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On the welfare dominance criteria using equivalence scales for different household sizes

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Abstract

Ebert (1999) proposed evaluating social welfare using an equivalence scale as weights for income units. In this paper, we extend his method to account for different numbers of income units and different numbers of individuals within each income unit. We illustrate the procedure by applying it to the distribution of retail sellers across Japan.

Key words: income distribution, equivalence scale, welfare dominance
JEL code: D31, D63, R12

1 Introduction

When comparing income distributions, it is important to consider differences in household size. Converting household income to individual income by using equivalence scales is a common technique. Ebert (1999) proposed an approach to evaluating social welfare using an equivalence scale as weights for income units. He showed that only such a formulation satisfies the transfer principle for individuals with different attributes.[1] Further, Ebert (1999) characterized his dominance criteria using a modified stochastic matrix.

In this paper, we extend this method to account for different compositions in income units and numbers of individuals within each income unit. It is often the case that income distributions are compared over time or across countries or regions, where the number of income units and the number of individuals within each income unit are different. We show that social welfare can be compared in this case as well using a slightly modified version of the Ebert (1999) formulation.

There is debate in the literature as to whether the welfare of each income unit should be weighted by an equivalence scale or by the number of individuals in an income unit (e.g., Glewwe, 1991; Trannoy, 2003; Shorrocks, 2004). Shorrocks (2004) proposed a method to weight each income unit by the number of individuals belonging to the unit in order to evaluate social welfare while

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[1] See also Lambert (2001).

adjusting individual incomes using equivalence scales. At the same time, he argued that such a method satisfies the compensation principle. We show that the method proposed by Shorrocks (2004) can also be verified using a modified doubly stochastic matrix.

The evaluation of social welfare using equivalence scales is informative not only for comparing individual incomes but also for comparing distributions of attributes other than income among regions and nations.[2] In such situations, it may be necessary to use equivalence scales to evaluate the distribution of these other-than-income attributes. For example, the distribution of public facilities such as libraries and medical institutions, which are partially non-rival in a region, requires welfare evaluation using an equivalence scale. In this paper, we show an application to the distribution of retail sellers among the various regions of Japan.

The next section presents the analytical framework and the criteria for welfare dominance used in the study. Section 3 illustrates the methodology of the study through an application to a regional economy. A brief summary concludes the paper.

2 Welfare Dominance Criteria Using Equivalent Income

2.1 Analytical Framework

Consider two societies, X and Y. Each of the societies consists of h_X and h_Y income units (e.g., households, regions, countries, etc.), respectively. The income of each unit in the two societies is represented by the vectors $\mathbf{x} = (x_1, \dots, x_{h_X})$ and $\mathbf{y} = (y_1, \dots, y_{h_Y})$. The number of individuals in each income unit is given by the vectors $\mathbf{n}^X = (n_1^X, \dots, n_{h_X}^X)$ and $\mathbf{n}^Y = (n_1^Y, \dots, n_{h_Y}^Y)$. Thus, the total population in each society is denoted by $n_\Sigma^X = \sum_{i=1}^{h_X} n_i^X$ and $n_\Sigma^Y = \sum_{i=1}^{h_Y} n_i^Y$, respectively. The equivalence scale for income unit *i* in society *X* is m_i^X. We define m_i^Y for society Y in the same way. The distribution of the equivalence scale in each society is represented by the vectors $\mathbf{m}^X = \left(m_1^X, \dots, m_{h_X}^X\right)$ and $\mathbf{m}^Y = \left(m_1^Y, \dots, m_{h_Y}^Y\right)$, respectively. The total equivalent population in society X is $m_\Sigma^X = \sum_{i=1}^{n_X} m_i^X$; similarly, for Y, it is m_Σ^Y. We allow that $m_\Sigma^X \neq m_\Sigma^Y$, $n_\Sigma^X \neq n_\Sigma^Y$, and $h_X \neq h_Y$. In summary, we denote by $(\mathbf{x}, \mathbf{n}^X, \mathbf{m}^X)$ and $(\mathbf{y}, \mathbf{n}^Y, \mathbf{m}^Y)$ the distributions in society *X* and *Y*, respectively.

