Toyama Math. J.
Vol. 40(2018·2019), 17-35

Intermediate pseudoconvexity for unramified Riemann domains over $\mathbb{C}^n$

Makoto ABE, Tadashi SHIMA and Shun SUGIYAMA

Abstract. We characterize the q-pseudoconvexity for unramified Riemann domains over $\mathbb{C}^n$, where $1 \leq q \leq n$, by the continuity property which holds for a class of maps whose projections to $\mathbb{C}^n$ are families of unidirectionally parameterized q-dimensional analytic balls written by polynomials of degree at most two.

1. Introduction

An unramified (Riemann) domain over $\mathbb{C}^n$ is a pair (D, π) of a second countable connected complex manifold D and a locally biholomorphic map $\pi : D \to \mathbb{C}^n$. According to Fritzsche–Grauert [4], we denote by $\breve{\partial} D$ the set of accessible boundary points of (D, π), by $\breve{D} = D \cup \breve{\partial} D$ the abstract closure of (D, π), and by $\breve{\pi} : \breve{D} \to \mathbb{C}^n$ the extension of π to $\breve{D}$.

We say that an upper semicontinuous function $u : D \to [-\infty, +\infty)$ is q-plurisubharmonic, where $1 \leq q \leq n$, if for every open set G of $\mathbb{C}^q$ and for every holomorphic map $f : G \to D$ the function $u \circ f : G \to [-\infty, +\infty)$ is subpluriharmonic in the sense of Fujita [5, 6]. We say that (D, π) is q-pseudoconvex if the function $-\ln d_D : D \to \mathbb{R}$ is q-plurisubharmonic, where d_D denotes the Euclidean boundary distance function of (D, π).

In this paper, we give a characterization of the q-pseudoconvexity for unramified domains over $\mathbb{C}^n$ by the continuity property which holds for

2010 *Mathematics Subject Classification.* 32F10, 32D26.
Key words and phrases. unramified Riemann domain, q-plurisubharmonic function, q-pseudoconvex domain.

a class of maps whose projections to $\mathbb{C}^n$ are families of unidirectionally parameterized q-dimensional analytic balls written by polynomials of degree at most 2. To be precise, we prove that an unramified domain (D, π) over $\mathbb{C}^n$ is q-pseudoconvex, where $1 \leq q \leq n$, if and only if the following condition is satisfied (see Theorem 4.1):

> Let $\breve{\lambda} : \overline{\mathbf{B}_q(0,1)} \times [0,1] \to \breve{D}$, where $\mathbf{B}_q(0,1)$ denotes the unit ball in $\mathbb{C}^q$, be a continuous map which satisfies the conditions that $\breve{\lambda}(\overline{\mathbf{B}_q(0,1)} \times [0,1] \setminus \{(0,1)\}) \subset D$, there exists a holomorphic map $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) : \mathbb{C}^{q+1} \to \mathbb{C}^n$ of the form $\lambda_\nu(z_1, z_2, \ldots, z_q, t) = P_\nu(z_1, z_2, \ldots, z_q) + c_\nu t$, where P_ν is a polynomial of variables $z_1, z_2, \ldots, z_q$ of degree at most 2 and $c_\nu \in \mathbb{C}$ for every $\nu = 1, 2, \ldots, n$, such that $H := \lambda(\mathbb{C}^{q+1})$ is a $(q+1)$-dimensional complex affine subspace of $\mathbb{C}^n$, the induced map $\lambda : \mathbb{C}^{q+1} \to H$ is biholomorphic, and $\breve{\pi} \circ \breve{\lambda} = \lambda$ on $\overline{\mathbf{B}_q(0,1)} \times [0,1]$. Then, we have that $\breve{\lambda}(0,1) \in D$.

As a corollary, we obtain a Lelong type characterization of a q-pseudoconvex unramified domain over $\mathbb{C}^n$ (see Corollary 5.2). On the other hand, in the case where $q = 1$, we obtain a characterization of a pseudoconvex unramified domain over $\mathbb{C}^n$, which generalizes Yasuoka [21, Theorem 2] and refines Sugiyama [17, Theorem 3.1] (see Corollary 5.3).

