Contact metric structures on 3-dimensional manifolds

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Preface

A differentiable manifold M^{2n+1} is said to have a contact structure or to be a contact manifold if there exists a 1-form η over M^{2n+1} such that $\eta \wedge (d\eta)^n \neq 0$. The condition $\eta \wedge (d\eta)^n \neq 0$ means that a contact manifold is orientable. It is known that a smooth hypersurface satisfying some conditions has a contact structure. As a special case S^{2n+1} is a contact manifold. When a contact form η is given on M^{2n+1}, there exists a system (ξ, φ, g) of a vector field ξ, a tensor field φ of type (1,1) and a Riemannian metric g, which called a contact metric structure.

On the other hand the notion of almost contact metric structures is a generalization of the notion of contact metric structures. An almost contact metric structure does not assume the condition $\eta \wedge (d\eta)^n \neq 0$. From the point of view of the Riemannian geometry of contact metric manifolds we consider K-contact structures.

This paper consists of three chapters. In Chapter 1 we mention the notion of an almost contact metric structure (φ, ξ, η, g) on M^{2n+1} and give its examples. Next we show that on an almost contact metric manifold M^{2n+1} we can construct a useful orthonormal basis called φ-basis. And we explain that on the almost contact metric manifold $\mathbf{R}^{2n+1}$ the sectional curvature of a vector X orthogonal to ξ and φX is equal to -3. Finally we show that on the Heisenberg group $H_{\mathbf{R}}$ identified with $\mathbf{R}^3$ left translation preserves η and g is a left invariant metric.

Chapter 2 we mention the notion of a contact metric structure $(\varphi,\ \xi,\ \eta,\ g)$ and give its examples. Remark that for a contact form η, ξ is unique but g and φ are not necessarily unique. Next we show that in Hopf's mapping $\pi : S^3 \longrightarrow S^2$ the value of $d\pi(\xi)$ is equal to 0. Moreower we mention the notion of K-contact structure. We consider the sectional curvature of K-contact manifold M^{2n+1}. Finally we check that the almost contact metric structure on $M^{2n} \times \mathbf{R}$ is not a contact metric structure.

It is known that every compact orientable 3-dimensional manifold has a contact structure. In Chapter 3 we consider 3-dimensional contact manifolds, especially S^3, $\mathbf{R^3}$ and T^3. We give a typical contact form η on S^3, $\mathbf{R^3}$ and T^3 respectively. Then we completely determine their contact metric structures. Next, we check that such contact metric structures are η-Einstein or not. If $M^3 = S^3$, (φ, ξ, η, g) is η-Einstein if and only if g is the standard metric. If $M^3 = \mathbf{R^3}$, all (φ, ξ, η, g) are η-Einstein. If $M^3 = T^3$, one parameter family of (φ, ξ, η, g) are η-Einstein. We check that such contact metric structures are Sasakian or not. If $M^3 = S^3$, (φ, ξ, η, g) is Sasakian if and only if g is the standard metric. If $M^3 = \mathbf{R^3}$, all (φ, ξ, η, g) are Sasakian. If $M^3 = T^3$, all (φ, ξ, η, g) are not Sasakian. We check that such contact metric structures are K-contact or not. If $M^3 = S^3$, (φ, ξ, η, g) is K-contact if and only if g is the standard metric. If $M^3 = \mathbf{R^3}$, all (φ, ξ, η, g) are K-contact. If $M^3 = T^3$, all (φ, ξ, η, g) are not K-contact.

Chapter 1

Almost contact metric manifolds

1.1 almost contact manifolds

We say M^{2n+1} has an almost contact structure or sometimes (φ, ξ, η)-structure if it admits a tensor field φ of type (1, 1), a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \tag{1.1}$$
$$\varphi^2(X) = -X + \eta(X)\xi, \tag{1.2}$$

for $X \in \mathfrak{X}(M^{2n+1})$.

Theorem 1.1.1. (cf.[3]) Suppose M^{2n+1} has a (φ, ξ, η)-structure. Then we have

$$\varphi(\xi) = 0, \tag{1.3}$$
$$\eta \circ \varphi = 0, \tag{1.4}$$
$$rank\ \varphi = 2n. \tag{1.5}$$

Proof First by substituting $X = \xi$ into (1.2), from (1.1) we get

$$\varphi(\varphi\xi) = 0. \tag{1}$$

Now we assume

$$\varphi\xi \neq 0. \tag{2}$$

We again substitute $X = \varphi\xi$ into (1.2) and get

$$\varphi^2(\varphi\xi) = -\varphi\xi + \eta(\varphi\xi)\xi. \tag{3}$$

In the left side of (3) we get from (1)

$$\varphi^2(\varphi\xi) = 0$$

and hence

$$\varphi\xi = \eta(\varphi\xi)\xi. \tag{4}$$

From (2) we get

$$\eta(\varphi\xi) \neq 0. \tag{5}$$

On the other hand using (4) we get from (5)

$$\varphi(\varphi\xi) = \varphi(\eta(\varphi\xi)\xi) = \{\eta(\varphi\xi)\}^2\xi \neq 0,$$

that is

$$\varphi(\varphi\xi) \neq 0.$$

This is a contradiction. Thus, $\varphi\xi = 0$.
Next by substituting φX into (1.2), we get

$$\eta(\varphi X)\xi = \varphi^2(\varphi X) + \varphi X. \tag{6}$$

Using (1.2) we compute the right side of (6)

$$\begin{aligned} \varphi^2(\varphi X) + \varphi X &= \varphi(\varphi^2 X) + \varphi X \\ &= \eta(X)\varphi\xi. \end{aligned}$$

Since $\varphi\xi = 0$, we get from (6)

$$\eta(\varphi X)\xi = 0$$

and hence $\eta(\varphi X) = 0$. Thus, $\eta \circ \varphi = 0$.
Finally for $X \in \mathrm{Ker}(\varphi)$ we get

$$\varphi^2 X = 0. \tag{7}$$

By substituting X into (1.2), from (7) we get $X = \eta(X)\xi$ and hence

$$\dim(\mathrm{Ker}(\varphi)) = 1$$

Thus, rank $\varphi = 2n + 1 - 1 = 2n$. □

Example

Proposition 1.1.2. (cf.[3]) Let η be the 1-form, ξ the characteristic vector field and φ the tensor field on $\mathbf{R}^{2n+1}$ defined by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \tag{1.6}$$

$$\xi = 2\frac{\partial}{\partial z}, \tag{1.7}$$

$$\varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}, \tag{1.8}$$

respectively. Then $(\mathbf{R}^{2n+1}, \varphi, \xi, \eta)$ is an almost contact manifold.

Proof First using (1.6), (1.7) we get

$$\eta(\xi) = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i)(2\frac{\partial}{\partial z}) = 1.$$

Next let $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial z}$ be natural basis on $\mathbf{R}^{2n+1}$.
Using (1.8), we get

$$\begin{aligned} \varphi^2(\frac{\partial}{\partial x^i}) &= \varphi(-\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i} - y^i\frac{\partial}{\partial z}, \\ (-I + \eta \otimes \xi)(\frac{\partial}{\partial x^i}) &= -\frac{\partial}{\partial x^i} + \frac{1}{2}(0 - y^i)2\frac{\partial}{\partial z} = -\frac{\partial}{\partial x^i} - y^i\frac{\partial}{\partial z}, \end{aligned}$$

and

$$\begin{aligned} \varphi^2(\frac{\partial}{\partial y^i}) &= \varphi(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}) = -\frac{\partial}{\partial y^i} + y^i \cdot 0 = -\frac{\partial}{\partial y^i}, \\ (-I + \eta \otimes \xi)(\frac{\partial}{\partial y^i}) &= -\frac{\partial}{\partial y^i} + \frac{1}{2}(0 - 0)2\frac{\partial}{\partial z} = -\frac{\partial}{\partial y^i}, \end{aligned}$$

where $i = 1, \cdots, n$.
Moreover we get

$$\begin{aligned} \varphi^2(\frac{\partial}{\partial z}) &= \varphi(0) = 0, \\ (-I + \eta \otimes \xi)(\frac{\partial}{\partial z}) &= -\frac{\partial}{\partial z} + \frac{\partial}{\partial z} = 0. \end{aligned}$$

Therefore (1.1) and (1.2) hold. □

1.2 almost contact metric manifolds

Definition 1.2.1. *If a manifold M^{2n+1} with an a almost contact structure (φ, ξ, η) admits a Riemannian metric satisfying*

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.9}$$

then g is called a compatible metric and (φ, ξ, η, g) is called an almost contact metric structure on M^{2n+1}.

Proposition 1.2.1. (cf.[3]) On an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$,

$$\eta(X) = g(X, \xi) \tag{1.10}$$

hold.

Proof By substituting $Y = \xi$ into (1.9), from (1.1), (1.3) we get

$$0 = g(\varphi X, \varphi\xi) = g(X, \xi) - \eta(X)\eta(\xi) = g(X, \xi) - \eta(X)$$

and hence (1.10). □

Proposition 1.2.2. (cf.[3]) M^{2n+1} is an almost contact metric manifold with (φ, ξ, η, g). U is a local coordinate neighborhood on M^{2n+1}.
(1) If X_1 is a unit vector field on U orthogonal to ξ, then φX_1 is a unit vector field orthogonal to both ξ and X_1.
(2) If X_2 is a unit vector field on U orthogonal to ξ, X_1 and φX_1, then φX_2 is a unit vector field orthogonal to ξ, X_1, X_2 and φX_1.
(3) We proceed in the same way as (1), (2). Then we can obtain a orthonormal basis $\{X_1, \cdots, X_n, \varphi X_1, \cdots, \varphi X_n, \xi\}$ on U.

Proof (1) First by substituting φX_1 into (1.10), from (1.4) we get

$$g(\varphi X_1, \xi) = \eta(\varphi X_1) = 0.$$

Next from (1.2) and the above equation we get

$$\begin{aligned} g(\varphi^2 X_1, \varphi X_1) &= g(-X_1 + \eta(X_1)\xi, \varphi X_1) \\ &= -g(X_1, \varphi X_1) + \eta(X_1) g(\xi, \varphi X_1) \\ &= -g(X_1, \varphi X_1). \end{aligned}$$

The other hand from (1.9), (1.4) we get

$$g(\varphi^2 X_1, \varphi X_1) = g(\varphi X_1, X_1) - \eta(\varphi X_1)\eta(X_1) = g(\varphi X_1, X_1).$$

And hence

$$g(\varphi X_1, X_1) = 0.$$

Finally computing $g(\varphi X_1, \varphi X_1)$ by (1.9), from the assumption and (1.10) we get

$$g(\varphi X_1, \varphi X_1) = g(X_1, X_1) - \eta(X_1)\eta(X_1) = g(X_1, X_1) - g(X_1, \xi)g(X_1, \xi) = 1.$$

(2) Similarly we can see that φX_2 is a unit vector field orthogonal to ξ, X_1 and X_2. We shall prove $g(\varphi X_2, \varphi X_1) = 0$. From (1.9) and the assumption we get

$$g(\varphi X_2, \varphi X_1) = g(X_2, X_1) - \eta(X_2)\eta(X_1) = 0.$$

(3) Suppose that $\{X_1, \cdots, X_k, \varphi X_1, \cdots, \varphi X_k, \xi\}$ is an orthonormal frame and X_{k+1} is a unit vector field orthogonal to $X_1, \cdots, X_k, \varphi X_1, \cdots, \varphi X_k$ and ξ. Similarly we can see that φX_{k+1} is a unit vector field orthogonal to $X_1, \cdots, X_k, X_{k+1}, \varphi X_1, \cdots, \varphi X_k, \xi$. □

Definition 1.2.2. *We call $\{X_1, \cdots, X_n, \varphi X_1, \cdots, \varphi X_n, \xi\}$ a φ-basis on M^{2n+1}.*

Example

Proposition 1.2.3. (cf.[3]) Let $\mathbf{R}^{2n+1}$ be an almost contact manifold with (φ, ξ, η) satisfying (1.6), (1.7) and (1.8). Let g be the Riemannian metric on $\mathbf{R}^{2n+1}$ defined by

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}((dx^i)^2 + (dy^i)^2) \tag{1.11}$$

and the matrix of components of g, namely

$$\frac{1}{4}\begin{pmatrix} \delta_{ij} + y^i y^j & 0 & -y^i \\ 0 & \delta_{ij} & 0 \\ -y^j & 0 & 1 \end{pmatrix}. \tag{1.12}$$

Then $\mathbf{R}^{2n+1}$ is an almost contact metric manifold.

Before the proof of this proposition, we prepare the following Lemma.

Lemma 1.2.4. *For $i = 1, \cdots, n$, put $X_i = 2\frac{\partial}{\partial y^i}, X_{n+i} = 2(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z})$ on $\mathbf{R}^{2n+1}$. Then $\{X_i, X_{n+i}, \xi\}_{i=1,\cdots,n}$ forms a φ-basis on $\mathbf{R}^{2n+1}$.*

Proof Using (1.12) we have

$$\begin{aligned} g(X_i, X_j) &= g(2\frac{\partial}{\partial y^i}, 2\frac{\partial}{\partial y^j}) = 4g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = \delta_{ij}, \\ g(X_{n+i}, X_{n+j}) &= g(2(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}), 2(\frac{\partial}{\partial x^j} + y^j \frac{\partial}{\partial z})) \\ &= \delta_{ij} + y^i y^j + y^j(-y^i) + y^i(-y^j) + y^i y^j = \delta_{ij}, \\ g(X_i, X_{n+j}) &= g(2\frac{\partial}{\partial y^i}, 2(\frac{\partial}{\partial x^j} + y^j \frac{\partial}{\partial z})) = 0 + y^j \cdot 0 = 0, \\ g(\xi, X_i) &= g(2\frac{\partial}{\partial z}, 2\frac{\partial}{\partial y^i}) = 0, \\ g(\xi, X_{n+i}) &= g(2\frac{\partial}{\partial z}, 2(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z})) = -y^i + y^i = 0. \end{aligned}$$

Hence $\{X_i, X_{n+i}, \xi\}_{i=1,\cdots,n}$ is an orthonormal basis on $\mathbf{R}^{2n+1}$. Moreover

$$\varphi(X_i) = \varphi(2\frac{\partial}{\partial y^i}) = 2\varphi(\frac{\partial}{\partial y^i}) = 2(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}) = X_{n+i}.$$

Therefore we can denote $\{X_i, X_{n+i}, \xi\}_{i=1,\cdots,n}$ by $\{X_i, \varphi(X_i), \xi\}_{i=1,\cdots,n}$. □

Now we prove Proposition 1.2.3.