Following Ebert (1999), we designate the valuation function for income unit *i* in society *X* as $m_i^X U(x_i / m_i^X)$, where U is a non-decreasing concave function of equivalent income. In order to compare the welfare of societies with different numbers of income units, we define social welfare as the sum of individual valuation per equivalent population. Then, the social welfare of society *X* is

[2] For example, Duan and Chen (2018) analyzed the energy consumption of countries around the world using a Lorenz curve and a Gini coefficient. Tang and Wang (2009) used a Lorenz curve and a Gini coefficient to study the land use structure changes in China. Cromley (2018) measured differences in grocery store accessibility in Akron, Ohio, between two population segments based on spatial Lorenz curves and related indicators. Lope and Dolgun (2020) compared the geography of the total and accessible tram services with the geography of the total and disabled population in Melbourne using a Gini coefficient and a Lorenz curve.

expressed as follows:

$$W(\mathbf{x}, \mathbf{m}^X) = \frac{1}{m_\Sigma^X} \sum_{i=1}^{h_X} m_i^X U\left(\frac{x_i}{m_i^X}\right). \tag{1}$$

In similar fashion, $W(\mathbf{y}, \mathbf{m}^Y)$ denotes the welfare of society *Y*. In the following, we represent by Ω_M the set of social valuation functions shown in (1).

2.2 Welfare Dominance Criteria

In comparisons of social welfare, the doubly stochastic matrix plays an important role. As is well known, when two income distributions $\mathbf{y}$ and $\mathbf{y}'$ are related via a doubly stochastic matrix $\mathbf{P}$ such that $\mathbf{yP} = \mathbf{y}'$, income inequality in $\mathbf{y}'$ is smaller than in $\mathbf{y}$ and social welfare in $\mathbf{y}'$ is higher under any S-concave social welfare function.[3] The basic idea in this paper is the same, but in order to consider social welfare based on equivalent individuals and income, we modify the doubly stochastic matrix.

Consider a matrix $\mathbf{Q}$ with non-negative elements $h_X \times h_Y$ having the following properties:

$$\mathbf{m}^X\mathbf{Q} = \mathbf{m}^Y, \tag{2}$$

$$\mathbf{Q}\mathbf{e}_{h_Y}^T = \frac{m_\Sigma^Y}{m_\Sigma^X}\mathbf{e}_{h_X}^T, \tag{3}$$

where $\mathbf{e}_n = (1, \dots, 1)$ is an *n*-dimensional row vector such that all elements are 1. In the following, we denote by $\Gamma_M(\mathbf{m}^X, \mathbf{m}^Y)$ the set of non-negative matrices satisfying (2) and (3). If $\mathbf{m}^X = \mathbf{m}^Y\mathbf{\Pi}$ holds by using an appropriate permutation matrix $\mathbf{\Pi}$, then $\mathbf{Q}$ is an *m*-stochastic matrix as defined by Ebert (1999). In addition, if $\mathbf{m}^X = \mathbf{m}^Y = \mathbf{e}$, then $\mathbf{Q}$ is a doubly stochastic matrix.

When the two distributions are related via the matrix $\mathbf{Q} \in \Gamma_M(\mathbf{m}^X, \mathbf{m}^Y)$ such that

$$\mathbf{xQ} \leq \mathbf{y}, \tag{4}$$

holds, we obtain the following proposition regarding social welfare.

Proposition 1 Consider two income distributions $(\mathbf{x}, \mathbf{n}^X, \mathbf{m}^X)$ and $(\mathbf{y}, \mathbf{n}^Y, \mathbf{m}^Y)$. The following two conditions are equivalent:

[3] See Marshall et al. (1979).

