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A remark on algebraic addition theorems

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Abstract. Let K be a subfield of the meromorphic function field on $\mathbf{C}^n$ which has the transcendence degree n over $\mathbf{C}$. There are two ways of stating an algebraic addition theorem for K. The author wrote the relation of these two ways in a book [Toroidal Groups, Yokohama Publishers, Inc., Yokohama, 2018]. However, the explanation was too rough. In this paper, the detailed complete proof is given.

1. Introduction

Let $\mathcal{M}(\mathbf{C}^n)$ be the field of meromorphic functions on $\mathbf{C}^n$. We studied and determined an algebraic function field $K(\subset \mathcal{M}(\mathbf{C}^n))$ of n variables over $\mathbf{C}$ ([1, 2]). This study makes Weierstrass' statement established and clear. Weierstrass' statement is stated in [6] as follows:

> Tout système de n fonctions (indépendantes) à n variables qui admet un théorème d'addition est une combinaison algébrique de n fonctions abéliennes (ou dégénérescences) à n arguments et aux mêmes périodes.

We consider the following condition (T) for a subfield K of $\mathcal{M}(\mathbf{C}^n)$.

(T) K is finitely generated over $\mathbf{C}$ and $\mathrm{Trans}_{\mathbf{C}}K = n$.

If K satisfies condition (T), then there exist $f_0, f_1, \ldots, f_n \in K$ such that $K = \mathbf{C}(f_0, f_1, \ldots, f_n)$ and $f_1, \ldots, f_n$ are algebraically independent

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over $\mathbf{C}$. To simplify the description, we write $\mathbf{C}(F) = \mathbf{C}(f_0, f_1, \dots, f_n)$ and $\mathbf{C}(F_1) = \mathbf{C}(f_1, \dots, f_n)$ setting $F := \{f_0, f_1, \dots, f_n\}$ and $F_1 := \{f_1, \dots, f_n\}$. Furthermore we write

$$\mathbf{C}(F(x), F(y)) = \mathbf{C}(f_0(x), f_1(x), \dots, f_n(x), f_0(y), f_1(y), \dots, f_n(y)),$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two independent tuples of n complex variables. We note that there are two ways of stating an algebraic addition theorem. The first one is the following.

Definition 1. *Let $K = \mathbf{C}(F)$ be as above. We say that K admits* (AAT) *if $f_j(x + y) \in \mathbf{C}(F(x), F(y))$ for any $j = 0, 1, \dots, n$.*

The above definition is independent of the choice of generators $f_0, f_1, \dots, f_n$ (cf. Lemma 6.2.2 in [3]).

An algebraic addition theorem stated in Weierstrass' original statement is slightly different. That is the following type.

Definition 2. *Let $f_1, \dots, f_n \in \mathcal{M}(\mathbf{C}^n)$ be algebraically independent over $\mathbf{C}$. We say that $f_1, \dots, f_n$ admit* (AAT*) *if $f_j(x + y)$ is algebraic over $\mathbf{C}(F_1(x), F_1(y))$ for all $j = 1, \dots, n$.*

We assume that algebraically independent functions $f_1, \dots, f_n \in \mathcal{M}(\mathbf{C}^n)$ admit (AAT*). Let $K/\mathbf{C}(F_1)$ be a finite algebraic extension. If $g_1, \dots, g_n \in K$ are algebraically independent, then $g_1, \dots, g_n$ admit (AAT*) (Lemma 6.2.6 in [3]). Therefore, (AAT*) for subfields K is defined as follows.

Definition 3. *Let K be a subfield of $\mathcal{M}(\mathbf{C}^n)$ satisfying condition* (T)*. We say that K admits* (AAT*) *if there exist $f_1, \dots, f_n \in K$ algebraically independent over $\mathbf{C}$ such that $f_1, \dots, f_n$ admit* (AAT*)*.*

The following proposition is immediate from the definitions.

Proposition 1 (Proposition 6.2.8 in [3]) *Let K be a subfield of $\mathcal{M}(\mathbf{C}^n)$ satisfying condition* (T)*. If K admits* (AAT)*, then it admits* (AAT*)*.*

The converse is the following theorem.