Proof We can easily get

$$\eta(X_i) = \eta(X_{n+i}) = 0, \tag{1.13}$$

$$\varphi(X_i) = X_{n+i}, \tag{1.14}$$

$$\varphi(X_{n+i}) = -X_i, \tag{1.15}$$

where $i = 1, \cdots, n$.
In proposition 1.1.2 we proved that the $\mathbf{R}^{2n+1}$ is an almost contact manifold. And then by using above equations and a φ-*basis* $\{X_i, \varphi(X_i), \xi\}$, we shall verify (1.9).
(1) For $X = X_i, Y = X_j$,

$$\begin{aligned} g(\varphi X_i, \varphi X_j) &= g(X_{n+i}, X_{n+j}) = \delta_{ij}, \\ g(X_i, X_j) - \eta(X_i)\eta(X_j) &= \delta_{ij} - 0 \cdot 0 = \delta_{ij}. \end{aligned}$$

(2) For $X = X_i, Y = X_{n+j}$,

$$\begin{aligned} g(\varphi X_i, \varphi X_{n+j}) &= g(X_{n+i}, -X_j) = 0, \\ g(X_i, X_{n+j}) - \eta(X_i)\eta(X_{n+j}) &= 0 - 0 \cdot 0 = 0. \end{aligned}$$

(3) For $X = X_i, Y = \xi$,

$$\begin{aligned} g(\varphi X_i, \varphi \xi) &= g(X_{n+i}, 0) = 0, \\ g(X_i, \xi) - \eta(X_i)\eta(\xi) &= 0 - 0 \cdot 1 = 0. \end{aligned}$$

(4) For $X = X_{n+i}, Y = X_{n+j}$,

$$\begin{aligned} g(\varphi X_{n+i}, \varphi X_{n+j}) &= g(-X_i, -X_j) = \delta_{ij}, \\ g(X_{n+i}, X_{n+j}) - \eta(X_{n+i})\eta(X_{n+j}) &= \delta_{ij} - 0 \cdot 0 = \delta_{ij}. \end{aligned}$$

(5) For $X = X_{n+i}, Y = \xi$,

$$\begin{aligned} g(\varphi X_{n+i}, \varphi \xi) &= g(-X_i, 0) = 0, \\ g(X_{n+i}, \xi) - \eta(X_{n+i})\eta(\xi) &= 0 - 0 \cdot 1 = 0. \end{aligned}$$

(6) For $X = \xi, Y = \xi$,

$$\begin{aligned} g(\varphi \xi, \varphi \xi) &= g(0, 0) = 0, \\ g(\xi, \xi) - \eta(\xi)\eta(\xi) &= 1 - 1 \cdot 1 = 0. \end{aligned}$$

Thus

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

holds for $X, Y \in \mathfrak{X}(\mathbf{R}^{2n+1})$.
Hence the $\mathbf{R}^{2n+1}$ is an almost contact metric manifold. □

Moreover we can easily obtain the following equations about the φ-*basis* $\{X_i, \varphi(X_i), \xi\}_{i=1,\cdots,n}$,

(1.16) $$[X_i, X_{n+j}] = 2\delta_{ij}\xi, \quad \text{others are equal to } 0.$$

Proposition 1.2.5. (cf.[3]) Let $\mathbf{R}^{2n+1}$ be the almost contact metric manifold defined in Proposition 1.2.3. Then the sectional curvature of a plane section spanned by a vector X orthogonal to ξ and φX is equal to -3.

Proof Let $\{X_1, \cdots, X_n, \varphi X_1, \cdots, \varphi X_n, \xi\}$ be a φ-basis. Using (1.7) and (1.16) we get

$$\begin{array}{lll} \nabla_{X_i} X_j = 0, & \nabla_{X_i} X_{n+j} = \delta_{ij}\xi, & \nabla_{X_i}\xi = -X_{n+i} \\ \nabla_{X_{n+i}} X_j = -\delta_{ji}\xi, & \nabla_{X_{n+i}} X_{n+j} = 0, & \nabla_{X_{n+i}}\xi = X_i \\ \nabla_{\xi} X_j = -X_{n+j}, & \nabla_{\xi} X_{n+j} = X_j, & \nabla_{\xi}\xi = 0 \end{array}$$

Then we get

$$\begin{array}{ll} R(X_i, X_j)X_k = 0, & R(X_i, X_j)X_{n+k} = -\delta_{jk}X_{n+i} + \delta_{ik}X_{n+j}, \\ R(X_i, X_{n+j})X_k = \delta_{kj}X_{n+i} + 2\delta_{ij}X_{n+k}, & R(X_i, X_{n+j})X_{n+k} = -\delta_{jk}X_j - 2\delta_{ij}X_k, \\ R(X_{n+i}, X_{n+j})X_k = -\delta_{kj}X_i + \delta_{ki}X_j, & R(X_{n+i}, X_{n+j})X_{n+k} = 0. \end{array}$$

For X orthogonal to ξ, we can put

$$X = \sum_{h=1}^{n} \alpha_h X_h + \sum_{h=1}^{n} \beta_h X_{n+h} \qquad \alpha_h, \beta_h \in C^\infty(\mathbf{R}^{2n+1}), \tag{1.17}$$

and put $Y = \sum_{h=1}^{n} \alpha_h X_h$, $Z = \sum_{h=1}^{n} \beta_h X_{n+h}$.
Then we have

$$\begin{aligned} R(Y, \varphi Y)\varphi Y &= -\sum_{i,j,k=1}^{n} \alpha_i\alpha_j\alpha_k(\delta_{ik}X_j + 2\delta_{ij}X_k), \\ R(Y, \varphi Y)\varphi Z &= -\sum_{i,j,k=1}^{n} \alpha_i\alpha_j\beta_k(\delta_{kj}X_{n+i} + 2\delta_{ij}X_{n+k}), \\ R(Y, \varphi Z)\varphi Y &= -\sum_{i,j,k=1}^{n} \alpha_i\beta_j\alpha_k(-\delta_{jk}X_{n+i} + \delta_{ik}X_{n+j}), \\ R(Y, \varphi Z)\varphi Z &= 0, \end{aligned}$$

$$
\begin{aligned}
R(Z,\varphi Y)\varphi Y &= 0,\\
R(Z,\varphi Y)\varphi Z &= -\sum_{i,j,k=1}^{n}\beta_i\alpha_j\beta_k(-\delta_{kj}X_i+\delta_{ki}X_j),\\
R(Z,\varphi Z)\varphi Y &= -\sum_{i,j,k=1}^{n}\beta_i\beta_j\alpha_k(\delta_{jk}X_i+2\delta_{ji}X_k),\\
R(Z,\varphi Z)\varphi Z &= -\sum_{i,j,k=1}^{n}\beta_i\beta_j\beta_k(\delta_{ki}X_{n+j}+2\delta_{ji}X_{n+k}).
\end{aligned}
$$

Now we compute $g(R(X,\varphi X)\varphi X,X)$ as follows:

$$
\begin{aligned}
&g(R(X,\varphi X)\varphi X,X)\\
&\quad= g(R(Y,\varphi Y)\varphi Y,Y+Z)\\
&\qquad+g(R(Y,\varphi Y)\varphi Z,Y+Z)\\
&\qquad+g(R(Y,\varphi Z)\varphi Y,Y+Z)\\
&\qquad+g(R(Z,\varphi Y)\varphi Z,Y+Z)\\
&\qquad+g(R(Z,\varphi Z)\varphi Y,Y+Z)\\
&\qquad+g(R(Z,\varphi Z)\varphi Z,Y+Z)\\
&\quad= -3\sum_{i,j=1}^{n}\alpha_i^2\alpha_j^2-6\sum_{i,j=1}^{n}\alpha_i^2\beta_j^2-3\sum_{i,j=1}^{n}\beta_i^2\beta_j^2\\
&\quad= -3(\sum_{i=1}^{n}\alpha_i^2+\sum_{i=1}^{n}\beta_i^2)^2\\
&\quad= -3g(X,X)^2.
\end{aligned}
$$

Next we compute $g(X,X)g(\varphi X,\varphi X)-g(X,\varphi X)^2$. Since X is orthogonal to ξ, $g(X,\varphi X)=0$. Hence by using (1.13) we get

$$
\begin{aligned}
&g(X,X)g(\varphi X,\varphi X)-g(X,\varphi X)^2\\
&\quad= g(X,X)\{g(X,X)-\eta(X)\eta(X)\}\\
&\quad= g(X,X)^2.
\end{aligned}
$$

Therefore

$$g(R(X,\varphi X)\varphi X,X)=-3\{g(X,X)g(\varphi X,\varphi X)-g(X,\varphi X)^2\}. \qquad \square$$

Proposition 1.2.6. (cf.[3]) Let $\mathbf{R}^3$ be the almost contact metric manifold defined in Proposition 1.2.3. We can identify $\mathbf{R}^3$ with the Heisenberg group

$$\mathbf{H}_{\mathbf{R}} = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbf{R} \right\}.$$

And then the followings hold
(1) Left translation preserves η,
(2) g is a left invariant metric on $\mathbf{H}_{\mathbf{R}}$.

Proof Let $A, Q \in \mathbf{H}_{\mathbf{R}}$ be the elements

$$A = \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \text{respectively.}$$

Then the left translation on $\mathbf{H}_{\mathbf{R}}$ by A is denoted by

$$AQ = \begin{pmatrix} 1 & y+b & z+bx+c \\ 0 & 1 & x+a \\ 0 & 0 & 1 \end{pmatrix}.$$

And then we define the map $\psi : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that

$$\psi(x, y, z) = (x+a, y+b, z+bx+c). \tag{1.18}$$

From (1.6) η is denoted as follows:

$$\eta = \frac{1}{2}(dz - ydx). \tag{1.19}$$

For $p \in \mathbf{R}^3$, we take a local coordinate (x, y, z). From (1.18) we get a Jacobian matrix of ψ at p as follows :

$$(J\psi)_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix}.$$

Hence we get

$$(d\psi)_p(\left(\frac{\partial}{\partial x}\right)_p) = \left(\frac{\partial}{\partial x}\right)_{\psi(p)} + b\left(\frac{\partial}{\partial z}\right)_{\psi(p)}, \tag{1.20}$$

$$(d\psi)_p(\left(\frac{\partial}{\partial y}\right)_p) = \left(\frac{\partial}{\partial y}\right)_{\psi(p)}, \tag{1.21}$$

$$(d\psi)_p(\left(\frac{\partial}{\partial z}\right)_p) = \left(\frac{\partial}{\partial z}\right)_{\psi(p)}. \tag{1.22}$$

First we shall prove (1). We check the equation

$$\eta_{\psi(p)} \circ (d\psi)_p(X) = \eta_p(X) \quad \text{for } X \in T_p(\mathbf{R^3}).$$

From (1.20), (1.21) and (1.22), using (1.19) we substitute $(\frac{\partial}{\partial x})_p, (\frac{\partial}{\partial y})_p, (\frac{\partial}{\partial z})_p$ into $\eta_{\psi(p)} \circ (d\psi)_p$ and η_p respectively.

Case 1 : $X = \left(\frac{\partial}{\partial x}\right)_p.$

$$\begin{aligned}
\eta_{\psi(p)} \circ (d\psi)_p \left(\frac{\partial}{\partial x}\right)_p &= \frac{1}{2}\{(dz)_{\psi(p)} - y(\psi(p))(dx)_{\psi(p)}\}\{\left(\frac{\partial}{\partial x}\right)_{\psi(p)} + b\left(\frac{\partial}{\partial z}\right)_{\psi(p)}\} \\
&= \frac{1}{2}(b - y(\psi(p)) = \frac{1}{2}\{b - (y+b)\} = -\frac{1}{2}y, \\
\eta_p \left(\frac{\partial}{\partial x}\right)_p &= \frac{1}{2}\{(dz)_p - y(p)(dx)_p\}\left(\frac{\partial}{\partial x}\right)_p = \frac{1}{2}(-y(p)) = -\frac{1}{2}y.
\end{aligned}$$

Case 2 : $X = \left(\frac{\partial}{\partial y}\right)_p.$

$$\begin{aligned}
\eta_{\psi(p)} \circ (d\psi)_p \left(\frac{\partial}{\partial y}\right)_p &= \frac{1}{2}\{(dz)_{\psi(p)} - y(\psi(p))(dx)_{\psi(p)}\}\left(\frac{\partial}{\partial y}\right)_{\psi(p)} = 0, \\
\eta_p \left(\frac{\partial}{\partial y}\right)_p &= \frac{1}{2}\{(dz)_p - y(p)(dx)_p\}\left(\frac{\partial}{\partial y}\right)_p = 0.
\end{aligned}$$

Case 3 : $X = \left(\frac{\partial}{\partial z}\right)_p.$

$$\begin{aligned}
\eta_{\psi(p)} \circ (d\psi)_p \left(\frac{\partial}{\partial z}\right)_p &= \frac{1}{2}\{(dz)_{\psi(p)} - y(\psi(p))(dx)_{\psi(p)}\}\left(\frac{\partial}{\partial z}\right)_{\psi(p)} = \frac{1}{2}, \\
\eta_p \left(\frac{\partial}{\partial z}\right)_p &= \frac{1}{2}\{(dz)_p - y(p)(dx)_p\}\left(\frac{\partial}{\partial z}\right)_p = \frac{1}{2}.
\end{aligned}$$

Thus $\psi^*\eta = \eta$ holds.