(i) There exists a $\mathbf{Q} \in \Gamma_M\ (\mathbf{m}^X, \mathbf{m}^Y)$ such that $\mathbf{xQ} \leq \mathbf{y}$.

(ii) $\mathrm{W}(\mathbf{y}, \mathbf{m}^Y) \geq \mathrm{W}(\mathbf{x}, \mathbf{m}^X)$ holds $\forall W \in \Omega_M$.

Proof See Appendix.

Proposition 1 is a generalization of the result of Ebert (1999). If part (i) of Proposition 1 holds as an equality, then $(1/m_\Sigma^Y)\mathbf{ye}_{h_Y}^T = (1/m_\Sigma^Y)\mathbf{xQe}_{h_Y}^T = (1/m_\Sigma^X)\mathbf{xe}_{h_X}^T$: i.e., the total income per-equivalent individual is equal between the two distributions.

Shorrocks (2004) argued that for welfare comparisons, social welfare should be calculated based on the equivalent income weighted by the number of household members.[4] Thus, we can represent social welfare as

$$\widetilde{W}(\mathbf{x}, \mathbf{m}^X, \mathbf{n}^X) = \frac{1}{n_\Sigma^X} \sum_{i=1}^{h_x} n_i^X U\left(\frac{x_i}{m_i^X}\right). \tag{5}$$

In the following, we denote by Ω_N the set of social evaluation functions shown in (5).

To compare social welfare, consider l matrix $\widetilde{\mathbf{Q}} \geq \mathbf{O}$ with non-negative elements having the following properties:

$$\mathbf{m}^X \widetilde{\mathbf{Q}} = \mathbf{m}^Y, \tag{6}$$

$$\widetilde{\mathbf{Q}}(\mathbf{d}^Y)^T = \frac{n_\Sigma^Y}{n_\Sigma^X}(\mathbf{d}^X)^T, \tag{7}$$

where

$$\mathbf{d}^Y = \left(\frac{n_1^Y}{m_1^Y}, \dots, \frac{n_{h_Y}^Y}{m_{h_Y}^Y}\right), \quad \mathbf{d}^X = \left(\frac{n_1^X}{m_1^X}, \dots, \frac{n_{h_X}^X}{m_{h_X}^X}\right). \tag{8}$$

We denote the set of $h_X \times h_Y$ non-negative matrices satisfying (6), (7), and (8) by $\Gamma_N(\mathbf{n}^X, \mathbf{n}^Y, \mathbf{m}^X, \mathbf{m}^Y)$. We then obtain the following result, similar to Proposition 1.

Proposition 2 Consider two distributions characterized by $(\mathbf{x}, \mathbf{n}^X, \mathbf{m}^X)$ and $(\mathbf{y}, \mathbf{n}^Y, \mathbf{m}^Y)$. The

[4] Decoster and Ooghe (2003) clarified the differences between Shorrocks' and Ebert's arguments and discussed their significance for empirical analysis.

following two conditions are equivalent:

(i) There exists $\widetilde{\mathbf{Q}} \in \Gamma_N(\mathbf{n}^X, \mathbf{n}^Y, \mathbf{m}^X, \mathbf{m}^Y)$ such that $\mathbf{x}\widetilde{\mathbf{Q}} \leq \mathbf{y}$.

(ii) $W(\mathbf{y}, \mathbf{n}^Y, \mathbf{m}^Y) \geq W(\mathbf{x}, \mathbf{n}^X, \mathbf{m}^X)$ holds $\forall W \in \Omega_N$.

Proof The proof follows the same procedure as in Proposition 1.

If $m_i^X = n_i^X$ and $m_i^Y = n_i^Y$ hold, that is, if equivalent income is defined in terms of income per household member, then $\widetilde{\mathbf{Q}}$ is equal to $\mathbf{Q}$.

