\,\xi^{0^\dagger}\sigma^\mu\partial_\mu\xi^0, \qquad (5.34) \\
\mathcal{L}^{\pm n}_{4D} &= \sum_{n=1}^{\infty}\int_0^{2\pi R} dx^5\, i\eta_n^2\left(f^{-n}\zeta^{-n^T}f^{+n}, \bar{\xi}^{+n^T}\right)\begin{pmatrix} -i\partial_5 & \sigma^\mu\partial_\mu \\ \bar{\sigma}^\mu\partial_\mu & i\partial_5 \end{pmatrix}\begin{pmatrix} f^{+n}\xi^{+n} \\ f^{-n}\bar{\zeta}^{-n^T}\end{pmatrix} \\
&= \sum_{n=1}^{\infty}\left(i\bar{\xi}^{-n}\bar{\sigma}^\mu\partial_\mu\xi^{-n} + i\bar{\zeta}^{+n^T}\sigma^\mu\partial_\mu\bar{\zeta}^{+n^T} - \frac{n}{R}\left(\zeta^{-n^T}\xi^{+n} + \bar{\xi}^{+n}\bar{\zeta}^{-n^T}\right)\right) \\
&= \sum_{n=1}^{\infty}\bar{\psi}^n\left(i\gamma^\mu\partial_\mu - \frac{n}{R}\right)\psi^n, \quad : \psi^n = \begin{pmatrix}\xi^{+n} \\ \bar{\zeta}^{-n^T}\end{pmatrix} \qquad (5.35)
\end{aligned}$$

with

$$\Psi_R = \sum_{n=0}^{\infty}\eta_n f^{-n}(x^5)\psi_R^{-n}(x^\mu), \quad \Psi_L = \sum_{n=0}^{\infty}\eta_n f^{+n}(x^5)\psi_L^{+n}(x^\mu). \qquad (5.36)$$

Note that in the both case, the KK modes have the both chirality while zero mode only has a left- or right-handed chirality. It is the desired result reflecting the effect of the Orbifolding of the space. We can expect the theory to be chiral asymmetric below the compactification scale $1/R$.

5.1.3 Mass spectrum of a bulk fermion (Fundamental)

So far, we have discussed a free bulk fermion. Now let's move on to a bulk fermion coupled to the bulk gauge field. Its action is given by

$$\mathcal{S} = \int d^4x \int_0^{2\pi R} dx^5 \bar{\Psi} \left(i\Gamma^M D_M \right) \Psi, \tag{5.37}$$

where we define the covariant derivative as $D_M = \partial_M + ig_5 A_M$. From the boundary conditions, we have four possible choices for the eigenvalues of the field. Let us first consider the case where $\eta_U = +1, \eta_P = +1$, then the case where $\eta_U = +1, \eta_P = -1$, and finally the case where $\eta_U = -1$.

We consider $\tilde{\Psi}_d$ as a bulk fermion in the fundamental **3** representation with $\eta_P = +1$. From boundary conditions Eq. (5.28), this bulk fermion can be rewritten in terms of the 4D chiralities as

$$\tilde{\Psi}_d = \begin{pmatrix} \tilde{\psi}^d_{1L} + \tilde{\psi}^d_{1R} \\ \tilde{\psi}^d_{2L} + \tilde{\psi}^d_{2R} \\ \tilde{\chi}^d_R + \tilde{\chi}^d_L \end{pmatrix} = \sum_{n=0}^{\infty} \eta_n \begin{pmatrix} f^{+n}\psi^{d,+n}_{1L} + f^{-n}\psi^{d,-n}_{1R} \\ f^{+n}\psi^{d,+n}_{2L} + f^{-n}\psi^{d,-n}_{2R} \\ f^{+n}\chi^{d,+n}_{R} - f^{-n}\chi^{d,-n}_{L} \end{pmatrix} \tag{5.38}$$

where (ψ_1, ψ_2) and χ denote the doublet and singlet of the $SU(2)_L$ gauge group. Since we are interested in the mass of the bulk fermion, we extract the part of the action in Eq. (5.37) that is related to the mass term:

$$\begin{aligned} \mathcal{S} &= \int d^4x \int_0^{2\pi R} dx^5 \bar{\tilde{\Psi}}_d \left(i\Gamma^M \partial_M - ig_5 \gamma^5 \langle A_5 \rangle \right) \tilde{\Psi}_d \\ &= \int d^4x \int_0^{2\pi R} dx^5 \left(\bar{\tilde{\Psi}}^1_d, \bar{\tilde{\Psi}}^2_d, \bar{\tilde{\Psi}}^3_d \right) \begin{pmatrix} i\Gamma^M\partial_M & 0 & 0 \\ 0 & i\Gamma^M\partial_M & -i\gamma^5\frac{\alpha}{R} \\ 0 & -i\gamma^5\frac{\alpha}{R} & i\Gamma^M\partial_M \end{pmatrix} \begin{pmatrix} \tilde{\Psi}^1_d \\ \tilde{\Psi}^2_d \\ \tilde{\Psi}^3_d \end{pmatrix} \end{aligned} \tag{5.39}$$

After a basis transformation using the diagonalization matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{5.40}$$

and integration for the fifth dimension direction, the 4D effective Lagrangian is given by

$$\mathcal{L}_{4D} = \sum_{n=-\infty}^{\infty} \bar{\tilde{\Psi}}^{(1)}_{d,n} \left(i\gamma^\mu\partial_\mu + \frac{n}{R} \right) \tilde{\Psi}^{(1)}_{d,n} + \sum_{n=-\infty}^{\infty} \bar{\tilde{\Psi}}^{(2)}_{d,n} \left(i\gamma^\mu\partial_\mu + \frac{n+\alpha}{R} \right) \tilde{\Psi}^{(2)}_{d,n}, \tag{5.41}$$

where the 4D mass eigenstates and these mass spectrums are defined as

$$\begin{aligned} \tilde{\Psi}^{(1)}_{d,n} &= \frac{\eta_n}{\sqrt{2}} \begin{cases} -\psi^{d,+n}_{1L} + \psi^{d,-n}_{1R} & (n>0) \\ 2\left(-\psi^{d,0}_{1L} + \psi^{d,0}_{1R} \right) & (n=0) \\ -\psi^{d,-n}_{1L} + \psi^{d,+n}_{1R} & (n<0) \end{cases} \qquad : \tilde{m}^{(1)}_{d,n} = -\frac{n}{R} \\ \tilde{\Psi}^{(2)}_{d,n} &= \frac{\eta_n}{\sqrt{2}} \begin{cases} -\psi^{d,+n}_{2L} + i\chi^{d,+n}_R + \psi^{d,-n}_{2R} - i\chi^{d,-n}_L & (n>0) \\ 2\left(\psi^{d,0}_{2R} - i\chi^{d,0}_L \right) & (n=0) \\ \psi^{d,-n}_{2L} - i\chi^{d,-n}_R + \psi^{d,+n}_{2R} - i\chi^{d,+n}_L & (n<0) \end{cases} \qquad : \tilde{m}^{(2)}_{d,n} = -\frac{n+\alpha}{R}. \end{aligned} \tag{5.42}$$

Therefore, the 4D effective theory contains the two types of fermion: infinite modes with mass spectrums $m_n = -n/R$ and $m_n = -(n+\alpha)/R$.

Then we consider Ψ_d as a bulk fermion in the fundamental **3** representation with $\eta_P = -1$. If it has the same quantum numbers as $\tilde{\Psi}_d$ except for the eigenvalues of Z_2 parity, then we can write a Z_2 invariant mass term ($\bar{\Psi}_d M \tilde{\Psi}_d$ + h.c.). From boundary conditions Eq. (5.28), this bulk fermion also can be rewritten in terms of the 4D chiralities as

$$\Psi_d = \begin{pmatrix} \psi^d_{1R} + \psi^d_{1L} \\ \psi^d_{2R} + \psi^d_{2L} \\ {\chi^d}_L + {\chi^d}_R \end{pmatrix} = \sum_{n=0}^{\infty} \eta_n \begin{pmatrix} f^{+n}\psi^{d,+n}_{1R} - f^{-n}\psi^{d,-n}_{1L} \\ f^{+n}\psi^{d,+n}_{2R} - f^{-n}\psi^{d,-n}_{2R} \\ f^{+n}\chi^{d,+n}_{L} + f^{-n}\chi^{d,-n}_{R} \end{pmatrix}. \tag{5.43}$$

After separation of variables, the 4D fields are the same as that of $\tilde{\Psi}_d$. The part of the action for Ψ_d that is related to mass is the rewriting of $\tilde{\Psi}_d$ to Ψ_d in Eq. (5.39). After a basis transformation using the same diagonalization matrix to (5.40) and integration for the fifth dimension direction, we find the 4D effective Lagrangian

$$\mathcal{L}_{4D} = \sum_{n=-\infty}^{\infty} \bar{\Psi}^{(1)}_{d,n} \left(i\gamma^\mu \partial_\mu - \frac{n}{R} \right) \Psi^{(1)}_{d,n} + \sum_{n=-\infty}^{\infty} \bar{\Psi}^{(2)}_{d,n} \left(i\gamma^\mu \partial_\mu - \frac{n+\alpha}{R} \right) \Psi^{(2)}_{d,n}, \tag{5.44}$$

where the 4D mass eigenstates and these mass spectrums are defined as

$$\begin{aligned}
\Psi^{(1)}_{d,n} &= \frac{\eta_n}{\sqrt{2}} \begin{cases} \psi^{d,+n}_{1R} + \psi^{d,-n}_{1L} & (n>0) \\ 2\left(\psi^{d,0}_{1R} + \psi^{d,0}_{1L}\right) & (n=0) \\ \psi^{d,+n}_{1R} + \psi^{d,-n}_{1L} & (n<0) \end{cases} && : m^{(1)}_{d,n} = \frac{n}{R} \\
\Psi^{(2)}_{d,n} &= \frac{\eta_n}{\sqrt{2}} \begin{cases} \psi^{d,+n}_{2R} + i\chi^{d,+n}_{L} + \psi^{d,-n}_{2L} + i\chi^{d,-n}_{R} & (n>0) \\ 2\left(\psi^{d,0}_{2R} + i\chi^{d,0}_{L}\right) & (n=0) \\ \psi^{d,-n}_{2R} + i\chi^{d,-n}_{L} - \psi^{d,+n}_{2L} - i\chi^{d,+n}_{R} & (n<0) \end{cases} && : m^{(2)}_{d,n} = \frac{n+\alpha}{R}.
\end{aligned} \tag{5.45}$$

Therefore, Ψ_d appears as the infinite modes with mass spectrums $m_n = n/R$ and $m_n = (n+\alpha)/R$ in the 4D effective theory.

Finally, let's discuss the behavior of the bulk fermion with $\eta_U = -1$. It means that the field is an antiperiodic function for S^1. Therefore, the Fourier decomposition is not done with a set of periodic functions but with that of antiperiodic ones:

$$\begin{aligned}
f^{+n}(x^5) &= \frac{1}{\sqrt{\pi R}} \cos \frac{\left(n+\frac{1}{2}\right) x^5}{R} \\
f^{-n}(x^5) &= \frac{1}{\sqrt{\pi R}} \sin \frac{\left(n+\frac{1}{2}\right) x^5}{R}
\end{aligned} \tag{5.46}$$

Since these functions also satisfy the orthonormal conditions, the only change that appears in the 4D effective theory after integration is that the mass shifts $n/R \to (n+1/2)/R$. In summary, when considering a field with $\eta_U = -1$, we only need to shift that mass with $\eta_U = +1$ by half an integer.

5.1.4 Mass spectrum of a bulk fermion (Symmetric)

Then what happens in the case of the symmetric $\bar{\mathbf{6}}$ representation? In general, when we consider fields belonging to different representations, the differences appear in the number of components and the coupling to another field. In other words, we expect that the α dependence of the mass eigenstates will change, and we will check it in the following. The bulk fermion in the symmetric $\bar{\mathbf{6}}$ representation has two subscripts for the $SU(3)_w$ gauge group and is symmetric for these two subscripts. These two subscripts can be represented as

$$\Psi^{ij} = \frac{1}{\sqrt{2}} \begin{array}{c} \\ \begin{pmatrix} \sqrt{2}\Psi_1 & \Psi_2 & \Psi_4 \\ \Psi_2 & \sqrt{2}\Psi_3 & \Psi_5 \\ \Psi_4 & \Psi_5 & \sqrt{2}\Psi_6 \end{pmatrix}^{i}_{j} \end{array} . \tag{5.47}$$

Since this field transforms for the gauge transformation U_3 of $SU(3)_w$ as

$$\Psi^{ij} \to \Psi^{i'j'} = \Psi^{ij} U_3^{ii'} U_3^{jj'}, \tag{5.48}$$

the covariant derivative that keeps it gauge invariant is given by $D_M = \partial_M + 2i\, g_5 A_M$. Thus, the action for the bulk fermion in symmetric representation is given by

$$\mathcal{S} = \int d^4x \int_0^{2\pi R} dx^5 \,\mathrm{Tr}\left[\bar{\Psi} i \Gamma^M (\partial_M + 2i\, g_5 A_M)\Psi\right]. \tag{5.49}$$

This action satisfies the following boundary conditions for Ψ:

$$\begin{aligned} \Psi(x_\mu, x_5) &= \hat{U}\Psi(x_\mu, x_5)\hat{U}^{\mathrm{T}} = \eta_U \Psi(x_\mu, x_5 + 2\pi R) \\ \Psi(x_\mu, x_5) &= \hat{P}\Psi(x_\mu, x_5)\hat{P}^{\mathrm{T}} = \eta_P \Psi(x_\mu, -x_5) \end{aligned} \tag{5.50}$$

There are four possible choices of the eigenvalues η_U, η_P as same as the fundamental case. First, we consider Ψ_u as the bulk fermion in symmetric representation with $\eta_P = +1$. From boundary conditions Eq. (5.50), It can be rewritten in terms of the 4D chiralities as

$$\begin{aligned} \Psi_u &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\,(\phi^u_{1R} + \phi^u_{1L}) & \phi^u_{2R} + \phi^u_{2L} & \psi^u_{1L} + \psi^u_{1R} \\ \phi^u_{2R} + \phi^u_{2L} & \sqrt{2}\,(\phi^u_{3R} + \phi^u_{3L}) & \psi^u_{2L} + \psi^u_{2R} \\ \psi^u_{1L} + \psi^u_{1R} & \psi^u_{2L} + \psi^u_{2R} & \sqrt{2}\,(\chi^u_R + \chi^u_L) \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \frac{\eta_n}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\left(f^{+n}\phi^{u,+n}_{1R} - f^{-n}\phi^{u,-n}_{1L}\right) & f^{+n}\phi^{u,+n}_{2R} - f^{-n}\phi^{u,-n}_{2L} & f^{+n}\psi^{u,+n}_{1L} + f^{-n}\psi^{u,-n}_{1R} \\ f^{+n}\phi^{u,+n}_{2R} - f^{-n}\phi^{u,-n}_{2L} & \sqrt{2}\left(f^{+n}\phi^{u,+n}_{3R} - f^{-n}\phi^{u,-n}_{3L}\right) & f^{+n}\psi^{u,+n}_{2L} + f^{-n}\psi^{u,-n}_{2R} \\ f^{+n}\psi^{u,+n}_{1L} + f^{-n}\psi^{u,-n}_{1R} & f^{+n}\psi^{u,+n}_{2L} + f^{-n}\psi^{u,-n}_{2R} & \sqrt{2}\left(f^{+n}\chi^{u,+n}_{R} - f^{-n}\chi^{u,-n}_{L}\right) \end{pmatrix}, \end{aligned} \tag{5.51}$$

where $\{\phi_1, \phi_2, \phi_3\}$, $\{\psi_1, \psi_2\}$ and χ denote triplet, doublet and singlet of the $SU(2)_L$ gauge group, respectively. After a basis transformation using the same diagonalization matrix to (5.40) and integration for the fifth dimension direction as similar to the fundamental case, we find the 4D effective Lagrangian

$$\begin{aligned} \mathcal{L}_{4D} =& \sum_{n=-\infty}^{\infty} \sum_{i=1,2} \bar{\Psi}^{(i)}_{u,n} \left(i\gamma^\mu \partial_\mu - m^{(i)}_{u,n}\right) \Psi^{(i)}_{u,n} \\ &+ \sum_{n=-\infty}^{\infty} \bar{\Psi}^{(3)}_{u,n} \left(i\gamma^\mu \partial_\mu - m^{(3)}_{u,n}(\alpha)\right) \Psi^{(3)}_{u,n} + \sum_{n=-\infty}^{\infty} \bar{\Psi}^{(5)}_{u,n} \left(i\gamma^\mu \partial_\mu - m^{(5)}_{u,n}(2\alpha)\right) \Psi^{(5)}_{u,n}, \end{aligned} \tag{5.52}$$

where each mass eigenstates and mass eigenvalues is defined as

$$\Psi^{(1)}_{u,n} = \frac{\eta_n}{\sqrt{2}} \begin{cases} \phi^{u,+n}_{1R} + \phi^{u,-n}_{1L} & (n>0) \\ 2\phi^{u,0}_{1R} & (n=0) \\ \phi^{u,-n}_{1R} + \phi^{u,+n}_{1L} & (n<0) \end{cases} \qquad : m^{(1)}_{u,n} = \frac{n}{R}$$

$$\Psi^{(2)}_{u,n} = \frac{\eta_n}{\sqrt{2}} \begin{cases} -\phi^{u,+n}_{3R} + \chi^{u,+n}_{R} - \phi^{u,-n}_{3L} + \chi^{u,-n}_{L} & (n>0) \\ 2\left(-\phi^{u,0}_{3R} + \chi^{u,0}_{R}\right) & (n=0) \\ -\phi^{u,-n}_{3R} + \chi^{u,-n}_{R} - \phi^{u,+n}_{3L} + \chi^{u,+n}_{L} & (n<0) \end{cases} \qquad : m^{(2)}_{u,n} = \frac{n}{R}$$

$$\Psi^{(3)}_{u,n} = \frac{\eta_n}{\sqrt{2}} \begin{cases} \phi^{u,+n}_{2R} + i\psi^{u,+n}_{1L} + \phi^{u,-n}_{2L} + i\psi^{u,-n}_{1R} & (n>0) \\ 2\left(\phi^{u,0}_{2R} + i\psi^{u,0}_{1L}\right) & (n=0) \\ \phi^{u,-n}_{2R} + i\psi^{u,-n}_{1L} - \phi^{u,+n}_{2L} - i\psi^{u,+n}_{1R} & (n<0) \end{cases} \qquad : m^{(3)}_{u,n} = \frac{n+\alpha}{R}$$

$$\Psi^{(5)}_{u,n} = \frac{\eta_n}{2} \begin{cases} \phi^{u,+n}_{3R} + \chi^{u,+n}_{R} + i\sqrt{2}\psi^{u,+n}_{2L} + \phi^{u,-n}_{3L} + \chi^{u,-n}_{L} + i\sqrt{2}\psi^{u,-n}_{2R} & (n>0) \\ 2\left(\phi^{u,0}_{3R} + \chi^{u,0}_{R} + i\sqrt{2}\psi^{u,0}_{2L}\right) & (n=0) \\ \phi^{u,-n}_{3R} + \chi^{u,-n}_{R} + i\sqrt{2}\psi^{u,-n}_{2L} - \phi^{u,+n}_{3L} - \chi^{u,+n}_{L} - i\sqrt{2}\psi^{u,+n}_{2R} & (n<0) \end{cases} \qquad : m^{(5)}_{u,n} = \frac{n+2\alpha}{R} \tag{5.53}$$

Therefore, we can see that Ψ_u appears in the 4D effective theory as two infinite particles with $m_{u,n} = n/R$, infinite particles with $m_{u,n} = (n+\alpha)/R$, and infinite particles with $m_{u,n} = (n+2\alpha)/R$. We can also read from it that there is a difference in the α dependence of the mass spectrums compared to the fundamental representation.