Next we shall prove (2). Let p be a point on $\mathbf{R^3}$. From (1.18) the

Riemannian metric g_p and $g_{\psi(p)}$ are

$$(1.23)\quad g_p = \begin{pmatrix} 1+y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}, \quad g_\psi(p) = \begin{pmatrix} 1+(y+b)^2 & 0 & -(y+b) \\ 0 & 1 & 0 \\ -(y+b) & 0 & 1 \end{pmatrix},$$

respectively.
Now we check the equation

$$(1.24)\qquad g_p(X,Y) = g_{\psi(p)}((d\psi)_p(X),(d\psi_p)(Y)) \quad \text{for } X,Y \in T_p(\mathbf{R^3}).$$

We substitute $(\frac{\partial}{\partial x})_p, (\frac{\partial}{\partial y})_p, (\frac{\partial}{\partial z})_p$ into the both side of (1.24). Using (1.23), from (1.20), (1.21) and (1.22) we get

Case 1′: $X = \left(\frac{\partial}{\partial x}\right)_p, Y = \left(\frac{\partial}{\partial y}\right)_p$.

$$\begin{aligned} g_p(\left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p) &= 0, \\ g_{\psi(p)}(\left(\frac{\partial}{\partial x}\right)_p, (d\psi)_p(\left(\frac{\partial}{\partial y}\right)_p) &= g_{\psi(p)}(\left(\frac{\partial}{\partial x}\right)_{\psi(p)} + b\left(\frac{\partial}{\partial z}\right)_{\psi(p)}, \left(\frac{\partial}{\partial y}\right)_{\psi(p)}) \\ &= 0 + 0 = 0. \end{aligned}$$

Case 2′: $X = \left(\frac{\partial}{\partial x}\right)_p, Y = \left(\frac{\partial}{\partial z}\right)_p$
.

$$\begin{aligned} g_p(\left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial z}\right)_p) &= -y, \\ g_{\psi(p)}((d\psi)_p((\frac{\partial}{\partial x})_p), (d\psi)_p((\frac{\partial}{\partial z})_p)) &= g_{\psi(p)}(\left(\frac{\partial}{\partial x}\right)_{\psi(p)} + b\left(\frac{\partial}{\partial z}\right)_{\psi(p)}, \left(\frac{\partial}{\partial z}\right)_{\psi(p)}) \\ &= -(y+b) + b = -y. \end{aligned}$$

Case 3′: $X = \left(\frac{\partial}{\partial x}\right)_p, Y = \left(\frac{\partial}{\partial x}\right)_p$
.

$$\begin{aligned} g_p(\left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial x}\right)_p) &= 1 + y^2, \\ g_{\psi(p)}(\left(\frac{\partial}{\partial x}\right)_p, (d\psi)_p(\left(\frac{\partial}{\partial x}\right)_p) &= 1 + (y+b)^2 - b(y+b) - b(y+b) + b^2 \\ &= 1 + y^2. \end{aligned}$$

Case 4′: $X = \left(\frac{\partial}{\partial y}\right)_p, Y = \left(\frac{\partial}{\partial z}\right)_p$

.

$$\begin{aligned} g_p(\left(\frac{\partial}{\partial y}\right)_p, \left(\frac{\partial}{\partial z}\right)_p) &= 0, \\ g_{\psi(p)}((d\psi)_p(\left(\frac{\partial}{\partial y}\right)_p, (d\psi)_p(\left(\frac{\partial}{\partial z}\right)_p) &= g_{\psi(p)}(\left(\frac{\partial}{\partial y}\right)_{\psi(p)}, \left(\frac{\partial}{\partial z}\right)_{\psi(p)}) \\ &= 0. \end{aligned}$$

Case 5′: $X = \left(\frac{\partial}{\partial y}\right)_p, Y = \left(\frac{\partial}{\partial y}\right)_p$

.

$$\begin{aligned} g_p(\left(\frac{\partial}{\partial y}\right)_p, \left(\frac{\partial}{\partial y}\right)_p) &= 1, \\ g_{\psi(p)}((d\psi)_p(\left(\frac{\partial}{\partial y}\right)_p, (d\psi)_p(\left(\frac{\partial}{\partial y}\right)_p) &= g_{\psi(p)}(\left(\frac{\partial}{\partial y}\right)_{\psi(p)}, \left(\frac{\partial}{\partial y}\right)_{\psi(p)}) \\ &= 1. \end{aligned}$$

Case 6′: $X = \left(\frac{\partial}{\partial z}\right)_p, Y = \left(\frac{\partial}{\partial z}\right)_p$

.

$$\begin{aligned} g_p(\left(\frac{\partial}{\partial z}\right)_p, \left(\frac{\partial}{\partial z}\right)_p) &= 1, \\ g_{\psi(p)}((d\psi)_p(\left(\frac{\partial}{\partial z}\right)_p, (d\psi)_p(\left(\frac{\partial}{\partial z}\right)_p) &= g_{\psi(p)}(\left(\frac{\partial}{\partial z}\right)_{\psi(p)}, \left(\frac{\partial}{\partial z}\right)_{\psi(p)}) \\ &= 1. \end{aligned}$$

Therefore (1.24) holds. □

Chapter 2

Contact metric manifolds

2.1 contact manifolds

Remark that, in this paper, the exterior differentiation $d\eta$ of a 1-form η is defined by

$$d\eta(X,Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X,Y])),$$

for $X, Y \in \mathfrak{X}(M)$.

In terms of a local coordinates $x^1, \cdots, x^{2n+1}$ of M^{2n+1}, if $\eta = \sum_{i=1}^{2n+1} \eta_i dx^i$, then $d\eta$ is expressed as

$$d\eta = \frac{1}{2} \sum_{i,j=1}^{2n+1} \frac{\partial \eta_i}{\partial x^j} dx^j \wedge dx^i.$$

Definition 2.1.1. *A $(2n+1)$-dimensional C^∞ manifold M is said to be a contact manifold if it carries a 1-form η such that*

$$\eta \wedge (d\eta)^n \neq 0. \tag{2.1}$$

The 1-form η is called a contact form on M. It is well known that there exists the unique vector field ξ satisfying

$$\eta(\xi) = 1, \tag{2.2}$$

$$d\eta(\xi, X) = 0, \tag{2.3}$$

for $X, Y \in \mathfrak{X}(M)$.
The pair (M, η) is called a contact manifold and the vector field ξ is called the characteristic vector field of η.

Example

Proposition 2.1.1. (cf.[3]) If η is the 1-form defined by (1.6) on $\mathbf{R}^{2n+1}$, then the pair $(\mathbf{R}^{2n+1}, \eta)$ is a contact manifold.

Proof Since $d\eta = -\frac{1}{4}\sum_{i=1}^{n} dy^i \wedge dx^i$, if we put $dy^i \wedge dx^i = \omega^i$,

$$d\eta = -\frac{1}{4}\sum_{i=1}^{n}\omega^i.$$

Clearly

$$\omega^i \wedge \omega^j = \omega^j \wedge \omega^i.$$

Then we have

$$\begin{aligned}(d\eta)^n &= (-\frac{1}{4}\sum_{i=1}^{n}\omega^i) \wedge (-\frac{1}{4}\sum_{i=1}^{n}\omega^i) \wedge \cdots \wedge (-\frac{1}{4}\sum_{i=1}^{n}\omega^i)\\ &= (-\frac{1}{4})^n \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} \omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_n}\\ &= (-\frac{1}{4})^n n!\ \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n.\end{aligned}$$

Hence

$$\begin{aligned}\eta \wedge (d\eta)^n &= \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i) \wedge (-\frac{1}{4})^n n!\ \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\\ &= \frac{1}{2}(\frac{1}{4})^n n!\ dz \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\\ &= \frac{1}{2}(\frac{1}{4})^n n!\ dz \wedge dy^1 \wedge dx^1 \wedge dy^2 \wedge dx^2 \wedge \cdots \wedge dy^n \wedge dx^n\\ &\neq 0.\end{aligned}$$

Therefore (2.1) holds. □

The following theorem was proved by J.Gray.

Theorem 2.1.2. (see[2]) Let $\iota : M^{2n+1} \longrightarrow \mathbf{R}^{2n+2}$ be a smooth hypersurface immersed in $\mathbf{R}^{2n+2}$. If no tangent space of M^{2n+1} contains the origin of $\mathbf{R}^{2n+2}$, then M^{2n+1} has a contact structure. That is, let $(x^1, \cdots, x^{2n+2})$ be cartesian coordinates on $\mathbf{R}^{2n+2}$. And we consider the 1-form α defined by

$$\alpha = x^1 dx^2 - x^2 dx^1 + \cdots + x^{2n+1} dx^{2n+2} - x^{2n+2} dx^{2n+1},$$

then $\eta = \iota^* \alpha$ is a contact form.

Corollary 1. (cf.[2]) S^{2n+1} is a contact manifold.

Using the above results we will show that the real projective space P^{2n+1} is a contact manifold. We consider a system of coordinate neighborhoods $\{(U_i^+, \psi_i^+), (U_i^-, \psi_i^-)\}_{i=1,\cdots,2n+2}$ on S^{2n+1} such that

$$U_i^+ = \{(x^1, \cdots, x^i, \cdots, x^{2n+2}) \in S^{2n+1} \mid x^i > 0\},$$

$$U_i^- = \{(x^1, \cdots, x^i, \cdots, x^{2n+2}) \in S^{2n+1} \mid x^i < 0\},$$

$$\psi_i^+(x^1, \cdots, x^i, \cdots, x^{2n+2}) = (x^1, \cdots, \hat{x^i}, \cdots, x^{2n+2}),$$

$${}_i^-(x^1, \cdots, x^i, \cdots, x^{2n+2}) = (x^1, \cdots, \hat{x^i}, \cdots, x^{2n+2}).$$

Lemma 2.1.3. *Let η be the 1-form given in Theorem 2.1.2. We define the map $F : S^{2n+1} \longrightarrow S^{2n+1}$ by*

$$F(p) = -p \quad for\ p \in S^{2n+1}. \tag{2.4}$$

Then $F^*\eta = \eta$ holds.

Proof First we consider $\iota : U_i^+ \longrightarrow \mathbf{R}^{2n+2}$. We set the local coordinate of $\mathbf{R}^{2n+2}$ $(z^1, \cdots, z^{2n+2})$. And we set the local coordinate of U_i^+ $(x^1, \cdots, x^{2n+1})$.

For $X_p \in T_p(U_i^+)(p \in U_i^+)$, we put

$$X_p = (\xi_1 \frac{\partial}{\partial x^1} + \cdots + \xi_{2n+1} \frac{\partial}{\partial x^{2n+1}})_p.$$

Since $\iota(p) = p$, we get

(2.5)

$$(z^1, \cdots, z^{i-1}, z^i, z^{i+1}, \cdots, z^{2n+2}) = (x^1, \cdots, x^{i-1}, \sqrt{1- \parallel x \parallel^2}, x^i, \cdots, x^{2n+1}),$$

where $(x^1)^2 + \cdots + (x^{2n+1})^2 = \| x \|^2$.
We put $\dfrac{1}{\sqrt{1 - \| x \|^2}} = \lambda$ and hence get

$$\begin{aligned} (d\iota)_p(X_p) &= (\xi_1 \frac{\partial}{\partial z^1} + \cdots + \xi_{i-1} \frac{\partial}{\partial z^{i-1}} - \lambda(\xi_1 x^1 + \cdots + \xi_{2n+1} x^{2n+1}) \frac{\partial}{\partial z^i} + \\ &\quad \xi_i \frac{\partial}{\partial z^{i+1}} + \cdots + \xi_{2n+1} \frac{\partial}{\partial z^{2n+2}})_{\iota(p)}. \end{aligned}$$

Moreover we put $\mu = \xi_1 x^1 + \cdots + \xi_{2n+1} x^{2n+1}$ and get

$$\begin{aligned} \eta_p(X_p) &= (\iota^* \alpha)_p(X_p)) = \alpha_{\iota(p)}((d\iota)_p(X_p))_{\iota(p)} \\ &= (z^1 dz^2 - z^2 dz^1 + \cdots + z^{2n+1} dz^{2n+2} - z^{2n+2} dz^{2n+1})_{\iota(p)} \\ &\quad (\xi_1 \frac{\partial}{\partial z^1} + \cdots + \xi_{i-1} \frac{\partial}{\partial z^{i-1}} - \lambda\mu \frac{\partial}{\partial z^i} + \xi_i \frac{\partial}{\partial z^{i+1}} + \cdots + \xi_{2n+1} \frac{\partial}{\partial z^{2n+2}})_{\iota(p)}. \end{aligned}$$

When $i = 2k - 1 \quad (k \in \{1, \cdots, n+1\})$,

$$\begin{aligned} &\eta_p(X_p) \\ &= (z^1 dz^2 - \cdots + z^{2k-1} dz^{2k} - z^{2k} dz^{2k-1} + z^{2k+1} dz^{2k+2} - \cdots - z^{2n+2} dz^{2n+1})_{\iota(p)} \\ &\quad (\xi_1 \frac{\partial}{\partial z^1} + \cdots - \lambda\mu \frac{\partial}{\partial z^{2k-1}} + \xi_{2k-1} \frac{\partial}{\partial z^{2k}} + \xi_{2k} \frac{\partial}{\partial z^{2k+1}} \cdots + \xi_{2n+1} \frac{\partial}{\partial z^{2n+2}})_{\iota(p)} \\ &= \xi_2 z^1 - \cdots + \xi_{2k-1} z^{2k-1} + \lambda\mu z^{2k} + \xi_{2k+1} z^{2k+1} - \cdots - \xi_{2n} z^{2n+2}. \end{aligned}$$

From (2.5) we get

$$\eta_p(X_p) = \xi_2 x^1 - \cdots + \xi_{2k-1} \frac{1}{\lambda} + \lambda\mu x^{2k-1} + \xi_{2k+1} x^{2k} - \cdots - \xi_{2n} x^{2n+1}.$$

When $i = 2k \quad (k \in \{1, \cdots, n+1\})$, similarly

$$\eta_p(X_p) = \xi_2 x^1 - \cdots - \lambda\mu x^{2k-1} - \xi_{2k-1} \frac{1}{\lambda} + \xi_{2k+1} x^{2k} - \cdots - \xi_{2n} x^{2n+1}.$$

Next we put the local coordinates of $F(U_i^+)$ $(y^1, \cdots, y^{2n+1})$ and get

$$(y^1, \cdots, y^{2n+1}) = (-x^1, \cdots, -x^{2n+1}) \tag{2.6}$$

We put $(dF)_p(X_p) = Y_{F(p)}$ and get

$$Y_{F(p)} = (-\xi_1 \frac{\partial}{\partial y^1} - \cdots - \xi_{2n+1} \frac{\partial}{\partial y^{2n+1}})_{F(p)}.$$

We consider the mapping $\iota : U_i^- \longrightarrow \mathbf{R}^{2n+2}$ and get
(2.7)
$$(z^1, \cdots, z^{i-1}, z^i, z^{i+1}, \cdots, z^{2n+2}) = (y^1, \cdots, y^{i-1}, -\sqrt{1- \| y \|^2}, y^i, \cdots, y^{2n+1}),$$

where $(y^1)^2 + \cdots + (y^{2n+1})^2 = \| y \|^2$.
We put $\dfrac{1}{\sqrt{1- \| y \|^2}} = \lambda'$ and hence get

$$\begin{aligned}
&(d\iota)_{F(p)}(Y_{F(p)}) \\
&= (\xi_1 \frac{\partial}{\partial z^1} - \cdots - \xi_{i-1}\frac{\partial}{\partial z^{i-1}} - \lambda'(\xi_1 y^1 + \cdots + \xi_{2n+1} y^{2n+1})\frac{\partial}{\partial z^i} - \xi_i \frac{\partial}{\partial z^{i+1}} - \\
&\quad \cdots - \xi_{2n+1}\frac{\partial}{\partial z^{2n+2}})_{\iota(F(p))}.
\end{aligned}$$

Moreover we put $\mu' = \xi_1 y^1 + \cdots + \xi_{2n+1} y^{2n+1}$, then

$$\begin{aligned}
&(F^*\eta)_p(X_p) = \eta_{F(p)}((dF)_p(X_p)) \\
&= (\iota^*\alpha)_{F(p)}((dF)_p(X_p)) = \alpha_{\iota(F(p))}((d\iota)_{F(p)}(Y_{F(p)})) \\
&= (z^1 dz^2 - z^2 dz^1 + \cdots + z^{2n+1} dz^{2n+2} - z^{2n+2} dz^{2n+1})_{\iota(F(p))} \\
&(-\xi_1 \frac{\partial}{\partial z^1} - \cdots - \xi_{i-1}\frac{\partial}{\partial z^{i-1}} - \lambda'\mu'\frac{\partial}{\partial z^i} - \xi_i\frac{\partial}{\partial z^{i+1}} - \cdots - \xi_{2n+1}\frac{\partial}{\partial z^{2n+2}})_{\iota(F(p))}.
\end{aligned}$$