2. Preliminaries

Let $n \in \mathbb{N}$. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{C}^n$, that is,

$$\|z\| := \left(\sum_{\nu=1}^{n} |z_\nu|^2 \right)^{1/2}$$

for every $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. We call the set

$$\mathbf{B}_n(c, r) := \{ z \in \mathbb{C}^n \mid \|z - c\| < r \}$$

the *open ball* of *radius* r with *center* c in $\mathbb{C}^n$, where $r \in (0, +\infty]$ and $c \in \mathbb{C}^n$. We call the set $\mathbf{B}_n(0,1)$ the *unit ball* in $\mathbb{C}^n$.

Proposition 2.1. *Let (D, π) be an unramified domain over $\mathbb{C}^n$. Let $a_k \in D$, $r_k \in (0, +\infty]$, and B_k a neighborhood of a_k in D such that $\pi(B_k) = \mathbf{B}_n(\pi(a_k), r_k)$ and $\pi|_{B_k} : B_k \to \pi(B_k)$ is biholomorphic for each $k = 1, 2$. Assume that $B_1 \cap B_2 \neq \emptyset$. Then, we have that $(\pi|_{B_1})^{-1} = (\pi|_{B_2})^{-1}$ on $\pi(B_1) \cap \pi(B_2)$ and the map $\pi|_{B_1 \cap B_2} : B_1 \cap B_2 \to \pi(B_1) \cap \pi(B_2)$ is biholomorphic.*

Proof. Let $Q_k := \mathbf{B}_n(\pi(a_k), r_k)$ for each $k = 1, 2$. For an arbitrary $a \in B_1 \cap B_2 \neq \emptyset$, we have that $\pi(a) \in Q_1 \cap Q_2$ and $(\pi|_{B_k})^{-1}(\pi(a)) = a$ for each $k = 1, 2$. Moreover, the set $Q_1 \cap Q_2$ is connected and therefore $(\pi|_{B_1})^{-1} = (\pi|_{B_2})^{-1}$ on $Q_1 \cap Q_2$ by the identity principle for liftings. Then, for every $z \in Q_1 \cap Q_2$, we have that

$$x := (\pi|_{B_1})^{-1}(z) = (\pi|_{B_2})^{-1}(z) \in B_1 \cap B_2$$

and $\pi(x) = z$. It follows that $\pi(B_1 \cap B_2) = Q_1 \cap Q_2$ and the map $\pi|_{B_1 \cap B_2} : B_1 \cap B_2 \to Q_1 \cap Q_2$ is biholomorphic. $\square$

Let (D, π) be an unramified domain over $\mathbb{C}^n$. For every point $a \in D$, the (*Euclidean*) *boundary distance* $d_D(a)$ of a is the supremum of all $r \in (0, +\infty]$ which satisfy the condition that there exists a neighborhood B of a in D such that $\pi(B) = \mathbf{B}_n(\pi(a), r)$ and the map $\pi|_B : B \to \mathbf{B}_n(\pi(a), r)$ is biholomorphic. The function $d_D : D \to (0, +\infty]$ is said to be the (*Euclidean*) *boundary distance function* of (D, π). For every $a \in D$ and for every $r \in (0, d_D(a)]$, there exists a unique neighborhood $B(a, r)$ of a in D such that $\pi(B(a, r)) = \mathbf{B}_n(\pi(a), r)$ and the map $\pi|_{B(a,r)} : B(a, r) \to \mathbf{B}_n(\pi(a), r)$ is biholomorphic. We call the set $B(a, r)$ the *open ball* in D of *radius* r with *center* a. The map $\pi : D \to \mathbb{C}^n$ is biholomorphic if and only if there exists $a \in D$ such that $d_D(a) = +\infty$. If π is not biholomorphic, then the function $d_D : D \to \mathbb{R}$ is continuous (see Jarnicki-Pflug [10, pp. 6–7]).

Proposition 2.2. *Let (D, π) be an unramified domain over $\mathbb{C}^n$. Let $a \in D$ and E a subset of the closure of $B(a, d_D(a))$ in D. Let $W := \bigcup_{x \in E} B(x, d_D(x))$. Then, the map $\pi|_W : W \to \pi(W)$ is biholomorphic.*