2.3 Minimum Incremental Expenditure

If we focus only on the existence of the dominance relation, the generalized Lorenz curve gives a simple and clear result. In such a case, it is not so significant to conduct the analysis via Proposition 1, which requires solving matrix inequalities. However, part (ii) of Proposition 1 gives the partial order for two distributions, and the dominance relation is not always observed. In this case, as well, we can gain further insights by examining (4).

Vectorizing the matrix inequality in (4) allows us to consider the following linear programming problem.[5]

Problem 1 For a given set of attributes, population, and equivalent population distributions, $(\mathbf{x}, \mathbf{n}^X, \mathbf{m}^X)$ and $(\mathbf{y}, \mathbf{n}^Y, \mathbf{m}^Y)$,

$$\min_{\mathbf{q}, \mathbf{z}^+, \mathbf{z}^-} \mathbf{w}\mathbf{z}^-, \tag{9}$$

subject to

$$\widetilde{\mathbf{A}} \begin{bmatrix} \mathbf{q} \\ \mathbf{z}^+ \\ \mathbf{z}^- \end{bmatrix} = \begin{bmatrix} \mathbf{y}^T \\ (\mathbf{m}^Y)^T \\ (m_\Sigma^Y / m_\Sigma^X)\mathbf{e}_{h_Y}^T \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} \boldsymbol{q} \\ \mathbf{z}^+ \\ \mathbf{z}^- \end{bmatrix} \geq \mathbf{0}, \tag{11}$$

where

[5] For the vectorizing procedure, see Rao and Mitra (1971).

$$\widetilde{\mathbf{A}} := \begin{bmatrix} \mathbf{I}_{h_Y} \otimes \mathbf{x} & \mathbf{I}_{h_Y} & -\mathbf{I}_{h_Y} \\ \mathbf{I}_{h_Y} \otimes \mathbf{m}^X & \mathbf{0} & \mathbf{0} \\ \mathbf{e}_{h_Y} \otimes \mathbf{I}_{h_X} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{12}$$

and $\mathbf{A} \otimes \mathbf{B}$ is the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$. Also, $\mathbf{q} \in \mathbb{R}^{h_Y h_X}$, $\mathbf{z}^+ \in \mathbb{R}^{h_Y}$, $\mathbf{z}^- \in \mathbb{R}^{h_Y}$ are variables, and $\mathbf{w} \in \mathbb{R}^{h_Y}_{++}$ is the coefficient vector. If the optimal solution of Problem 1 is zero, then we can construct a matrix $\mathbf{Q} \in \Gamma_M(\mathbf{m}^X, \mathbf{m}^Y)$ by dividing the solution vector $\mathbf{q}$ into h_X-dimensional column vectors such that $\mathbf{q} = \mathrm{vec}\mathbf{Q}$, and $\mathbf{xQ} + \mathbf{z}^+ = \mathbf{y}$ holds.

Furthermore, the optimal solution to Problem 1 can be interpreted as the minimum additional amount of an attribute that would result in society Y dominating society X in the sense of Proposition 1. Note that the coefficient vector for the objective function can be thought of as the shadow price of the attributes that will improve social welfare. Thus, we can define the minimum incremental expenditure (MIE) for welfare dominance in society Y relative to society X as follows:

$$E_M(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \min_{\mathbf{z}^-}\{\mathbf{wz}^- : \mathbf{z}^- \geq \mathbf{0}, W(\mathbf{x}, \mathbf{m}^X) \leq W(\mathbf{y} + (\mathbf{z}^-)^T, \mathbf{m}^Y), \forall W \in \Omega_M\}. \tag{13}$$

If part (i) of Proposition 1 holds, the implication is that $E_M(\mathbf{w}, \mathbf{x}, \mathbf{y}) = 0$; i.e., society Y is more desirable than society X under the social evaluation function belonging to Ω_M. In contrast, if $E_M(\mathbf{w}, \mathbf{x}, \mathbf{y}) > 0$, then there exists a social evaluation function, $W \in \Omega_M$, that makes society X more desirable than society Y. The MIE, $E_M(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \mathbf{wz}^{-*}$, represents the attributes required for society Y to dominate society X as evaluated by the shadow price $\mathbf{w}$.