Theorem 1 (Proposition 6.2.9 in [3]) *Let K be a subfield of $\mathcal{M}(\mathbf{C}^n)$ satisfying condition* (T)*. If K admits* (AAT*)*, then there exists an algebraic extension $\widetilde{K}$ of K such that $\widetilde{K}$ admits* (AAT)*.*

The statement of the above theorem is correct. However, the proof in [3] was written too roughly. Recently, Baro, de Vicente and Otero [4] published an extension result for maps admitting an algebraic addition theorem. We note that a similar argument is needed to complete the proof of the above theorem. In [4] they considered germs of meromorphic maps. Therefore, they had to investigate carefully domains of convergence. Our objects are meromorphic functions on $\mathbf{C}^n$. Then, some parts of their argument are not needed in our case. Although Theorem 1 is considered as a corollary of the main theorem of [4], we will give the detailed proof of Theorem 1 modifying their argument.

2. Proof of Theorem 1

We assume that a subfield K of $\mathcal{M}(\mathbf{C}^n)$ satisfies condition (T) and admits (AAT*) through this section. We set $K = \mathbf{C}(F)$, where $F = \{f_0, f_1, \dots, f_n\}$ and $f_1, \dots, f_n$ are algebraically independent over $\mathbf{C}$.

Lemma 1. *There exists an open dense subset Ω of $\mathbf{C}^n$ such that for any $f \in K$, $f(x+a)$ is algebraic over K for all $a \in \Omega$.*

Proof. Let P_{f_i} be the polar set of f_i for $i = 0, 1, \dots, n$. We set $P_F := \bigcup_{i=0}^{n} P_{f_i}$. Since K admits (AAT*), $f_j(x+y)$ is algebraic over $\mathbf{C}(F_1(x), F_1(y))$ for any $j = 1, \dots, n$. Then there exists a non-zero polynomial of the minimal degree

$$P_j(X) = \sum_{k=0}^{m_j} A_k^{(j)}(x,y) X^k$$

with $A_k^{(j)}(x,y) \in \mathbf{C}[f_1(x), \dots, f_n(x), f_1(y), \dots, f_n(y)]$ such that $A_0^{(j)}(x,y)$, $A_1^{(j)}(x,y), \dots, A_{m_j}^{(j)}(x,y)$ have no common divisor except constants and $P_j(f_j(x+y)) = 0$. We define

$$N_j := \{a \in \mathbf{C}^n \setminus P_F; A_k^{(j)}(x,a) = 0, k = 0, 1, \dots, m_j\}$$

and $N := \bigcup_{j=1}^{n} N_j$. If we set $\Omega := \mathbf{C}^n \setminus (P_F \cup N)$, then Ω is an open dense subset of $\mathbf{C}^n$. Take any $a \in \Omega$. By the definition of Ω, $f_j(x+a)$ is algebraic over K for any $j = 1, \ldots, n$. Since $f_0(x+a)$ is algebraic over $\mathbf{C}(F_1(x+a))$, it is algebraic over K. $\square$

Lemma 2. *For any $f \in K$, $f(-x)$ is algebraic over K.*

Proof. It suffices to show that $f_j(-x)$ is algebraic over K for any $j = 0, 1, \ldots, n$.

By the assumption, $f_j(x+y)$ is algebraic over $\mathbf{C}(F(x), F(y))$. Therefore we have $\mathrm{Trans}_{\mathbf{C}}\mathbf{C}(F(x), F(y), F(x+y)) = 2n$. On the other hand we have $\mathrm{Trans}_{\mathbf{C}}\mathbf{C}(F(x), F(x+y)) = 2n$. Hence $f_j(y)$ is algebraic over $\mathbf{C}(F(x), F(x+y))$. Take $a \in \Omega$. Let $y = -x + a$. Then $f_j(-x+a)$ is algebraic over $\mathbf{C}(F(x), F(a)) = K$. Substituting $x+a$ for x, we obtain that $f_j(-x)$ is algebraic over $\mathbf{C}(F(x+a))$. By Lemma 1, $f_i(x+a)$ is algebraic over K for all $i = 0, 1, \ldots, n$. Thus we conclude that $f_j(-x)$ is algebraic over K. $\square$