Proof. We have only to prove that the map $\pi|_W : W \to \pi(W)$ is injective. Let $d := d_D$, $B := B(a, d(a))$, and $Q := \mathbf{B}_n(\pi(a), d(a))$. Let $y_1, y_2 \in W$

and assume that $z_0 := \pi(y_1) = \pi(y_2)$. Then, there exists $x_k \in E$ such that $y_k \in B(x_k, d(x_k))$ for each $k = 1, 2$. Since $B \cap B(x_k, d(x_k)) \neq \emptyset$, we have that $\left(\pi|_{B(x_k, d(x_k))} \right)^{-1} = \left(\pi|_B \right)^{-1}$ on $Q \cap \mathbf{B}_n(\pi(x_k), d(x_k))$ by Proposition 2.1. Let $c := (d(x_2)\pi(x_1) + d(x_1)\pi(x_2)) \,/\, (d(x_1) + d(x_2))$. Since $\pi(x_1), \pi(x_2) \in \overline{Q}$ and $z_0 \in \mathbf{B}_n(\pi(x_1), d(x_1)) \cap \mathbf{B}_n(\pi(x_2), d(x_2)) \neq \emptyset$, we have that $c \in Q \cap \mathbf{B}_n(\pi(x_1),\, d(x_1)) \cap \mathbf{B}_n(\pi(x_2),\, d(x_2))$. Then,

$$\left(\pi|_B\right)^{-1}(c) \in B \cap B(x_1,\, d(x_1)) \cap B(x_2,\, d(x_2))$$

and therefore $B(x_1,\, d(x_1)) \cap B(x_2,\, d(x_2)) \neq \emptyset$. By Proposition 2.1, we have that $\left(\pi|_{B(x_1,\, d(x_1))} \right)^{-1} = \left(\pi|_{B(x_2,\, d(x_2))} \right)^{-1}$ on $\mathbf{B}_n(\pi(x_1),\, d(x_1)) \cap \mathbf{B}_n(\pi(x_2),\, d(x_2))$. It follows that $y_1 = \left(\pi|_{B(x_1,\, d(x_2))} \right)^{-1}(z_0) = \left(\pi|_{B(x_2,\, d(x_2))} \right)^{-1}(z_0) = y_2$. Thus, we proved that $\pi|_W$ is injective. □

Let D be a complex manifold. According to Fujita [5, 6], an upper semicontinuous function $u : D \to [-\infty, +\infty)$ is said to be *subpluriharmonic* if for every relatively compact open set G of D and for every real-valued pluriharmonic function h defined on a neighborhood of $\overline{G}$, the inequality $u \leq h$ on ∂G implies the inequality $u \leq h$ on $\overline{G}$.[1] If D is holomorphically spreadable, then, by Vâjâitu [20, Proposition 2], an upper semicontinuous function $u : D \to [-\infty, +\infty)$ is subpluriharmonic if and only if there exists an open covering $\{U_i\}_{i \in I}$ of D such that u is subpluriharmonic on U_i for every $i \in I$.

Proposition 2.3 (Słodkowski [16, Lemma 4.4]) *Let D be an open set of $\mathbb{C}^n$ and $u : D \to [-\infty, +\infty)$ an upper semicontinuous function. If u is not subpluriharmonic, then there exist $c \in D$, $r \in (0, d_D(c))$, $K > 0$, and a function f holomorphic in a neighborhood of $\overline{\mathbf{B}_n(c, r)}$ such that $u(c) + \Re(f(c)) = 0$, and $u + \Re(f) \leq -K \|z - c\|^2$ on $\overline{\mathbf{B}_n(c, r)}$.*[2]

Let D be a complex manifold of dimension n. Let q be an integer such that $1 \leq q \leq n$. Then, an upper semicontinuous function

[1] The subpluriharmonic functions on an open set D of $\mathbb{C}^n$ exactly coincide with the $(n-1)$-plurisubharmonic functions on D in the sense of Hunt–Murray [9, Definition 2.3] (see Fujita [6, Proposition 2]).

[2] We can choose f to be a polynomial of degree at most 2 (see Abe–Sugiyama [1]). However, we do not use this refinement to prove Theorem 4.1.

$u : D \to [-\infty, +\infty)$ is said to be *q-plurisubharmonic* if for every open set G of $\mathbb{C}^q$ and for every holomorphic map $f : G \to D$ the function $u \circ f : G \to [-\infty, +\infty)$ is subpluriharmonic.[3] An upper semicontinuous function $u : D \to [-\infty, +\infty)$ is q-plurisubharmonic if and only if there exists an open covering $\{U_i\}_{i\in I}$ of D such that u is q-plurisubharmonic on U_i for every $i \in I$.