In addition, from the optimal vector $\mathbf{z}^{-*}$, we can identify how much the attributes of any unit need to increase in order to achieve welfare dominance. Moreover, when $E_M(\mathbf{w}, \mathbf{x}, \mathbf{y}) > 0$, we can construct a social evaluation function that evaluates society X as more desirable by solving the dual problem of Problem 1.[6]

We can also consider the optimization problem for Proposition 2 in a similar way. We can define the MIE to be welfare dominance in weighted with the number of individuals as follows:

$$E_N(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \min_{\mathbf{z}^-}\{\mathbf{wz}^- : \mathbf{z}^- \geq \mathbf{0}, W(\mathbf{x}, \mathbf{n}^X, \mathbf{m}^X) \leq W(\mathbf{y} + (\mathbf{z}^-)^T, \mathbf{n}^Y, \mathbf{m}^Y), \forall W \in \Omega_N\}. \tag{14}$$

3 Illustrative Applications

We can illustrate the method of Proposition 1 with a simple example in which we apply the procedure described in the previous section to an analysis involving the distribution of retailers among Japanese cities. In order to illustrate the method's broad applicability, we purposely chose an

[6] See appendix.

example that does not focus on income inequality.

In recent years, the number of retailers in Japan has fallen due to the continuing decline in the country's population. In particular, many small retailers located in rural areas have closed, with no successors to assume ownership. As the population of Japan ages, the decrease in the number of retailers will increasingly affect the accessibility of shopping for the elderly.

In our example, we use the sales floor space of retailers in each city as representative of the convenience of shopping in that city. Since there is partial nonrivalry in the retail industry, we use the square root of the population to define the sales floor space per equivalent population as the convenience of shopping in the city. That is, $\theta = 0.5$ in $m_i = (n_i)^\theta$. Following the framework of Proposition 1, we evaluate the sales floor space per equivalent population and weight each city by its equivalent population.

It should be noted that, while the target of the study is Japanese cities, 23 cities in the Tokyo metropolis were excluded from the sample because these cities have special functions as part of Japan's capital. Each ordinance-designated city is treated as a single city and is not divided into wards. In all, 792 cities were included in the analysis.[7] The period of study consists of two years: FY2011 and FY2016. The population of each city is based on the census populations in 2010 and 2015.

Table 1. Descriptive statistics

	Year	Mean	Max	Min	Std	# of cities
Sales floor space (m^2)	2011	145,149	2,675,573	1,745	241,507	792
	2016	148,617	2,846,212	828	254,530	792
Population	2010	135,985	3,688,773	4,387	251,136	792
	2015	135,058	3,724,844	3,585	254,497	792
Square root of population	2010	317	1,921	66	188	792
	2015	314	1,930	60	190	792
Sales floor space per square root of population	2011	352	1,660	26	205	792
	2016	358	1,819	14	219	792

Source: Statistics Bureau of Japan. *Economic Census for Business Activity, Population Census*

Table 1 summarizes the descriptive statistics. Comparing the two periods, the minimum sales floor is smaller in 2016 than in 2011, so that the distribution of 2016 does not dominate that of 2011. In other words, the dominance relation in the sense of Proposition 1 does not hold.