When $i = 2k-1 \quad (k \in \{1, \cdots, n+1\})$,

$$\begin{aligned}
&(F^*\eta)_p(X_p) \\
&= (z^1 dz^2 - \cdots + z^{2k-1}dz^{2k} - z^{2k}dz^{2k-1} + z^{2k+1}dz^{2k+2} - \cdots - z^{2n+2}dz^{2n+1})_{\iota(F(p))} \\
&\quad (-\xi_1\frac{\partial}{\partial z^1} - \cdots - \lambda'\mu'\frac{\partial}{\partial z^{2k-1}} - \xi_{2k-1}\frac{\partial}{\partial z^{2k}} - \xi_{2k}\frac{\partial}{\partial z^{2k+1}} - \cdots - \xi_{2n+1}\frac{\partial}{\partial z^{2n+2}})_{\iota(F(p))} \\
&= -\xi_2 z^1 + \cdots + \xi_{2k-3} z^{2k-2} - \xi_{2k-1} z^{2k-1} + \lambda'\mu' z^{2k} - \xi_{2k+1} z^{2k+1} + \cdots + \xi_{2n} z^{2n+2}.
\end{aligned}$$

Then from (2.7)

$$(F^*\eta)_p(X_p) = -\xi_2 y^1 + \cdots + \xi_{2k-3} y^{2k-2} + \xi_{2k-1}\frac{1}{\lambda'} + \lambda'\mu' y^{2k-1} - \xi_{2k+1} y^{2k} + \cdots + \xi_{2n} y^{2n+2}.$$

From (2.6)

$$(F^*\eta)_p(X_p) = \xi_2 x^1 - \cdots - \xi_{2k-3} x^{2k-2} + \xi_{2k-1}\frac{1}{\lambda'} - \lambda'\mu' x^{2k-1} + \xi_{2k+1} x^{2k} - \cdots - \xi_{2n} x^{2n+2}.$$

Since $\lambda' = \lambda, \mu' = \mu$,

$$\begin{aligned}(F^*\eta)_p(X_p) &= \xi_2 x^1 - \cdots - \xi_{2k-3}x^{2k-2} + \xi_{2k-1}\frac{1}{\lambda} + \lambda\mu x^{2k-1} + \xi_{2k+1}x^{2k} - \cdots - \xi_{2n}x^{2n+2} \\ &= \eta_p(X_p).\end{aligned}$$

When $i = 2k \quad (k \in \{1, \cdots, n+1\})$, similarly

$$\begin{aligned}(F^*\eta)_p(X_p) &= \xi_2 x^1 - \cdots - \lambda\mu x^{2k-1} - \xi_{2k-1}\frac{1}{\lambda} + \xi_{2k+1}x^{2k} - \cdots - \xi_{2n}x^{2n+1} \\ &= \eta_p(X_p).\end{aligned}$$

Therefore we get $F^*\eta = \eta$. □

Next we consider a natural projection $\pi : \mathbf{R}^{2n+2} - \{\mathbf{0}\} \longrightarrow P^{2n+1}$. We define a open set W_i of $\mathbf{R}^{2n+2} - \{\mathbf{0}\}$ and the open set V_i of P^{2n+1} such that

$$W_i = \{(x_1, \cdots, x_i, \cdots, x_{2n+2}) | x_i \neq 0\}, \tag{2.8}$$

$$V_i = \pi(W_i), \tag{2.9}$$

for $i = 1, \cdots, 2n+2$.

Moreover we define the homeomorphism $\sigma_i : V_i \to \mathbf{R}^{2n+1}$ and then get the following Lemma.

Lemma 2.1.4. *Let $\{(V_i, \sigma_i)\}_{i=1,\cdots,2n+2}$ be a system of coordinate neighborhood on P^{2n+1}. Then in the natural projection $\pi : S^{2n+1} \to P^{2n+1}$ the followings hold.*

$$\pi : U_i^+ \to V_i \text{ is a } C^\infty \text{diffeomorphism}, \tag{2.10}$$

$$\pi : U_i^- \to V_i \text{ is a } C^\infty \text{diffeomorphism}, \tag{2.11}$$

$$\pi(x) = \pi(-x) \quad \text{for } x \in U_i^+. \tag{2.12}$$

Theorem 2.1.5. (cf.[9]) P^{2n+1} is a contact manifold.

Proof For $x \in U_i^+$ we put $\pi(x) = l$ and then from (2.12) get

$$\pi(x) = \pi(-x) = l \tag{2.13}$$

Since (2.10) and (2.11) hold, for$X_l \in T_l(V_i)$ there exists a unique $Y_x \in T_x(U_i^+)$ and a unique $Y_{-x} \in T_{-x}(U_i^-)$ such that

$$(d\pi)_x(Y_x) = (d\pi)_{-x}(Y_{-x}) = X_l. \tag{2.14}$$

From (2.13) the following equation holds

$$\pi = \pi \circ F. \tag{2.15}$$

And then from (2.14) and (2.15) we get

$$\begin{aligned} X_l &= (d\pi)_x(Y_x) = d(\pi \circ F)_x(Y_x) = (d\pi)_{F(x)}((dF)_x(Y_x)) \\ &= (d\pi)_{-x}((dF)_x(Y_x)). \end{aligned}$$

Hence also from (2.14)

$$(d\pi)_{-x}(Y_{-x}) = (d\pi)_{-x}((dF)_x(Y_x)).$$

Since $(d\pi)_{-x} : T_{-x}(U_i^-) \to T_l(V_i)$ is an injection, we get

$$Y_{-x} = (dF)_x(Y_x). \tag{2.16}$$

From Lemma 2.1.3 the following equation holds

$$(F^*\eta)_x(Y_x) = \eta_x(Y_x) \quad for\ x \in (U_i^+). \tag{2.17}$$

On the other hand from (2.16) we get

$$(F^*\eta)_x(Y_x) = \eta_{F(x)}((dF)_x(Y_x)) = \eta_{-x}(Y_{-x}).$$

And then from (2.17) we get

$$\eta_{-x}(Y_{-x}) = \eta_x(Y_x).$$

Hence we can define the 1-form $\bar{\eta}$ on P^{2n+1} such that for $l \in P^{2n+1}$, $X_l \in T_l(P^{2n+1})$

$$\bar{\eta}(X_l) = \eta_x(Y_x),$$

where $\pi(x) = l,\ x \in S^{2n+1},\ (d\pi)_x(Y_x) = X_l$.
Thus, P^{2n+1} is a contact manifold. □

The value of $d\pi(\xi)$ in Hopf's mapping π

The following lemma is well known.

Lemma 2.1.6. *Let* $S^3 = \{(z^1, z^2) \in \mathbf{C}^2 \,\big|\, |z^1|^2 + |z^2|^2 = 1\}$ *and* $S^2 = \{(x, y, z) \in \mathbf{R}^3 \,\big|\, x^2 + y^2 + z^2 = 1\}$*. Then in Hopf's mapping* $\pi : S^3 \longrightarrow S^2$ *the following equations hold*

$$x = 2\mathrm{Re}(\overline{z^1} \cdot z^2), \quad y = 2\mathrm{Im}(\overline{z^1} \cdot z^2), \quad z = |z^2|^2 - |z^1|^2.$$

Proposition 2.1.7. *We consider Hopf's mapping* $\pi : S^3 \longrightarrow S^2$*. When we put* $\xi = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} - x^4\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^4}$, $d\pi(\xi) = 0$ *holds.*

Proof For $p \in S^3$, let the local coordinates of p (z^1, z^2) such that $z^1 = x^1 + ix^2$, $z^2 = x^3 + iz^4$.
We consider the C^∞curve $c(t)$ on S^3 defined by

$$c(t) = (e^{it}z^1, e^{it}z^2) \quad (t \in \mathbf{R})$$

Then we get

$$c(t) = (x^1 \cos t - x^2 \sin t, x^1 \sin t + x^2 \cos t, x^3 \cos t - x^4 \sin t, x^3 \sin t + x^4 \cos t).$$

Hence we get

$$\left.\frac{dc}{dt}\right|_{t=0} = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} - x^4\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^4} = \xi$$

Next we put the local coordinates of S^2 (x, y, z). Since $\pi(c(t)) = \pi(e^{it}z^1, e^{it}z^2)$, from Lemma 2.1.6 we get

$$\begin{aligned} x &= 2\mathrm{Re}(\overline{e^{it}z^1} \cdot e^{it}z^2) = 2\mathrm{Re}(\overline{z^1} \cdot z^2), \\ y &= 2\mathrm{Im}(\overline{e^{it}z^1} \cdot e^{it}z^2) = 2\mathrm{Im}(\overline{z^1} \cdot z^2), \\ z &= |e^{it}z^2|^2 - |e^{it}z^1|^2 = |z^2|^2 - |z^1|^2. \end{aligned}$$

And then we get

$$\pi(c(t)) = (2\mathrm{Re}(\overline{z^1} \cdot z^2), 2\mathrm{Im}(\overline{z^1} \cdot z^2), |z^2|^2 - |z^1|^2).$$

Therefore

$$\left.\frac{d(\pi \circ c)}{dt}\right|_{t=0} = (0, 0, 0),$$

that is $d\pi(\xi) = 0$. □

2.2 contact metric manifolds

Let η be a contact form on M. A Riemannian metric g is said to be an associated metric if there exists a tensor field φ of type (1,1) satisfying

$$d\eta(X,Y) = g(X,\varphi Y), \tag{2.18}$$
$$\eta(X) = g(X,\xi), \tag{2.19}$$
$$\varphi^2 = -I + \eta \otimes \xi. \tag{2.20}$$

Definition 2.2.1. *The structure (φ, ξ, η, g) satisfying (2.1),(2.18),(2.19) and (2.20) is called a contact metric structure and a manifold M^{2n+1} with a contact metric structure (φ, ξ, η, g) is said to be a contact metric manifold.*

Theorem 2.2.1. (cf.[4]) *The following equations hold on a contact metric structure (φ, ξ, η, g)*

$$\varphi\xi = 0, \tag{2.21}$$
$$\eta \circ \varphi = 0, \tag{2.22}$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.23}$$

Proof First for $X \in \mathfrak{X}(M^{2n+1})$, using (2.18) and (2.3) we get

$$g(X, \varphi\xi) = d\eta(X, \xi) = 0. \tag{1}$$

Substituting $X = \varphi\xi$ into (1), we get

$$g(\varphi\xi, \varphi\xi) = 0$$

and hence $\varphi\xi = 0$.
Next using (2.19), (2.18), and (2.3) we get

$$\eta \circ \varphi(X) = g(\xi, \varphi X) = d\eta(\xi, X) = 0$$

. Thus $\eta \circ \varphi = 0$. Finally using (2.18), (2.20) and (2.19) we get

$$\begin{aligned} g(\varphi X, \varphi Y) &= -d\eta(Y, \varphi X) = -g(Y, \varphi^2 X) = -g(Y, -X + \eta(X)\xi) \\ &= g(X,Y) - \eta(X)g(Y,\xi) \\ &= g(X,Y) - \eta(X)\eta(Y) \end{aligned}$$

Thus, $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$. □

Example

Proposition 2.2.2. (cf.[3]) *Let* $\mathbf{R}^{2n+1}$ *be the almost contact metric manifold defined in Proposition 1.2.3. Then* $\mathbf{R}^{2n+1}$ *is a contact metric manifold.*

Proof We already proved that the $\mathbf{R}^{2n+1}$ satisfies (2.1), (1,10) i.e.(2.19) and (2.20). And then we must prove that (2.18) holds on it. Using (1.13),(1.14),(1.15) and (1.16) we compute as follows :

(1) $X = X_i, Y = X_j$

$$\begin{aligned}\frac{1}{2}(X_i\eta(X_j) - X_j\eta(X_i) - \eta([X_i, X_j])) &= 0\\ g(X_i, \varphi X_j) &= g(X_i, X_{n+j}) = 0\end{aligned}$$

(2) $X = X_i, Y = X_{n+j}$

$$\begin{aligned}\frac{1}{2}(X_i\eta(X_{n+j}) - X_{n+j}\eta(X_i) - \eta([X_i, X_{n+j}])) &= -\delta_{ij}\eta(\xi) = -\delta_{ij}\\ g(X_i, \varphi X_j) &= g(X_i, -X_j) = -\delta_{ij}\end{aligned}$$

(3) $X = X_i, Y = \xi$

$$\begin{aligned}\frac{1}{2}(X_i\eta(\xi) - \xi\eta(X_i) - \eta([X_i, \xi])) &= 0\\ g(X_i, \varphi\xi) &= g(X_i, 0) = 0\end{aligned}$$

(4) $X = X_{n+i}, Y = X_{n+j}$

$$\begin{aligned}\frac{1}{2}(X_{n+i}\eta(X_{n+j}) - X_{n+j}\eta(X_{n+i}) - \eta([X_{n+i}, X_{n+j}])) &= 0\\ g(X_{n+i}, \varphi X_{n+j}) &= g(X_{n+i}, -X_j)\\ &= 0\end{aligned}$$

(5) $X = X_{n+i}, Y = \xi$

$$\begin{aligned}\frac{1}{2}(X_{n+i}\eta(\xi) - \xi\eta(X_{n+i}) - \eta([X_{n+i}, \xi])) &= 0\\ g(X_{n+i}, \varphi\xi) &= g(X_{n+i}, 0) = 0\end{aligned}$$

(6) $X = \xi, Y = \xi$

$$\begin{aligned}\frac{1}{2}(\xi\eta(\xi) - \xi\eta(\xi) - \eta([\xi, \xi])) &= 0\\ g(\xi, \varphi\xi) &= g(\xi, 0) = 0\end{aligned}$$

Therefore (2.18) holds. □

Definition 2.2.2. *On a contact metric manifold* $(M^{2n+1}, \varphi, \xi, \eta, g)$ *we define the operators* l *and* h *by*

$$lX = R(X,\xi)\xi, \quad h = \frac{1}{2}\mathcal{L}_\xi\varphi. \tag{2.24}$$

Clearly the (1,1)-type tensors h and l are symmetric.

Proposition 2.2.3. (cf.[4]) *h and l satisfy the following equations*

$$h\xi = 0,\ l\xi = 0,\ Trh = 0,\ Trh\varphi = 0 \text{ and } h\varphi = -\varphi h. \tag{2.25}$$

Proof From (2.21) we get

$$h\xi = \frac{1}{2}(\mathcal{L}_\xi\varphi)(\xi) = \frac{1}{2}([\xi, \varphi\xi] - \varphi[\xi,\xi]) = 0$$

and

$$l\xi = R(\xi,\xi)\xi = 0.$$

We will prove $Trh = 0$. Let $\{e_1 = \xi, e_2, \cdots, e_{2n+1}\}$ be the orthonormal basis on M^{2n+1}. For $i \in \{1, 2, \cdots, 2n+1\}$ we put $h(e_i) = \sum_{j=1}^{2n+1} a_{ij}e_j$ and hence get $a_{ii} = g(h(e_i), e_i)$.