Lemma 3. *There exist a finite subset $\mathcal{C}$ of $\Omega \cup \{0\}$ with $0 \in \mathcal{C}$ and $\mathcal{C} = -\mathcal{C}$, and a finite number of functions $A_0, A_1, \ldots, A_N \in \mathbf{C}(\{F(x+a), F(y+a); a \in \mathcal{C}\})$ satisfying the following conditions.*
(a) For any $f \in K$, $f(x+y)$ is algebraic over $\mathbf{C}(A_0, A_1, \ldots, A_N)$.
(b) For any $j = 0, 1, \ldots, N$ we have

$$A_j(x, y) = A_j(x+a, y-a) \tag{1}$$

for any $a \in \mathbf{C}^n$.

Proof. Take any $i = 0, 1, \ldots, n$. We set $\mathcal{S}_0^{(i)} := \{0\}$ and $K_0^{(i)} := \mathbf{C}(F(x), F(y))$. Let

$$P_0(X) = X^{\ell_0+1} + \sum_{j=0}^{\ell_0} A_{0,j}^{(i)}(x, y) X^j, \quad A_{0,j}^{(i)}(x, y) \in K_0^{(i)},$$

be the minimal polynomial of $f_i(x+y)$ over $K_0^{(i)}$. If all of $A_{0,j}^{(i)}$ satisfy (1), then we denote $\mathcal{C}^{(i)} := \mathcal{S}_0^{(i)}$ and $A_j^{(i)}(x, y) := A_{0,j}^{(i)}(x, y)$ for $j = 0, 1, \ldots, \ell_0$. Otherwise, there exists $a_1 \in \mathbf{C}^n$ such that

$$Q_0(X) := P_0(X) - \left(X^{\ell_0+1} + \sum_{j=0}^{\ell_0} A_{0,j}^{(i)}(x+a_1, y-a_1) X^j \right) \neq 0.$$

Since Ω is dense in $\mathbf{C}^n$, we may assume $a_1 \in \Omega$. We set $\mathcal{S}_1^{(i)} := \mathcal{S}_0^{(i)} \cup \{a_1, -a_1\}$ and $K_1^{(i)} := \mathbf{C}(\{F(x+a), F(y+a); a \in \mathcal{S}_1^{(i)}\})$ in this case. Then $K_0^{(i)} \subset K_1^{(i)}$ and $f_i(x+y)$ is algebraic over $K_1^{(i)}$. We take the minimal polynomial

$$P_1(X) = X^{\ell_1+1} + \sum_{j=0}^{\ell_1} A_{1,j}^{(i)}(x,y) X^j$$

of $f_i(x+y)$ over $K_1^{(i)}$. Since $\deg Q_0 < \ell_0 + 1$, we have $\deg P_1 < \deg P_0$. If all of $A_{1,j}^{(i)}$ satisfy (1), then we set $\mathcal{C}^{(i)} := \mathcal{S}_1^{(i)}$ and $A_j^{(i)} := A_{1,j}^{(i)}$ for $j = 0, 1, \ldots, \ell_1$.

If it is not the case, we repeat the above procedure. Then we obtain sequences $\{\mathcal{S}_k^{(i)}\}$, $\{K_k^{(i)}\}$ and $\{Q_k\}$ such that $\mathcal{S}_k^{(i)} = \mathcal{S}_{k-1}^{(i)} \cup \{a+a_k, a-a_k; a \in \mathcal{S}_{k-1}^{(i)}\}$ for some $a_k \in \Omega$ with $Q_{k-1}(X) \neq 0$ and $K_k^{(i)} = \mathbf{C}(\{F(x+a), F(y+a); a \in \mathcal{S}_k^{(i)}\})$. If $Q_{k-1}(X) \neq 0$, then $\deg Q_k < \deg Q_{k-1}$. Therefore, this procedure stops by a finite number of steps. Let s be the first number with $Q_s(X) = 0$. Then for the minimal polynomial

$$P_s(X) = X^{\ell_s+1} + \sum_{j=0}^{\ell_s} A_{s,j}^{(i)}(x,y) X^j$$

of $f_i(x+y)$ over $K_s^{(i)}$, the coefficients $A_{s,0}^{(i)}, A_{s,1}^{(i)}, \ldots, A_{s,\ell_s}^{(i)}$ satisfy (1). We set $\mathcal{C}^{(i)} := \mathcal{S}_s^{(i)}$ and $A_j^{(i)} := A_{s,j}^{(i)}$ for $j = 0, 1, \ldots, N_i$, where we write $N_i = \ell_s$. Let $\mathcal{C} := \bigcup_{i=0}^n \mathcal{C}^{(i)}$ and $\{A_0, A_1, \ldots, A_N\} := \bigcup_{i=0}^n \{A_0^{(i)}, A_1^{(i)}, \ldots, A_{N_i}^{(i)}\}$. Then we obtain the desired conclusion. $\square$