In the following, 2011 is set as society X, and 2016 is set as society Y. Let the value of the coefficient vector be $\mathbf{w}_0 = 100 * \mathbf{e}_{h_Y}$. Solving Problem 1, we obtain $E_M(\mathbf{w}_0, \mathbf{x}, \mathbf{y}) = 132{,}263$. That is, society Y does not dominate society X in the sense of Proposition 1.[8] In addition, from the

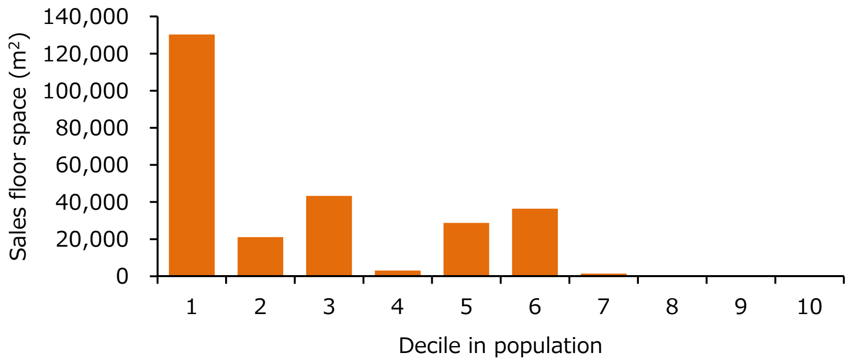
[7] The municipal data used here are reconstructed with municipalities as of March 31, 2020. Municipalities in Japan can be divided into cities, towns and villages. Towns and villages are smaller in population size. In 2020, 932 of Japan's 1742 municipalities were towns and villages.

[8] Instead, solving Problem 1 by exchanging years yields a positive optimal solution. Therefore, there is no one-sided dominance relationship between the two distributions.

optimal vector, $\mathbf{z}^{-*}$, we obtain $\mathbf{e}\mathbf{z}^{-*} = 26{,}452$, which implies that for the 2016 distribution to be desirable in terms of $W \in \Omega_M$, the sales floor needs to be increased by approximately 26,452m^2.

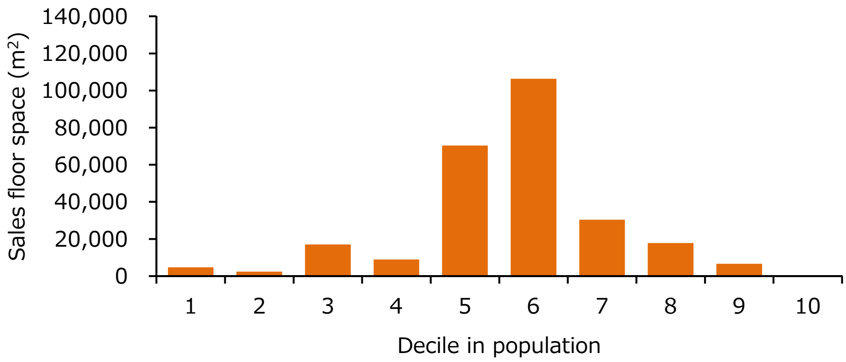
Figure 1 shows the required sales floor space for each decile of a city's population size. It is clear from the figure that the decrease in sales floor space in small cities since 2011 has become a bottleneck in improving the social evaluation function. In most cases, however, the attribute increments required to improve social welfare at minimal cost are not unique. In addition, the amount by which the attribute of any given unit should be increased depends on the shadow price.

Figure 1. Sales floor space required for welfare dominance (2016 for [?] 2011)