Using (2.18) we compute Trh as follows:

$$\begin{aligned}
(1)\ \ Trh &= \sum_{i=1}^{2n+1} a_{ii} = \sum_{i=1}^{2n+1} g(h(e_i), e_i) = \sum_{i=1}^{2n+1} g(\frac{1}{2}(\mathcal{L}_\xi\varphi)(e_i), e_i) \\
&= \sum_{i=1}^{2n+1} \frac{1}{2} g([\xi, \varphi e_i] - \varphi[\xi, e_i], e_i) \\
&= \frac{1}{2}\{\sum_{i=1}^{2n+1} g([\xi, \varphi e_i], e_i) + \sum_{i=1}^{2n+1} g([\xi, e_i], \varphi e_i)\}.
\end{aligned}$$

Now we take another orthonormal basis $\{e'_1, e'_2, \cdots, e'_{2n+1}\}$ such that $e'_1 = \xi, e'_i = \varphi e_i$ for $i \in \{2, 3, \cdots, 2n+1\}$. Similarly using (2.19),(2.20) we compute

Trh

$$
\begin{aligned}
(2)\quad Trh &= \frac{1}{2}\{\sum_{i=1}^{2n+1} g([\xi,\varphi e_i'],e_i') + \sum_{i=1}^{2n+1} g([\xi,e_i'],\varphi e_i')\} \\
&= \frac{1}{2}\{\sum_{i=1}^{2n+1} g([\xi,\varphi^2 e_i],\varphi e_i) + \sum_{i=1}^{2n+1} g([\xi,\varphi e_i],\varphi^2 e_i)\} \\
&= -\frac{1}{2}\{\sum_{i=1}^{2n+1} g([\xi,\varphi e_i],e_i) + \sum_{i=1}^{2n+1} g([\xi,e_i],\varphi e_i)\}.
\end{aligned}
$$

Since (1) = (2), we have $Trh = 0$.
Next we will prove $Trh\varphi = 0$. For $i \in \{1, 2, \cdots, 2n+1\}$ we put $h\varphi(e_i) = \sum_{j=1}^{2n+1} b_{ij} e_j$. Using (2.18), (2.19) we compute $Trh\varphi$ as follows:

$$
\begin{aligned}
(3)\quad Trh\varphi &= \sum_{i=1}^{2n+1} b_{ii} = \sum_{i=1}^{2n+1} g(h\varphi(e_i),e_i) = \sum_{i=1}^{2n+1} g(\frac{1}{2}(\mathcal{L}_\xi\varphi)(\varphi e_i),e_i) \\
&= \frac{1}{2}\sum_{i=1}^{2n+1}\{-g([\xi,e_i],e_i) + g([\xi,\eta(e_i)\xi],e_i) + g([\xi,\varphi e_i],\varphi e_i)\} \\
&= \frac{1}{2}\sum_{i=1}^{2n+1}\{-g([\xi,e_i],e_i) + g([\xi,\varphi e_i],\varphi e_i)\}
\end{aligned}
$$

Similarly using (2.19), (2.20) we compute

$$
\begin{aligned}
(4)\qquad Trh\varphi &= \frac{1}{2}\sum_{i=1}^{2n+1}\{-g([\xi,e_i'],e_i') + g([\xi,\varphi e_i'],\varphi e_i')\} \\
&= -\frac{1}{2}\sum_{i=1}^{2n+1}\{-g([\xi,\varphi e_i],\varphi e_i) + g([\xi,\varphi^2 e_i],\varphi^2 e_i)\} \\
&= -\frac{1}{2}\sum_{i=1}^{2n+1}\{-g([\xi,e_i],e_i) + g([\xi,\varphi e_i],\varphi e_i)\}
\end{aligned}
$$

Since (3) = (4), we have $Trh\varphi = 0$.

Finally we prove $h\varphi = -\varphi h$. Now,using (2.2), (2.1) we get

$$\begin{aligned} d\eta(\xi, X) &= \frac{1}{2}\{\xi(\eta(X)) - X(\eta(\xi)) - \eta([\xi, X])\} \\ &= \frac{1}{2}\{\xi(\eta(X)) - \eta([\xi, X])\} = 0 \end{aligned}$$

and hence

$$\xi(\eta(X)) = \eta([\xi, X]) \tag{5}$$

We compute $h\varphi(X), -\varphi h(X)$ respectively.

$$\begin{aligned} h\varphi(X) &= \frac{1}{2}(\mathcal{L}_\xi\varphi)(\varphi X) = \frac{1}{2}([\xi, \varphi^2 X] - \varphi[\xi, \varphi X]) \\ &= \frac{1}{2}([\xi, -X + \eta(X)\xi] - \varphi[\xi, \varphi X]) \\ &= -\frac{1}{2}([\xi, X] + \varphi[\xi, \varphi X] - (\xi\eta(X))\xi) \end{aligned}$$

$$\begin{aligned} -\varphi h(X) &= -\varphi(\frac{1}{2}(\mathcal{L}_\xi\varphi)(X)) = -\frac{1}{2}(\varphi[\xi, \varphi X] - \varphi^2[\xi, X]) \\ &= -\frac{1}{2}([\xi, X] + \varphi[\xi, \varphi X] - \eta([\xi, X])\xi) \end{aligned}$$

From (5) we get $h\varphi = -\varphi h$. □

The following formulas are known. (cf. [2], [5]).

(2.26) $$\nabla_X\xi = -\varphi X - \varphi hX \quad (\text{and hence } \nabla_\xi\xi = 0),$$

(2.27) $$\nabla_\xi\varphi = 0,$$

(2.28) $$Trl = g(Q\xi, \xi) = 2n - Trh^2,$$

(2.29) $$\varphi l\varphi - l = 2(\varphi^2 + h^2),$$

(2.30) $$\nabla_\xi h = \varphi - \varphi l - \varphi h^2,$$

where Q is the Ricci operator and ∇ the Riemannian connection of g.

2.3 K-contact structures

Definition 2.3.1. *A vector field X on a Riemannian manifold M^{2n+1} is called a Killing vector field if X satisfies $\mathcal{L}_X g = 0$,that is,*

$$\mathcal{L}_X(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$$

for any vector fields Y and Z on M^{2n+1}, where $\mathcal{L}_X$ denotes the Lie differentiation with respect to X.

Definition 2.3.2. *Let M^{2n+1} be a contact metric manifold with (φ, ξ, η, g). If ξ is a Killing vector field, then we call the (φ, ξ, η, g) a K-contact structure.*

Proposition 2.3.1. (cf.[2]) If a contact metric manifold M^{2n+1} with (φ, ξ, η, g) is a K-contact manifold, then the following equation holds.

$$\nabla_X \xi = -\varphi X. \tag{2.31}$$

Proof For $X, Y \in \mathfrak{X}(M^{2n+1})$ using (2.18) we get

$$\begin{aligned} g(X, \varphi Y) &= d\eta(X, Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X, Y])\} \\ &= \frac{1}{2}\{X\eta(Y) - \eta(\nabla_X Y) - Y\eta(X) + \eta(\nabla_Y X)\} \\ &= \frac{1}{2}\{Xg(Y, \xi) - g(\nabla_X Y, \xi) - Yg(X, \xi) + g(\nabla_Y X, \xi)\} \\ &= \frac{1}{2}\{g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi) - g(\nabla_X Y, \xi) \\ &\quad - g(\nabla_Y X, \xi) - g(X, \nabla_Y \xi) + g(\nabla_Y X, \xi)\} \\ &= \frac{1}{2}\{g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)\} \\ &= \frac{1}{2}\{g(Y, \nabla_X \xi) + g(Y, \nabla_X \xi)\} \quad \text{(since } \xi \text{ is Killing)} \\ &= g(Y, \nabla_X \xi). \end{aligned}$$

Then we get $g(X, \varphi Y) = g(Y, \nabla_X \xi)$ and hence $g(Y, \nabla_X \xi) + g(Y, \varphi X) = 0$. Thus

$$g(\nabla_X \xi + \varphi X, Y) = 0 \tag{2.32}$$

By substituting $Y = \nabla_X \xi + \varphi X$ into the above equation, we get

$$\nabla_X \xi = -\varphi X.$$

□

Example

Proposition 2.3.2. (see [3]) *Let* $\mathbf{R}^{2n+1}$ *be the contact metric manifold defined in Proposition 2.2.2. Then* $\mathbf{R}^{2n+1}$ *is a K-contact manifold.*

Proof We must prove that $\mathcal{L}_\xi g$ is equal to 0. Since $\mathcal{L}_\xi g$ is symmetric, using (2.16) we compute as follows:

(1) $X = X_i, Y = X_j$

$$\begin{aligned}\mathcal{L}_\xi g(X_i, X_j) &= -g([\xi, X_i], X_j) - g(X_i, [\xi, X_j]) \\ &= -g(0, X_j) - g(X_i, 0) = 0\end{aligned}$$

(2) $X = X_i, Y = X_{n+j}$

$$\begin{aligned}\mathcal{L}_\xi g(X_i, X_{n+j}) &= -g([\xi, X_i], X_{n+j}) - g(X_i, [\xi, X_{n+j}]) \\ &= -g(0, X_{n+j}) - g(X_i, 0) = 0\end{aligned}$$

(3) $X = X_{n+i}, Y = X_{n+j}$

$$\begin{aligned}\mathcal{L}_\xi g(X_{n+i}, X_{n+j}) &= -g([\xi, X_{n+i}], X_{n+j}) - g(X_{n+i}, [\xi, X_{n+j}]) \\ &= -g(0, X_{n+j}) - g(X_{n+i}, 0) = 0\end{aligned}$$

(4) $X = \xi, Y = X_i$

$$\begin{aligned}\mathcal{L}_\xi g(\xi, X_i) &= -g([\xi, \xi], X_i) - g(\xi, [\xi, X_i]) \\ &= -g(0, X_i) - g(\xi, 0) = 0\end{aligned}$$

(5) $X = \xi, Y = X_{n+i}$

$$\begin{aligned}\mathcal{L}_\xi g(\xi, X_{n+i}) &= -g([\xi, \xi], X_{n+i}) - g(\xi, [\xi, X_{n+i}]) \\ &= -g(0, X_{n+i}) - g(\xi, 0) = 0\end{aligned}$$

(6) $X = \xi, Y = \xi$

$$\begin{aligned} \mathcal{L}_\xi g(\xi,\xi) &= -g([\xi,\xi],\xi) - g(\xi,[\xi,\xi]) \\ &= -g(0,\xi) - g(\xi,0) = 0 \end{aligned}$$

Therefore $\mathcal{L}_\xi g$ is equal to 0. □

Proposition 2.3.3. (cf.[2]) *Let M^{2n+1} be a K-contact manifold with structure tensors (φ,ξ,η,g). Then the sectional curvature of any plane section containing ξ is equal to 1.*

Proof Let X be a unit vector field orthogonal to ξ. Then

$$\begin{aligned} R(\xi,X)\xi &= \nabla_\xi\nabla_X\xi - \nabla_X\nabla_\xi\xi - \nabla_{[\xi,X]}\xi \\ &= -\nabla_\xi\varphi X + \varphi[\xi,X] \quad (\ from\ (2.31)\ and\ (2.26)\ ) \\ &= -\nabla_\xi\varphi X + \varphi(\nabla_\xi X - \nabla_X\xi) \quad (\ since\ T^\nabla = 0) \\ &= -\varphi\nabla_X\xi \quad (\ from\ (2.27)\ ) \\ &= \varphi^2 X \quad (\ from\ (2.31)\ ) \\ &= -X + \eta(X)\xi = -X + g(X,\xi)\xi = -X \end{aligned}$$

and hence

$$\begin{aligned} g(R(\xi,X)X,\xi) &= -g(R(\xi,X)\xi,X) \\ &= -g(-X,X) = g(X,X) = 1. \end{aligned}$$

□

Corollary 2. *On $\mathbf{R}^{2n+1}$ defined in Proposition 2.3.2, the sectional curvature of any plane section containing ξ is equal to 1.*

Remark that we shall show that there exist some almost contact metric structures which are not contact metric structure.

Example

Proposition 2.3.4. (cf.[3]) *Let (M^{2n},J,G) be an almost Hermitian manifolds with local coordinates $x^1,\cdots,x^{2n}$ and let t be the coordinate on $\mathbf{R}$. We* *define η, ξ, g, φ on the $M^{2n} \times \mathbf{R}$ as follows :*

$$\begin{aligned} (2.33) \quad \eta &= f dt, \quad \text{where } f \text{ is some non- vanishing function}, \\ (2.34) \quad \xi &= \frac{1}{f}\frac{\partial}{\partial t}, \\ (2.35) \quad g &= G + \eta \otimes \eta, \\ (2.36) \quad \varphi\xi &= 0, \quad \varphi X = JX \quad \text{for } X \text{ orthogonal to } \xi. \end{aligned}$$

Then (φ, ξ, η, g) is an almost contact metric structure which is not a contact metric structure.

Proof We can see that the following equations hold

$$g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}) = 0, \tag{2.37}$$

$$g(\xi, J(\frac{\partial}{\partial x^i})) = 0. \tag{2.38}$$

Because from (2.35)

$$\begin{aligned} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}) &= g(\frac{\partial}{\partial x^i} + 0, 0 + \frac{\partial}{\partial t}) \\ &= G(\frac{\partial}{\partial x^i}, 0) + \eta(0) \cdot \eta(\frac{\partial}{\partial t}) = 0. \end{aligned}$$

And we put $J(\frac{\partial}{\partial x^i}) = \sum_{j=1}^{2n} \alpha_i^j \frac{\partial}{\partial x^j}$, then from (2.27)

$$g(J(\frac{\partial}{\partial x^i}), \xi) = g(\sum_{j=1}^{2n} \alpha_i^j \frac{\partial}{\partial x^j}, \frac{1}{f}\frac{\partial}{\partial t}) = \frac{1}{f}\sum_{j=1}^{2n} \alpha_i^j g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial t}) = 0.$$

Therefore $J(\frac{\partial}{\partial x^i})$ is orthogonal to ξ.