Next, we prepare $\tilde{\Psi}_u$ with $\eta_P = -1$ as a pair of Ψ_u. It allows us to add a mass term $\mathrm{Tr}[\bar{\Psi}_u M \tilde{\Psi}_u]$ + h.c. to the Lagrangian, just as in the fundamental representation case. From boundary conditions Eq. (5.50), $\tilde{\Psi}_u$ also can be rewritten in terms of the 4D chiralities as

$$\begin{aligned} \tilde{\Psi}_u &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\left(\tilde{\phi}^u_{1L} + \tilde{\phi}^u_{1R}\right) & \tilde{\phi}^u_{2L} + \tilde{\phi}^u_{2R} & \tilde{\psi}^u_{1R} + \tilde{\psi}^u_{1L} \\ \tilde{\phi}^u_{2L} + \tilde{\phi}^u_{2R} & \sqrt{2}\left(\tilde{\phi}^u_{3L} + \tilde{\phi}^u_{3R}\right) & \tilde{\psi}^u_{2R} + \tilde{\psi}^u_{2L} \\ \tilde{\psi}^u_{1R} + \tilde{\psi}^u_{1L} & \tilde{\psi}^u_{2R} + \tilde{\psi}^u_{2L} & \sqrt{2}\left(\tilde{\chi}^u_L + \tilde{\chi}^u_R\right) \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \frac{\eta_n}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\left(f^{+n}\phi^{u,+n}_{1L} + f^{-n}\phi^{u,-n}_{1R}\right) & f^{+n}\phi^{u,+n}_{2L} + f^{-n}\phi^{u,-n}_{2R} & f^{+n}\psi^{u,+n}_{1R} - f^{-n}\psi^{u,-n}_{1L} \\ f^{+n}\phi^{u,+n}_{2L} + f^{-n}\phi^{u,-n}_{2R} & \sqrt{2}\left(f^{+n}\phi^{u,+n}_{3L}{}_L + f^{-n}\phi^{u,-n}_{3R}\right) & f^{+n}\psi^{u,+n}_{2R} - f^{-n}\psi^{u,-n}_{2L} \\ f^{+n}\psi^{u,+n}_{1R} - f^{-n}\psi^{u,-n}_{1L} & f^{+n}\psi^{u,+n}_{2R} - f^{-n}\psi^{u,-n}_{2L} & \sqrt{2}\left(f^{+n}\chi^{u,+n}_{L} + f^{-n}\chi^{u,-n}_{R}\right) \end{pmatrix}. \end{aligned} \tag{5.54}$$

After the same calculations as in Ψ_u, the 4D effective Lagrangian for $\tilde{\Psi}_u$ is given by

$$\begin{aligned} \mathcal{L}_{4D} &= \sum_{n=0}^{\infty} \sum_{i=1,2} \bar{\tilde{\Psi}}^{(i)}_{u,n} \left(i\gamma^\mu\partial_\mu - \tilde{m}^{(i)}_{u,n}\right) \tilde{\Psi}^{(i)}_{u,n} \\ &+ \sum_{n=-\infty}^{\infty} \bar{\tilde{\Psi}}^{(3)}_{u,n} \left(i\gamma^\mu\partial_\mu - \tilde{m}^{(3)}_{u,n}(\alpha)\right) \tilde{\Psi}^{(3)}_{u,n} + \sum_{n=-\infty}^{\infty} \bar{\tilde{\Psi}}^{(5)}_{u,n} \left(i\gamma^\mu\partial_\mu - \tilde{m}^{(5)}_{u,n}(2\alpha)\right) \tilde{\Psi}^{(5)}_{u,n}. \end{aligned} \tag{5.55}$$

We omit each mass eigenstate in this case because they can be easily derived from the previous discussion. As a result, mass eigenstates with masses opposite in sign to the case of Ψ_u appear.

Let us now consider a low-energy effective theory below the compactification scale $1/R$, and discuss whether it can reproduce the SM. Since only zero modes appear at this energy scale, we

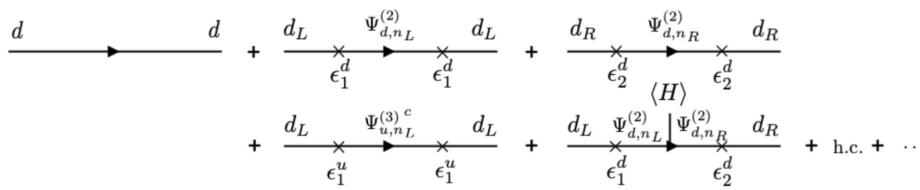

Figure 5.1: The corrected propagator for d

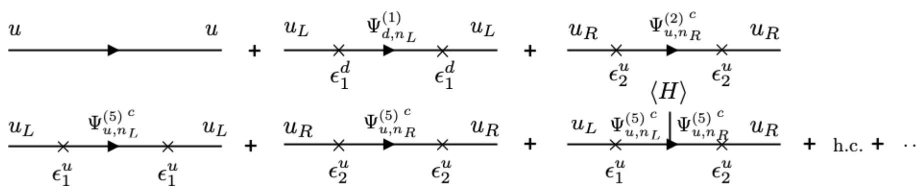

Figure 5.2: The corrected propagator for u

will focus our discussion on the zero mode of each bulk field. Because the masses of the zero-mode bulk fields are typically zero or integer multiples of α/R, we will see that some ingenuity is required to create a fermion mass hierarchy in the SM. It is essentially due to the fact that the gauge coupling controls both the bulk gauge field and bulk fermions. In this model, the brain fermions play the role of the SM fermions. Therefore, we introduce a mixing term $\bar{\Psi}M\tilde{\Psi}$ with a mixing mass M sufficiently larger than $1/R$ to avoid the zero modes of the bulk fermions appearing at low energy.

5.1.5 Mass spectrum of a brane fermion

Finally, we will discuss the mass spectrums of the brane fermions. The brane fermion is massless before the symmetry breaking and acquires the mass through coupling to the bulk fermion after that. To discuss that process concretely, we summarize the mass eigenstates of the bulk fermions in the form of a two-component vector $(\Psi_n^{(i)}, \tilde{\Psi}_n^{(i)})_a^{\mathrm{T}}$. Then the projector for this vector can be represented in momentum space using the following 2×2 matrix:

$$K_{a,n}^{(i)} = \begin{pmatrix} \not{p} - m_{a,n}^{(i)}(\alpha) & M_a \\ M_a & \not{p} + m_{a,n}^{(i)}(\alpha) \end{pmatrix} \tag{5.56}$$

The propagator $\Delta_{a,n}^{(i)}$ for each mass eigenstates of the bulk fermion is obtained by taking the inverse of this matrix as

$$\Delta_{a,n}^{(i)} = \frac{i}{p^2 - m_{a,n}^{(i)\,2}(\alpha) - M_a^2} \begin{pmatrix} \not{p} + m_{a,n}^{(i)}(\alpha) & -M_a \\ -M_a & \not{p} - m_{a,n}^{(i)}(\alpha) \end{pmatrix}. \tag{5.57}$$

Now that we are ready to discuss the mass of the brane fermion. Let us start by rewriting the bulk-brane mixing term in Eq. 5.8 with $\Psi_{a,n}^{(i)}$. After integrating out the fifth direction, bulk-brane mixing terms in the 4D effective Lagrangian is rewritten as

$$\mathcal{L}_{4D} \supset \sum_{n=0}^{\infty} \frac{1}{2} \frac{\eta_n \xi_{1,n}}{\sqrt{\pi R}} \left(e_1^d \bar{Q}_L \psi_R^{d,+n} + e_1^u \bar{Q}_R^c \psi_L^{u,+n} + \text{h.c.} \right) \\ + \sum_{n=0}^{\infty} \frac{1}{2} \frac{\eta_n \xi_{1,n}}{\sqrt{\pi R}} \left(e_2^d \bar{d}_R \chi_L^{d,+n} + e_2^u \bar{u}_L^c \chi_R^{u,+n} + \text{h.c.} \right). \tag{5.58}$$

Note that the odd function vanishes on the brane, $x_5 = 0, \pi R$, the relationship between $\psi^{a(\pm n)}$,$\chi^{a(\pm n)}$ and $\Psi_{a,n}^{(i)}$ can be read as

$$\mathcal{L}_{4D} \supset \frac{1}{\pi R} \sum_{n=-\infty}^{\infty} \left[-i\epsilon_1^u \xi_{1,n} \left(\bar{d}_R^c \Psi_{u,nL}^{(3)} - \bar{u}_R^c \Psi_{u,nL}^{(5)} \right) + \frac{\epsilon_2^u}{\sqrt{2}} \xi_{2,n} \bar{u}_L^c \left(\Psi_{u,nR}^{(5)} + \Psi_{u,nR}^{(2)} \right) \right. \\ \left. + \epsilon_1^d \xi_{1,n} \left(\bar{u}_L \Psi_{d,nR}^{(1)} + \bar{d}_L \Psi_{d,nR}^{(2)} \right) - i\epsilon_2^d \xi_{2,n} \, \bar{d}_R \Psi_{d,nL}^{(2)} + \text{h.c.} \right]. \tag{5.59}$$

where the coefficient $\xi_{i,n}$ denotes the fifth-dimensional wave function on the brane:

$$\xi_{i,n} = \cos\left(\frac{n x_i^5}{R}\right) = \begin{cases} 1 & : x_i^5 = 0 \\ (-1)^n & : x_i^5 = \pi R \end{cases} \tag{5.60}$$

For the following calculations, we define a new field $\Psi_{a,n}^{(i)'}$ modified by a phase from $\Psi_{a,n}^{(i)}$

$$\begin{aligned} \Psi_{d,nR}^{(1)'} &= \Psi_{d,nR}^{(1)} & \Psi_{u,nL}^{(3)'} &= -i\Psi_{u,nL}^{(3)} \\ \Psi_{d,nR}^{(2)'} &= \Psi_{d,nR}^{(2)} & \Psi_{u,nL}^{(5)'} &= -i\Psi_{u,nL}^{(5)} \\ \Psi_{d,nL}^{(2)'} &= -i\Psi_{d,nL}^{(2)} & \Psi_{u,nR}^{(5)'} + \Psi_{u,nR}^{(2)'} &= -\left(\Psi_{u,nR}^{(5)} + \Psi_{u,nR}^{(2)}\right) \end{aligned} \tag{5.61}$$

Then, Eq. (5.59) is rewritten as

$$\mathcal{L}_{4D} \supset \frac{1}{\pi R} \sum_{n=-\infty}^{\infty} \left[\epsilon_1^u \xi_{1,n} \left(\bar{d}_R^c \Psi_{u,nL}^{(3)'} - \bar{u}_R^c \Psi_{u,nL}^{(5)'} \right) - \frac{\epsilon_2^u}{\sqrt{2}} \xi_{2,n} \bar{u}_L^c \left(\Psi_{u,nR}^{(5)'} + \Psi_{u,nR}^{(2)'} \right) \right. \\ \left. + \epsilon_1^d \xi_{1,n} \left(\bar{u}_L \Psi_{d,nR}^{(1)'} + \bar{d}_L \Psi_{d,nR}^{(2)'} \right) + \epsilon_2^d \xi_{2,n} \, \bar{d}_R \Psi_{d,nL}^{(2)'} + \text{h.c.} \right]. \tag{5.62}$$

Only $\Psi_{d,n}^{(2)'}$ and $\Psi_{u,n}^{(5)'}$ couple to brane fields with both left and right chirality, and this coupling induces the mass term. The other $\Psi_{a,n}^{(i)'}$ are only coupled to either the left or right chirality, which induces corrections to the kinetic operators K^u and K^d for the brane fermions $u = u_R + u_L$ and $d = d_R + d_L$. The propagators for u, d after adding the correction for bulk-brane mixing are given by

$$\Delta_u = i\left[\not{p}\mathrm{Re}\left(1+\frac{\epsilon_1^{d\,2}}{x^d}P_L f_0(x^d,0)+\frac{\epsilon_2^{u2}}{2x^u}P_R f_0(x^u,0)+\left(\frac{\epsilon_1^{u2}}{x^u}P_L+\frac{\epsilon_2^{u2}}{2x^u}P_R\right)f_0(x^u,2\alpha)\right)\right.$$
$$\left.-\frac{1}{\pi R}\mathrm{Im}\,\frac{\epsilon_1^u\epsilon_2^u}{\sqrt{2}}\left(P_L+P_R\right)f_\delta(x^u,\alpha)\right]^{-1} \tag{5.63}$$

$$\Delta_d = i\left[\not{p}\mathrm{Re}\left(1+\left(\frac{\epsilon_1^{d2}}{x^d}P_L+\frac{\epsilon_2^{d2}}{x^d}P_R\right)f_0(x^d,\alpha)+\frac{\epsilon_1^{u2}}{x^u}P_L f_0(x^u,\alpha)\right)\right.$$
$$\left.-\frac{1}{\pi R}\mathrm{Im}\,\epsilon_1^d\epsilon_2^d\left(P_L+P_R\right)f_\delta(x^d,\alpha)\right]^{-1} \tag{5.64}$$

To calculate the infinite sum in the above equation, we moved to Euclidean space and used the dimensionless momentum variables $x=\pi Rp$ and $x^a=\pi R\sqrt{p^2+M_a^2}$. It allows us to rewrite the infinite sum as follows:

$$f_0(x,\alpha)=\sum_{n=-\infty}^{\infty}\frac{1}{x+i\pi(n+\alpha)}=\coth(x+i\pi\alpha)$$
$$f_1(x,\alpha)=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{x+i\pi(n+\alpha)}=\sinh^{-1}(x+i\pi\alpha) \tag{5.65}$$

The function $f_\delta(x,\alpha)$ can also be interpreted as representing the propagation of a bulk fermion between two fixed points separated by a distance $\delta\pi R$. This expression can also be summarized as

$$f_\delta(x,\alpha)=\sum_{k=-\infty}^{\infty}\mathrm{e}^{-|2k+\delta|(x+i\pi\alpha)} \tag{5.66}$$

By taking the inverse of Eq. (5.64), (5.63), the kinetic operators are given by

$$K_u=\not{p}\mathrm{Re}\left[1+\frac{\epsilon_1^{d\,2}}{x^d}P_L f_0(x^d,0)+\frac{\epsilon_2^{u2}}{2x^u}P_R f_0(x^u,0)+\left(\frac{\epsilon_1^{u2}}{x^u}P_L+\frac{\epsilon_2^{u2}}{2x^u}P_R\right)f_0(x^u,2\alpha)\right]$$
$$-\frac{1}{\pi R}\mathrm{Im}\left[\frac{\epsilon_1^u\epsilon_2^u}{\sqrt{2}}f_\delta(x^u,\alpha)\right], \tag{5.67}$$

$$K_d=\not{p}\mathrm{Re}\left[1+\left(\frac{\epsilon_1^{d\,2}}{x^d}P_L+\frac{\epsilon_2^{d\,2}}{x^d}P_R\right)f_0(x^d,\alpha)+\frac{\epsilon_1^{u2}}{x^u}P_L f_0(x^u,\alpha)\right]$$
$$-\frac{1}{\pi R}\mathrm{Im}\left[\epsilon_1^d\epsilon_2^d f_\delta(x^d,\alpha)\right]. \tag{5.68}$$

The mixing between the brain and bulk fermions should affect the mass of the bulk fermion as same as the brane fermion case. However, if the mass induced for the brane fermion is sufficiently smaller than that of the bulk fermion, we can neglect the mixing corrections to the bulk fermion. In this case, the physical masses of the brane fermions are given by

$$m^a=\frac{m_0^a}{\sqrt{Z_1^aZ_2^a}}, \tag{5.69}$$

where the normalization factors Z_i^a and mass parameters m_0^a are defined as

$$
\begin{aligned}
m_0^u &= \frac{\epsilon_1^u \epsilon_2^u}{\sqrt{2}\pi R}\mathrm{Im} f_\delta(\lambda^u, 2\alpha), \\
m_0^d &= \frac{\epsilon_1^d \epsilon_2^d}{\pi R}\mathrm{Im} f_\delta(\lambda^d, \alpha), \\
Z_i^u &= 1 + \delta_{i1}\frac{\epsilon_1^{d^2}}{\lambda^d}\mathrm{Re} f_0(\lambda^d, 0) + \delta_{i2}\frac{\epsilon_2^{u^2}}{2\lambda^u}\mathrm{Re} f_0(\lambda^u, 0) + \frac{\epsilon_i^{u^2}}{2^{\delta_{i2}}\lambda^u}\mathrm{Re} f_0(\lambda^u, 2\alpha), \\
Z_i^d &= 1 + \frac{\epsilon_i^{d^2}}{\lambda^d}\mathrm{Re} f_0(\lambda^d, \alpha) + \delta_{i1}\frac{\epsilon_1^{u^2}}{\lambda^u}\mathrm{Re} f_0(\lambda^u, \alpha).
\end{aligned} \tag{5.70}
$$

5.1.6 One-loop effective potential

As mentioned above, the Higgs potential in GHU is flat at the tree level due to the higher dimensional gauge symmetry. Then, the quantum corrections induce its shape. In this part, we discuss the 1-loop contribution to the Higgs potential of each particle in the GHU model with flat extra dimension. As used in Sec. 2.3, the effective potential at the one-loop level is given by

$$
V_{\mathrm{eff}}(\alpha) = V_0 + \sum_I \frac{\sigma_I}{2}\int \frac{d^4 p_E}{(2\pi)^4} \ln\left(p_E^2 + m_I^2(\alpha)\right), \tag{5.71}
$$

where V_0 is a constant and the sum runs over all 4D fields whose masses depend on α. Let's start with the contribution of the bulk fields to the effective potential. Looking at the 5D bulk field from the perspective of 4D, an infinite number of mass eigenstates (zero and KK modes) appear. After the EWSB, the masses of these modes are typically given by

$$
m^2(\alpha) = M^2 + m_n^2(\alpha)\ , \quad m_n(\alpha) = \frac{n + q\alpha}{R}, \tag{5.72}
$$

where M denotes the 5D mass of the bulk field and $m_n(\alpha)$ is the mass induced by the oscillation energy on the fifth dimension and the spontaneous symmetry breaking. The coefficient q is the charge related to the coupling with the Higgs doublet, and determined from the $SU(3)_w$ representation of the bulk field. Then, we represent Eq. (5.71) as

$$
\begin{aligned}
V_{\mathrm{bulk}}(\alpha) &= -\frac{1}{2}\sum_I (-)^{F_I}\int_0^\infty \frac{dt}{t}\int \frac{d^4 p_E}{(2\pi)^4}\,\mathrm{e}^{-t\left(p_E^2 + M_I^2(\alpha)\right)} \\
&= -\frac{1}{2}\sum_I (-)^{F_I}\int_0^\infty \frac{dt}{t}\int \frac{d\Omega_4}{(2\pi)^4}\frac{1}{2}\int_0^\infty dp_E^2\, p_E^2\,\mathrm{e}^{-t\left(p_E^2 + M_I^2(\alpha)\right)} \\
&= -\frac{1}{32\pi^2}\sum_I (-)^{F_I}\int_0^\infty \frac{dt}{t^3}\,\mathrm{e}^{-tM_I^2(\alpha)},
\end{aligned} \tag{5.73}
$$

In the first line, we use the following relation in the limit $\epsilon \to 0$:

$$
\Gamma(\epsilon)\int \frac{dp}{M(p)^\epsilon} = \int_0^\infty du\, u^{\epsilon-1}\int dp\,\mathrm{e}^{-uM(p)} \tag{5.74}
$$

For simplicity, we start with the case of $M = 0$. After variable transformation at $t = 1/l$, we can rewrite the effective potential by using the Poisson summation formula B as

$$
\begin{aligned}
V_{\text{bulk}}(\alpha) &= -\sum_I (-)^{F_I} \frac{1}{32\pi^2} \int_0^\infty dl\, l \left(\sum_n \mathrm{e}^{-\frac{(n+q\alpha)^2}{R^2 l}} \right) \\
&= -\sum_I (-)^{F_I} \frac{R}{32\pi^{\frac{3}{2}}} \sum_n \mathrm{e}^{2\pi i n q\alpha} \int_0^\infty dl\, l^{\frac{3}{2}}\, \mathrm{e}^{-\pi^2 l n^2 R^2}
\end{aligned}
\tag{5.75}
$$

where we perform the sum over the KK expansion (n) from the sum over all mass eigenstates. From this integral form, we can see that the contribution for the zero modes is divergent. However, since this contribution does not depend on α, it can be treated as a constant term in the potential. Therefore, we can ignore this term by appropriately choosing the reference point of the potential. A more detailed discussion is provided in Ref. [45]. By changing the integration variable to $l' = \pi^2 l n^2 R^2$, we can perform the momentum integration for the KK mode contribution as

$$
\int_0^\infty dl\, l^{\frac{2}{3}}\, \mathrm{e}^{-\pi^2 l n^2 R^2} = (\pi n R)^{-5}\, \Gamma(\frac{5}{2}). \tag{5.76}
$$

Therefore, the one-loop contribution from bulk field with $M = 0$ is given by

$$
\begin{aligned}
V_{\text{bulk}}(\alpha) &= -\sum_I (-)^{F_I} \frac{\Gamma(\frac{5}{2})}{32\pi^{\frac{13}{2}}} R \sum_{n\neq 0} \frac{1}{(nR)^5}\, \mathrm{e}^{2\pi i n q\alpha} \\
&= -\sum_I (-)^{F_I} \frac{1}{32\pi^2} \frac{1}{(\pi R)^4} \frac{3}{4} \sum_{n\neq 0} \frac{1}{n^5} \big\{ \cos(2\pi n q\alpha) + i \sin(2\pi n q\alpha) \big\} \\
&= -\sum_I (-)^{F_I} \frac{1}{32\pi^2} \frac{1}{(\pi R)^4} \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n^5} \cos(2\pi n q\alpha).
\end{aligned}
\tag{5.77}
$$

In the last line, we rewrites the sum using the property of the even and odd functions. Then we will move on our discussion for $M \neq 0$. The modified part is integral over l, and then we find

$$
\begin{aligned}
\int_0^\infty dl\, l^{\frac{3}{2}}\, \mathrm{e}^{-\frac{M^2}{l} - (\pi n R)^2 l} &= \frac{1}{(\pi n R)^5} \cdot \left(\mathrm{e}^{-2\pi n R M} \frac{3\sqrt{\pi}}{4} (1 + 2\pi n R M + (2\pi n R M)^2) \right) \\
&= \frac{3\sqrt{\pi}}{4(\pi R)^5} \frac{1}{n^3}\, \mathrm{e}^{-2n\lambda} \left(\frac{1}{n^2} + \frac{2\lambda}{n} + \frac{4\lambda^2}{3} \right).
\end{aligned}
\tag{5.78}
$$

Thus, the one-loop contribution from bulk field with $M \neq 0$ is given by

$$
V_{\text{bulk}}(\alpha) = -\sum_I (-)^{F_I} \frac{1}{32\pi^2} \frac{1}{(\pi R)^4} \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}\, \mathrm{e}^{-2n\lambda} \cos(2\pi n q\alpha) \left(\frac{1}{n^2} + \frac{2\lambda}{n} + \frac{4\lambda^2}{3} \right). \tag{5.79}
$$

It can reproduce the above result in the limit $M = 0$. On the other hand, when the 5D mass parameter λ is large, the contribution is exponentially suppressed. As a result, the bulk field with a large 5D mass is less contribute to the Higgs potential.