Proof of Theorem 1. Let $\mathcal{C}$ and $A_0, A_1, \ldots, A_N \in \mathbf{C}(\{F(x+a), F(y+a); a \in \mathcal{C}\})$ be as in Lemma 3. Take $b \in \mathbf{C}^n$ such as $A_j(x,b) \in \mathcal{M}(\mathbf{C}^n)$ for all $j = 0, 1, \ldots, N$. We define $\Omega_0 := \bigcap_{c \in \mathcal{C}} (\Omega - b - c)$. Then Ω_0 is also an open dense subset of $\mathbf{C}^n$. Let $B_j(x) := A_j(x,b)$ for $j = 0, 1, \ldots, N$. We define two fields $\widetilde{K}$ and L by

$$\widetilde{K} := \mathbf{C}(\{B_j(x+a), B_j(-x+a); a \in \Omega_0, j = 0, 1, \ldots, N\})$$

and

$$L := \mathbf{C}(\{F(x+c), F(-x+c); c \in \mathcal{C}\}).$$

We show that $\widetilde{K} \subset L$ and any $f \in \widetilde{K}$ is algebraic over K. From (1) it follows that for any $a \in \mathbf{C}^n$

$$A_j(x+a,b) = A_j(x,a+b) \tag{2}$$

for $j = 0, 1, \ldots, N$. Let $a \in \Omega_0$. Since $a+b+c \in \Omega$ for any $c \in \mathcal{C}$, we have $B_j(x+a) = A_j(x,a+b) \in \mathbf{C}(\{F(x+c); c \in \mathcal{C}\})$. By Lemma 1, any element of $\mathbf{C}(\{F(x+c); c \in \mathcal{C}\})$ is algebraic over K. Because of $\mathcal{C} = -\mathcal{C}$, we have that $B_j(-x+a) \in \mathbf{C}(\{F(-x+c); c \in \mathcal{C}\})$ and any element of $\mathbf{C}(\{F(-x+c); c \in \mathcal{C}\})$ is algebraic over $\mathbf{C}(F(-x))$. Since $f_j(-x)$ is algebraic over K for $j = 0, 1, \ldots, n$ (Lemma 2), any element of $\mathbf{C}(\{F(-x+c); c \in \mathcal{C}\})$ is algebraic over K.

Next we show that $\widetilde{K}$ admits (AAT). Since $\mathrm{Trans}_{\mathbf{C}}\widetilde{K} = n$, we can take $g_0, g_1, \ldots, g_n \in \widetilde{K}$ such that $\widetilde{K} = \mathbf{C}(G)$ and $g_1, \ldots, g_n$ are algebraically independent over $\mathbf{C}$. Let $f \in \widetilde{K}$. It is obvious by the definition of $\widetilde{K}$ that $f(x+a) \in \widetilde{K}$ for any $a \in \Omega_0$. We define $g(x,y) := f(x+y)$. Then, $g(x,a) \in \mathbf{C}(G(x))$ for any $a \in \Omega_0$. Similarly, we have $g(a,y) \in \mathbf{C}(G(y))$ for any $a \in \Omega_0$. It follows from Theorem 3 in [5] (the proof is the same as that of Theorem 6.6.5 in [3]) that $f(x+y) = g(x,y) \in \mathbf{C}(G(x), G(y))$ on $\Omega_0 \times \Omega_0$. Since $\Omega_0 \times \Omega_0$ is an open dense subset of $\mathbf{C}^n \times \mathbf{C}^n$, it holds on $\mathbf{C}^n \times \mathbf{C}^n$ by the uniqueness theorem. $\square$

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