(1) $\eta(\xi) = 1$

From (2.33),(2.34)

$$\eta(\xi) = f dt(\frac{1}{f}\frac{\partial}{\partial t}) = 1$$

(2) $\varphi^2 = -I + \eta \otimes \xi$

For $X \in \mathfrak{X}(M^{2n} \times \mathbf{R})$ we put $X = \sum_{i=1}^{2n} \alpha_i \frac{\partial}{\partial x^i} + \beta \frac{\partial}{\partial t}$ and get

$$\eta(X) = \eta(\sum_{i=1}^{2n} \alpha_i \frac{\partial}{\partial x^i} + \beta \frac{\partial}{\partial t}) = \eta(\beta \frac{\partial}{\partial t})$$

And then

$$\begin{aligned}
\varphi^2(X) &= \varphi\{\varphi(\sum_{i=1}^{2n} \alpha_i \frac{\partial}{\partial x^i} + \beta \frac{\partial}{\partial t})\} = \varphi\{(\sum_{i=1}^{2n} \alpha_i \varphi(\frac{\partial}{\partial x^i}) + \beta \varphi(\frac{\partial}{\partial t})\} \\
&= \varphi\{(\sum_{i=1}^{2n} \alpha_i \varphi(\frac{\partial}{\partial x^i})\} = \varphi\{(\sum_{i=1}^{2n} \alpha_i J(\frac{\partial}{\partial x^i})\} = \{(\sum_{i=1}^{2n} \alpha_i \varphi(J(\frac{\partial}{\partial x^i}))\} \\
&= \sum_{i=1}^{2n} \alpha_i J^2(\frac{\partial}{\partial x^i}) \quad \text{(because } J(\frac{\partial}{\partial x^i}) \text{ is orthogonal to } \xi) \\
&= -\sum_{i=1}^{2n} \alpha_i \frac{\partial}{\partial x^i} \quad \text{(because } (M^{2n}, J, G) \text{ is an almost Hermitian manifolds)} \\
&= -X + \beta \frac{\partial}{\partial t} = -X + f\beta\xi = -X + \eta(\beta \frac{\partial}{\partial t})\xi \\
&= -X + \eta(X)\xi.
\end{aligned}$$

Therefore $\varphi^2 = -I + \eta \otimes \xi$ holds.

(3) $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$

For $X, Y \in \mathfrak{X}(M^{2n} \times \mathbf{R})$ we put

$X = X' + X'',\ Y = Y' + Y''$, where $X', Y' \in \mathfrak{X}(M^{2n}),\ X'', Y'' \in \mathfrak{X}(\mathbf{R})$.

And then

$$\begin{aligned}
g(\varphi X, \varphi Y) &= g(\varphi(X' + X''), \varphi(Y' + Y'')) = g(\varphi X' + \varphi X'', \varphi Y' + \varphi Y'')) = g(\varphi X', \varphi Y') \\
&= g(JX', JY') \quad \text{(because } X', Y' \in \mathfrak{X}(M^{2n})) \\
&= G(JX', JY') \\
&= G(X', Y') \quad \text{(because } (M^{2n}, J, G) \text{ is an almost Hermitian manifold)} \\
&= g(X, Y) - \eta(X'')\eta(Y'') = g(X, Y) - \eta(X)\eta(Y).
\end{aligned}$$

Therefore $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ holds.
(4) $\eta \wedge (d\eta)^n$

Since from (2.33) $d\eta = \dfrac{1}{2} df \wedge dt$, $\eta \wedge (d\eta) = \dfrac{1}{2} f dt \wedge df \wedge dt = 0$.
Thus

$$\eta \wedge (d\eta)^n \equiv 0.$$

Therefore (φ, ξ, η, g) is not a contact metric structure. □

Chapter 3

3-dimensional contact metric manifolds

We denote by ∇ the Riemannian connection of g and by R the Riemannian curvature tensor, which is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for $X, Y \in \mathfrak{X}(M^{2n+1})$.
The Ricci tensor $Ric(X,Y)$ is defied by

$$Ric(X,Y) = \sum_{i=1}^{2n+1} g(R(X_i, X)Y, X_i), \tag{3.1}$$

for $X, Y \in \mathfrak{X}(M^{2n+1})$,where $X_1, \cdots, X_{2n+1}$ is a local orthonormal frame field of M^{2n+1}.
The Ricci operator Q is defined by

$$Ric(X,Y) = g(QX, Y),$$

for $X, Y \in \mathfrak{X}(M^{2n+1})$.

Definition 3.0.3. *A contact metric structure is said to be η-Einstein if*

$$Q = pI + q\eta \otimes \xi$$

holds, where p, q are some smooth functions on M^{2n+1}.

Remark that the above equation is equivalent to

$$Ric(X,Y) = pg(X,Y) + qg(\xi,X)g(\xi,Y), \tag{3.2}$$

for $X, Y \in \mathfrak{X}(M^{2n+1})$.

Definition 3.0.4. *A contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be Sasakian if M^{2n+1} satisfies*

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{3.3}$$

for $X, Y \in \mathfrak{X}(M^{2n+1})$.
From Definition 2.3.1. and Definition 2.3.2. $(M^{2n+1}, \varphi, \xi, \eta, g)$ is K-contact manifold if and only if the following equation holds

$$g(X, \nabla_Y \xi) + g(\nabla_X \xi, Y) = 0, \tag{3.4}$$

for $X, Y \in \mathfrak{X}(M^{2n+1})$.

3.1 S^3 with the contact form η

Let $(x^1, \cdots, x^{2n+2})$ be Cartesian coordinates on the (2n+2)-dimensional Euclidean space $\mathbf{R}^{2n+2}$. We consider the 1-form α on $\mathbf{R}^{2n+2}$ defined by

$$\alpha = x^1 dx^2 - x^2 dx^1 + \cdots + x^{2n+1} dx^{2n+2} - x^{2n+2} dx^{2n+1} \tag{3.5}$$

and the inclusion mapping

$$\iota : S^{2n+1} \to \mathbf{R}^{2n+2}. \tag{3.6}$$

From Theorem 2.1.2., $\eta = \iota^* \alpha$ is a contact form on S^{2n+1}, i.e., $\eta \wedge (d\eta)^n \neq 0$ holds on S^{2n+1}. By using (3.5) we get

$$d\alpha = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \cdots + dx^{2n+1} \wedge dx^{2n+2}. \tag{3.7}$$

Throughout this section, we consider this contact form η on S^3. Then from (2.2) and (2.3), the characteristic vector field ξ is determined by

$$d\iota(\xi) = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}.$$

We take the independent vector fields $X_1, X_2, X_3 = \xi$ on S^3 such that

$$d\iota(X_1) = -x^3\frac{\partial}{\partial x^1} + x^4\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} - x^2\frac{\partial}{\partial x^4}, \tag{3.8}$$

$$d\iota(X_2) = -x^4\frac{\partial}{\partial x^1} - x^3\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^3} + x^1\frac{\partial}{\partial x^4}, \tag{3.9}$$

$$d\iota(X_3) = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} - x^4\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^4}. \tag{3.10}$$

g and φ of S^3

Let g be a Riemannian metric on (S^3, η) which satisfies (2.19). We put $g_{ij} = g(X_i, X_j)$ and $a = g_{11},\ b = g_{12} = g_{21},\ c = g_{22}$.

By using $\eta = \iota^*\alpha$, from (2.19) we get

$$g_{13} = g(X_1, X_3) = \eta(X_1) = 0,$$

$$g_{23} = g(X_2, X_3) = \eta(X_2) = 0$$

and from (2.2) get

$$g_{33} = g(X_3, X_3) = \eta(X_3) = 1.$$

Then, the 3×3 matrix (g_{ij}) is of the form

$$(g_{ij}) = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.11}$$

where $a, b, c \in C^\infty(S^3)$.

Since $\det(g_{ij}) > 0$, we get $ac - b^2 > 0$. Moreover, since $X_1 \neq 0, X_2 \neq 0$, we get $a = g(X_1, X_1) > 0, c = g(X_2, X_2) > 0$.

Conversely, let g be a tensor field of type (0,2) defined by (3.11). If $a > 0, c > 0$ and $ac - b^2 > 0$ holds, then g is a Riemannian metric satisfying (2.19).

Thus we have the following.

Proposition 3.1.1. *If a Riemannian metric g on (S^3, η) satisfies (2.19), then (3.11) and the following hold*

$$a > 0, c > 0 \quad \textit{and} \quad ac - b^2 > 0. \tag{3.12}$$

Conversely, let g be a tensor field of type (0,2) on (S^3, η) defined by (3.11). If g satisfies (3.12), then g is a Riemannian metric on (S^3, η) and satisfies (2.19).

Next, let φ be a tensor field of type (1,1) satisfying (2.18). Then, we have

$$\begin{aligned} \varphi(X_1) &= \frac{b}{ac-b^2}X_1 + \frac{-a}{ac-b^2}X_2, \\ \varphi(X_2) &= \frac{c}{ac-b^2}X_1 + \frac{-b}{ac-b^2}X_2, \\ \varphi(X_3) &= 0, \end{aligned}$$

where $a > 0, c > 0, ac - b^2 > 0$.
Because, by using $\eta = \iota^*\alpha$ from (3.7) we get

$$d\eta(X_i, X_j) = (dx^1 \wedge dx^2 + dx^3 \wedge dx^4)(d\iota(X_i), d\iota(X_j)).$$

And then from (3.8), (3.9), (3.10) we have

$$d\eta(X_1, X_2) = 1,\ d\eta(X_2, X_1) = -1,\ \text{others are equal to 0.}$$

Now, we put $\varphi(X_j) = \sum_{k=1}^{3} \varphi_{kj} X_k \quad (j = 1, 2, 3)$. Since $g(X_i, \varphi X_j) = \sum_{k=1}^{3} g_{ik}\varphi_{kj}$, from (2.18) we get

$$(g_{ij})(\varphi_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \frac{1}{ac-b^2} \begin{pmatrix} b & c & 0 \\ -a & -b & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.13}$$

where $a > 0, c > 0, ac - b^2 > 0$.

Proposition 3.1.2. *Let (φ, ξ, η, g) be given by (3.11), (3.12) and (3.13) on S^3. If (φ, ξ, η, g) is a contact metric structure, then the following equation holds*

$$ac - b^2 = 1. \tag{3.14}$$

Conversely, if (3.14) holds, then (φ, ξ, η, g) is a contact metric structure on S^3.

Proof. From (3.13) we get

$$(\varphi_{ij})^2 = \frac{1}{b^2 - ac}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

By putting $\psi = -I + \eta \otimes \xi$, we get the following equation

$$\psi(X_j) = -X_j + \eta(X_j)X_3 \qquad (j = 1, 2, 3).$$

By substituting $j = 1, 2, 3$ into the above equation, we get

$$\begin{aligned} \psi(X_1) &= -X_1 + \eta(X_1)X_3 = -X_1, \\ \psi(X_2) &= -X_2 + \eta(X_2)X_3 = -X_2, \\ \psi(X_3) &= -X_3 + \eta(X_3)X_3 = 0. \end{aligned}$$

Now, we put

$$\psi(X_j) = \psi_{1j}X_1 + \psi_{2j}X_2 + \psi_{3j}X_3 = \sum_{i=1}^{3} \psi_{ij}X_i.$$

By substituting $j = 1, 2, 3$ into the above equation, from the above result we get

$$\begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

If (φ, ξ, η, g) is a contact metric structure, by using (3.15) and (3.16) from (2.20) we get (3.14).

Conversely, if (3.14) holds, we can get (2.20). □

Corollary 3. *φ is denoted by the following matrix*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \begin{pmatrix} b & c & 0 \\ -a & -b & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad a > 0, c > 0, \ ac - b^2 = 1.$$

Curvature tensors

In this section, we assume a, b, c are constant. By using (g_{ij}) which satisfies (3.11),(3.12) and (3.14), from the basis $X_1, X_2, X_3 = \xi$, we can generate the orthonormal basis Y_1, Y_2, Y_3 on (S^3, g), that is

$$Y_1 = X_3,\ Y_2 = \frac{1}{\sqrt{a}}X_1,\ Y_3 = -\frac{\sqrt{a}b}{a}X_1 + \sqrt{a}X_2.$$

And then we get

$$X_3 = Y_1,\ X_1 = \sqrt{a}Y_2,\ X_2 = \frac{b}{\sqrt{a}}Y_2 + \frac{1}{\sqrt{a}}Y_3.$$

By computing $[X_i, X_j]$ from (3.8), (3.9), (3.10) and the above equations, we have

$$[X_1, X_2] = -2X_3,\ [X_2, X_3] = -2X_1,\ [X_3, X_1] = -2X_2$$

and

$$[Y_1, Y_2] = -\frac{2b}{a}Y_2 - \frac{2}{a}Y_3,\ [Y_1, Y_3] = \frac{2(a^2+b^2)}{a}Y_2 + \frac{2b}{a}Y_3,\ [Y_2, Y_3] = -2Y_1.$$

From the above results, we get

$$\begin{array}{ll}
2g(\nabla_{Y_2}Y_2, Y_1) = -\frac{4b}{a}, & 2g(\nabla_{Y_3}Y_3, Y_1) = \frac{4b}{a}, \\
2g(\nabla_{Y_1}Y_2, Y_3) = \frac{2(a-a^2-b^2-1)}{a}, & 2g(\nabla_{Y_1}Y_3, Y_2) = \frac{2(a^2+b^2-a+1)}{a}, \\
2g(\nabla_{Y_2}Y_3, Y_1) = \frac{2(a^2+b^2-a-1)}{a}, & 2g(\nabla_{Y_2}Y_1, Y_2) = \frac{4b}{a}, \\
2g(\nabla_{Y_2}Y_1, Y_3) = \frac{2(a-a^2-b^2+1)}{a}, & 2g(\nabla_{Y_3}Y_1, Y_2) = \frac{2(1-a-a^2-b^2)}{a}, \\
2g(\nabla_{Y_3}Y_1, Y_3) = -\frac{4b}{a}, & 2g(\nabla_{Y_3}Y_2, Y_1) = \frac{2(a^2+b^2+a-1)}{a},
\end{array}$$

others are equal to 0.