Proposition 2.4 (Fujita [6, Theorem 2]) *Let q and n be integers such that $1 \le q \le n$. Let D be an open set of $\mathbb{C}^n$ and $u : D \to [-\infty, +\infty)$ an upper semicontinuous function. Then, the following two conditions are equivalent.*

(1) *u is q-plurisubharmonic.*

(2) *u is $(q-1)$-plurisubharmonic in the sense of Hunt–Murray* [9, Definition 2.5]*, that is, for every q-dimensional complex affine subspace L of $\mathbb{C}^n$ the function $u|_{D\cap L}$ is subpluriharmonic.*

Corollary 2.5. *Let q and n be integers such that $1 \le q \le n$. Let (D, π) be an unramified domain over $\mathbb{C}^n$ and $u : D \to [-\infty, +\infty)$ an upper semicontinuous function. Then, the following two conditions are equivalent.*

(1) *u is q-plurisubharmonic.*

(2) *For every q-dimensional complex affine subspace L of $\mathbb{C}^n$ the function $u|_{\pi^{-1}(L)}$ is subpluriharmonic.*

Proof. There exists an open covering $\{U_i\}_{i\in I}$ of D such that $\pi|_{U_i} : U_i \to \pi(U_i)$ is biholomorphic for every $i \in I$. Then, by Proposition 2.4, condition (2) is equivalent to the one that u is q-plurisubharmonic on U_i for every $i \in I$, which is equivalent to condition (1). □

Let D be a complex manifold of dimension n. Let q be an integer such that $1 \le q \le n$. We say that D is *weakly q-pseudoconvex* if there exists

[3] The q-plurisubharmonic functions on an open set D of $\mathbb{C}^n$ exactly coincide with the pseudoconvex functions of order $n - q$ on D in the sense of Fujita [5, 6] and with the weakly q-plurisubharmonic functions on D in the sense of Popa-Fischer [15].

an exhaustion function $u : D \to [-\infty, +\infty)$ which is q-plurisubharmonic on D.[4] If D is a second countable complex manifold of dimension n with no compact connected components, then, by Greene–Wu [7] (see also Ohsawa [13]), D is n-complete, that is, there exists a $\mathscr{C}^\infty$ strictly subpluriharmonic exhaustion function on D and therefore D is weakly n-pseudoconvex.

Proposition 2.6. *Let q, m, and n be integers such that $1 \leq q \leq m \leq n$. Let D be a complex manifold of dimension n and E an m-dimensional closed complex submanifold of D. If D is weakly q-pseudoconvex, then E is also weakly q-pseudoconvex.*

Proof. Since D is weakly q-pseudoconvex, there exists a q-plurisubharmonic exhaustion function $u : D \to [-\infty, +\infty)$. Then, the function $u|_E : E \to [-\infty, +\infty)$ is a q-plurisubharmonic exhaustion function of E. □

Let (D, π) be an unramified domain over $\mathbb{C}^n$. Let q be an integer such that $1 \leq q \leq n$. We say that (D, π) is *q-pseudoconvex* if the function $-\ln d_D : D \to [-\infty, +\infty)$ is q-plurisubharmonic on D.[5]

Proposition 2.7 (Matsumoto [12, Theorem 2]) *Let q and n be integers such that $1 \leq q \leq n$. Let (D, π) be an unramified domain over $\mathbb{C}^n$. Then, the following two conditions are equivalent.*

(1) *D is weakly q-pseudoconvex.*

(2) *(D, π) is q-pseudoconvex.*

According to Fritzsche–Grauert [4, pp. 101–102], we recall some definitions related to the abstract boundary of an unramified domain (D, π) over $\mathbb{C}^n$. Let $\{x_k\}$ be a sequence of points in D which satisfies the following three conditions:

---

[4] A weakly 1-pseudoconvex manifold is nothing but a weakly pseudoconvex manifold in the sense of Demailly [3].

[5] If D is a connected open set of $\mathbb{C}^n$, then (D, i), where i denotes the inclusion, is q-pseudoconvex if and only if D is $(q-1)$-pseudoconvex in $\mathbb{C}^n$ in the sense of Słodkowski [16, Definition 4.1]. On the other hand, our definition of q-pseudoconvexity is different from that of Ohsawa [13].

- $\{x_k\}$ has no subsequence which converges in D.
- There exists $b \in \mathbb{C}^n$ such that $\lim_{k\to\infty} \pi(x_k) = b$ in $\mathbb{C}^n$.
- For every connected neighborhood V of b in $\mathbb{C}^n$ there exists $k_0 \in \mathbb{N}$ such that for every $k, l \geq k_0$ the point x_k can be joined to the point x_l by a path $\gamma : [0,1] \to D$ with $(\pi \circ \gamma)([0,1]) \subset V$.