Source: Author's calculation

Figure 2. Required incremental sales floor space under alternative shadow prices



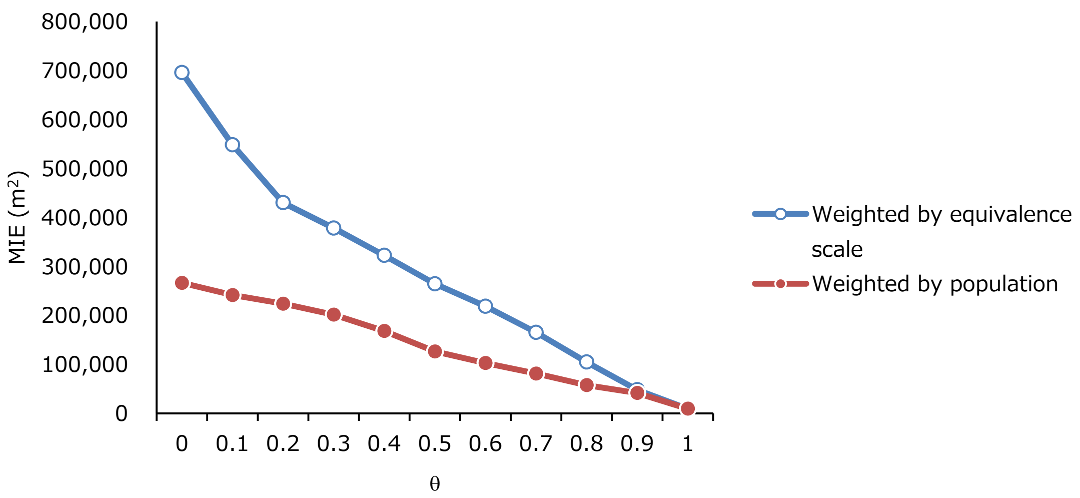
Source: Author's calculation

How does the required increase in sales floor space change when shadow prices differ? The increase in retail space may be easier to achieve in densely populated areas due to higher profitability. Therefore, we set the shadow price for each city to be proportional to the inverse of its population density in 2015, with the average value being 100. That is, $\mathbf{w} = \delta(1/d_1, \dots, 1/d_{h_Y})$ and $\delta = 100 * h_Y\left(\sum_{i=1}^{h_Y}(1/d_i)\right)^{-1}$, where d_i is the population density of city i. Figure 2 shows the increase in sales floor space needed to achieve welfare dominance under this shadow price, by city population size. In contrast to Figure 1, Figure 2 shows that an increase in sales floor space is necessary for cities with larger populations. Thus, by solving Problem 1 under appropriate shadow prices, we can obtain a desirable policy direction taking cost into account.

Now, to what extent does the MIE for welfare dominance expressed in equation (13) depend on the choice of equivalence scale? Figure 3 shows $\mathbf{ez}^{-*}$ when the value of θ is varied from 0 to 1 in increments of 0.1 for an equivalence scale such as $m_i = (n_i)^{\theta}$. Figure 3 also shows the MIE under a social valuation function weighted by population size for various values of θ. This case corresponds to (14), in line with Shorrocks (2004).

From Figure 3, we can see that the larger the value of θ, the smaller is the MIE.[9] As θ increases, the social evaluation function gives relative importance to the welfare of cities with large populations. The MIE becomes smaller as θ increases because of the larger shortage of sales floor space in smaller cities. In addition, when weighted by population size, the MIE is smaller in cities with larger populations under the distribution discussed here because it reflects differences in population size more strongly than the equivalence scale.

Figure 3. Minimum incremental expenditure for various equivalence scales



Source: Author's calculation.

4 Remarks

This paper generalizes the welfare dominance criterion of Ebert (1999) in terms of the number

[9] To be precise, the value in Figure 3 is the incremental sales floor space required for welfare dominance, equal to MIE/100.

of income units and the number of individuals within an income unit. His formulation using the M-stochastic matrix is not only theoretically significant but also practically useful. By varying the coefficients of the objective function or adding further constraints in a linear programming problem, we can make the analysis more conducive to solving policy issues.