And then we have

$$\begin{array}{ll}
\nabla_{Y_1}Y_1 = 0, & \nabla_{Y_1}Y_2 = \frac{a-a^2-b^2-1}{a}Y_3, \\
\nabla_{Y_1}Y_3 = \frac{-a+a^2+b^2+1}{a}Y_2, & \nabla_{Y_2}Y_1 = \frac{2b}{a}Y_2 + \frac{a-a^2-b^2+1}{a}Y_3, \\
\nabla_{Y_2}Y_2 = -\frac{2b}{a}Y_1, & \nabla_{Y_2}Y_3 = \frac{-a+a^2+b^2-1}{a}Y_1, \\
\nabla_{Y_3}Y_1 = \frac{-a-a^2-b^2+1}{a}Y_2 - \frac{2b}{a}Y_3, & \nabla_{Y_3}Y_2 = \frac{a+a^2+b^2-1}{a}Y_1, \\
\nabla_{Y_3}Y_3 = \frac{2b}{a}Y_1. &
\end{array}$$

Next, we put

$$\alpha = 1 - a - c, \tag{3.17}$$

$$\beta = \frac{2b}{a}, \tag{3.18}$$

$$\gamma = \alpha + \frac{2}{a}. \tag{3.19}$$

By using the above equations and (3.14), we have

$$[Y_1, Y_2] = -\beta Y_2 - (\gamma - \alpha)Y_3,\ [Y_1, Y_3] = (2 - \alpha - \gamma)Y_2 + \beta Y_3,\ [Y_2, Y_3] = -2Y_1$$

and

$$\begin{array}{lll} \nabla_{Y_1} Y_1 = 0, & \nabla_{Y_1} Y_2 = \alpha Y_3, & \nabla_{Y_1} Y_3 = -\alpha Y_2, \\ \nabla_{Y_2} Y_1 = \beta Y_2 + \gamma Y_3, & \nabla_{Y_2} Y_2 = -\beta Y_1, & \nabla_{Y_2} Y_3 = -\gamma Y_1, \\ \nabla_{Y_3} Y_1 = (\gamma - 2)Y_2 - \beta Y_3, & \nabla_{Y_3} Y_2 = -(\gamma - 2)Y_1, & \nabla_{Y_3} Y_3 = \beta Y_1. \end{array}$$

Hence we have

$$\begin{aligned} R(Y_1, Y_2)Y_1 &= (\alpha^2 - 2\gamma\alpha - 4)Y_2 + 2\alpha\beta Y_3, \\ R(Y_1, Y_2)Y_2 &= (-\alpha^2 + 2\gamma\alpha + 4)Y_1, \\ R(Y_1, Y_2)Y_3 &= -2\alpha\beta Y_1, \\ R(Y_1, Y_3)Y_1 &= 2\alpha\beta Y_2 + (\alpha^2 + 2\gamma\alpha - 4\alpha - 4)Y_3, \\ R(Y_1, Y_3)Y_2 &= -2\alpha\beta Y_1, \\ R(Y_1, Y_3)Y_3 &= (-\alpha^2 - 2\gamma\alpha + 4\alpha + 4)Y_1, \\ R(Y_2, Y_3)Y_1 &= 0, \\ R(Y_2, Y_3)Y_2 &= (-\alpha^2 + 4\alpha + 4)Y_3, \\ R(Y_2, Y_3)Y_3 &= (\alpha^2 - 4\alpha - 4)Y_2. \end{aligned}$$

From the above result, by using (3.1) we get

$$\big(Ric(Y_i, Y_j)\big) = \begin{pmatrix} -2\alpha^2 + 4\alpha + 8 & 0 & 0 \\ 0 & 2\gamma\alpha - 4\alpha & -2\alpha\beta \\ 0 & -2\alpha\beta & -2\gamma\alpha \end{pmatrix} \tag{3.20}$$

$$= \begin{pmatrix} -2(a+c)^2 + 10 & 0 & 0 \\ 0 & 2(a+c-1)(a+c+1-\frac{2}{a}) & \frac{4b}{a}(a+c-1) \\ 0 & \frac{4b}{a}(a+c-1) & -2(a+c-1)(a+c-1-\frac{2}{a}) \end{pmatrix}.$$

Proposition 3.1.3. *Let $(S^3, \varphi, \xi, \eta, g)$ be the contact metric manifold determined by Proposition 3.1.2. and we assume that a, b, c are constant.*
(1) $(S^3, \varphi, \xi, \eta, g)$ is η-Einstein if and only if $b = 0, a = c = 1$, that is, $(S^3, \eta, \xi, g, \varphi)$ is the standard 3-dimensional sphere.
(2) $(S^3, \varphi, \xi, \eta, g)$ is Sasakian if and only if $b = 0, a = c = 1$, that is, $(S^3, \varphi, \xi, \eta, g)$ is the standard 3-dimensional sphere.
(3) $(S^3, \varphi, \xi, \eta, g)$ is K-contact if and only if $b = 0, a = c = 1$, that is, $(S^3, \varphi, \xi, \eta, g)$ is the standard 3-dimensional sphere.

Proof. (1) If S^3 is η-Einstein, by substituting $(Y_i, Y_j) = (Y_1, Y_1), (Y_2, Y_2)$ into (3.2), from (3.20) we get

$$\begin{aligned} Ric(Y_i, Y_j) &= 2\alpha(\gamma - 2)g(Y_i, Y_j) \\ &\quad + 2(-\alpha^2 - \gamma\alpha + 4\alpha + 4)g(Y_1, Y_i)g(Y_1, Y_j). \end{aligned} \tag{3.21}$$

Moreover, we substitute $(Y_i, Y_j) = (Y_1, Y_2), (Y_1, Y_3), (Y_2, Y_3), (Y_3, Y_3)$ into (3.21) and hence get

$$\alpha \neq 0, \quad \beta = 0, \text{ i.e., } b = 0, \quad a = c = 1.$$

Conversely, if $b = 0, \quad a = c = 1$, (3.21) holds.
(2) If S^3 is Sasakian, by substituting $(X, Y) = (Y_1, Y_2), (Y_1, Y_3)$ into (3.3) we get

$$\begin{aligned} \alpha\beta &= 0, \\ \alpha^2 - 2\gamma\alpha - 4 &= -1, \\ \alpha^2 + 2\gamma\alpha - 4\alpha - 4 &= -1. \end{aligned}$$

Therefore, we get

$$\alpha \neq 0, \quad \beta = 0, \text{ i.e., } b = 0, \quad a = c = 1.$$

Conversely, if $b = 0, \quad a = c = 1$, (3.3) holds.
(3) If S^3 is K-contact, by substituting $(X, Y) = (Y_i, Y_j)$ into (3.4), we get

$$\begin{aligned} \frac{4b}{a} &= 0, \\ \frac{2(-a^2 - b^2 + 1)}{a} &= 0. \end{aligned}$$

And then we have

$$b = 0, \quad a = c = 1.$$

Conversely, if $b = 0,\ a = c = 1$, (3.4) holds. □

Remark. If (S^3, g) is a contact metric manifold which does not satisfy $b = 0,\ a = c = 1$, then (S^3, g) is neither η-Einstein nor Sasakian, K-contact.

3.2 $\mathbf{R}^3$ with the contact form η

Let η be the 1-form on $\mathbf{R}^3$ defined by

$$\eta = \frac{1}{2}(dx^3 - x^2 dx^1). \tag{3.22}$$

Then we get

$$\eta \wedge d\eta = \frac{1}{8}(dx^1 \wedge dx^2 \wedge dx^3) \neq 0, \tag{3.23}$$

i.e., η is a contact form on $\mathbf{R}^3$.

And from (2.2), (2.3) we get

$$\xi = 2\frac{\partial}{\partial x^3}. \tag{3.24}$$

g and φ of $\mathbf{R}^3$

Let g be a Riemannian metric on $(\mathbf{R}^3, \eta)$ which satisfies (2.19). We put $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $a = g_{11},\ b = g_{12} = g_{21},\ c = g_{22}$. By using (3.22) and (3.24), from (2.19) we have the following matrix

$$(g_{ij}) = \begin{pmatrix} a & b & -\frac{1}{4}x^2 \\ b & c & 0 \\ -\frac{1}{4}x^2 & 0 & \frac{1}{4} \end{pmatrix}, \tag{3.25}$$

where $a, b, c \in C^\infty(\mathbf{R}^3)$.

Since $\det(g_{ij}) > 0$, we get

$$(a - \frac{1}{4}(x^2)^2)c - b^2 > 0. \tag{3.26}$$

Moreover, since $\frac{\partial}{\partial x^1} \neq 0$ and $\frac{\partial}{\partial x^2} \neq 0$, we get $a = g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) > 0, c = g(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}) > 0$.

Conversely, let g be a tensor field of type (0,2) defined by (3.25). If $a > 0, c > 0$ and (3.26) hold, then we get $g_{11} > 0, g_{22} > 0, \det(g_{ij}) > 0$ and hence

(3.27) $$ac - b^2 > 0.$$

And then g is a Riemannian metric satisfying (2.19).

Because, let λ be an eigenvalue of (g_{ij}), λ satisfies the following equation

(3.28) $$16\lambda^3 - 4(4a + 4c + 1)\lambda^2 + (4a + 4c + 16ac - 16b^2 - (x^2)^2)\lambda + c(x^2)^2 - 4(ac - b^2) = 0.$$

We put the left side of (3.28) by $f(\lambda)$. Then from (3.26) we have

(3.29) $$f(0) = c(x^2)^2 - 4(ac - b^2) < 0.$$

The differential of $f(\lambda)$ is

$$f'(\lambda) = 48\lambda^2 - 8(4a + 4c + 1)\lambda + (4a + 4c + 16ac - 16b^2 - (x^2)^2).$$

On the other hand by using (3.26) and (3.27) we get

(3.30) $$4a + 4c + 16ac - 16b^2 > \frac{4(ac - b^2)}{c} > (x^2)^2.$$

Therefore, if a discriminant of the quadratic equation $f'(\lambda) = 0$ of λ is non-negative, from (3.30) $f'(\lambda) = 0$ has a positive number. And hence from (3.29), λ is a positive number. Also, if a discriminant of $f'(\lambda) = 0$ is negative, from (3.29) λ is a positive number. Moreover, we can see that g satisfies (2.19).

Thus we have the following.

Proposition 3.2.1. *If a Riemannian metric g on $(\mathbf{R}^3, \eta)$ satisfies (2.19), then (3.25) and the following holds*

(3.31) $$a > 0, \ c > 0, \quad (a - \frac{1}{4}(x^2)^2)c - b^2 > 0.$$

Conversely, let g be a tensor field of type (0,2) on $(\mathbf{R}^3, \eta)$ defined by (3.25). If g satisfies (3.31), then g is a Riemannian metric on $(\mathbf{R}^3, \eta)$ and satisfies (2.19).

Next, we denote the left side of (3.26) by G, i.e.,

$$(a - \frac{1}{4}(x^2)^2)c - b^2 = G. \tag{3.32}$$

Let φ be a tensor field of type (1,1) satisfying (2.18). We put

$$\varphi(\frac{\partial}{\partial x^j}) = \sum_{k=1}^{3} \varphi_{kj} \frac{\partial}{\partial x^k} \qquad (j = 1, 2, 3)$$

.

Corollary 4. *If (g_{ij}) defined by (3.25) satisfies (3.31), then*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \frac{1}{4G} \begin{pmatrix} b & c & 0 \\ -a + \frac{1}{4}(x^2)^2 & -b & 0 \\ x^2 b & x^2 c & 0 \end{pmatrix} \tag{3.33}$$

holds, where $a > 0,\ c > 0,\ (a - \frac{1}{4}(x^2)^2)c - b^2 > 0$.

Proof. By substituting $X = \frac{\partial}{\partial x^i},\ Y = \frac{\partial}{\partial x^j}$ into (2.18), we get

$$\begin{pmatrix} a & b & -\frac{1}{4}x^2 \\ b & c & 0 \\ -\frac{1}{4}x^2 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\det(g_{ij}) > 0$, we get (3.33). □

Proposition 3.2.2. *(φ, ξ, η, g) is defined by (3.25), (3.31) and (3.33) on $\mathbf{R}^3$. If (φ, ξ, η, g) is a contact metric structure, then*

$$G = \frac{1}{16}. \tag{3.34}$$

holds. Conversely, if (3.34) holds, then (φ, ξ, η, g) is a contact metric structure on $\mathbf{R}^3$.

Proof. If (φ, ξ, η, g) is a contact metric structure, then (2.20) holds. By substituting (3.33), (3.22) and (3.24) into (2.20), we get

$$\frac{1}{16G^2} \begin{pmatrix} b & c & 0 \\ -a + \frac{1}{4}(x^2)^2 & -b & 0 \\ x^2 b & x^2 c & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -x^2 & 0 & 0 \end{pmatrix}.$$

Then we have (3.34).
Conversely, if (3.34) holds, we can get (2.20). This completes the proof. □

Corollary 5. *φ is denoted by the following matrix*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = 4 \begin{pmatrix} b & c & 0 \\ -a + \frac{1}{4}(x^2)^2 & -b & 0 \\ x^2 b & x^2 c & 0 \end{pmatrix},$$

where $a > 0,\ c > 0,\ (a - \frac{1}{4}(x^2)^2)c - b^2 = \frac{1}{16}$.

Curvature tensors

In this section, we assume that b and $c > 0$ are constant, and a is given by

$$a = \frac{1}{4}(x^2)^2 + \frac{1}{c}(b^2 + \frac{1}{16}).$$

We put $X_1 = \xi,\ X_2 = \dfrac{\partial}{\partial x^1},\ X_3 = \dfrac{\partial}{\partial x^2}$ on $(\mathbf{R}^3, g)$. By using g that satisfies (3.25), (3.31) and (3.34), from the basis X_1, X_2, X_3 we can generate the orthonormal basis Y_1, Y_2, Y_3 on $(\mathbf{R}^3, g)$, that is

$$Y_1 = 2\frac{\partial}{\partial x^3},\ Y_2 = \alpha(\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}),\ Y_3 = 4(-\alpha b\frac{\partial}{\partial x^1} + \frac{1}{\alpha}\frac{\partial}{\partial x^2} - \alpha b x^2\frac{\partial}{\partial x^3}),$$

where $\alpha = \dfrac{4\sqrt{c}}{\sqrt{16b^2+1}}$. Then we get

$$[Y_1, Y_2] = 0,\ [Y_2, Y_3] = -2Y_1,\ [Y_1, Y_3] = 0.$$

Then we may see that

$$\begin{aligned} &2g(\nabla_{Y_1}Y_2, Y_3) = 2, \quad &&2g(\nabla_{Y_1}Y_3, Y_2) = -2, \quad &&2g(\nabla_{Y_2}Y_1, Y_3) = 2, \\ &2g(\nabla_{Y_2}Y_3, Y_1) = -2, \quad &&2g(\nabla_{Y_3}Y_1, Y_2) = -2, \quad &&2g(\nabla_{Y_3}Y_2, Y_1) = 2, \end{aligned}$$

and the others are equal to 0. Therefore, we have

$$\begin{aligned} &\nabla_{Y_1}Y_1 = 0, \quad &&\nabla_{Y_1}Y_2 = Y_3, \quad &&\nabla_{Y_1}Y_3 = -Y_2, \quad &&\nabla_{Y_2}Y_1 = Y_3, \quad &&\nabla_{Y_2}Y_2 = 0, \\ &\nabla_{Y_2}Y_3 = -Y_1, \quad &&\nabla_{Y_3}Y_1 = -Y_2, \quad &&\nabla_{Y_3}Y_2 = Y_1, \quad &&\nabla_{Y_3}Y_3 = 0. \end{aligned}$$

Hence we get

$$\begin{array}{lll} R(Y_1,Y_2)Y_1=-Y_2, & R(Y_1,Y_2)Y_2=Y_1, & R(Y_1,Y_2)Y_3=0,\\ R(Y_1,Y_3)Y_1=-Y_3, & R(Y_1,Y_3)Y_2=0, & R(Y_1,Y_3)Y_3=Y_1,\\ R(Y_2,Y_3)Y_1=0, & R(Y_2,Y_3)Y_2=3Y_3, & R(Y_2,Y_3)Y_3=-3Y_2. \end{array}$$

Using (3.1) we have

$$\big(Ric(Y_i,Y_j)\big)=\begin{pmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{pmatrix}. \tag{3.35}$$

Proposition 3.2.3. $(\mathbf{R}^3,g)$ *is η-Einstein, Sasakian and K-contact.*

Proof. Substituting $(Y_i,Y_j)=(Y_1,Y_1),(Y_2,Y_2)$ into (3.2), from (3.35) we get

$$Ric(Y_i,Y_j)=-2g(Y_i,Y_j)+4g(Y_1,Y_i)g(Y_1,Y_j).$$

Moreover, we can see that if $(Y_i,Y_j)=(Y_1,Y_2),(Y_1,Y_3),(Y_2,Y_3)$ and (Y_3,Y_3), then the above equation holds. Therefore, $(\mathbf{R}^3,g)$ is η-Einstein.