Next, we discuss the contribution of the brane fermions to the effective potential. Unlike the bulk field, the mass eigenstates of the brane fermions are finite in number, but each mass has a complex form. By using the normalization factors and mass parameters in Eq. (5.70), the one-loop contributions from the brane fermions are defined as

$$V_a = -12 \cdot \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \ln\left(-p_E^2 Z_1^a Z_2^a + (m^a(\alpha))^2\right), \tag{5.80}$$

where the factor 12 is from the spin, color, and particle/antiparticle degrees of freedom. It can be rewritten in terms of $x = \pi R p$ as

$$\begin{aligned} V_a &= -12 \cdot \frac{1}{16\pi^2} \int_0^\infty dx\, x^3 \frac{1}{(\pi R)^4} \ln\left(Z_1^a Z_2^a \frac{x^2}{(\pi R)^2} + m^a(\alpha)^2 \right) \\ &= -\frac{3}{4\pi^2 (\pi R)^4} \int_0^\infty dx\, x^3 \left[\ln\left(Z_1^a Z_2^a + \frac{(\pi R)^2}{x^2} m^a(\alpha)^2 \right) - \ln \frac{x^2}{(\pi R)^4} \right]. \end{aligned} \tag{5.81}$$

The second term in this integral diverges. But since it does not depend on α, we can ignore this term as in the bulk field case. Therefore, one-loop contributions from u and d are given by

$$\begin{aligned} V_u(\alpha) =& \frac{-3}{4\pi^6 R^4} \int_0^\infty dx\, x^3 \ln\left[\prod_{i=1}^{2} \mathrm{Re}\left[1 + \delta_{i1} \frac{\epsilon_1^{d^2}}{x^d} f_0(x^d, 0) + \delta_{i2} \frac{\epsilon_2^{u^2}}{2x^u} f_0(x^u, 0) + \frac{\epsilon_i^{u^2}}{2^{\delta_{i2}} x^u} f_0(x^u, 2\alpha) \right] \right. \\ &\left. + \prod_{i=1}^{2} \mathrm{Im}\left[\frac{\epsilon_i^{u^2}}{2^{\delta_{i2}} x} f_\delta(x^u, \alpha) \right] \right], \end{aligned} \tag{5.82}$$

$$\begin{aligned} V_d(\alpha) =& \frac{-3}{4\pi^6 R^4} \int_0^\infty dx\, x^3 \ln\left[\prod_{i=1}^{2} \mathrm{Re}\left[1 + \frac{\epsilon_1^{d^2}}{x^d} f_0(x^d, \alpha) + \delta_{i1} \frac{\epsilon_1^{u^2}}{x^u} f_0(x^u, \alpha) \right] \right. \\ &\left. + \prod_{i=1}^{2} \mathrm{Im}\left[\frac{\epsilon_i^{d^2}}{x} f_\delta(x^d, \alpha) \right] \right]. \end{aligned} \tag{5.83}$$

The net effective potential is the sum of these contributions. When this effective potential has a minimum at $\alpha \neq 0$, electroweak symmetry breaking occurs. This alternative set of procedures to the Higgs mechanism is called the Hosotani mechanism. However, it is non-trivial that the symmetry breaking occurs. The symmetry breaking requires the properly prepared particles and model parameters.

5.2 SU(3) model with 5D Lorentz symmetry relaxed

So far, we have considered the Toy $SU(3)$ model. Although this model can reproduce the masses of most SM particles, it suffers from several problems: (1) When the extended electroweak symmetry is $SU(3)_w$ only, the Weinberg angle does not agree with the experimental value. (2) The masses of the top quark and Higgs cannot be reproduced. (3) The configuration of the Toy SU(3) model expects the KK particle much too light, which is inconsistent with the collider experiments. To solve these problems, following two extensions are proposed as the realistic GHU model with a flat extra dimension: SU(3) model with large representation, SU(3) model with 5D Lorentz symmetry relaxed. Our study will be based on the latter model.

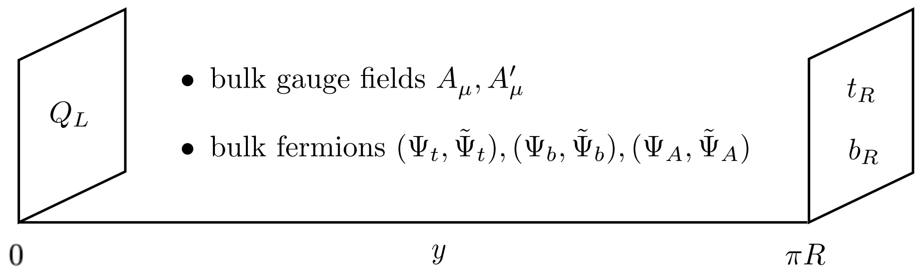

Figure 5.3: Field configuration on the S^1/Z_2 orbifold.

In this model, the electroweak symmetry is extended to $SU(3)_w \times U(1)'$. With the additional $U(1)$ symmetry, we can construct the hypercharge generators by a linear combination of two $U(1)$ generators as $Y = t_8/\sqrt{3} + t'$. The gauge field of $U(1)_Y$ and its orthogonal gauge field of $U(1)_X$ are defined by

$$A_Y = \frac{g' A^8 + \sqrt{3} g A'}{\sqrt{3g^2 + g'^2}}, \quad A_X = \frac{\sqrt{3} g A^8 + g' A'}{\sqrt{3g^2 + g'^2}}. \tag{5.84}$$

Similarly, the gauge coupling constant of $U(1)_Y$ is defined by the combination of the two gauge coupling constants of $SU(3)_w$ and $U(1)'$ as $g_Y = \sqrt{3} g g'/\sqrt{3g^2 + g'^2}$. With the introduction of the additional gauge coupling constant g' for $U(1)'$, the Weinberg angle is modified as

$$\sin^2 \theta_W = \frac{g_Y^2}{g^2 + g_Y^2} = \frac{3}{4 + 3g^2/g'^2}\,. \tag{5.85}$$

By determining the appropriate ratio of the two gauge coupling constants, we can reproduce the experimental value of the Weinberg angle. We briefly discuss an additional gauge field A_X that does not appear in the SM. In contrast to the SM gauge group, the $U(1)_X$ gauge group has a quantum anomaly on the brane due to the presence of the brane fermions. Then, the A_X acquires a brane localized mass M_X whose natural value is the cut-off scale of the model. Therefore, A_X can be neglected for the low-energy effective theory we discuss. Details are discussed in Ref. [34, 44]. The gauge anomalies coming from brane localized fermions is also discussed in Ref [44, 46, 47]

Now we move on to the discussion of the matter sector. This model has two changes from the Toy $SU(3)$ model: The first is the addition of an anti-periodic bulk fermion pair to adjust the shape of the Higgs potential. The second is the introduction of a 5D Lorentz symmetry breaking parameter in the fifth component of the covariant derivative to reproduce the top quark and Higgs masses. According to Ref. [34], we localize left- and right-handed brane fermions on the different brane ($\delta = 1$ set up) to reproduce the Higgs mass. The field configuration on the S^1/Z_2 orbifold is shown in Fig. 5.3 Focusing on the third-generation quark, which gives a dominant contribution to the Higgs potential, the matter Lagrangian in Eq. (5.6) is modified

Table 5.2: Matter contents and those quantum numbers. The hypercharge of $(\Psi_A, \tilde{\Psi}_A)$ is chosen such that $(\Psi_A, \tilde{\Psi}_A)$ do not mix with the bulk or brane fermions. The color factor is denoted by C_F.

Fields	$SU(3)_c \times SU(3)_w$	periodicity (η)	$SU(3)_c \times SU(2)_L \times U(1)_Y$	C_F
$(\Psi_t, \tilde{\Psi}_t)$	$(\mathbf{3}, \bar{\mathbf{6}})$	periodic (0)	$(\mathbf{3},\mathbf{1})_{2/3} + (\mathbf{3},\mathbf{2})_{1/6} + (\mathbf{3},\mathbf{3})_{-1/3}$	3
$(\Psi_b, \tilde{\Psi}_b)$	$(\mathbf{3}, \mathbf{3})$	periodic (0)	$(\mathbf{3},\mathbf{1})_{-1/3} + (\mathbf{3},\mathbf{2})_{1/6}$	3
$(\Psi_A, \tilde{\Psi}_A)$	$(\mathbf{1}, \mathbf{6})$	antiperiodic (1)	$(\mathbf{1},\mathbf{1})_X + (\mathbf{1},\mathbf{2})_{X+1/2} + (\mathbf{1},\mathbf{3})_{X+1}$	1
Q_L			$(\mathbf{3},\mathbf{2})_{1/6}$	3
t_R			$(\mathbf{3},\mathbf{1})_{2/3}$	3
b_R			$(\mathbf{3},\mathbf{1})_{-1/3}$	3

by these changes as

$$\begin{aligned}\mathcal{L}_{\text{matter}} = \sum_{j=t,b,A} &\left\{ \bar{\Psi}_j \left(i\not{D}_4 - k_j D_5 \gamma^5 \right) \Psi_j + \bar{\tilde{\Psi}}_j \left(i\not{D}_4 - \tilde{k}_j D_5 \gamma^5 \right) \tilde{\Psi}_j + \left(\bar{\Psi}_j M_j \tilde{\Psi}_j + \text{h.c.} \right) \right\} \\ &+ \delta(y-0) \left\{ \bar{Q}_L i\not{D}_4 Q_L + \left(e_1^b \bar{Q}_L \psi_b + e_1^t \bar{Q}_R^c \psi_t + \text{h.c.} \right) \right\} \\ &+ \delta(y-\pi R) \left\{ \bar{t}_R i\not{D}_4 t_R + \bar{b}_R i\not{D}_4 b_R + \left(e_2^b \bar{b}_R \chi_b + e_2^t \bar{t}_L^c \chi_t + \text{h.c.} \right) \right\} ,\end{aligned} \tag{5.86}$$

where the matter contents are summarized in Table 5.2. For the sake of simplicity, we take $k_j = \tilde{k}_j$. Thus, there are totally 10 model parameters: $k_t, k_b, k_A, \lambda^t, \lambda^b, \lambda^A, \epsilon_1^t, \epsilon_2^t, \epsilon_1^b$ and ϵ_2^b. In principle, similar Lorentz symmetry breaking effects can be also introduced in the bulk gauge sector. Since the deviation of the triple Higgs boson coupling is caused mainly by the fermionic sector, we set the gauge and Higgs counterparts to the unity in the following analysis for simplicity in this thesis.

5.2.1 Mass spectrums

To estimate the quantum effect on the Higgs potential, we need to clarify the α dependence on all particles. In this part, we will discuss the mass spectrums contained in this model from the 4D perspective. The introduction of the Lorentz symmetry breaking parameters can be interpreted as a change in the compactification scale perceived by each particle as $1/R \to k_j/R$. Therefore, this effect can be incorporated by replacing the parameter in the Toy model as follows:

$$R \to R/k \ , \quad \lambda \to \lambda/k \ , \quad \epsilon \to \epsilon/k \tag{5.87}$$

In the limit $M_X \to \infty$ where we ignore the effect of A_X, the masses of the weak bosons are given by

$$m_{W^\pm}^{(n)}(\alpha) = \frac{n+\alpha}{R}, \quad m_Z^{(n)}(\alpha) = \frac{n+\alpha \sec\theta_W}{R}. \tag{5.88}$$

By modifing the Weinberg angle, we can reproduce the SM masses: $m_{W^\pm}^{(0)} = \alpha/R$ and $m_Z^{(0)} = \sec\theta_W \alpha/R$. For the bulk fermions, we can ignore the bulk-brane mixing effect as mentioned in Sec 5.1. Then the mass spectrums of bulk fermions are given by

$$m_j^{(n)}(q\alpha) = \sqrt{\left(k_j \frac{n+\eta/2+q\alpha}{R} \right)^2 + \left(\frac{\lambda^j}{\pi R} \right)^2}, \tag{5.89}$$

where the parameter $\eta = 0\,(1)$ stands for periodic (antiperiodic) bulk fermions. The charge related to the coupling with the Higgs doublet q is determined from the $SU(3)_w$ indices of the bulk fermions: $q = 0, 1$ for the fundamental representation and $q = 0, 1, 2$ for the symmetric representation. Thus we have following mass spectrums which depend on α: $m_t^{(n)}(\alpha), m_t^{(n)}(2\alpha), m_b^{(n)}(\alpha), m_A^{(n)}(\alpha)$ and $m_A^{(n)}(2\alpha)$. For the brane fermions, normalization factors and mass parameters defined in Eq. (5.70) are modified as

$$\begin{aligned}
m_0^t &= \frac{k_t}{\sqrt{2}\pi R}\frac{\epsilon_1^t\epsilon_2^t}{k_t^2}\mathrm{Im}f_1(\lambda^t/k_t, 2\alpha), \\
m_0^b &= \frac{k_b}{\pi R}\frac{\epsilon_1^b\epsilon_2^b}{k_b^2}\mathrm{Im}f_1(\lambda^b/k_b, \alpha). \\
Z_i^t &= 1 + \delta_{i1}\frac{(\epsilon_1^b)^2}{k_b\lambda^b}\mathrm{Re}f_0(\lambda^b/k_b, 0) + \delta_{i2}\frac{(\epsilon_2^t)^2}{2k_t\lambda^t}\mathrm{Re}f_0(\lambda^t/k_t, 0) + \frac{(\epsilon_i^t)^2}{2^{\delta_{i2}}k_t\lambda^t}\mathrm{Re}f_0(\lambda^t/k_t, 2\alpha), \\
Z_i^b &= 1 + \frac{(\epsilon_i^b)^2}{k_b\lambda^b}\mathrm{Re}f_0(\lambda^b/k_b, \alpha) + \delta_{i1}\frac{(\epsilon_1^t)^2}{k_t\lambda^t}\mathrm{Re}f_0(\lambda^t/k_t, \alpha).
\end{aligned} \tag{5.90}$$

Finally, we check how the mass of the top quark is given in $\alpha \ll 1$. In such a limit, the physical masses of the brane top and bottom are expanded as

$$\begin{aligned}
m_{\mathrm{phys}}^t &\simeq \frac{2k_t\alpha}{\sqrt{2}R}\frac{\frac{\epsilon_1^t\epsilon_2^t}{k_t^2}\frac{\coth(\lambda^t/k_t)}{\sinh(\lambda^t/k_t)}}{\sqrt{\left(1 + \frac{\epsilon_1^{b2}}{k_b\lambda^b}\coth\left(\frac{\lambda^b}{k_b}\right) + \frac{\epsilon_1^{t2}}{k_t\lambda^t}\coth\left(\frac{\lambda^t}{k_t}\right)\right)\left(1 + \frac{\epsilon_2^{t2}}{k_t\lambda^t}\coth\left(\frac{\lambda^t}{k_t}\right)\right)}} + \mathcal{O}(\alpha^3)\,, \\
m_{\mathrm{phys}}^b &\simeq \frac{k_b\alpha}{R}\frac{\frac{\epsilon_1^b\epsilon_2^b}{k_b^2}\frac{\coth(\lambda^b/k_b)}{\sinh(\lambda^b/k_b)}}{\sqrt{\left(1 + \frac{\epsilon_1^{b2}}{k_b\lambda^b}\coth\left(\frac{\lambda^b}{k_b}\right) + \frac{\epsilon_1^{t2}}{k_t\lambda^t}\coth\left(\frac{\lambda^t}{k_t}\right)\right)\left(1 + \frac{\epsilon_2^{b2}}{k_b\lambda^b}\coth\left(\frac{\lambda^b}{k_b}\right)\right)}} + \mathcal{O}(\alpha^3)\,,
\end{aligned} \tag{5.91}$$

where we use following approximation:

$$\begin{aligned}
\mathrm{Im}f_\delta(b, \alpha) &\simeq -\pi\frac{\coth(b)}{\sinh(b)}\alpha + \mathcal{O}(\alpha^3) \\
\frac{1}{\sqrt{a + \mathrm{Re}\coth(c + i\pi\alpha)}} &\simeq \frac{1}{\sqrt{a + \coth(c)}} + \mathcal{O}(\alpha^2)
\end{aligned} \tag{5.92}$$

For the sake of simplicity, we take $\epsilon_{1,2}^t \ll 1$ and $\epsilon_{1,2}^b = 0$, then it is reduced as

$$m_{\mathrm{phys}}^t \simeq \sqrt{2}k_t m_W\frac{\lambda^t/k_t}{\sinh\lambda^t/k_t} < \sqrt{2}k_t m_W\,, \tag{5.93}$$

where we use $m_W = \alpha/R$. Therefore, if we keep the 5D Lorentz symmetry ($k_t = 1$), the mass of the top quark is smaller than the experimental value. Moreover, for $\alpha \ll 1$, we can approximate the masses of the brane fermions in a form linear to the VEV (α), as in the SM. Therefore, ignoring the higher-order terms in α, the contribution to the triple Higgs boson coupling is given by

$$\frac{\partial^3}{\partial\alpha^3}V_a = \frac{3}{\alpha}\frac{\partial^2}{\partial\alpha^2}V_a - \frac{3}{\alpha^2}\frac{\partial}{\partial\alpha}V_a - 3\frac{m_a^4}{\pi^2 v^3}\left(\frac{g_4 R}{2}\right)^{-3}\,, \tag{5.94}$$

as in the SM case.

5.2.2 One-loop effective potential

Next, we discuss how the 1-loop contribution of each particle to the Higgs potential can be formalized using the mass spectrums defined above. For the bulk gauge fields, it takes the form of

$$V_V(\alpha) = -\frac{9}{64\pi^6 R^4} \sum_{n=1}^{\infty} \frac{1}{n^5} \cos(2\pi n\alpha). \tag{5.95}$$

Then, the total contribution from the gauge sector is summarized as

$$V_g(\alpha) = 2V_V(\alpha) + V_V(\alpha \sec\theta_W). \tag{5.96}$$

Similarly, for the bulk fermions, it takes the form of

$$V_{\Psi_j}(q\alpha) = \frac{3k_j^4 C_F}{8\pi^6 R^4} \sum_{n=1}^{\infty} \frac{(\sigma_S)^n}{n^5} \left[1 + 2n\frac{\lambda^j}{k_j} + \frac{4}{3}n^2 \frac{(\lambda^j)^2}{k_j^2}\right] \mathrm{e}^{-2n\lambda^j/k_j} \cos(2\pi nq\alpha), \tag{5.97}$$

where $\sigma_S = (-1)^\eta$ and C_F stands for the color factor. Then, the total contribution from the bulk fermion pairs is summarized as

$$V_f(\alpha) = V_{\Psi_t}(\alpha) + V_{\Psi_t}(2\alpha) + V_{\Psi_b}(\alpha) + V_{\Psi_A}(\alpha) + V_{\Psi_A}(2\alpha). \tag{5.98}$$

Finally, for the brane fermions, Eq. (5.82) and Eq. (5.83) are modified as

$$\begin{aligned} V_t(\alpha) =& \frac{-C_F}{4\pi^6 R^4} \int_0^\infty dx\, x^3 \ln\left[\prod_{i=1}^{2} \mathrm{Re}\left[1 + \delta_{i1}\frac{(\epsilon_1^b)^2}{k_b x^b} f_0\left(\frac{x^b}{k_b}, 0\right) + \delta_{i2}\frac{(\epsilon_2^t)^2}{2k_t x^t} f_0\left(\frac{x^t}{k_t}, 0\right)\right.\right. \\ &\left.\left. + \frac{(\epsilon_1^t)^2}{2^{\delta_{i2}} k_t x^t} f_0\left(\frac{x^t}{k_t}, 2\alpha\right)\right] + \prod_{i=1}^{2} \mathrm{Im} \frac{(\epsilon_i^t)^2}{2^{\delta_{i2}} k_t x}\left[f_1\left(\frac{x^t}{k_t}, 2\alpha\right)\right]\right], \end{aligned} \tag{5.99}$$

and

$$\begin{aligned} V_b(\alpha) =& \frac{-C_F}{4\pi^6 R^4} \int_0^\infty dx\, x^3 \ln\left[\prod_{i=1}^{2} \mathrm{Re}\left[1 + \frac{(\epsilon_i^b)^2}{k_b x^b} f_0\left(\frac{x^b}{k_b}, \alpha\right) + \delta_{i1}\frac{(\epsilon_1^t)^2}{k_t x^t} f_0\left(\frac{x^t}{k_t}, \alpha\right)\right]\right. \\ &\left. + \prod_{i=1}^{2} \mathrm{Im}\left[\frac{(\epsilon_i^b)^2}{k_b x} f_1\left(\frac{x^b}{k_b}, \alpha\right)\right]\right]. \end{aligned} \tag{5.100}$$

These contributions are finite for any α, respectively. Therefore, although GHU is a non-renormalizable higher dimensional theory, it predicts a finite Higgs mass because the effective potential is finite at the one-loop level. Putting the above contributions together, we get the total one-loop effective potential,

$$V_{\text{eff}}(\alpha) = V_g(\alpha) + V_f(\alpha) + V_t(\alpha) + V_b(\alpha). \tag{5.101}$$

The minimum value α_0 of this potential is obtained by the tadpole condition,

$$\left(\frac{gR}{2}\right) \left.\frac{\partial V_{\text{eff}}}{\partial \alpha}\right|_{\alpha=\alpha_0} = 0. \tag{5.102}$$

The mass squared of the Higgs boson is given by the second derivatives of this potential:

$$m_h^2 = \left(\frac{gR}{2}\right)^2 \left.\frac{\partial^2 V_{\text{eff}}(\alpha)}{\partial \alpha^2}\right|_{\alpha=\alpha_0} \tag{5.103}$$

Similarly, the triple Higgs boson coupling is given by the third derivatives of that:

$$\lambda_{hhh} = \left(\frac{gR}{2}\right)^3 \left.\frac{\partial^3 V_{\text{eff}}(\alpha)}{\partial \alpha^3}\right|_{\alpha=\alpha_0} \tag{5.104}$$

The compactification scale $1/R$ is determined by using the mass relation of the W boson, $m_{W^\pm}^{(0)} = \alpha_0/R$. In order to avoid the experimental constraints for KK particles, $1/R$ should be sufficiently large. In other words, α_0 should be very small ($\alpha_0 \ll 1$). It should be emphasized that tuning of model parameters is required to obtain the very small α_0. The reason is that the Higgs potential in GHU is generated only by radiative corrections and is sensitive to model parameters. Therefore, it is nontrivial whether the Higgs field develops a nonzero VEV as mentioned before. We also have to adjust the mass of the Higgs boson to its experimental value by further tuning the model parameters. Namely, the mass of the discovered Higgs boson and the tadpole condition strongly constrain the shape of the Higgs potential. It plays s significant role in our analysis.