Appendix Proof of Proposition 1

i) ⇒ ii) follows immediately from the fact that U is a non-decreasing concave function. Suppose now that (4) has no non-negative solution. Note that the matrix inequality (4) can be expressed in vector form as

$$\mathbf{Ap} \leq \mathbf{b}, \tag{A.1}$$

where

$$\mathbf{A} := \begin{bmatrix} \mathbf{I}_{h_Y} \otimes \mathbf{x} \\ \mathbf{I}_{h_Y} \otimes \mathbf{m}^X \\ -\mathbf{I}_{h_Y} \otimes \mathbf{m}^X \\ \mathbf{e}_{h_Y} \otimes \mathbf{I}_{n_X} \\ -\mathbf{e}_{h_Y} \otimes \mathbf{I}_{n_X} \end{bmatrix}, \quad \mathbf{p} = \mathrm{vec}\mathbf{Q}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{y}^T \\ (\mathbf{m}^Y)^T \\ -(\mathbf{m}^Y)^T \\ (m_\Sigma^Y / m_\Sigma^X)\mathbf{e}_{n_Y}^T \\ -(m_\Sigma^Y / m_\Sigma^X)\mathbf{e}_{n_Y}^T \end{bmatrix}.$$

From the theorem of the alternative for nonnegative solutions of linear inequalities, we can confirm that the following system of inequalities,

$$\mathbf{vA} \geq \mathbf{0}, \tag{A.2}$$

and

$$\mathbf{vb} < 0, \tag{A.3}$$

has a nonnegative solution $\mathbf{v}$ (e.g., Gale, 1961). Let us decompose the vector of the non-negative solution as $\mathbf{v}^* := (\mathbf{a}^*, \mathbf{c}^+, \mathbf{c}^-, \mathbf{z}^+, \mathbf{z}^-)$, where $\mathbf{a}^*, \mathbf{c}^+, \mathbf{c}^- \in \mathbb{R}_+^{h_Y}$, and $\mathbf{z}^+, \mathbf{z}^- \in \mathbb{R}_+^{h_X}$. If we use this notation to write the inequalities (A.2) and (A.3), we can express them as follows:

$$a_i^* x_j + c_i^* m_j^X + z_j^* \geq 0, \tag{A.4}$$

$$\sum_{i=1}^{h_Y}\left(a_i^* y_i + c_i^* m_i^Y\right) + \frac{m_\Sigma^Y}{m_\Sigma^X}\sum_{j=1}^{h_X} z_j < 0, \tag{A.5}$$

for $i \in \{1, \dots, h_Y\}$, and $j \in \{1, \dots, h_X\}$, where $c_i^* = c_i^+ - c_i^-$ and $z_j^* = z_j^+ - z_j^-$. Let $\mathcal{V} = \left\{(a_1^*, c_1^*), \dots, \left(a_{h_Y}^*, c_{h_Y}^*\right)\right\}$ be the set of the pair of coefficients (a_k^*, c_k^*). From (A.4) and (A.5), we obtain

$$\sum_{j=1}^{h_X}\left[m_j^X \times \min_{(a_k^*, c_k^*)\in\mathcal{V}}\left\{a_k^* \frac{x_j}{m_j^X} + c_k^*\right\} + z_j^*\right] \geq 0, \tag{A.6}$$

$$\sum_{i=1}^{h_Y}\left[m_i^Y \times \min_{(a_k^*, c_k^*)\in\mathcal{V}}\left\{a_k^* \frac{y_i}{m_i^Y} + c_k^*\right\}\right] + \frac{m_\Sigma^Y}{m_\Sigma^X}\sum_{j=1}^{h_X} z_j < 0. \tag{A.7}$$

Here, we consider the following valuation function:

$$U\left(\frac{y_i}{m_i^Y}\right) = \min_{(a_k^*, c_k^*)\in\mathcal{V}}\left\{a_k^* \frac{y_i}{m_i^Y} + c_k^*\right\}. \tag{A.8}$$

We can verify that U in (A.8) is nondecreasing concave in (y_i/m_i^Y). Thus, if $\mathbf{xQ} \nleq \mathbf{y}$, $\forall \mathbf{Q} \in \Gamma_M(\mathbf{m}^X, \mathbf{m}^Y)$, then we can find a social evaluation function that determines society X to be more desirable than society Y.

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