We say that two such sequences $\{x_k\}$ and $\{y_k\}$ are equivalent if they satisfy the following two conditions:

- $\lim_{k\to\infty} \pi(x_k) = \lim_{k\to\infty} \pi(y_k) = b$.
- For every connected neighborhood V of b in $\mathbb{C}^n$ there exists $k_0 \in \mathbb{N}$ such that for every $k, l \geq k_0$ the point x_k can be joined to the point y_l by a path $\gamma : [0,1] \to D$ with $(\pi \circ \gamma)([0,1]) \subset V$.

An *accessible boundary point* of (D, π) is an equivalence class $\xi = [\{x_k\}]$ of such sequences. We denote by $\breve{\partial} D$ the set of all accessible boundary points of (D, π). We call the set $\breve{\partial} D$ the *abstract boundary* of (D, π) and the set $\breve{D} := D \cup \breve{\partial} D$ the *abstract closure* of (D, π). For every $\xi = [\{x_k\}] \in \breve{\partial} D$, we define a neighborhood system $\{\breve{U}\}$ of ξ in $\breve{D}$ as follows:

Take an arbitrary connected open set U in D such that $x_k \in U$ except for finitely many k. Then, let $\breve{U}$ be the union of U and the set of all accessible boundary points $\eta = [\{y_k\}]$ such that $y_k \in U$ except for finitely many k and $\lim_{k\to\infty} \pi(y_k) \in \overline{\pi(U)}$.

In this way, the set $\breve{D}$ becomes a regular space and the map $\breve{\pi} : D \to \mathbb{C}^n$ defined by

$$\breve{\pi}(x) := \begin{cases} \pi(x) & \text{if } x \in D, \\ \lim_{k\to\infty} x_k & \text{if } x = [\{x_k\}] \end{cases}$$

is continuous.

3. Lemmata

Lemma 3.1. *Let q and n be integers such that $1 \leq q \leq n$. Let (D, π) be a q-pseudoconvex unramified domain over $\mathbb{C}^n$. Let $\breve{\lambda} : \overline{\mathbf{B}_q(0,1)} \times [0,1] \to \breve{D}$ be a continuous map which satisfies the following two conditions:*

- $\breve{\lambda}((\overline{\mathbf{B}_q(0,1)} \times [0,1)) \cup (\partial \mathbf{B}_q(0,1) \times \{1\})) \subset D$.
- *The map* $\breve{\lambda}(\,\cdot\,,t) : \mathbf{B}_q(0,1) \to D$ *is holomorphic for every* $t \in [0,1)$.

Then, we have that $\breve{\lambda}(\overline{\mathbf{B}_q(0,1)} \times \{1\})) \subset D$.

Proof. Since $\breve{\lambda}(\partial \mathbf{B}_q(0,1) \times \{1\}) \subset D$, there exists $r \in (0,1)$ such that $\breve{\lambda}(\{r \le \|z\| \le 1\} \times \{1\}) \subset D$. By Proposition 2.7, there exists an exhaustion function $u : D \to [-\infty, +\infty)$ which is q-plurisubharmonic on D. Then, we have that

$$K := \max_{\{r \le \|z\| \le 1\} \times [0,1]} u \circ \breve{\lambda} < +\infty.$$

Let $t \in [0,1)$. Since the map $\breve{\lambda}(\,\cdot\,,t) : \mathbf{B}_q(0,1) \to D$ is holomorphic, the function $(u \circ \breve{\lambda})(\cdot, t) : \mathbf{B}_q(0,1) \to [-\infty, +\infty)$ is subpluriharmonic. Therefore, by the maximum principle for subpluriharmonic functions (see Fujita [5, Proposition 1]), we have that $\left(u \circ \breve{\lambda}\right)(z,t) \le K$ for every $z \in \overline{\mathbf{B}_q(0,1)}$. It follows that $\breve{\lambda}(\overline{\mathbf{B}_q(0,1)} \times [0,1)) \subset \{u \le K\}$. Since $\{u \le K\}$ is a compact set in D, we have that

$$\breve{\lambda}(z,1) = \lim_{t \to 1-0} \breve{\lambda}(z,t) \in \{u \le K\} \subset D$$