5.3 Experimental constraints

Before the discussion of the triple Higgs boson coupling, we check several constraints on the model parameters. In general, GHU should be aware of the experimental limit on the masses of the top quark and the Higgs boson, and the compactification scale. In the earlier study performed in Ref. [34], the Lorentz-violating parameters k_j are introduced to increase the masses of the top quark and the Higgs boson. In Refs. [34,35], the ρ parameter and the $Zb_L\bar{b}_L$ coupling are investigated and turned out to be strongly affected by the KK modes of the bulk fields, such as the sine modes of $A_X^{(n)}$ and the lightest modes of the bulk fermions, typically 1st Ψ_t mode. An improved analysis of electroweak observables in a slightly refined $SU(3)_w$ GHU model gives the lower bound on the compactification scale as $1/R \gtrsim 5$ TeV for $m_h = 125$ GeV [35]. Since the works of Refs. [34,35] were done before the discovery of the Higgs boson in 2012, we revisit the $SU(3)_w$ model to investigate viable parameter regions.

First, let us discuss the necessity of the antiperiodic fermion. Even if there was no antiperiodic fermion, it seems to be possible to satisfy the experimental constraints, because there are a lot of model parameters. We will mainly check the two experimental constraints, compactification scale and mass of the top boson. Then, we analyze the potential minimum, α_0, by taking the following model parameter regions:

$$\begin{cases} 0.7 < k_t < 3 \\ 0.7 < k_b < 3 \end{cases}, \quad \begin{cases} 0.5 < \lambda^t < 1.5 \\ 5 < \lambda^b =< 7 \end{cases}, \quad \begin{cases} 0.5 < \epsilon_{1,2}^t < 9.5 \\ 0.5 < \epsilon_{1,2}^b < 9.5 \end{cases} \tag{5.105}$$

Since 5D mass parameters λ^i control the scale of the brane fermion masses roughly as $m^i \propto e^{-\lambda^i}$, we set $\lambda^t = 1$ and $\lambda^b = 6.9$. The numerical results is shown in Fig. 5.4. From the analytical expression (5.93), the parameter region $k_t < 1$ is excluded because it cannot satisfy the constraint of the top mass. In the remaining parameter regions, we predict very small

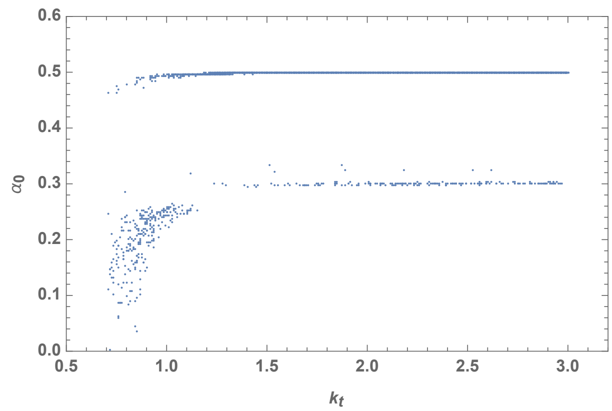

Figure 5.4: Potential minimum in the model without the antiperiodic fermion

compactification scale that is experimentally unacceptable. In particular, Figure 5.4 shows that the $V_{\Psi_t}(2\alpha)$, with a minimum at $\alpha \simeq 0.3$, or $V_{\Psi_t}(\alpha) + V_{\Psi_b}(\alpha)$, with a minimum at $\alpha = 0.5$, mainly contribute to the Higgs potential. Therefore, we need the contribution of the antiperiodic fermion to cancel these contributions.

Next, we perform a random scan of model parameters, and numerically calculate the masses of the top quark and the Higgs boson as well as the compactification scale from the effective potential defined above. According to the previous study [34], we scan the following model parameter regions :

$$\begin{cases} 1.5 < k_t < 2.5, \\ 1.25 < k_b < 2.25, \\ 1.1 \times k_t < k_A < 1.5 \times k_t, \end{cases} \quad \begin{cases} 0.5 < \lambda^t < 1.5, \\ 5 < \lambda^b < 7, \\ 0.75 < \lambda^A < 3.5, \end{cases} \quad \begin{cases} 0.75 < \epsilon^t_{1,2} < 7.5, \\ 2 < \epsilon^b_{1,2} < 7. \end{cases} \tag{5.106}$$

Figure 5.5 shows the numerical results of the predicted values of the top quark mass m_t and the Higgs boson mass m_h. The left (right) scatter plot shows cases for compactification scales larger than 800 GeV (5 TeV). Point colors are assigned to each range of the Lorentz violating parameter k_t: Blue points stands for $2.25 < k_t < 2.5$, green for $2 < k_t < 2.25$, red for $1.75 < k_t < 2$, and black for $1.5 < k_t < 1.75$. The left plot corresponds to the results presented in Ref. [34]. The effective potential roughly scales as the fourth power of k's. This is reflected in the plots that show the mass of the Higgs boson is positively correlated with k_t. As the compactification scale $1/R$ is increased, a finer tuning is required for obtaining the correct weak-scale VEV. Therefore, as shown in the right panel of Fig. 5.5, the number of allowed parameter sets is smaller if we impose the experimental lower bound on the compactification scale, $1/R > 5$ TeV [35]. The bottom line is that there is still enough room for reproducing the measured values of the masses of the top and Higgs boson for $1.5 < k_t < 2$. However, since this analysis uses the top quark and Higgs masses at the weak scale, an RGE analysis is necessary to ensure that these masses are reproduced correctly at the compactification scale. It is future work.

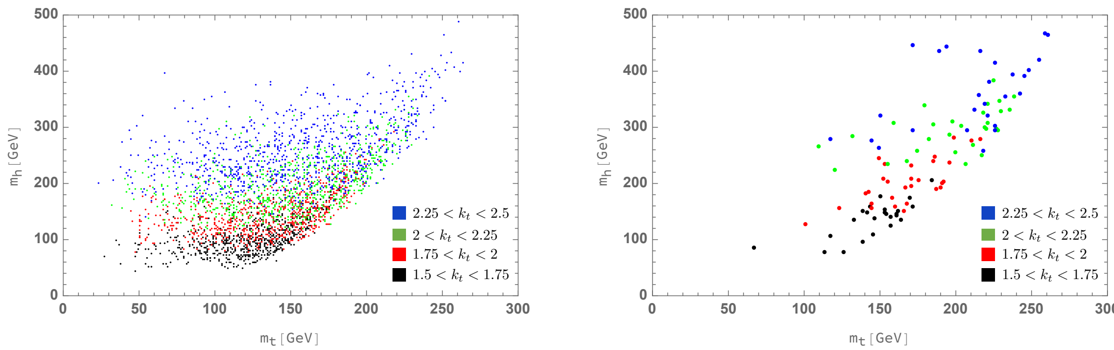

Figure 5.5: Scatter plots of the predicted values of the top quark mass m_t and the Higgs boson mass m_h for $1/R > 800$ GeV (left) and for $1/R > 5$ TeV (right). Point colors are assigned to each range of the Lorentz violating parameter k_t: Blue points stands for $2.25 < k_t < 2.5$, green for $2 < k_t < 2.25$, red for $1.75 < k_t < 2$, and black for $1.5 < k_t < 1.75$.

5.4 Analysis of the triple Higgs boson coupling

After imposing the experimental constraints mentioned above, we will discuss the deviation of the triple Higgs boson coupling λ_{hhh} from the SM value $\lambda_{hhh}^{\rm SM}$. Before moving on to the calculations, we should mention the experimental accuracy of λ_{hhh}. The Higgs pair production at the LHC Run2 imposes constraints on the triple Higgs boson coupling as $-5.0 < \lambda_{hhh}/\lambda_{hhh}^{\rm SM} < 12.0$ for ATLAS [48] and $-11.8 < \lambda_{hhh}/\lambda_{hhh}^{\rm SM} < 18.8$ for CMS [49] at the 95% confidence level. It is expected that the accuracy of the triple Higgs boson coupling measurement will be drastically improved in future collider experiments. At the High-Luminosity LHC, $0.52 < \lambda_{hhh}/\lambda_{hhh}^{\rm SM} < 1.5$ and $0.57 < \lambda_{hhh}/\lambda_{hhh}^{\rm SM} < 1.5$ with and without systematic uncertainties ($1\ \sigma$), respectively [16]. At the ILC with a center-of-mass energy of $\sqrt{s} = 1$ TeV and integrated luminosity of $L = 4\ \mathrm{ab}^{-1}$, a precision of 10% is estimated [50]. At the CLIC with $\sqrt{s} = 3$ TeV and $L = 5\ \mathrm{ab}^{-1}$, the triple Higgs boson coupling will be measured with a relative uncertainty of -8% to 11% ($1\ \sigma$) [51]. In particular, the lepton colliders have a cleaner background than the proton colliders, which allows accurate determination of λ_{hhh}.

In the following, we define the deviation of the triple Higgs boson coupling as

$$\Delta\lambda = \frac{\lambda_{hhh} - \lambda_{hhh}^{\rm SM}}{\lambda_{hhh}^{\rm SM}}. \tag{5.107}$$

where $\lambda_{hhh}^{\rm SM}$ only include the top quark effect as a dominant one-loop correction. Taking into account the scale dependence of top quark and Higgs masses, we use the model parameters in the range of 152 GeV $< m_t <$ 182 GeV and 110 GeV $< m_h <$ 140 GeV in our analysis. Fig. 5.6 shows the compactification scale dependence on the deviation of the triple Higgs boson coupling. The orange band denotes the $1\ \sigma$ accuracy expected at the ILC. From Fig. 5.6, we can see that the deviation $\Delta\lambda$ is primarily characterized by the compactification scale $1/R$. At very large compactification scales, the deviation of the triple Higgs boson coupling almost vanishes. Namely, we expect

$$\lim_{1/R\to\infty} \Delta\lambda = 0. \tag{5.108}$$

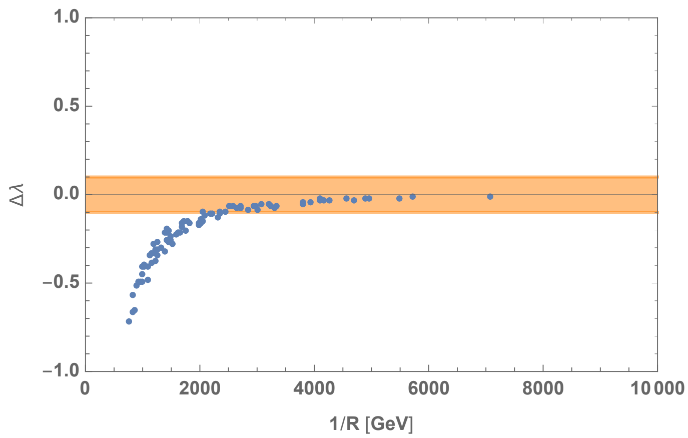

Figure 5.6: Compactification scale dependence of the deviation of the triple Higgs boson coupling $\Delta\lambda$. The orange band denotes the 1 σ accuracy expected at the ILC.

If a significant deviation is observed at the future colliders, we need the additional extension to our GHU model compared with the current constraint on the compactification scale, $1/R >$ 5 TeV.

We next investigate which field contributions have a significant impact on the deviation. For a moment, we decompose the deviation into parts that derive from each field as

$$\Delta\lambda_{hhh} = \sum_i \Delta\lambda_i = \sum_i \frac{\delta\lambda_i - \lambda_{hhh}^{\mathrm{SM}}}{\lambda_{hhh}^{\mathrm{SM}}}, \tag{5.109}$$

where $\delta\lambda_i$ denotes the contribution from a field i,

$$\delta\lambda_i = \left.\frac{\partial^3 V_i}{\partial\alpha^3}\right|_{\alpha=\alpha_0} - \frac{3}{\alpha}\left.\frac{\partial^2 V_i}{\partial\alpha^2}\right|_{\alpha=\alpha_0} + \frac{3}{\alpha^2}\left.\frac{\partial V_i}{\partial\alpha}\right|_{\alpha=\alpha_0}. \tag{5.110}$$

As a result, the deviation mainly consists of the following three contributions: $\Delta\lambda_{\Psi_t(2\alpha)}, \Delta\lambda_t$ and $\Delta\lambda_{\Psi_A(2\alpha)}$. In Fig. 5.7, green points shows the each contribution to the deviation $\Delta\lambda_i$ for $\Psi_t(2\alpha)$ (upper left panel), t (upper right), $\Psi_A(2\alpha)$ (lower left), and their sum (lower right),

$$\Delta\lambda_{\Psi_t(2\alpha)+t+\Psi_A(2\alpha)} = \frac{\delta\lambda_{\Psi_t(2\alpha)} + \delta\lambda_t + \delta\lambda_{\Psi_A(2\alpha)} - \lambda_{hhh}^{\mathrm{SM}}}{\lambda_{hhh}^{\mathrm{SM}}}. \tag{5.111}$$

For comparison, the total deviation parameter $\Delta\lambda$ is also plotted (red). Particles with larger q values imply the stronger couplings to the Higgs field, and give dominant contributions to the deviation of the triple Higgs boson coupling. $\Psi_t(2\alpha)$ has the lightest bulk fermion mode and largest q and thus give the largest contribution to the deviation.

From the above numerical results, the deviation of the triple Higgs boson coupling converges to zero as the compactification scale increases. It means that the shape of our GHU Higgs potential is close to the SM Higgs potential at around the VEV for small α_0 although their tree-level formulas are completely different. Here, we analyze the shape of the Higgs potential at small α to investigate this mystery. First, we consider the contribution of the brane

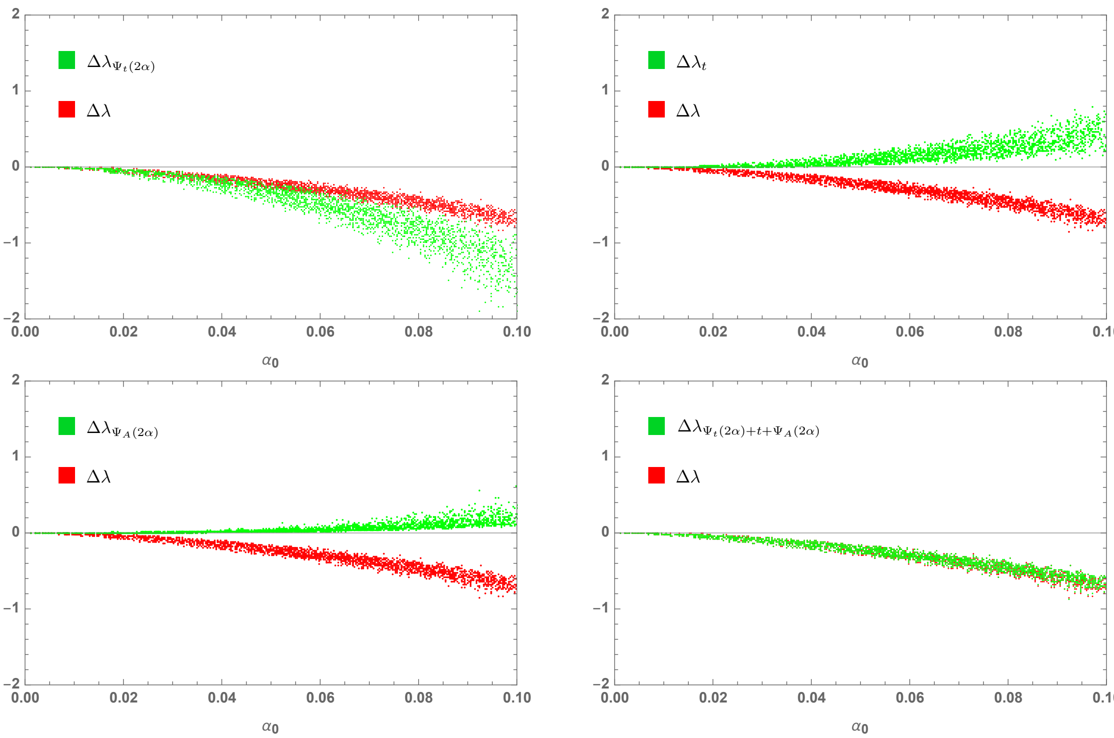

Figure 5.7: Each contribution to the deviation $\Delta\lambda_i$ for $\Psi_t(2\alpha)$ (upper left panel), t (upper right), $\Psi_A(2\alpha)$ (lower left), and their sum (lower right). For comparison, the total deviation $\Delta\lambda$ is also plotted (red).

fermions to the effective potential. Although the field-dependent masses of the brane fermions are complicated functions of α as discussed in Sec. 5.1, they can be approximated by linear functions of α for $\alpha \ll 1$ as is the case for the fermions in the SM. Thus they are consistent with SM contributions. For the bulk contributions, we find the following representations by expanding on α:

$$\begin{aligned} V_g(q\alpha) &\simeq \frac{1}{R^4}\left\{C_1^g q^2\alpha^2 + C_2^g q^4\alpha^4 + C_3^g q^4\alpha^4 \ln(q^2\alpha^2) + C_4^g q^6\alpha^6 + \mathcal{O}(\alpha^6)\right\} \\ V_f(q\alpha) &\simeq \frac{1}{R^4}\left\{C_1^\Psi q^2\alpha^2 + C_2^\Psi q^4\alpha^4 + C_4^\Psi q^6\alpha^6\right\} \end{aligned} \tag{5.112}$$

where the coefficients C's are the dimensionless coefficients formed by model parameters and naturally have a same order. The logarithmic term in (5.112) comes from the fourth derivative of V_g:

$$\begin{aligned} \frac{\partial^4 V_g}{\partial\alpha^4} &\propto \ln(1-\mathrm{e}^{-2i\pi q\alpha}) + \ln(1-\mathrm{e}^{2i\pi q\alpha}) \\ &\sim \ln(2i\pi q\alpha) + \ln(-2i\pi q\alpha) = 2\ln(2\pi) + \ln(q^2\alpha^2) \end{aligned} \tag{5.113}$$

Since it is originated from the zero modes of the bulk gauge fields, it takes the same value as that of the SM gauge fields. On the other hand, this term not appears in (5.112) due to the

5D mass parameters of the bulk fermions:

$$\frac{\partial^4 V_f}{\partial \alpha^4} \to \ln(1 - \mathrm{e}^{2\frac{k}{\lambda} - 2i\pi q\alpha}) + \ln(1 - \mathrm{e}^{2\frac{k}{\lambda} + 2i\pi q\alpha}) \sim 2\ln(1 - \mathrm{e}^{2\frac{k}{\lambda}}) \tag{5.114}$$

Keeping this in mind, the GHU Higgs potential can be expanded with respect to α as

$$V_{\text{eff}}(\alpha) = \left(\frac{1}{R}\right)^4 \left(-\widetilde{A}\alpha^2 + \widetilde{B}\alpha^4 + \widetilde{C}\alpha^4 \ln\frac{\alpha^2}{\alpha_0^2} + \widetilde{D}\alpha^6 + \mathcal{O}(\alpha^8)\right), \tag{5.115}$$

where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ are dimensionless parameters that are functions of the model parameters. Let us focus on the region where α/R is around the weak scale for a successful EWSB. Using the relation $\alpha/R = g\phi/2$, Eq. (5.115) is rewritten with respect to ϕ as

$$V_{\text{eff}}(\phi) \simeq -\left(\frac{1}{R}\right)^2 A\phi^2 + B\phi^4 + C\phi^4 \ln\frac{\phi^2}{v^2} + \left(\frac{1}{R}\right)^{-2} D\phi^6, \tag{5.116}$$

where

$$A = \left(\frac{g}{2}\right)^2 \widetilde{A}, \quad B = \left(\frac{g}{2}\right)^4 \widetilde{B}, \quad C = \left(\frac{g}{2}\right)^4 \widetilde{C}, \quad D = \left(\frac{g}{2}\right)^6 \widetilde{D}. \tag{5.117}$$

From the tadpole and Higgs mass conditions for the Higgs potential, we find

$$0 = -\left(\frac{1}{R}\right)^2 \frac{A}{2v^2} + B + \frac{C}{2} + \frac{3}{2}\left(\frac{1}{R}\right)^{-2} Dv^2, \tag{5.118}$$

and

$$m_h^2 = 4\left(\frac{1}{R}\right)^2 A + 8Cv^2 + 12\left(\frac{1}{R}\right)^{-2} Dv^4. \tag{5.119}$$

Namely, two of the four coefficients are rewritten by the experimental values. There remain the two independent parameters. There must be a gap of at least 10 times between the compactness scale and the VEV. Therefore, for the coefficient of the first term to be on the EW scale, parameter A must be very small due to the cancellation between the model parameters. It is called the little hierarchy problem. In our GHU model, this cancellation is accomplished by introducing the additional bulk fields (Ψ_A, $\tilde{\Psi}_A$). Eliminating the parameters A and B using the tadpole and mass conditions, Eqs. (5.118) and (5.119), the triple Higgs boson coupling becomes an implicit function of the parameters A and B and is written as

$$\lambda_{hhh} = \frac{3}{v}\left[m_h^2 + \frac{16}{3}Cv^2 + 16\left(\frac{1}{R}\right)^{-2} Dv^4\right]. \tag{5.120}$$

Since the term proportional to $\phi^4 \ln\phi^2$ arises from loop diagrams in which only SM like particles are involved, the value of the coefficient C is surprisingly the same as that in the SM. Therefore, the difference with the SM only appears in the third term. Then the its deviation from the SM takes the form of

$$\Delta\lambda = \frac{48Dv^3}{\lambda_{hhh}^{\text{SM}}}\left(\frac{1}{R}\right)^{-2}. \tag{5.121}$$

Namely, in the cases when the ϕ^6 term and higher order ones are negligibly small, the shape of the potential and the triple Higgs boson coupling should be close to the SM ones irrespective of the origin of the potential. In deriving the Analytical expression for the effective potential, we have not assumed specific bulk fields or Lorentz-violating parameters. We emphasize that such a conclusion is applicable to a wide range of GHU models with a flat extra dimension that reduces to the Higgs sector with one Higgs doublet below the compactification scale. It has not been manifested in earlier works. Therefore, if a significant deviation of the triple Higgs boson coupling is observed at the future collider experiments, we have to extensively modify the GHU framework.