Next, we shall check whether $(\mathbf{R}^3,g)$ satisfies (3.3), i.e.,

$$R(Y_i,Y_j)Y_1=g(Y_1,Y_j)Y_i-g(Y_1,Y_i)Y_j.$$

for $i,j=1,2,3$. From values $R(Y_i,Y_j)Y_k$ of the curvature tensor, we may see that the above equation holds. Therefore, $(\mathbf{R}^3,g)$ is Sasakian.

Finally, we shall check whether $(\mathbf{R}^3,g)$ satisfies (3.4), i.e.,

$$2g(Y_i,\nabla_{Y_j}Y_1)+2g(\nabla_{Y_i}Y_1,Y_j)=0 \quad (i,j=1,2,3).$$

From the calculation of $2g(\nabla_{Y_i}Y_j,Y_k)$ we may see that the above equation holds. Therefore, $(\mathbf{R}^3,g)$ is K-contact. □

3.3 T^3 with the contact form η

Let η be the 1-form on T^3 defined by

$$\eta=\cos nx^3dx^1+\sin nx^3dx^2 \qquad n\in\mathbf{N}. \tag{3.36}$$

Then we get

$$\eta \wedge d\eta = -\frac{1}{2} n dx^1 \wedge dx^2 \wedge dx^3 \neq 0, \tag{3.37}$$

i.e, η is a contact form on T^3.
From (2.2), (2.3) we get

$$\xi = \cos nx^3 \frac{\partial}{\partial x^1} + \sin nx^3 \frac{\partial}{\partial x^2}. \tag{3.38}$$

Let g be a Riemannian metric on (T^3, η) which sarisfies (2.19). We put $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $a = g_{11}, b = g_{12} = g_{21}, c = g_{22}$.
By using (3.36) and (3.38), from (2.19) we get

$$a \cos nx^3 + b \sin nx^3 = \cos nx^3 \tag{3.39}$$
$$b \cos nx^3 + c \sin nx^3 = \sin nx^3 \tag{3.40}$$
$$g_{31} \cos nx^3 + g_{32} \sin nx^3 = 0. \tag{3.41}$$

Proposition 3.3.1. *(3.39), (3.40) and (3.41) hold if and only if there exist $\beta, \alpha, g_{33} \in C^\infty(T^3)$ which satisfy the following matrix (g_{ij})*

$$(g_{ij}) = \begin{pmatrix} \beta \sin^2 nx^3 + 1 & -\beta \sin nx^3 \cos nx^3 & -\alpha \sin nx^3 \\ -\beta \sin nx^3 \cos nx^3 & \beta \cos^2 nx^3 + 1 & \alpha \cos nx^3 \\ -\alpha \sin nx^3 & \alpha \cos nx^3 & g_{33} \end{pmatrix}. \tag{3.42}$$

Proof. If (3.39) and (3.40) hold, there exist $l, k \in \mathbf{R}$ which satisfy the following equations

$$a - 1 = k(-\sin nx^3), \tag{3.43}$$
$$b = k \cos nx^3, \tag{3.44}$$
$$b = l(-\sin nx^3), \tag{3.45}$$
$$c - 1 = l \cos nx^3. \tag{3.46}$$

From (3.44) and (3.45) we get

$$k \cos nx^3 = l(-\sin nx^3).$$

When $\cos nx^3 \neq 0$ and $\sin nx^3 \neq 0$ hold, we get

$$\frac{k}{-\sin nx^3} = \frac{l}{\cos nx^3}.$$

By putting $\beta = \dfrac{k}{-\sin nx^3} = \dfrac{l}{\cos nx^3}$, from (3.41) we get (3.42). Moreover, (3.42) includes the case that either $\cos nx^3 = 0$ or $\sin nx^3 = 0$ holds.

Conversely, we can see that (g_{ij}) satisfies (3.39), (3.40) and (3.41). □

g and φ of T^3

We define the matrix B

$$B = \begin{pmatrix} \beta\sin^2 nx^3 + 1 & -\beta\sin nx^3\cos nx^3 & -\alpha\sin nx^3 \\ -\beta\sin nx^3\cos nx^3 & \beta\cos^2 nx^3 + 1 & \alpha\cos nx^3 \\ -\alpha\sin nx^3 & \alpha\cos nx^3 & g_{33} \end{pmatrix}. \tag{3.47}$$

Proposition 3.3.2. *Let g be the tensor field of type (0,2) on (T^3, η) defined by the matrix B, where $g_{ij} = g(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j})$ and $B = (g_{ij})$.*

g is a Riemannian metric satisfying (2.19) if and only if the following conditions hold

$$(1+\beta)g_{33} - \alpha^2 > 0\ ,\ g_{33} > 0. \tag{3.48}$$

Proof. From Proposition 3.3.1, g satisfies (2.19). If g is a Riemannian metric, since $\det(g_{ij}) > 0$,we get

$$\det(B) = (1+\beta)g_{33} - \alpha^2 > 0.$$

Next, we put an eigenvalue of $B = \lambda$ and $g(\lambda) = \det(B - \lambda I)$. Then, we get

$$g(\lambda) = (1-\lambda)\{\lambda^2 - (1+\beta+g_{33})\lambda + (1+\beta)g_{33} - \alpha^2\}.$$

One of solution in $g(\lambda) = 0$ is equal to 1. The other solutions are in the following equation

$$\lambda^2 - (1+\beta+g_{33})\lambda + (1+\beta)g_{33} - \alpha^2 = 0. \tag{3.49}$$

By putting a discriminant of the above equation = D, we get

$$D = \{g_{33} - (1+\beta)\}^2 + 4\alpha^2 \geq 0.$$

Since λ are positive definite, from (3.49) we get $g_{33} > 0$.

Conversely if (3.48) holds, we can see that $g_{11} > 0$, $g_{22} > 0$, $\det(B) > 0$ and an eigenvalue of B are positive definite. □

Next, let φ a tensor field of type (1,1) satisfying (2.18). We put

$$\varphi(\frac{\partial}{\partial x^j}) = \sum_{k=1}^{3} \varphi_{kj} \frac{\partial}{\partial x^k} \qquad (j = 1, 2, 3)$$

.

Corollary 6. *If (g_{ij}) defined by the matrix B satisfies (3.48), then the following equation holds.*

(3.50)
$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix}$$

$$= \frac{n}{2|B|} \begin{pmatrix} -\alpha \sin^2 nx^3 & \alpha \sin nx^3 \cos nx^3 & g_{33} \sin nx^3 \\ \alpha \sin nx^3 \cos nx^3 & -\alpha \cos^2 nx^3 & -g_{33} \cos nx^3 \\ -(1+\beta) \sin nx^3 & (1+\beta) \cos nx^3 & \alpha \end{pmatrix},$$

where $(1+\beta)g_{33} - \alpha^2 > 0$, $g_{33} > 0$.

Proof. Since $\det(g_{ij}) > 0$, from (2.18) we get (3.50). □

We put

(3.51)
$$\rho = \det(B) = (1+\beta)g_{33} - \alpha^2.$$

Proposition 3.3.3. (φ, ξ, η, g) *is given by (3.47), (3.48) and (3.50) on* T^3. *If* (φ, ξ, η, g) *is a contact metric structure, then*

(3.52)
$$n^2 = 4\rho.$$

holds. Conversely, if (3.52) holds, then (φ, ξ, η, g) *is a contact metric structure on* T^3.

Proof. If (φ, ξ, η, g) is a contact metric structure, by substituting (3.50), (3.36) and (3.38) into (2.20) we get

$$\frac{n^2}{4\rho} \begin{pmatrix} -\sin^2 nx^3 & \sin nx^3 \cos nx^3 & 0 \\ \sin nx^3 \cos nx^3 & -\cos^2 nx^3 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\sin^2 nx^3 & \sin nx^3 \cos nx^3 & 0 \\ \sin nx^3 \cos nx^3 & -\cos^2 nx^3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence we get (3.52).

Conversely, if (3.52) holds, we can get (2.20). This completes the proof. □

Corollary 7. *φ is denoted by the following matrix*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \frac{2}{n} \begin{pmatrix} -\alpha \sin^2 nx^3 & \alpha \sin nx^3 \cos nx^3 & g_{33} \sin nx^3 \\ \alpha \sin nx^3 \cos nx^3 & -\alpha \cos^2 nx^3 & -g_{33} \cos nx^3 \\ -(1+\beta) \sin nx^3 & (1+\beta) \cos nx^3 & \alpha \end{pmatrix},$$

where $(1+\beta) g_{33} - \alpha^2 = \frac{n^2}{4}, \quad g_{33} > 0.$

Curvature tensors

In this section we assume β, α, g_{33} are constant. We take the following basis on (T^3, g),

$$X_1 = \xi = \cos nx^3 \frac{\partial}{\partial x^1} + \sin nx^3 \frac{\partial}{\partial x^2}, \; X_2 = -\sin nx^3 \frac{\partial}{\partial x^1} + \cos nx^3 \frac{\partial}{\partial x^2}, \; X_3 = \frac{\partial}{\partial x^3}.$$

By using g that satisfies (3.47),(3.48) and (3.52), from the above basis we get the following orthonormal basis Y_1, Y_2, Y_3 on (T^3, g),

$$Y_1 = \xi = \cos nx^3 \tfrac{\partial}{\partial x^1} + \sin nx^3 \tfrac{\partial}{\partial x^2}, \qquad Y_2 = \gamma(-\sin nx^3 \tfrac{\partial}{\partial x^1} + \cos nx^3 \tfrac{\partial}{\partial x^2}),$$
$$Y_3 = \mu(\lambda \sin nx^3 \tfrac{\partial}{\partial x^1} - \lambda \cos nx^3 \tfrac{\partial}{\partial x^2} + \tfrac{\partial}{\partial x^3}),$$

where

$$\gamma = \frac{1}{\sqrt{1+\beta}}, \tag{3.53}$$

$$\lambda = \gamma^2 \alpha, \tag{3.54}$$

$$\mu = \frac{2}{n\gamma}. \tag{3.55}$$

For simplicity we put

$$-\frac{1}{\gamma^2} = a. \tag{3.56}$$

Then we get

$$[Y_1, Y_2] = 0, \; [Y_1, Y_3] = 2aY_2, \; [Y_2, Y_3] = 2Y_1.$$

We have

$$
\begin{array}{lll}
2g(\nabla_{Y_1}Y_2, Y_3) = -2a-2, & 2g(\nabla_{Y_1}Y_3, Y_2) = 2a+2, & 2g(\nabla_{Y_2}Y_1, Y_3) = -2a-2, \\
2g(\nabla_{Y_2}Y_3, Y_1) = 2a+2, & 2g(\nabla_{Y_3}Y_1, Y_2) = -2a+2, & 2g(\nabla_{Y_3}Y_2, Y_1) = 2a-2,
\end{array}
$$

the others are equal to 0.

Thus, we get

$$
\begin{array}{lll}
\nabla_{Y_1}Y_1 = 0, & \nabla_{Y_1}Y_2 = -(a+1)Y_3, & \nabla_{Y_1}Y_3 = (a+1)Y_2, \\
\nabla_{Y_2}Y_1 = -(a+1)Y_3, & \nabla_{Y_2}Y_2 = 0, & \nabla_{Y_2}Y_3 = (a+1)Y_1, \\
\nabla_{Y_3}Y_1 = -(a-1)Y_2, & \nabla_{Y_3}Y_2 = (a-1)Y_1, & \nabla_{Y_3}Y_3 = 0.
\end{array}
$$

Hence we have

$$
\begin{array}{ll}
R(Y_1, Y_2)Y_1 = -(a+1)^2Y_2, & R(Y_1, Y_2)Y_2 = (a+1)^2Y_1, \\
R(Y_1, Y_2)Y_3 = 0, & R(Y_1, Y_3)Y_1 = (a+1)(3a-1)Y_3, \\
R(Y_1, Y_3)Y_2 = 0, & R(Y_1, Y_3)Y_3 = -(a+1)(3a-1)Y_1, \\
R(Y_2, Y_3)Y_1 = 0, & R(Y_2, Y_3)Y_2 = -(a+1)(a-3)Y_3, \\
R(Y_2, Y_3)Y_3 = (a+1)(a-3)Y_2. &
\end{array}
$$

From the above result, by using (3.1), (3.56) and (3.53) we get

$$
\big(Ric(X_i, X_j)\big) = 2\beta \begin{pmatrix} -2-\beta & 0 & 0 \\ 0 & 2+\beta & 0 \\ 0 & 0 & -\beta \end{pmatrix}. \tag{3.57}
$$

Proposition 3.3.4. *(1) (T^3, g) is η-Einstein if and only if $\beta = 0$ holds.*
(2) (T^3, g) is not Sasakian.
(3) (T^3, g) is not K-contact.

Proof. (1) If (T^3, g) is η-Einstein, then from (3.2) the following equation holds for any $i, j = 1, 2, 3$

$$
Ric(Y_i, Y_j) = pg(Y_i, Y_j) + qg(Y_1, Y_i)g(Y_1, Y_j). \tag{3.58}
$$

By substituting $(Y_i, Y_j) = (Y_1, Y_2), (Y_2, Y_2)$ into (3.58), from (3.57) we get

$$
Ric(Y_i, Y_j) = 2\beta(2+\beta)g(Y_i, Y_j) - 4\beta(2+\beta)g(Y_1, Y_i)g(Y_1, Y_j). \tag{3.59}
$$

Moreover, we substitute $(Y_i, Y_j) = (Y_3, Y_3)$ into (3.59) and get

$$
-2\beta^2 = 2\beta(2+\beta).
$$

Since (5.13) implies $1 + \beta \neq 0$, $\beta = 0$ holds.

Conversely, if $\beta = 0$, (3.59) holds.

(2) If (T^3, g) is Sasakian, then from (3.3) and (2.19) the following equation holds for any $i, j = 1, 2, 3$

$$R(Y_i, Y_j)Y_1 = g(Y_1, Y_j)Y_i - g(Y_1, Y_i)Y_j. \tag{3.60}$$

By substituting $(Y_i, Y_j) = (Y_1, Y_2), (Y_1, Y_3)$ into (3.60), we get

$$a = 0. \tag{3.61}$$

But since (3.56) implies $a < 0$, (3.61) does not hold. Therefore, (T^3, g) is not Sasakian.

(3) If (T^3, g) is K-contact, then from (3.4) the following equation holds for any $i, k = 1, 2, 3$

$$2g(Y_k, \nabla_{Y_i} Y_1) + 2g(\nabla_{Y_k} Y_1, Y_i) = 0. \tag{3.62}$$

By substituting $(Y_k, Y_i) = (Y_3, Y_2)$ into (3.62), we get

$$a = 0. \tag{3.63}$$

Similarly, since $a < 0$, (3.63) does not hold. Therefore, (T^3, g) is not K-contact. □

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