for every $z \in \overline{\mathbf{B}_q(0,1)}$. □

Lemma 3.2. *Let q, m, and n be integers such that $1 \le q \le m \le n$. Let (D, π) be an unramified domain over $\mathbb{C}^n$. Let H be an m-dimensional complex affine subspace of $\mathbb{C}^n$ such that $\pi^{-1}(H)$ is weakly q-pseudoconvex. Let $\breve{\lambda} : \overline{\mathbf{B}_q(0,1)} \times [0,1] \to \breve{D}$ be a continuous map such that $\breve{\lambda}(\overline{\mathbf{B}_q(0,1)} \times [0,1] \setminus \{(0,1)\}) \subset D \cap \pi^{-1}(H)$ and the map $\breve{\lambda}(\,\cdot\,,t) : \mathbf{B}_q(0,1) \to D$ is holomorphic for every $t \in [0,1)$. Then, we have that $\breve{\lambda}(0,1) \in D$.*

Proof. Let $F := \overline{\mathbf{B}_q(0,1)} \times [0,1]$. Let Z be the connected component of $\pi^{-1}(H)$ which includes the connected set $\breve{\lambda}(F \setminus \{(0,1)\})$. Then, the closed complex subspace Z of D with the induced map $\pi_{Z,H} : Z \to H$ is an unramified domain over $H \cong \mathbb{C}^m$. By assumption, $\pi^{-1}(H)$ is weakly q-pseudoconvex and therefore Z is q-pseudoconvex by Proposition 2.7. Suppose that $\breve{\lambda}(0,1) \in \breve{\partial} D$. Take arbitrary sequences $\left\{(z^{(k)}, s^{(k)})\right\}$ and $\left\{(w^{(k)}, t^{(k)})\right\}$ in $F \setminus \{(0,1)\}$ which converge to $(0,1)$ in F. Then, both two sequences $\left\{\breve{\lambda}(z^{(k)}, s^{(k)})\right\}$ and $\left\{\breve{\lambda}(w^{(k)}, t^{(k)})\right\}$ have no convergent subsequence in Z and we have that

$$\lim_{k\to\infty} \pi(\breve{\lambda}(z^{(k)}, s^{(k)})) = \lim_{k\to\infty} \pi(\breve{\lambda}(w^{(k)}, t^{(k)})) = \breve{\pi}(\breve{\lambda}(0,1))$$

in $\mathbb{C}^n$. For every $r > 0$, there exist $\rho \in (0,1)$ and $\delta \in (0,1)$ such that $\left(\breve{\pi} \circ \breve{\lambda}\right)(\mathbf{B}_q(0,\rho) \times (1-\delta, 1]) \subset \mathbf{B}_n(\breve{\pi}(\breve{\lambda}(0,1)), r)$. There exists $k_0 \in \mathbb{N}$ such that $(z^{(k)}, s^{(k)})$, $(w^{(k)}, t^{(k)}) \in \mathbf{B}_q(0,\rho) \times (1-\delta, 1]$ for every $k \geq k_0$. Since the set $\mathbf{B}_q(0,\rho) \times (1-\delta, 1]$ is convex, the path

$$\gamma : [0,1] \to Z, \quad \gamma(\tau) := \breve{\lambda}((1-\tau)\, z^{(k)} + \tau w^{(l)}, (1-\tau)\, s^{(k)} + \tau t^{(l)}),$$

joins $\breve{\lambda}(z^{(k)}, s^{(k)})$ to $\breve{\lambda}(w^{(l)}, t^{(l)})$ and satisfies the condition that $(\pi \circ \gamma)([0,1]) \subset \mathbf{B}_n(\breve{\pi}(\breve{\lambda}(0,1)), r)$ for $k, l \geq k_0$. Therefore, the sequences $\left\{\breve{\lambda}(z^{(k)}, s^{(k)})\right\}$ and $\left\{\breve{\lambda}(w^{(k)}, t^{(k)})\right\}$ are equivalent each other and determine a unique point $\xi \in \breve{\partial} Z$ and we have that

$$\lim_{k\to\infty} \breve{\lambda}(z^{(k)}, s^{(k)}) = \lim_{k\to\infty} \breve{\lambda}(w^{(k)}, t^{(k)}) = \xi$$

in $\breve{Z}$. It follows that $\lim_{(z,t)\to(0,1)} \breve{\lambda}(z,t) = \xi$ in $\breve{Z}$ although $\breve{\lambda}(F \setminus \{(0,1)\}) \subset Z$. This contradicts Lemma 3.1. Thus, we proved that $\breve{\lambda}(0,1) \in D$. $\square$

Lemma 3.3. *Let (D,π) be an unramified domain over $\mathbb{C}^n$ such that π is not biholomorphic. Then, for every $a \in D$, there exist $b^{(0)} \in \partial\mathbf{B}_n(\pi(a), d_D(a))$ and $\xi^{(0)} \in \breve{\partial} D$ such that*

$$\lim_{z\to b^{(0)}} \left(\pi|_{B(a,d_D(a))}\right)^{-1}(z) = \xi^{(0)}$$

in $\breve{D}$.