Chapter 6

Two Higgs doublet model from the extra dimensions

In this chapter, we will introduce the 2HDM originated from the extra dimension as research that utilizes the extra dimension. The two-Higgs-doublet model (2HDM) is a simple extension of the standard model (SM) which only adds one another Higgs doublet, but it has rich phenomenology not expected in the SM. In particular, various studies have been performed to realize the electroweak baryogenesis scenario [52–55], muon $g-2$ [56–60] and neutrino mass [61–63]. The following three assumptions are conventionally imposed in the 2HDM: (1) softly broken Z_2 symmetry, (2) CP invariance in the Higgs potential, and (3) custodial symmetry or $m_{H^\pm} = m_A$. However, these assumptions can not be justified in the framework of the 2HDM, and their origin is left to some paradigms. For example, it is well known that the Type-II 2HDM appears as a low-energy effective theory of the Minimal Supersymmetric Standard Model [64]. There is also work on extending electroweak symmetry to naturally derive these three assumptions for all four types of the 2HDMs [65]. Nevertheless, the relationship between each type of the 2HDM and the paradigms is not well understood. In our study, we propose a model in which each type of the 2HDM appears as an effective theory by introducing an extra dimension without imposing the Z_2 symmetry. Namely, we have to avoid the dangerous FCNC by using physics of extra dimensions instead of the Z_2 symmetry. To achieve it, we use the technique, which localizes the field on the extra space proposed in Ref. [66]. With this technique, the coupling between fields localized at different points in the extra space can be exponentially suppressed by the extra-dimensional integration. This idea has been applied to some studies: suppression of the proton decay [66–68] and explanation of the fermion mass hierarchy [69, 70]. We will use this idea to either one of the two Higgs couplings with ϕ_1, ϕ_2 to avoid the dangerous FCNC. First, we define the kink configurations used for field localization in Sec. A. In Sec. B, we discuss the localization of fermions and the Higgs in 5D. Then we will construct a toy model for the 5D case and investigate the problems. In order to make a realistic model, we will move to the 6D space-time. In Sec. C, we discuss the localization of the field in 6D. Finally, we build a viable model and see how the problems in 5D can be solved in Sec. D. Throughout this chapter, we do not assume a concrete extra-dimensional structure such as S^1/Z_2 in GHU but only consider its length and boundary conditions on edge.

6.1 Kink configuration

As preparation for discussing field localization, we will introduce kink coordination. A kink is a nontrivial 1+1 dimensional soliton that has different values at both ends of the space. As a simple example, we will show a kink that changes only in the extra-dimensional direction in the 5D spacetime. It is usually described by the following 5D action:

$$S_S = \frac{1}{g_Y^2} \int d^4x \; dx_5 \left[\frac{1}{2} \left(\partial_M S \right)^2 - \frac{\lambda^2}{2} \left(S^2 - v^2 \right)^2 \right] \tag{6.1}$$

Since the scalar potential of eq.(6.1) has the two minimum at $S = \pm v$, the kink has multiple vacuum states. The equation of motion for $S(x_5)$ is derived from eq.(6.1) as

$$\partial_5^2 S(x_5) = 2\lambda^2 \left(S(x_5)^2 - v^2 \right) S(x_5) \; . \tag{6.2}$$

If we consider the length of the extra dimension to be infinite at this instant and impose the boundary condition $S(\infty) = +v, S(-\infty) = -v$, the general solution to this equation is given by

$$S(x_5) = v \tanh[\lambda v(x_5 - l)] \; , \tag{6.3}$$

where l represents the point in which the two ground states meet. By inserting this solution into Eq. (6.1), we find the energy density of the kink $\mathcal{E}(x_5)$ as

$$S = -\int d^4x \; dx_5 \mathcal{E}(x_5) \; , \quad \mathcal{E}(x_5) = \left(\frac{v^4}{g_Y^2} \right) \frac{\lambda^2}{\cosh^4[\lambda v(x_5 - l)]} . \tag{6.4}$$

It means that kink energies are concentrated at $x_5 = l$ because the potential wall is crossed where the two ground states meet. We use this energy to localize the field. Since the behavior of the kink at $x_5 = l$ is significant in the following discussion, we approximate the general solution as

$$S(x_5) \simeq 2\mu^2 (x_5 - l) \; , \tag{6.5}$$

where $\mu^2 = \lambda v^2/2$ denotes the typical kink scale. This approximate representation is valid when the size of the extra dimension is finite.

[61]

6.2 Field Localization in 5D

Now let us discuss how the fermions and scalar fields are localized in 5D spacetime using the kink. We first consider the one 5D fermion Ψ coupled to the kink $S(y)$, described by the action

$$S_\Psi = \int d^4x \, dy \, \bar{\Psi} \left[i\Gamma^\mu \partial_\mu + i\Gamma^5 \partial_y + y_S S(y) \right] \Psi \; , \tag{6.6}$$

where we use the kink coupling y_S and following 4×4 gamma matrices in 5D that satisfy $\{\Gamma^M, \Gamma^N\} = g^{MN} \equiv \mathrm{diag}(+1, -1, -1, -1, -1)$:

$$\Gamma^\mu = \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \; , \quad \Gamma^5 = i\gamma^5 = i \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{6.7}$$

By using the separation of variables for Ψ

$$\Psi(x_\mu, y) = \sum_{n=0}^{\infty} \psi_R^{(n)}(x_\mu)\,\chi_n^{(+)}(y) + \sum_{n=0}^{\infty} \psi_L^{(n)}(x_\mu)\,\chi_n^{(-)}(y) \ , \tag{6.8}$$

we can separate the Dirac equation derived from the action (6.6) into the chirality of the fermion $\psi_{R,L}^{(n)}$. For example, multiplying the Dirac equation by the right-hand chirality operator $P_R = (1+\gamma^5)/2$ from the left, we find

$$i\Gamma^\mu \partial_\mu \left(\sum_n \psi_L^{(n)}(x_\mu)\,\chi_n^{(-)}(y) \right) = \left[-i\Gamma^5 \partial_y + y_S S(y) \right] \left(\sum_n \psi_R^{(n)}(x_\mu)\,\chi_n^{(+)}(y) \right) \ . \tag{6.9}$$

Since this equation holds identically for any $\psi^{(n)}(x_\mu)$ and $\chi_n(y)$, we have

$$\begin{aligned} i\Gamma^\mu \partial_\mu \psi_L^{(n)}(x_\mu)\,\chi_n^{(-)}(y) &= m_n \psi_R^{(n)}(x_\mu)\,\chi_n^{(-)}(y) \ , \\ \left[-i\Gamma^5 \partial_y + y_S S(x_5) \right] \psi_R^{(n)}(x_\mu)\,\chi_n^{(+)}(y) &= m_n \psi_R^{(n)}(x_\mu)\,\chi_n^{(-)}(y) \ . \end{aligned} \tag{6.10}$$

Considering the same way for P_L, we get the two sets of equations for $\psi^{(n)}$ and χ_n respectively as a result:

$$\begin{cases} Q_+ \chi_n^{(+)}(y) = m_n \chi_n^{(-)}(y) \\ Q_- \chi_n^{(-)}(y) = m_n \chi_n^{(+)}(y) \end{cases} , \quad \begin{cases} i\gamma^\mu \partial_\mu \psi_R^{(n)} = m_n \psi_L^{(n)} \\ i\gamma^\mu \partial_\mu \psi_L^{(n)} = m_n \psi_R^{(n)} \end{cases} \tag{6.11}$$

where we define $Q_\pm \equiv \mp \partial_y + y_S S(y) = Q_\mp^\dagger$ by using $\gamma^5 \psi_{R/L} = \pm \psi_{R/L}$. Furthermore, equations for $\chi_n^{(\pm)}$ can be summarized in the form of matrix equation

$$\hat{Q} \begin{pmatrix} \chi_n^{(+)} \\ \chi_n^{(-)} \end{pmatrix} = m_n \begin{pmatrix} \chi_n^{(+)} \\ \chi_n^{(-)} \end{pmatrix} \ , \quad \text{where } \hat{Q} \equiv \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix} = i\sigma^2 \partial_y + \sigma^1 y_S S(y) \ . \tag{6.12}$$

Then, we find the Schrödinger equation for the wave-functions $\chi_n^{(\pm)}$

$$\begin{aligned} H^{(\pm)} \chi_n^{(\pm)} &= m_n^2 \chi_n^{(\pm)} \ , \\ H^{(\pm)} &= Q_\pm^\dagger Q_\pm = -\partial_y^2 + y_S^2 S^2(y) \pm y_S \partial_y S(y) \ . \end{aligned} \tag{6.13}$$

By using the approximating kink, Eq. (6.13) becomes

$$\left[-\partial_y^2 + 4y_S^2 \mu^4 (y-l)^2 \pm 2y_S \mu^2 \right] \chi_n^{(\pm)}(y) = m_n^{\pm 2} \chi_n^{(\pm)}(y) \,. \tag{6.14}$$

This differential equation is the same as the Schrödinger equation of a Harmonic oscillator,

$$\frac{d^2 \psi_n(x)}{dx^2} - m^2 \omega^2 \psi_n(x) = -2mE_n \psi_n(x) \tag{6.15}$$

with a solution

$$\begin{aligned} \psi_n(x) &= N H_n(x) \exp\left(-\frac{m\omega}{2} x^2 \right) \ , \\ E_n &= \omega \left(n + \frac{1}{2} \right) \ , \end{aligned} \tag{6.16}$$

where H_n is the Hermite polynomial, N is the normalization factor, and $n = 0, 1, 2, \cdots$. Using this well-known result, if $y_S\mu^2 > 0$, we find

$$\begin{aligned} \chi_n^{(\pm)}(y) &= N H_n(y) \exp\left[-y_S\mu^2(y-l)^2\right] , \\ m_n^{\pm 2} &= 4 y_S \mu^2 \left(n + \frac{1}{2} \pm \frac{1}{2}\right) , \end{aligned} \tag{6.17}$$

as the solution for Eq. (6.14). Thus, for $y_S\mu^2 > 0$, the left-handed fermion only has a massless mode $m_n = 0$ and its wave function is given by the Gaussian centered at $y = l$:

$$\chi_0^{(-)}(y) = \left(\frac{2y_S\mu^2}{\pi}\right)^{1/4} \exp\left[-y_S\mu^2(y-l)^2\right] \tag{6.18}$$

On the other hand, for $y_S\mu^2 < 0$, the equations for the right-handed and left-handed fermions are exchanged and right-handed fermion only has a massless mode. As we have seen in Sec. 4.3, the free 5D fermions have mixed left and right chirality due to the 5D Lorentz symmetry. However, thanks to the kink coupling, chiral asymmetry can be reproduced at the low energy. Therefore, we can localize the chiral fermion at any point in the extra space. Then, we will mention the validity of the approximation of the kink. For the non-approximating kink, Eq. (6.13) becomes

$$\begin{aligned} 0 &= \left[\partial_y^2 - y_S^2 S^2(y) \mp y_S(\partial_y S(y)) + \omega_n^2\right] \chi_n^{(\pm)}(y) \\ &= \partial_y^2 \chi_n^{(\pm)}(y) + \left(\omega_n^2 - y_S^2 v^2 + \frac{v^2(y_S^2 \mp y_S\lambda)}{\cosh^2[\lambda v(y-l)]}\right) \chi_n^{(\pm)}(y) . \end{aligned} \tag{6.19}$$

This equation has a same form to the Gauss's hypergeometric differential equation. After complicated calculations performed in [], we obtain similar results to the approximating kink case. For $y_S/\lambda > 0$ ($y_S/\lambda < 0$), the left-handed (right-handed) zero-mode wave function is normalized as

$$\chi_0(y) = \frac{(\lambda v)^{1/2}}{\pi^{1/4}} \sqrt{\frac{\Gamma(s+1/2)}{\Gamma(s)}} \frac{1}{\cosh^s[\lambda v(y-l)]} , \tag{6.20}$$

where we use $s = \mp y_S/\lambda$. On the other hand, right-handed (left-handed) one is not normalizable. Compared to Eq. (6.17), this wave function is more widely spread on the extra space as shown in Fig. 6.1.

Then we extend this discussion to the scalar fields such as the Higgs fields. The action for the scalar field Φ coupled to the background kink S is given by

$$S_\Phi = \int d^4x\, dy \left(\frac{1}{2}\left(\partial_\mu\Phi\partial^\mu\Phi - \partial_y\Phi\partial_y\Phi\right) - \frac{1}{2}\lambda_S\Phi^2S^2 - V(\Phi)\right) . \tag{6.21}$$

The equation of motion for Φ is given by

$$\partial_\mu\partial^\mu\Phi - \partial_y^2\Phi + \lambda_S S^2\Phi + M^2\Phi = 0 , \tag{6.22}$$

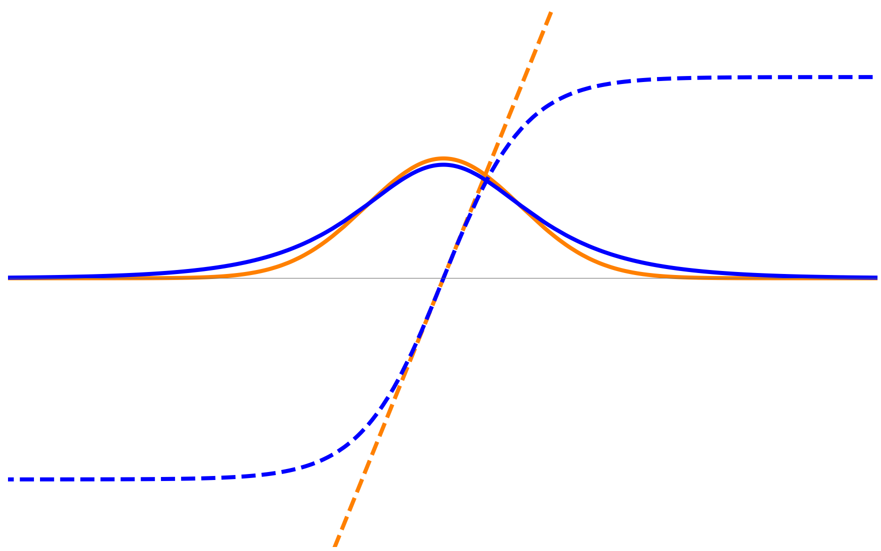

Figure 6.1: Localized wave functions by using the approximating (orange) and non-approximating (blue) kinks.

where we replace $V(\Phi)$ with $\frac{1}{2}M^2\Phi^2$ to keep only the linear terms in Eq. (6.22). By using the separation of variables for Φ

$$\Phi(x_\mu, y) = \sum_{n=0}^{\infty} \phi^{(n)}(x_\mu)\, \chi_\Phi^n(y) \ , \tag{6.23}$$

as in the case of the fermions, we can separate the Eq. (6.22) for Φ as follow:

$$\begin{aligned} 0 &= \partial_\mu \partial^\mu \phi^{(n)} + m_n^2 \phi^{(n)} \\ 0 &= \partial_y^2 \chi_\Phi^n - \lambda_S S^2(y) - (M^2 - m_n^2)\chi_\Phi^n \end{aligned} \tag{6.24}$$

By using the approximating kink as in the case of the fermion, the equation for χ_Φ^n becomes

$$\left[-\partial_y^2 + 4\lambda_S \mu^4 (y-l)^2\right] \chi_\Phi^n = (m_n^2 - M^2)\chi_\Phi^n \, . \tag{6.25}$$

In contrast to the fermion case, it does not contain a first derivative term. This equation is also the same as the Schrödinger equation of a Harmonic oscillator. Thus, for $\lambda_S > 0$, the solution of this equation is given by

$$\begin{aligned} \chi_\Phi^n(y) &= N H_n(y) \exp\left[-\sqrt{\lambda_S \mu^4}(y-l)^2\right] \ , \\ m_n^2 - M^2 &= 4\sqrt{\lambda_S}\mu^2 \left(n + \frac{1}{2}\right) \ . \end{aligned} \tag{6.26}$$

In particular, the zero-mode wave function and its mass are given by

$$\begin{aligned} \chi_\Phi^0(y) &= \left(\frac{2\sqrt{\lambda_S}\mu^2}{\pi}\right)^{1/4} \exp\left[-\sqrt{\lambda_S}\mu^2(y-l)^2\right] \ , \\ m_0^2 &= M^2 + 2\sqrt{\lambda_S}\mu^2 \ . \end{aligned} \tag{6.27}$$

Because of the difference in the equation, the right-hand side of mass relation in Eq. (6.26) does not become zero for $n=0$. If there is a cancellation between the M^2 and $2\sqrt{\lambda_S}\mu^2$, we can get the mass parameter m_0^2 below the electroweak scale. Thus, only ϕ_0 remains in the 4D effective theory.