Proof. Let $d := d_D$, $B := B(a, d(a))$, $Q := \mathbf{B}_n(\pi(a), d(a))$, and $\sigma := \left(\pi|_B\right)^{-1} : Q \to B$. Take an arbitrary $b \in \partial Q$. First, we consider the case where there exists a sequence $\left\{w^{(k)}\right\} \subset Q$ which converges to b in $\mathbb{C}^n$ such that the sequence $\left\{\sigma(w^{(k)})\right\}$ converges to some ξ in D. Then, we have that $B \cap B(\xi, d(\xi)) \neq \emptyset$ and

$$\pi(\xi) = \lim_{k\to\infty} \pi(\sigma(w^{(k)})) = \lim_{k\to\infty} w^{(k)} = b.$$

Therefore, by Proposition 2.1, we have that

$$\lim_{z\to b}\sigma(z)=\lim_{z\to b}\left(\pi|_{B(\xi,d(\xi))}\right)^{-1}(z)=\left(\pi|_{B(\xi,d(\xi))}\right)^{-1}(b)=\xi$$

in D. Next, we consider the case other than the above. Take arbitrary sequences $\{z^{(k)}\}$ and $\{w^{(k)}\}$ in Q which converge to b in $\mathbb{C}^n$. Then, both two sequences $\{\sigma(z^{(k)})\}$ and $\{\sigma(w^{(k)})\}$ have no convergent subsequence in D and we have that

$$\lim_{k\to\infty}\pi(\sigma(z^{(k)}))=\lim_{k\to\infty}\pi(\sigma(w^{(k)}))=b$$

in $\mathbb{C}^n$. For every $r>0$, there exists $k_0\in\mathbb{N}$ such that $z^{(k)},w^{(k)}\in\mathbf{B}_n(b,r)$ for every $k\geq k_0$. Since $Q\cap\mathbf{B}_n(b,r)$ is convex, the path

$$\gamma:[0,1]\to D,\quad \gamma(t):=\sigma((1-t)\,z^{(k)}+tw^{(l)}),$$

joins $\sigma(z^{(k)})$ to $\sigma(w^{(l)})$ and satisfies the condition that $(\pi\circ\gamma)([0,1])\subset\mathbf{B}_n(b,r)$ for $k,l\geq k_0$. Therefore, the sequences $\{\sigma(z^{(k)})\}$ and $\{\sigma(w^{(k)})\}$ are equivalent each other, determine a point $\xi\in\breve{\partial}D$, and we have that

$$\lim_{k\to\infty}\sigma(z^{(k)})=\lim_{k\to\infty}\sigma(w^{(k)})=\xi$$

in $\breve{D}$. It follows that $\lim_{z\to b}\sigma(z)=\xi$ in $\breve{D}$. Thus, we proved that, for every $b\in\partial Q$, there exists a unique $\xi(b)\in\breve{D}$ such that $\lim_{z\to b}\sigma(z)=\xi(b)$ in $\breve{D}$. Seeking a contradiction, suppose that $\xi(b)\in D$ for every $b\in\partial Q$. Then, by Proposition 2.2, the map $\pi|_W:W\to\pi(W)$ is biholomorphic, where

$$W:=B\cup\left(\bigcup_{b\in\partial Q}B(\xi(b),d(\xi(b)))\right).$$

Since $\overline{Q}\subset\pi(W)$, there exists $\delta>0$ such that $\mathbf{B}_n(\pi(a),d(a)+\delta)\subset\pi(W)$. Then, $U:=(\pi|_W)^{-1}(\mathbf{B}_n(\pi(a),d(a)+\delta))$ is a neighborhood of a and the map $\pi:U\to\mathbf{B}_n(\pi(a),d(a)+\delta)$ is biholomorphic, which contradicts the definition of $d(a)$. It follows that there exists $b^{(0)}\in\partial Q$ such that $\xi^{(0)}:=\xi(b^{(0)})\in\breve{\partial}D$. □

By a similar argument in Yasuoka [21, pp. 143–144], we can prove the following lemma (see Sugiyama [18]).