6.3 Higgs couplings in 5D

In the previous section, we learned that fermions and scalar fields are localized on the extra space. In order to make a difference between the Yukawa coupling with the two Higgs doublets, these two fields should be separated. Therefore, we use two kinks with different centers to localize ϕ_1^0 and ϕ_2^0 to $y=l_1$ and $y=l_2$, respectively. For simplicity, we unify the width of the wave function to $1/\mu$. We also define the ratio of the extra dimension scale $1/2L$ to the kink scale ϵ as $\epsilon\mu = 1/2L$. From these remarks, we prepare the following fifth dimensional wave functions:

$$\begin{aligned}
\phi_1^0(x_\mu) : \chi_{\Phi_1}^0 &= \left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2(y-l_1)^2\right] \quad \text{with} \quad m_1^2 = M_1^2 + 2\mu^2 \ , \\
\phi_2^0(x_\mu) : \chi_{\Phi_2}^0 &= \left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2(y-l_2)^2\right] \quad \text{with} \quad m_2^2 = M_2^2 + 2\mu^2 \ , \\
u_R^0(x_\mu) : \chi_{U_R}^0 &= \left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2(y-l_u)^2\right] \quad \text{with} \quad m_{u_R^0}^2 = 0 \ , \\
d_R^0(x_\mu) : \chi_{D_R}^0 &= \left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2(y-l_d)^2\right] \quad \text{with} \quad m_{d_R^0}^2 = 0 \ , \\
e_R^0(x_\mu) : \chi_{E_R}^0 &= \left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2(y-l_e)^2\right] \quad \text{with} \quad m_{e_R^0}^2 = 0 \ , \\
q_L^0(x_\mu) : \chi_{Q_L}^0 &= \sqrt{\epsilon\mu} \quad \text{with} \quad m_{q_L^0}^2 = 0 \ , \\
l_L^0(x_\mu) : \chi_{L_L}^0 &= \sqrt{\epsilon\mu} \quad \text{with} \quad m_{l_L^0}^2 = 0 \ ,
\end{aligned} \tag{6.28}$$

The right-handed fermions are localized on either of l_1 or l_2. The left-handed fermions are not coupled to the kink for simplicity, and therefore the zero mode is flat on the extra space. By using above representation, the Yukawa sector in 4D effective theory is given by

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} \supset\ & y_{u\Phi_1}\bar{q}_L^0\phi_1^{0*}u_R^0 + y_{d\Phi_1}\bar{q}_L^0\phi_1^0 d_R^0 + y_{e\Phi_1}\bar{l}_L^0\phi_1^0 e_R^0 \\
&+ y_{u\Phi_2}\bar{q}_L^0\phi_2^{0*}u_R^0 + y_{d\Phi_2}\bar{q}_L^0\phi_2^0 d_R^0 + y_{e\Phi_2}\bar{l}_L^0\phi_2^0 e_R^0 + \text{h.c.} \ ,
\end{aligned} \tag{6.29}$$

where the 4D effective Yukawa couplings $y_{f\Phi_i}$ are defined by the extra dimensional integration as

$$y_{f\Phi_i} = y'_{f\Phi_i}\int dy\, \chi_{F_L}^0\chi_{\Phi_i}^0\chi_{F_R} = y'_{f\Phi_i}\sqrt{\epsilon\mu}\int dy\, \chi_{\Phi_i}^0\chi_{F_R}^0 \ . \tag{6.30}$$

$y'_{f\Phi_i}$ is a 5D Yukawa couplings which have a mass dimension $m^{1/2}$. Note that there are only two patterns for the integration in Eq. (6.30): (a) ϕ_i and f_R are localized at the same point. (b) ϕ_i

	ϕ_1	ϕ_2	q_L	l_L	u_R	d_R	e_R
Type-I	+	−	+	+	−	−	−
Type-II	+	−	+	+	−	+	+
Type-X	+	−	+	+	−	−	+
Type-Y	+	−	+	+	−	+	−

	ϕ_1^0	ϕ_2^0	q_L^0	l_L^0	u_R^0	d_R^0	e_R^0
Type-I	S_1	S_2	−	−	S_2	S_2	S_2
Type-II	S_1	S_2	−	−	S_2	S_1	S_1
Type-X	S_1	S_2	−	−	S_2	S_2	S_1
Type-Y	S_1	S_2	−	−	S_2	S_1	S_2

Table 6.1: Assignment of the Z_2 parity for each type of the 2HDM (left). Corresponding kink couplings (right) in 5D case.

and f_R are localized at different points. In particular, the latter is exponentially suppressed by the extra-dimensional integration:

$$\begin{aligned} &(a)\ :\ y_{f\phi_i} = y'_{f\phi_i}\sqrt{\epsilon\mu} \\ &(b)\ :\ y_{f\phi_i} = y'_{f\phi_i}\sqrt{\epsilon\mu}\,\exp\left[-\frac{1}{2}\mu^2 l^2\right] \end{aligned} \tag{6.31}$$

Here, we use the relative distance $l = |l_1 - l_2|$. Therefore, the Yukawa coupling ratio between $y_{f\phi_1}$ and $y_{f\phi_2}$ take the form of $(b)/(a)$ or $(a)/(b)$. When U_R couples to S_2, the Yukawa coupling ratio for u_R^0 is given by

$$r^u_{\phi_1/\phi_2} \equiv \frac{y_{u\phi_1}}{y_{u\phi_2}} = \frac{y'_{u\phi_1}}{y'_{u\phi_2}}\exp\left[-\frac{1}{2}\mu^2 l^2\right]\,. \tag{6.32}$$

It means that the fermions strongly couple with Higgs doublet that localize at the same point, and the dangerous FCNC can be avoided if the Higgs doublets are separated enough from each other. For example, for $\mu l = 5$ and $y'_{u\phi_1} = y'_{u\phi_2}$, we find

$$r^u_{\phi_1/\phi_2} \simeq 10^{-6} < \frac{m_u}{m_t} \tag{6.33}$$

where we use the mass ratio of the top to up quark as a guide for the upper limit on the acceptable FCNC. Therefore we can easily avoid the dangerous FCNC. In addition that, the classification of the four types of 2HDM by Z_2 parity can correspond to the assignment of the kink coupling as shown in Table 6.1. Fig. 6.2 also shows the field configuration on the extra space in each situation.

Let us now discuss the effect of the above configuration on the Higgs potential. Since our model does not impose a softly broken Z_2 symmetry, the Higgs potential in 5D takes the general form of

$$\begin{aligned} V(\Phi_1,\Phi_2) = {}& M_1^2\Phi_1^\dagger\Phi_1 + M_2^2\Phi_2^\dagger\Phi_2 - \left(M_3^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right) \\ &+ \frac{1}{2}\lambda'_1(\Phi_1^\dagger\Phi_1)^2 + \frac{1}{2}\lambda'_2(\Phi_2^\dagger\Phi_2)^2 + \lambda'_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda'_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) \\ &+ \left(\frac{1}{2}\lambda'_5(\Phi_1^\dagger\Phi_2)^2 + \lambda'_6(\Phi_1^\dagger\Phi_1)(\Phi_1^\dagger\Phi_2) + \lambda'_7(\Phi_2^\dagger\Phi_2)(\Phi_1^\dagger\Phi_2) + \text{h.c.}\right)\,, \end{aligned} \tag{6.34}$$

where λ'_i is the coupling constants in the Higgs potential which have a mass dimension m^{-1}. We should define $M_1^2, M_2^2 \gg M_3^2$ to treat M_3^2 as a perturbation in Eq. (6.22). Since the scale of

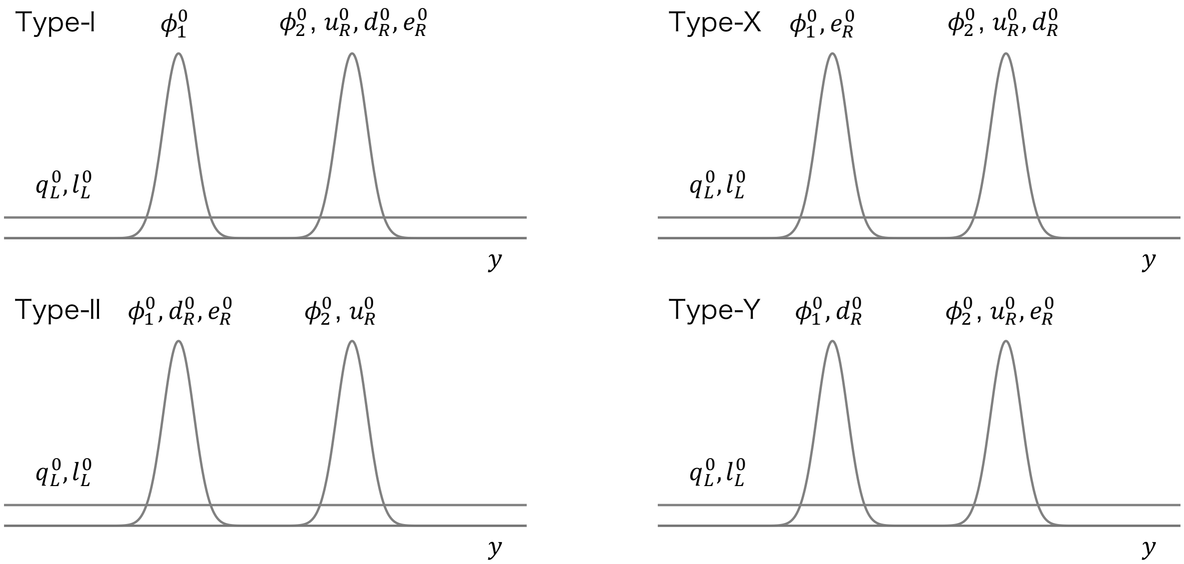

Figure 6.2: Arrangement of the wave functions for the 1st generation fermions and Higgs doublets on a extra space in the case of the Type-I (Upper left) ,Type-X (Upper right) Type-II (Lower left) and Type-Y (Lower right).

the extra dimension is sufficiently larger than the EW scale, the Higgs VEVs are defined after the extra dimensional integration. After separating the variables for $\Phi_{1,2}$, we can find the 4D Higgs potential for the zero-modes of the Higgs doublets ϕ_1^0, ϕ_2^0

$$\begin{aligned} V(\phi_1^0, \phi_2^0) &= m_1^2 \phi_1^{0\dagger} \phi_1^0 + m_2^2 \phi_2^{0\dagger} \phi_2^0 - \left(m_3^2 \phi_1^{0\dagger} \phi_2^0 + \text{h.c.} \right) \\ &+ \frac{1}{2}\lambda_1 (\phi_1^{0\dagger}\phi_1^0)^2 + \frac{1}{2}\lambda_2 (\phi_2^{0\dagger}\phi_2^0)^2 + \lambda_3 (\phi_1^{0\dagger}\phi_1^0)(\phi_2^{0\dagger}\phi_2^0) + \lambda_4 (\phi_1^{0\dagger}\phi_2^0)(\phi_2^{0\dagger}\phi_1^0) \\ &+ \left(\frac{1}{2}\lambda_5 (\phi_1^{0\dagger}\phi_2^0)^2 + \lambda_6 (\phi_1^{0\dagger}\phi_1^0)(\phi_1^{0\dagger}\phi_2^0) + \lambda_7 (\phi_2^{0\dagger}\phi_2^0)(\phi_1^{0\dagger}\phi_2^0) + \text{h.c.} \right) . \end{aligned} \tag{6.35}$$

where we use $m_i^2 = M_{\Phi_i}^2 + 2\mu^2$. The mass parameter m_3 and coupling constants λ_i are modified by the extra-dimensional integration as follow:

$$\begin{aligned} m_3^2 &= M_3^2 \int dy\ \chi_{\Phi_1}^0 \chi_{\Phi_2}^0 \\ \lambda_i &= \lambda_i' \int dy\ \chi_{\Phi_j}^0 \chi_{\Phi_k}^0 \chi_{\Phi_l}^0 \chi_{\Phi_m}^0 \quad (j,k,l,m = 1,2) \end{aligned} \tag{6.36}$$

By using the above configuration, we calculate them as

$$
\begin{aligned}
m_3^2 &= M_3^2 \exp\left(-\frac{1}{2}\mu^2 l^2\right) , \\
\lambda_i &= \lambda_i' \frac{\mu}{\sqrt{\pi}} \quad (i = 1, 2) , \\
\lambda_j &= \lambda_j' \frac{\mu}{\sqrt{\pi}} \exp\left(-\mu^2 l^2\right) \quad (j = 3, 4, 5) , \\
\lambda_k &= \lambda_k' \frac{\mu}{\sqrt{\pi}} \exp\left(-\frac{3}{4}\mu^2 l^2\right) \quad (k = 6, 7) .
\end{aligned} \tag{6.37}
$$

Except for λ_1, λ_2, those coefficients are exponentially suppressed similarly to the Yukawa coupling. In the following, we will check the mass eigenstates of the Higgs bosons in the above situation. Zero-mode 4D Higgs doublets are characterized as follow:

$$
\phi_1^0 = \begin{pmatrix} w_1^{0+} \\ \frac{1}{\sqrt{2}}\left(h_1^0 + i z_1^0 + v_1\right) \end{pmatrix} , \quad \phi_2^0 = \begin{pmatrix} w_2^{0+} \\ \frac{1}{\sqrt{2}}\left(h_2^0 + i z_2^0 + v_2\right) \end{pmatrix} \tag{6.38}
$$

The two vacuum expectation values $v_{1,2}$ are defined by the minimum of the classical Higgs potential:

$$
\begin{aligned}
\left.\frac{\partial V}{\partial \phi_1^0}\right|_{\sqrt{2}\phi_1^0 = v_1} = 0 \quad \therefore -m_1^2 = \frac{v_2}{v_1}\left(-m_3^2 + \frac{v_2^2 \lambda_7}{2}\right) + \frac{1}{2}v_1^2\lambda_1 + \frac{1}{2}v_2^2\left(\lambda_3 + \lambda_4 + \lambda_5\right) + \frac{3}{2}v_1 v_2 \lambda_6 \\
\left.\frac{\partial V}{\partial \phi_2^0}\right|_{\sqrt{2}\phi_2^0 = v_2} = 0 \quad \therefore -m_2^2 = \frac{v_1}{v_2}\left(-m_3^2 + \frac{v_2^2 \lambda_6}{2}\right) + \frac{1}{2}v_2^2\lambda_2 + \frac{1}{2}v_1^2\left(\lambda_3 + \lambda_4 + \lambda_5\right) + \frac{3}{2}v_1 v_2 \lambda_7
\end{aligned} \tag{6.39}
$$

We can get the mass matrix of the Higgs doublets from second derivatives of the Higgs potential:

$$
M_{ij} = \left.\frac{\partial V}{\partial \phi_i \partial \phi_j^\dagger}\right|_{\sqrt{2}\phi_{1,2} = v_{1,2}} \tag{6.40}
$$

For the CP odd Higgs, it is given by

$$
\begin{aligned}
M_{z_{1,2}^0} &= \frac{2m_3^2 - 2v_1 v_2 \lambda_5 - v_1^2 \lambda_6 - v_2^2 \lambda_7}{4} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \\
&\to \frac{1}{2}\left(\frac{m_3^2}{c_\beta s_\beta} - \lambda_5 v^2 - \lambda_6 v^2 \frac{c_\beta}{s_\beta} - \lambda_7 v^2 \frac{s_\beta}{c_\beta}\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} m_A^2 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{6.41}
$$

For the charged Higgs, it is given by

$$
\begin{aligned}
M_{w_{1,2}^0} &= \frac{2m_3^2 - v_1 v_2\left(\lambda_4 + \lambda_5\right) - v_1^2 \lambda_6 - v_2^2 \lambda_7}{2} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \\
&\to \left(m_A^2 + \frac{1}{2}\left(\lambda_5 - \lambda_4\right) v^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} m_{H^\pm}^2 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{6.42}
$$

Thus, we obtain the three Goldstone bosons and CP odd Higgs A, charged Higgs $H^{\pm}$ with mass spectrums, $m_A, m_{H^{\pm}}$:

$$\begin{aligned} m_A^2 &= \left(\frac{m_3^2}{c_\beta s_\beta} - \lambda_5 v^2 - \lambda_6 v^2 \frac{c_\beta}{s_\beta} - \lambda_7 v^2 \frac{s_\beta}{c_\beta}\right) \\ m_{H^\pm}^2 &= m_A^2 + \frac{1}{2}\left(\lambda_5 - \lambda_4\right) v^2 \ . \end{aligned} \tag{6.43}$$

Finally, for the CP even Higgs, the mass matrix is given by

$$\begin{aligned} M_{h^0} &= \frac{1}{2}\begin{pmatrix} A & C \\ C & B \end{pmatrix} \ , \quad \tan 2\alpha = \frac{2C}{A-B} \\ A &= \frac{1}{2}\left(2m_1^2 + 3\lambda_1 v_1^2 + v_2^2 \lambda_{345} + 6\lambda_6 v_1 v_2\right) \\ B &= \frac{1}{2}\left(2m_2^2 + 3v_2^2\lambda_2 + v_1^2\lambda_{345} + 6v_1 v_2 \lambda_7\right) \\ C &= \frac{1}{2}\left(-2m_3^2 + 2v_1 v_2 \lambda_{345} + 3(v_1^2\lambda_6 + v_2^2\lambda_7)\right) \ , \end{aligned} \tag{6.44}$$

where $\lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5$. Then, mass eigenstate is defined by the rotation angle α as follow:

$$\begin{aligned} H &= -\cos(\alpha) h_1^0 - \sin(\alpha) h_2^0 \\ h &= \sin(\alpha) h_1^0 - \cos(\alpha) h_2^0 \end{aligned} \tag{6.45}$$

As mentioned in Sec. 3.1, small m_3 value means that the additional Higgs bosons and SM Higgs become degenerate in the EW scale. Therefore, the 5D Toy model predicts the too light masses of the additional Higgs bosons due to the spatial separation of the two Higgs doublets. Even a slight rearrangement of the field does not fundamentally solve this problem, then we move to the 6D space-time.

6.4 Field Localization in 6D

In this section, we extend our discussion of the field localization in the 5D case to the 6D space-time (x_μ, y, z). For simplicity, we will use two kinks that only change in the different extra-dimensional direction: $S_1(y)$ and $S_2(z)$. We first consider the 6D fermion. The action for the 6D fermion Ψ coupled with kink $S_1(y)$ is given by

$$S_\Psi = \int d^6x \bar{\Psi}\left(i\Gamma^M \overleftrightarrow{\partial_M} - M - y_1 S_1(y)\right)\Psi \ , \tag{6.46}$$

where we use the following 8×8 6D gamma matrixes:

$$\begin{aligned} \Gamma^0 &= \gamma^0 \otimes \sigma^1 \ , \quad \Gamma^i = \gamma^i \otimes \sigma^1 \ (i = 1,2,3) \\ \Gamma^5 &= i\gamma^5 \otimes \sigma^1 \ , \quad \Gamma^6 = iI_4 \otimes \sigma^2 \end{aligned} \tag{6.47}$$

In Eq. (6.46, we define $\bar{\Psi} = \Psi^\dagger \Gamma^0$ and derivative which multiplies both Ψ and $\bar{\Psi}$ like $\overleftrightarrow{\partial_M} = (\overrightarrow{\partial_M} - \overleftarrow{\partial_M})/2$. The 6D fermon is an 8-component spinor that can be decomposed into 2-

component spinors by $\Gamma_7 = -I_4 \otimes \sigma_3$ and $\Gamma_{R/L} = \gamma^5 \otimes I_2$ as

$$P_{R/L}P_{\pm}\Psi = P_{\pm}P_{R/L}\Psi = \begin{pmatrix} \Psi_{L-} \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ \Psi_{R-} \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ \Psi_{L+} \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Psi_{R+} \end{pmatrix} , \tag{6.48}$$

where $P_{\pm}$ and $P_{R/L}$ imply the 6D and 4D chiral operators. Applying the variational principle to this action and extracting the surface term, we find

$$\left(i\Psi^{\dagger}\Gamma^0\Gamma^5\delta\Psi - i\delta\Psi^{\dagger}\Gamma^0\Gamma^5\Psi\right)_{y=\pm L_y} = 0 \ , \ \left(i\Psi^{\dagger}\Gamma^0\Gamma^6\delta\Psi - i\delta\Psi^{\dagger}\Gamma^0\Gamma^6\Psi\right)_{z=\pm L_z} = 0 \ , \tag{6.49}$$

where L_y and L_z are half length of the extra space for y and z direction. We have to determine the boundary conditions to satisfy this equation. After separation of variables and basis transformation

$$\Psi = \sum_i \sum_{n_y,n_z} \psi_i^{(n_y,n_z)}(x_\mu) f_i^{(n_y)}(y) g_i^{(n_z)}(z) \ , \quad \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} , \tag{6.50}$$

where sums run over $i = 1L, 1R, 2L, 2R$ and $n_y, n_z = 0, 1, 2, \cdots$, $\psi_i^{(n_y,n_z)}$ and $g_i^{(n_z)}$ obey the following free field equations:

$$(\partial^\mu\partial_\mu + m_y^2 + m_z^2)\psi_i^{(n_y,n_z)} = 0 \ , \quad (\partial_z^2 + m_z^2)g_i^{(n_z)} = 0 \tag{6.51}$$

On the other hand, the equation for $f_i^{(n_y)}$ has a same form to the Schrödinger equation of a Harmonic oscillator due to the kink coupling as

$$\begin{aligned} &(\partial_y^2 - (2\mu_\Psi^2(y - l_\Psi))^2 + 2\mu_\Psi^2 + m_y^2) f_{1R,2L}^{(n_y)}(y) = 0 \quad \therefore m_y^2 = 4\mu_\Psi^2(n_y + 1) \ , \\ &(\partial_y^2 - (2\mu_\Psi^2(y - l_\Psi))^2 - 2\mu_\Psi^2 + m_y^2) f_{1L,2R}^{(n_y)}(y) = 0 \quad \therefore m_y^2 = 4\mu_\Psi^2 n_y \ , \end{aligned} \tag{6.52}$$

where we define $\mu_\Psi^2 = y_1\mu_1^2$ and $l_\Psi = l_1 + M/2\mu_\Psi^2$. The resulting equation is the same as that for fermions in five dimensions, and it is found that two two-component spinors with different 4D chirality, Ψ_{1L}, Ψ_{2R}, are possible to have a massless zero mode. The wave function $f_i^{(n_y)}$ satisfy the condition (6.49) on $y = \pm L_y$ brane: $f_i^{(n_y)}(\pm L_y) = 0$. Whether they are massless or not is left to the boundary conditions on $g_i^{(n)}$. The general solution of the wave function in z-direction is given by solving the equation of motion for $g_i^{(n_z)}$:

$$\partial_z^2 g_i^{(n_z)} + m_z^2 g_i^{(n_z)} = 0 \ , \quad \therefore g_i^{(n_z)}(z) = C_1 \cos m_z z + C_2 \sin m_z z \ . \tag{6.53}$$

Imposing the Neumann and Dirichlet BC's on $g_i^{(n_z)}$, we obtain

$$\begin{aligned} &\text{Neuman BC}: \ g_i^{(n_z)}(z) = \frac{1}{2L_z}, \ g_i^{(n_z)}(z) = \frac{1}{L_z}\cos(m_z z) \text{ with } m_z = \frac{\pi n_z}{L_z} \ , \\ &\text{Dirichlet BC}: \ g_i^{(n_z)}(z) = \frac{1}{L_z}\sin(m_z z) \text{ with } m_z = \frac{\pi n_z}{L_z} \ . \end{aligned} \tag{6.54}$$

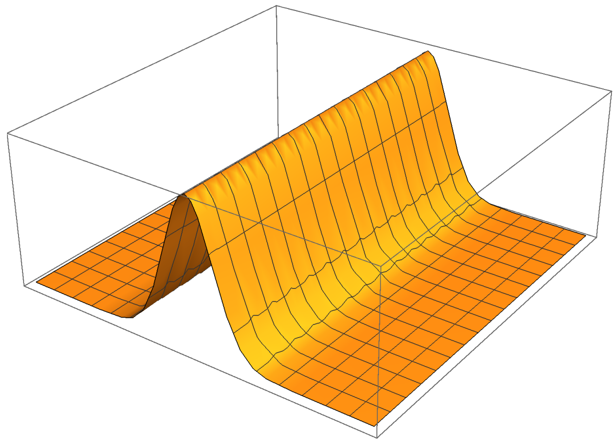

Figure 6.3: Zero mode wave function of the 6D fermion in the extra spaces

The massless zero mode only appears in the Neuman BC. Therefore, by imposing different boundary conditions on Ψ_{1L} and Ψ_{2R}, we can obtain only one massless chiral fermion. This choice satisfies the condition (6.49) on $z = \pm L_y$ brane Putting these results together, we find the zero-mode wave function for the 6D fermion and corresponding 4D mass spectrum

$$\begin{aligned} \chi_i^{(0,0)} &= \frac{1}{\sqrt{2L_z}} \left(\frac{2\mu_\Psi^2}{\pi}\right)^{1/4} \exp\left[-\mu_\Psi^2 (y - l_\Psi)^2\right] , \quad (i = 1L \text{ or } 2R) \\ m^2(0,0) &= 0 \quad \text{with} \quad m^2(n_y, n_z) = 4\mu_\Psi^2 n_y + \left(\frac{\pi n_z}{L_z}\right)^2 . \end{aligned} \tag{6.55}$$

Note that the 6D mass parameter M shifts center of the wave function: $l_1 \to l_1 + M/2\mu_\Psi^2$. We obtain similar result from the 6D fermion coupled with $S_2(z)$ as

$$\begin{aligned} \chi_i^{(0,0)} &= \frac{1}{\sqrt{2L_y}} \left(\frac{2\mu_\Psi^2}{\pi}\right)^{1/4} \exp\left[-\mu_\Psi^2 (z - l'_\Psi)^2\right] , \quad (i = 4L \text{ or } 4R) \\ m^2(0,0) &= 0 \quad \text{with} \quad m^2(n_y, n_z) = 4\mu_\Psi^2 n_z + \left(\frac{\pi n_y}{L_y}\right)^2 , \end{aligned} \tag{6.56}$$

where we use $l'_\Psi = l_2 + M/2\mu_\Psi^2$ and another basis transformation

$$\begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} . \tag{6.57}$$

Therefore, upon coupling with $S_1(y)$ or $S_2(z)$, the 6D fermions are localized like a band on the two-dimensional extra space as shown in Fig. 6.3.

Then, we consider the localization of scalars. The action for a 6D scalar Φ coupled with $S_1(y)$ is given by

$$S_\Phi = \int d^6x \left(\frac{1}{2} g^{MN} \left(\partial_M \Phi \partial_N \Phi\right) - \frac{1}{2}\lambda_{\Phi S} \Phi^2 S_1^2(y) - V(\Phi)\right) , \tag{6.58}$$

where we use the metric $g^{MN} = \mathrm{diag}(+1, -1, -1, -1, -1, -1)$. Applying the variational principle to this action, we find the requirement

$$(\partial_y \Phi)\delta\Phi|_{y=\pm L_y} = 0\ , \quad (\partial_z \Phi)\delta\Phi|_{z=\pm L_z} = 0 \tag{6.59}$$

There are two possible boundary conditions (BC's) that satisfy the above requirement:

- Neuman BC: $\partial\Phi|_{\mathrm{bound}} = 0$
- Dirichlet BC: $\Phi|_{\mathrm{bound}} = 0 \to \delta\Phi|_{\mathrm{bound}} = 0$ (Fixed value at the boundary)

The equation of motion for Φ derived from Eq. (6.58) is

$$\partial_\mu \partial^\mu \Phi - \partial_y^2 \Phi - \partial_z^2 \Phi + \lambda_{\Phi S} S_y^2 \Phi + M^2 \Phi = 0\,, \tag{6.60}$$

where we replace $V(\Phi)$ with $M^2\Phi^2/2$ to keep only the linear terms. By using the separation of variables

$$\Phi = \sum_{n_y, n_z} \phi^{(n_y,n_z)}(x_\mu) f^{(n_y)}(y) g^{(n_z)}(z)\ , \tag{6.61}$$

we derive equation of motion for $\phi^{(n_y,n_z)}$, $f^{(n_y)}$ and $g^{(n_z)}$ as follows:

$$\begin{cases} \partial_\mu \partial^\mu \phi^{(n_y,n_z)}(x_\mu) + (m_y^2 + m_z^2)\phi^{(n_y,n_z)}(x_\mu) = 0 \\ \partial_y^2 f^{(n_y)}(y) - \lambda_{\Phi S} S_y^2 f^{(n_y)}(y) = (M^2 - m_y^2) f^{(n_y)}(y) \\ \partial_z^2 g^{(n_z)}(z) + m_z^2 g^{(n_z)}(z) = 0 \end{cases}$$

For the approximating form of the kink, the wave function in y-direction obeys an identical equation to the Schrödinger equation of a Harmonic oscillator. Thus, wave function in y-direction is given by

$$f^{(n_y)}(y) = N H_{n_y}(y) \exp\Big(-\mu_\Phi^2 (y - l_1)^2 \Big)\ , \quad m_y^2 = M^2 + 4\mu_\Phi^2 \left(n_y + \frac{1}{2} \right)\ . \tag{6.62}$$

where we use $\mu_\Phi^2 = \sqrt{\lambda_{\Phi S}\mu_y^4}$ and $n_y = 0, 1, 2, \cdots$. The general solution of the wave function in z-direction is given by solving the equation of motion for $g^{(n_z)}(z)$:

$$\partial_z^2 g^{(n_z)}(z) + m_z^2 g^{(n_z)}(z) = 0 \qquad \therefore\, g^{(n_z)}(z) = C_1 \cos m_z z + C_2 \sin m_z z\ . \tag{6.63}$$

Imposing the Neumann and Dirichlet BC's for $h(z)$, we obtain

$$\begin{aligned} &\text{Neuman BC}: \ g^{(0)}(z) = \frac{1}{2L_z},\ g^{(n_z)}(z) = \frac{1}{L_z}\cos(m_z z) \text{ with } m_z = \frac{\pi n_z}{L_z}\ , \\ &\text{Dirichlet BC}: \ g^{(n_z)}(z) = \frac{1}{L_z}\sin(m_z z) \text{ with } m_z = \frac{\pi n_z}{L_z}\ , \end{aligned} \tag{6.64}$$

where $n_z = 1, 2, \cdots$. The zero mode (lowest energy mode) wave function only appears in the Neuman BC. Putting these results together, we find the zero-mode wave function for the 6D scalar and corresponding 4D mass spectrum

$$\begin{aligned} \chi_\Phi^{(0,0)} &= \frac{1}{\sqrt{2L_z}} \left(\frac{2\mu_\Phi^2}{\pi} \right)^{1/4} \exp\left[-\mu_\Phi^2 (y - l_1)^2 \right]\ , \\ m^2(0,0) &\ \text{ with } \ m^2(n_y, n_z) = M^2 + 4\mu_\Phi^2 \left(n_y + \frac{1}{2} \right) + \left(\frac{\pi n_z}{L_z} \right)^2\ . \end{aligned} \tag{6.65}$$

Therefore, we get the wave function localized as Gaussian in the y-direction and flat in the z-direction, as in the fermion case. The similar result holds for Φ coupled with S_z.

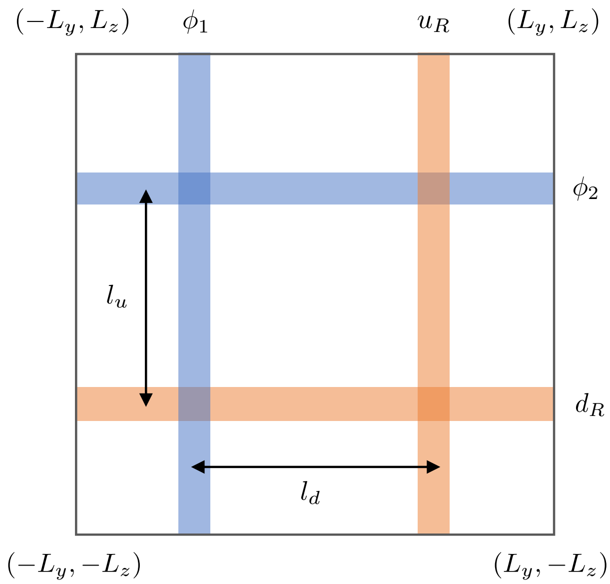

Figure 6.4: Example of the field localization in the y, z plane

6.5 Higgs couplings in 6D

Using the either kink of $S_1(y), S_2(z)$ that only cange in the y or z direction, we found that the 6D scalars and fermions are localized on extra spaces. In the following, we will check whether we can actually reproduce the required Higgs couplings. As a concrete example, we concider the arrangement of scalars and fermions as shown in Fig. 6.4. For simplicity, we unify $L_i = L, \mu_i = \mu$ and put $1/2L = \epsilon\mu$. The zero-mode wave function for Φ_1, U_R coupled with $S_1(y)$, Φ_2, D_R coupled with $S_2(z)$ and Q_L coupled with no kink are given by

$$
\begin{aligned}
\phi_1^0 : \chi_{\Phi_1}^{(0,0)} &= \sqrt{\epsilon\mu}\left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2 y^2\right] \quad \text{with} \quad m^2_{\phi_1^{0,0}} = M_1^2 + 2\mu^2 \ , \\
\phi_2^0 : \chi_{\Phi_2}^{(0,0)} &= \sqrt{\epsilon\mu}\left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2 z^2\right] \quad \text{with} \quad m^2_{\phi_2^{0,0}} = M_2^2 + 2\mu^2 \ , \\
u_R^0 : \chi_{U_R}^{(0,0)} &= \sqrt{\epsilon\mu}\left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2 (y-l_y)^2\right] \quad \text{with} \quad m^2_{u_R^{0,0}} = 0 \ , \\
d_R^0 : \chi_{D_R}^{(0,0)} &= \sqrt{\epsilon\mu}\left(\frac{2\mu^2}{\pi}\right)^{1/4} \exp\left[-\mu^2 (z-l_z)^2\right] \quad \text{with} \quad m^2_{d_R^{0,0}} = 0 \ , \\
q_L^0 : \chi_{Q_L}^{(0,0)} &= \sqrt{\epsilon\mu}\sqrt{\epsilon\mu} \quad \text{with} \quad m^2_{q_L^{0,0}} = 0 \ ,
\end{aligned} \tag{6.66}
$$

where we assume $M_3 \ll 1/L_{y,z}$ for the mixing term $M_3^2\Phi_1\Phi_2$ in the Higgs sector to keep the equation of motions for the extra dimensions. By using these representation, the 4D Higgs

	ϕ_1	ϕ_2	q_L	l_L	u_R	d_R	e_R
Type-I	$+$	$-$	$+$	$+$	$-$	$-$	$-$
Type-II	$+$	$-$	$+$	$+$	$-$	$+$	$+$
Type-X	$+$	$-$	$+$	$+$	$-$	$-$	$+$
Type-Y	$+$	$-$	$+$	$+$	$-$	$+$	$-$

	ϕ_1^0	ϕ_2^0	q_L^0	l_L^0	u_R^0	d_R^0	e_R^0
Type-I	S_1	S_2	$-$	$-$	S_1	S_1	S_1
Type-II	S_1	S_2	$-$	$-$	S_1	S_2	S_2
Type-X	S_1	S_2	$-$	$-$	S_1	S_1	S_2
Type-Y	S_1	S_2	$-$	$-$	S_1	S_2	S_1

Table 6.2: Type classification of the 2HDMs by the kink coupling in 6D case

couplings for down-type quark are calculated as

$$\begin{aligned} Y_1 \bar{Q}_L \Phi_1 D_R \to \bar{q}_L^0 \phi_1^0 d_R^0 \cdot Y_1 \int dy\ \epsilon\mu \left(\frac{2\mu^2}{\pi}\right)^{1/4} \mathrm{e}^{-\mu^2 y^2} \int dz\ \epsilon\mu \left(\frac{2\mu^2}{\pi}\right)^{1/4} \mathrm{e}^{-\mu^2 (z-l_z)^2} \\ = \bar{q}_L^0 \phi_1^0 d_R^0 \cdot Y_1 \left(\epsilon\mu\right)^2 \left(\frac{2\pi}{\mu^2}\right)^{1/2} = \bar{q}_L^0 \phi_1^0 d_R^0 \cdot Y_1 \sqrt{2\pi\mu^2}\epsilon^2 \ , \end{aligned} \tag{6.67}$$

and

$$\begin{aligned} Y_2 \bar{Q}_L \Phi_2 D_R \to \bar{q}_L^0 \phi_2^0 d_R^0 \cdot Y_2 \int_{-\epsilon\mu/2}^{+\epsilon\mu/2} dy\ (\epsilon\mu)^{3/2} \int dz \sqrt{\epsilon\mu} \left(\frac{2\mu^2}{\pi}\right)^{1/2} \mathrm{e}^{-\mu^2 z^2 - \mu^2 (z-l_z)^2} \\ = \bar{q}_L^0 \phi_1^0 d_R^0 \cdot Y_2\ \epsilon\mu\ \exp\left[-\frac{\mu^2 l_z^2}{2}\right] \ , \end{aligned} \tag{6.68}$$

where Y_1 and Y_2 denote 6D Yukawa couplings between down-type quark and Φ_1, Φ_2. The exponential suppression in the second line reflects the result of the overlap integration in the z direction. Therefore, the ratio between the 4D Higgs couplings are given by

$$\frac{y_2}{y_1} = \frac{Y_2}{Y_1} \frac{1}{\sqrt{2\pi\epsilon}} \exp\left[-\frac{\mu^2 l_z^2}{2}\right] \ . \tag{6.69}$$

Considering $\epsilon \ll 1$ from the relation between the extra-dimensional scale and the kink scale, $y_2 > y_1$ in the limit of $l_y \to 0$, while $y_1 \gg y_2$ in the limit of $l_y \to L_y$ $(1/\epsilon\mu)$ due to the exponential suppression. In other words, we can avoid the dangerous FCNC by separating Φ_2 and D_R sufficiently in this case. Since Higgs coupling is suppressed by the integration when right-handed fermions and scalars are parallel, but not when they are orthogonal, we can replace the Z_2 parity assignment with a kink coupling assignment as shown in Table 6.2.

The remainig issue is the effect of the extra dimensional integration on the Higgs potential in the 4D effective theory. To evaluate it, we start from the 6D action for the Higgs sector

$$\begin{aligned} S_{\Phi_1,\Phi_2} = \int d^6x \left\{ \frac{1}{2}\partial_M \Phi_1^\dagger \partial^M \Phi_1 - \frac{1}{2}\lambda_{\Phi_1 S_1} \Phi_1^\dagger \Phi_1 S_1^2 \right. \\ \left. + \frac{1}{2}\partial_M \Phi_2^\dagger \partial^M \Phi_2 - \frac{1}{2}\lambda_{\Phi_2 S_2} \Phi_2^\dagger \Phi_2 S_2^2 - V(\Phi_1, \Phi_2) \right\} \ , \end{aligned} \tag{6.70}$$

with the 6D Higgs potential

$$
\begin{aligned}
V(\Phi_1,\Phi_2) = {} & M_1^2\Phi_1^\dagger\Phi_1 + M_2^2\Phi_2^\dagger\Phi_2 - \left(M_3'^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right) \\
& + \frac{1}{2}\lambda_1'(\Phi_1^\dagger\Phi_1)^2 + \frac{1}{2}\lambda_2'(\Phi_2^\dagger\Phi_2)^2 + \lambda_3'(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4'(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) \\
& + \left(\frac{1}{2}\lambda_5'(\Phi_1^\dagger\Phi_2)^2 + \lambda_6'(\Phi_1^\dagger\Phi_1)(\Phi_1^\dagger\Phi_2) + \lambda_7'(\Phi_2^\dagger\Phi_2)(\Phi_1^\dagger\Phi_2) + \text{h.c.}\right) , \qquad (6.71)
\end{aligned}
$$

which contains λ_6, λ_7 terms due to the absence of the Z_2 symmetry. As mentioned above, we make the hierarchy between these mass parameters $M_1^2, M_2^2 \gg M_3^2$ by hand. Then, the zero-mode wave functions are represented as in Eq. (6.66). After extra dimensional integrarion, the 4D Higgs potential in the low-energy effective theory is given by

$$
\begin{aligned}
V(\phi_1^0,\phi_2^0) = {} & m_1^2\phi_1^{0\dagger}\phi_1^0 + m_2^2\phi_2^{0\dagger}\phi_2^0 - \left(m_3^2\phi_1^{0\dagger}\phi_2^0 + \text{h.c.}\right) \\
& + \frac{1}{2}\lambda_1(\phi_1^{0\dagger}\phi_1^0)^2 + \frac{1}{2}\lambda_2(\phi_2^{0\dagger}\phi_2^0)^2 + \lambda_3(\phi_1^{0\dagger}\phi_1^0)(\phi_2^{0\dagger}\phi_2^0) + \lambda_4(\phi_1^{0\dagger}\phi_2^0)(\phi_2^{0\dagger}\phi_1^0) \\
& + \left(\frac{1}{2}\lambda_5(\phi_1^{0\dagger}\phi_2^0)^2 + \lambda_6(\phi_1^{0\dagger}\phi_1^0)(\phi_1^{0\dagger}\phi_2^0) + \lambda_7(\phi_2^{0\dagger}\phi_2^0)(\phi_1^{0\dagger}\phi_2^0) + \text{h.c.}\right) , \qquad (6.72)
\end{aligned}
$$

where the mass parameters and Higgs self couplings are calculated as

$$
\begin{aligned}
m_1^2 &= \int dy \int dz \left\{\partial_y f_1 \partial_y f_1 g_1^2 + \partial_z g_1 \partial_z g_1 f_1^2 + \left(\lambda_{\Phi_1 S_1} S_1^2 + M_1^2\right) f_1^2 g_1^2\right\} \\
&= M_1^2 + 2\mu^2 , \\
m_2^2 &= \int dy \int dz \left\{\partial_y f_2 \partial_y f_2 g_2^2 + \partial_z g_2 \partial_z g_2 f_2^2 + \left(\lambda_{\Phi_2 S_2} S_2^2 + M_2^2\right) f_2^2 g_2^2\right\} \\
&= M_2^2 + 2\mu^2 , \\
m_3^2 &= M_3^2 \int dy\ f_1 f_2 \int dz\ g_1 g_2 = M_3^2 \epsilon\mu \sqrt{\frac{2\mu^2}{\pi}} \int dy\ \mathrm{e}^{-\mu^2 y^2} \int dz\ \mathrm{e}^{-\mu^2 z^2} \\
&= \sqrt{2\pi}\epsilon M_3^2 , \qquad (6.73)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_1 &= \lambda_1' \frac{2\epsilon^2\mu^4}{\pi} \int dy\ \mathrm{e}^{-4\mu^2 y^2} \int_{-1/2\epsilon\mu}^{+1/2\epsilon\mu} dz = \lambda_1' \frac{\epsilon\mu^2}{\sqrt{\pi}} , \\
\lambda_2 &= \lambda_2' \frac{2\epsilon^2\mu^4}{\pi} \int dz\ \mathrm{e}^{-4\mu^2 z^2} \int_{-1/2\epsilon\mu}^{+1/2\epsilon\mu} dy = \lambda_2' \frac{\epsilon\mu^2}{\sqrt{\pi}} , \\
\lambda_j &= \lambda_j' \frac{2\epsilon^2\mu^4}{\pi} \int dy\ \mathrm{e}^{-2\mu^2 y^2} \int dz\ \mathrm{e}^{-2\mu^2 z^2} = \lambda_j'(\epsilon\mu)^2 , \quad (j = 3,4,5) \\
\lambda_6 &= \lambda_6' \frac{2\epsilon^2\mu^4}{\pi} \int dy\ \mathrm{e}^{-3\mu^2 y^2} \int dz\ \mathrm{e}^{-\mu^2 z^2} = \lambda_6' \frac{2(\epsilon\mu)^2}{\sqrt{3}} , \\
\lambda_7 &= \lambda_7' \frac{2\epsilon^2\mu^4}{\pi} \int dy\ \mathrm{e}^{-\mu^2 y^2} \int dz\ \mathrm{e}^{-3\mu^2 z^2} = \lambda_7' \frac{2(\epsilon\mu)^2}{\sqrt{3}} . \qquad (6.74)
\end{aligned}
$$

They are not exponentially suppressed, in contrast to the 5D case. It is because the wave functions of ϕ_1^0 and ϕ_2^0 are not parallel. However, there is a milder Hierarchy linear to ϵ: $\lambda_{1,2} > \lambda_{3,4,5,6,7}$. Therefore, we can reproduce the viable Higgs couplings in such a 6D model. Although there is still a complaint about placing the kink coupling by hand, we can construct the four types of viable 2HDMs originated from the extra dimension. In contrast to the usual 2HDMs, our model has a generic Higgs potential not constrained by the softly broken Z_2 symmetry. Therefore, we expect to distinguish this model from the usual 2HDM by examining the physics as λ_6 and λ_7 affect it.

Chapter 7

Conclusion

We have studied the effects of the extra dimensions in Higgs physics. While the Higgs sector of the SM is so far consistent with the experiments, this is also true for various models that extend the Higgs sector. Therefore, in order to understand the Higgs sector properly, we need to investigate the possible extensions of the Higgs sector and distinguish those models by determining the nature of the Higgs sector accurately by the future collider experiments.

We first focused on the structure of the Higgs potential in GHU and analyzed the triple Higgs boson coupling. Since the triple Higgs boson coupling has been analyzed by previous studies for the warped model and $SU(3)$ model with large representation, we consider the $SU(3)$ model with 5D Lorentz symmetry relaxed. This model introduces a 5D Lorentz symmetry breaking to reproduce the masses of the Higgs and top quark. As a result, the deviation of the triple Higgs boson coupling from the SM is characterized by the compactification scale corresponding to the size of the extra dimension. It was also predicted that the deviation would be less than 10% within the compactification scale allowed by the experiment of about $1/R > 5$ TeV. Based on this result, we have investigated the shape of the Higgs potential and found that around the vacuum, it rapidly approaches that of the SM potential as the compactification scale increases. Furthermore, We have also indicated that the behavior of the potential and the triple Higgs boson coupling to the compactification scale is also applicable to the other GHU models with a flat extra dimension.

Then we have proposed a model that reproduces the viable Higgs couplings in the 2HDM without imposing the Z_2 symmetry by introducing extra dimensions. Reference [66] pointed out that the fermions could be localized on the extra space by introducing the coupling with a kink. This mechanism was often used, in particular, to resolve the flavor hierarchy of the fermions. We applied this mechanism to the Higgs sector in the 2HDM. By localizing the right-handed fermions and Higgs doublets on the extra space, we have exponentially suppressed either one of the Yukawa couplings to the two Higgs doublets. In the 5D case, since the two Higgs doublets are spatially separated, the parameters in the Higgs sector are also exponentially suppressed. Therefore, it was hard to avoid restrictions from experiments such as the mass of the additional Higgs bosons. However, vertically crossing the two Higgs doublets in the 6D case, we have found an arrangement that simultaneously avoids both the FCNC and Higgs potential problems. We have also seen that the Z_2 parity assignment, which classify the 2HDM into four types, corresponds to the kink coupling assignment as shown in Table 6.1. In addition, the Higgs potential of this model is not restricted by the Z_2 symmetry, and terms such as λ_6, λ_7 remain. Therefore, in principle, it would be possible to distinguish this model from the 2HDM with Z_2 by measuring this effect with a triple Higgs boson coupling or something.

Appendix A

Clifford algebra of $SO(4), SO(6)$

In this chapter, we consider a general extension to the Clifford algebra of $SO(4)$ to $SO(6)$ in order to define a six-dimensional gamma matrix. The Clifford algebra Γ^i of $SO(N)$ is defined as satisfying the anticommutation relation $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$. For example, the Clifford algebra of $SO(3)$ is given by the Pauli matrix $\sigma^i, (i = 1, 2, 3)$, and the Clifford algebra of $SO(N)$ can also be defined by the direct product of Pauli matrices, as follows.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1}$$

A.1 SO(4)

The Clifford algebra of $SO(4)$ can be represented as a $2^2 \times 2^2$ matrix by the following five matrices.

$$\begin{aligned} \Gamma_4^1 &= \sigma^1 \otimes \sigma^1 \\ \Gamma_4^2 &= \sigma^2 \otimes \sigma^1 \\ \Gamma_4^3 &= \sigma^3 \otimes \sigma^1 \\ \Gamma_4^4 &= \sigma^0 \otimes \sigma^2 \\ \Gamma_4^5 &= \sigma^0 \otimes \sigma^3 \end{aligned} \qquad \sigma^i \otimes \sigma^1 = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Γ^5 can be also represented as $\Gamma^5 = -\Gamma^1\Gamma^2\Gamma^3\Gamma^4$. This is checked from the calculation of the direct product

$$\begin{aligned} -\Gamma_4^1\Gamma_4^2\Gamma_4^3\Gamma_4^4 &= -\sigma^1\sigma^2\sigma^3\sigma^0 \otimes \sigma^1\sigma^1\sigma^1\sigma^2 \\ &= -i\sigma^3\sigma^3 \otimes \sigma^0(i\sigma^3) = \sigma^0 \otimes \sigma^3 \ , \end{aligned} \tag{A.2}$$

where we use $\sigma^1\sigma^2 = i\sigma^3$. In fact, since we consider the Minkowski space of $SO(1,3)$, the anticommutation relation to be satisfied by the gamma matrix is $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \equiv 2\mathrm{diag}(+1, -1, -1, -1)$. The gamma matrix in Weil representation (see Peskin) is given by

$$\gamma^0 = \sigma^0 \otimes \sigma^1 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \ , \quad \gamma^i = i\sigma^i \otimes \sigma^2 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (i = 1, 2, 3) \ . \tag{A.3}$$

I_N implies the $N \times N$ identity matrix. Using the remaining gamma matrix

$$\gamma^5 = -\sigma^0 \otimes \sigma^3 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3 \ , \tag{A.4}$$

we can define the 4D chirality for the four-component spinor ψ and decompose it into the two-component Weil spinors ψ_L, ψ_R:

$$\frac{1+\gamma^5}{2}\psi = P_R\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \ , \quad \frac{1-\gamma^5}{2}\psi = P_L\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad \text{where } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{A.5}$$

A.2 SO(6)

The Clifford algebra of $SO(6)$ can be expressed as an extension of $SO(4)$ by a $2^3 \times 2^3$ matrix as follows:

$$\begin{aligned} \Gamma_6^1 &= \sigma^1 \otimes \sigma^1 \otimes \sigma^1 \\ \Gamma_6^2 &= \sigma^2 \otimes \sigma^1 \otimes \sigma^1 \\ \Gamma_6^3 &= \sigma^3 \otimes \sigma^1 \otimes \sigma^1 \\ \Gamma_6^4 &= \sigma^0 \otimes \sigma^2 \otimes \sigma^1 \\ \Gamma_6^5 &= \sigma^0 \otimes \sigma^3 \otimes \sigma^1 \\ \Gamma_6^6 &= \sigma^0 \otimes \sigma^0 \otimes \sigma^2 \\ \Gamma_6^7 &= \sigma^0 \otimes \sigma^0 \otimes \sigma^3 \end{aligned} \tag{A.6}$$

The gamma matrix for the Minkowski space of $SO(1,5)$ is the 8×8 matrix satisfying $\{\Gamma^M, \Gamma^N\} = 2g^{MN} \equiv 2\text{diag}(+1,-1,-1,-1,-1)$ where $M, N = 0,1,2,3,5,6$. It can be analogized from above representations as follows:

$$\begin{aligned} \Gamma^0 &= \sigma^0 \otimes \sigma^1 \otimes \sigma^1 = \gamma^0 \otimes \sigma^1 \\ \Gamma^i &= i\sigma^i \otimes \sigma^2 \otimes \sigma^1 = \gamma^i \otimes \sigma^1 \ (i = 1,2,3) \\ \Gamma^5 &= -i\sigma^0 \otimes \sigma^3 \otimes \sigma^1 = i\gamma^5 \otimes \sigma^1 \\ \Gamma^6 &= i\sigma^0 \otimes \sigma^0 \otimes \sigma^2 = iI_4 \otimes \sigma^2 \end{aligned} \tag{A.7}$$

The remaining gamma matrix becomes a six-dimensional chiral operator that decomposes the eight-component spinor into the four-component spinors as in the $SO(1,3)$ case

$$\Gamma^7 = -\sigma^0 \otimes \sigma^0 \otimes \sigma^3 = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6 \ , \tag{A.8}$$

where

$$\begin{aligned} \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6 &= (i)^2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 I_4 \otimes \sigma^1\sigma^1\sigma^1\sigma^1\sigma^1\sigma^2 \\ &= i\gamma^5\gamma^5 \otimes i\sigma^3 = -I_4 \otimes \sigma^3 \ . \end{aligned} \tag{A.9}$$

Then we can define the 6D chirality as follow:

$$\frac{1+\Gamma^7}{2}\Psi = P_+\Psi = \begin{pmatrix} 0 \\ \Psi_+ \end{pmatrix} \ , \quad \frac{1-\Gamma^7}{2}\Psi = P_-\Psi = \begin{pmatrix} \Psi_- \\ 0 \end{pmatrix} \quad \Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} \tag{A.10}$$

We can also define the 4D chirality by using the operator $\Gamma_{R/L} = \gamma^5 \otimes \sigma^0$:

$$\frac{1+\Gamma_{R/L}}{2}\Psi = P_R\Psi = \begin{pmatrix} 0 \\ \Psi_R \\ 0 \\ \Psi_R \end{pmatrix} \quad , \quad \frac{1-\Gamma_{R/L}}{2}\Psi = P_L\Psi = \begin{pmatrix} \Psi_L \\ 0 \\ \Psi_L \\ 0 \end{pmatrix} \tag{A.11}$$

Since these two operators are commutative, $[\Gamma^7, \Gamma_{R/L}] = 0$, the combination of them decomposes the eight-component spinor into the four two-component spinors:

$$P_{R/L}P_{\pm}\Psi = P_{\pm}P_{R/L}\Psi = \begin{pmatrix} \Psi_{L-} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ \Psi_{R-} \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 0 \\ \Psi_{L+} \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Psi_{R+} \end{pmatrix} \tag{A.12}$$

Appendix B

Poisson summation formula

In the calculation of Eq. (5.75), we use the Poisson summation formula that relates the Fourier series coefficients of the periodic summation of a function to values of the function's continuous Fourier transform. For example, let us consider the following infinite sum of the function:

$$h(x) = \sum_{n=-\infty}^{\infty} f(x+n) \tag{B.1}$$

Since this fuction is obviously a periodic function, $h(x) = h(x+1)$, we can expand it by Fourier series as follow:

$$h(x) = \sum_{n=-\infty}^{\infty} a_n \, \mathrm{e}^{2\pi i n x} \tag{B.2}$$

Hence, the Fourier series coefficients a_n are calculated as

$$\begin{aligned} a_n &= \int_0^1 dx \, h(x) \, \mathrm{e}^{-2\pi i n x} \\ &= \sum_{m=-\infty}^{\infty} \int_0^1 dx \, f(x+m) \, \mathrm{e}^{-2\pi i n x} \\ &= \sum_{m=-\infty}^{\infty} \int_m^{m+1} dy \, f(y) \, \mathrm{e}^{-2\pi i n (y-m)} \\ &= \int_{-\infty}^{\infty} dy \, f(y) \, \mathrm{e}^{-2\pi i n y} = \hat{f}(n) \ . \end{aligned} \tag{B.3}$$

Namely, Fourier series coefficients a_n are rewritten by the Fourier transformation of $f(n)$. Then, we find the following relation:

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \, \mathrm{e}^{2\pi i n x} \tag{B.4}$$

For $f(x+n) = \exp[(x+n)/a]^2$ as in our calculation, Equation (B.4) becomes

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-(\frac{x+n}{a})^2} = \sum_{n=-\infty}^{\infty} \sqrt{\pi a} \, \mathrm{e}^{-(\pi n a)^2} \mathrm{e}^{2\pi i n x} \ , \tag{B.5}$$

where we use the Fourier transformation of the Gaussian

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha n^2} \mathrm{e}^{2\pi i n w} \, dn = \sqrt{\frac{\pi}{\alpha}} \, \mathrm{e}^{-\frac{\pi^2 w^2}{\alpha}} \ . \tag{B.6}$$

Appendix C

Field localization by the general kink

C.1 Fermion case

In this chapter, we will follow the process up to Eq. (6.20) in detail. First we transform Eq. (6.19) into the Gauss's hypergeometric differential equation. Putting $\xi = \tanh[\lambda v(y-l)]$ and using the notation

$$\begin{aligned} \epsilon^2 &= \frac{-\omega_n^2 + y_S^2 v^2}{\lambda^2 v^2} \\ s(s+1) &= \frac{y_S^2 \mp y_S \lambda}{\lambda^2} \ , \quad s = \frac{1}{2}\left(-1 + \sqrt{1 + 4\frac{y_S^2 \mp y_S\lambda}{\lambda^2}}\right) = \mp \frac{y_S}{\lambda} \ , \end{aligned} \tag{C.1}$$

Eq. (6.19) becomes

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\chi}{d\xi}\right] + \left[s(s+1) - \frac{\epsilon^2}{1-\xi^2}\right]\chi = 0 \ , \tag{C.2}$$

where we also use

$$\frac{d^2\chi}{dy^2} = \frac{d\xi}{dy}\frac{d}{d\xi}\left(\frac{d\xi}{dy}\frac{d\chi}{d\xi}\right) = v^2\lambda^2(1-\xi^2)\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\chi}{d\xi}\right] \ . \tag{C.3}$$

The first term of Eq. (C.2) is transformed as

$$\begin{aligned} \frac{d}{d\xi}\left[(1-\xi^2)\frac{d\chi}{d\xi}\right] &= \frac{d}{d\xi}\left[(1-\xi^2)\frac{d}{d\xi}\Big((1-\xi^2)^{\frac{\epsilon}{2}}\rho(\xi)\Big)\right] \\ &= \frac{d}{d\xi}\left[(1-\xi^2)^{\frac{\epsilon}{2}+1}\rho'(\xi) - \epsilon\,\xi(1-\xi^2)^{\frac{\epsilon}{2}}\rho(\xi)\right] \\ &= (1-\xi^2)^{\frac{\epsilon}{2}}\Big\{(1-\xi^2)\rho''(\xi) - 2\xi\left(\frac{\epsilon}{2}+1\right)\rho'(\xi) \\ &\qquad - \xi\epsilon\rho'(\xi) - \epsilon\rho(\xi) + \epsilon^2\xi^2(1-\xi^2)^{-1}\rho(\xi)\Big\} \ . \end{aligned} \tag{C.4}$$

Thus, Eq. (C.2) is rewritten as

$$\begin{aligned}
&\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\chi}{d\xi}\right]+\left[s(s+1)-\frac{\epsilon^2}{1-\xi^2}\right]\chi \\
&=(1-\xi^2)^{\frac{\epsilon}{2}}\Big((1-\xi^2)\rho''(\xi)-2\xi(\epsilon+1)\rho'(\xi)+s(s+1)\rho(\xi)-\epsilon(\epsilon+1)\rho(\xi)\Big) \\
&=(1-\xi^2)^{\frac{\epsilon}{2}}\Big((1-\xi^2)\rho''(\xi)-2\xi(\epsilon+1)\rho'(\xi)-(\epsilon-s)(\epsilon+s+1)\rho(\xi)\Big) \\
&=(1-\xi^2)^{\frac{\epsilon}{2}}\Big(u(1-u)\rho''(u)+(\epsilon+1)(1-2u)\rho'(u)-(\epsilon-s)(\epsilon+s+1)\rho(u)\Big)=0\ ,
\end{aligned} \tag{C.5}$$

where we use

$$\rho''(\xi)=\left(-\frac{1}{2}\right)^2\rho''(u)\ ,\qquad \rho'(\xi)=-\frac{1}{2}\rho'(u)\qquad :\ u=\frac{1}{2}(1-\xi)\ . \tag{C.6}$$

It has a same form as the Gauss' hypergeometric differential equation

$$x(1-x)y''+(\gamma-(\alpha+\beta+1)x)y'-\alpha\beta y=0\ , \tag{C.7}$$

with a solution represented by a hypergeometric function

$${}_2F_1(\alpha,\beta;\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!}=1+\frac{\alpha\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\cdots\ . \tag{C.8}$$

By making the substitution $\chi=(1-\xi^2)^{\epsilon/2}\rho(\xi)$ and temporarily changing the variable to $u=\frac{1}{2}(1-\xi)$, we obtain

$$u(1-u)\rho''(u)+(\epsilon+1)(1-2u)\rho'(u)-(e-s)(e+s+1)\rho(u)=0\ . \tag{C.9}$$

The solution finite for $\xi=1$ (for $x=\infty$) is

$$\chi_n^{(\pm)}\propto(1-\xi^2)^{\epsilon/2}\,{}_2F_1\left[\epsilon-s,\epsilon+s+1;\epsilon+1;\frac{1}{2}(1-\xi)\right]\ . \tag{C.10}$$

The hypergeometric function F diverge for $\xi=-1$ as seen in Eq. (C.8), unless we take $\epsilon-s=-n$ (or $\epsilon+s+1=-n$). When we take $\epsilon-s=-n$ (or $\epsilon+s+1=-n$), the hypergeometric function F become polynomial of degree n, which is finite for $\xi=-1$:

$$\chi_n^{(\pm)}\propto(1-\xi^2)^{(s-n)/2}\,{}_2F_1\left[-n,2s-n+1;s-n+1;\frac{1}{2}(1-\xi)\right]\ . \tag{C.11}$$

Then, the 4D mass spectrums ω_n are determined by $s-\epsilon=-n$ or

$$\frac{-\omega_n^2+y_S^2v^2}{\lambda^2v^2}=(s-n)^2\qquad\therefore\ \omega_n^2=v^2\lambda^2\left(2ns-n^2\right)\ . \tag{C.12}$$

For $s>0$, the zero-mode wave function χ_0 can be normalized as follow:

$$\begin{aligned}
\int_{-\infty}^{\infty}\chi_0\chi_0^*\,dy&=\int_{-\infty}^{\infty}\frac{N^2dy}{\cosh^{2s}[\lambda v(y-l)]}=2\int_0^{\infty}\frac{N^2dy}{\cosh^{2s}[\lambda v(y-l)]} \\
&=\frac{N^2}{\lambda v}\int_0^1 t^{s-1}(1-t)^{-1/2}dt=\frac{N^2}{\lambda v}\frac{\Gamma(s)\Gamma(1/2)}{\Gamma(s+1/2)}=1\ \therefore N=\frac{(\lambda v)^{1/2}}{\pi^{1/4}}\sqrt{\frac{\Gamma(s+1/2)}{\Gamma(s)}}\ ,
\end{aligned} \tag{C.13}$$

where we take $t = \mathrm{sech}^2[\lambda v(y-l)]$ and use

$$\frac{dt}{dy} = \mathrm{sech}^2[\lambda v(x_5 - l)]\tanh[\lambda v(y-l)] \quad \therefore\ dy = t^{-1}(1-t)^{-1/2}dt\ . \tag{C.14}$$

It implies that only either right-handed or left-handed fermion can be normalized. As denoted in Sec. 6.2 , for $y_S/\lambda > 0$ ($y_S/\lambda < 0$), the left-handed (right-handed) zero-mode wave function is normalized as

$$\chi_0(y) = \frac{(\lambda v)^{1/2}}{\pi^{1/4}}\sqrt{\frac{\Gamma(s+1/2)}{\Gamma(s)}}\frac{1}{\cosh^s[\lambda v(y-l)]}\ , \tag{C.15}$$

while right-handed (left-handed) one is not normalizable.

C.2 Scalar case

Next, we consider the same process for the 5D scalar. For the non-approximating kink $S(y) = v\tanh[\lambda v(y-l)]$, the equation of motion for χ_Φ^n is turned into

$$\begin{aligned} 0 &= \left[\partial_y^2 - \lambda_S S^2(y) + m_n^2\right]\chi_\Phi^n(y) \\ &= \partial_y^2\chi_\Phi^n(y) + \left(m_n^2 - \lambda_S v^2 + \frac{\lambda_S v^2}{\cosh^2[\lambda v(y-l)]}\right) \end{aligned} \tag{C.16}$$

The only difference from the fermion case is the absence of the $\partial_y S(y)$ term. Putting $\xi = \tanh[\lambda v(y-l)]$ and using the notation

$$\begin{aligned} \epsilon^2 &= \frac{-m_n^2 + \lambda_S v^2}{\lambda^2 v^2} \\ s(s+1) &= \frac{\lambda_S}{\lambda^2}\ , \quad s = \frac{1}{2}\left(-1 + \sqrt{1 + 4\frac{\lambda_S}{\lambda^2}}\right)\ , \end{aligned} \tag{C.17}$$

we obtain

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\chi_\Phi}{d\xi}\right] + \left[s(s+1) - \frac{\epsilon^2}{1-\xi^2}\right]\chi_\Phi = 0\ . \tag{C.18}$$

We can solve this equation in the same way as in the fermion case. Then, the n-th mode wave function χ_Φ^n is also given by

$$\chi_\Phi^n \propto (1-\xi^2)^{(s-n)/2}\,{}_2F_1\left[-n, 2s-n+1; s-n+1; \frac{1}{2}(1-\xi)\right]\ . \tag{C.19}$$

The 4D mass spectrums m_n are determined by $s - \epsilon = -n$:

$$\frac{-m_n^2 + \lambda_S v^2}{\lambda^2 v^2} = (s-n)^2 \quad \therefore\ m_n^2 = \lambda^2 v^2[(2n+1)s - n^2] \tag{C.20}$$

In particular, the zero-mode wave function is also normalized as

$$\chi_\Phi^0(y) = \frac{(\lambda v)^{1/2}}{\pi^{1/4}}\sqrt{\frac{\Gamma(s+1/2)}{\Gamma(s)}}\frac{1}{\cosh^s[\lambda v(y-l)]}\ . \tag{C.21}$$

The only difference from the fermion case is a mass spectrum, and the wave function has the same shape.

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