Lemma 3.4. *Let $c \in \mathbb{C}^n$, $r > 0$, and $f \in \mathscr{O}(\mathbf{B}_n(c,r))$ with $\Im(f(c)) = 0$. Let*

$$P(z) := \sum_{|\alpha|\leq 2} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}\, \mathrm{e}^f}{\partial z^\alpha}(c)\,(z-c)^\alpha\,, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),$$

for every $z \in \mathbb{C}^n$. Then, for every $\varepsilon \in (0, \mathrm{e}^{-\Re(f(c))})$, there exist $\rho \in (0,r)$, $\delta > 0$, and $M > 0$ such that

$$\ln|P(z) - t| \leq \Re(f(z)) - \varepsilon t + M\,\|z\|^3$$

for every $(z,t) \in \overline{\mathbf{B}_n(0,\rho)} \times [0,\delta]$.

4. Theorem

Theorem 4.1. *Let q and n be integers such that $1 \leq q \leq n$. Let (D,π) be an unramified domain over $\mathbb{C}^n$. Then, the following two conditions are equivalent.*

(1) *(D,π) is q-pseudoconvex.*

(2) *Let $\breve{\lambda} : \overline{\mathbf{B}_q(0,1)} \times [0,1] \to \breve{D} = D \cup \breve{\partial} D$ be a continuous map which satisfies the following two conditions:*

- *$\breve{\lambda}(\overline{\mathbf{B}_q(0,1)} \times [0,1] \setminus \{(0,1)\}) \subset D$.*
- *There exists a holomorphic map $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) : \mathbb{C}^{q+1} \to \mathbb{C}^n$ of the form*

$$\lambda_\nu(z_1, z_2, \ldots, z_q, t) = P_\nu(z_1, z_2, \ldots, z_q) + c_\nu t,$$

where $P_\nu(z_1, z_2, \ldots, z_q)$ is a polynomial of $z_1, z_2, \ldots, z_q$ of degree at most 2 and $c_\nu \in \mathbb{C}$ for every $\nu = 1, 2, \ldots, n$, such that the image $H := \lambda(\mathbb{C}^{q+1})$ is a $(q+1)$-dimensional complex affine subspace of $\mathbb{C}^n$, the induced map $\lambda : \mathbb{C}^{q+1} \to H$ is biholomorphic, and $\breve{\pi} \circ \breve{\lambda} = \lambda$ on $\overline{\mathbf{B}_q(0,1)} \times [0,1]$.

Then, we have that $\breve{\lambda}(0,1) \in D$.

Remark 4.2. The expression of λ in the hypothesis in condition (2) does not depend on a complex affine transformation of coordinates of $\mathbb{C}^n$.

Remark 4.3. If $q = n$, then there does not exist such λ that satisfies the hypothesis in condition (2) and therefore every unramified domain (D, π) over $\mathbb{C}^n$ satisfies condition (2).

Proof of Theorem 4.1.

(1) $\to$ (2). The assertion is a direct consequence of Lemma 3.1.

(2) $\to$ (1). We have only to prove the assertion when $1 \leq q \leq n-1$ and π is not biholomorphic. Seeking a contradiction, suppose that (D, π) is not q-pseudoconvex. By Corollary 2.5, there exists a q-dimensional complex affine subspace L of $\mathbb{C}^n$ such that the function $-\ln d$ is not subpluriharmonic on $\pi^{-1}(L)$, where $d := d_D$. Take a connected component Z of $\pi^{-1}(L)$ such that $(-\ln d)|_Z$ is not subpluriharmonic on Z. The closed complex submanifold Z of D with the induced map $\pi_{Z,L} : Z \to L$ is an unramified domain over $L \cong \mathbb{C}^q$. By a translation and by a unitary transformation of coordinates $w_1, w_2, \dots, w_n$ of $\mathbb{C}^n$, we may assume that $L = \{w_{q+1} = w_{q+2} = \cdots = w_n = 0\}$. Then, the functions $\pi_1|_Z, \pi_2|_Z, \dots, \pi_q|_Z$ give a system of local coordinates of Z near any point of Z, where $\pi_\nu := w_\nu \circ \pi$ for every $\nu = 1, 2, \dots, n$. By Proposition 2.3, there exist $a \in Z$, $r \in (0, d(a))$, $K > 0$, and a function f holomorphic near $\overline{B^{(Z)}(a, r)}$ such that $-\ln d(a) + \Re(f(a)) = 0$, $\Im(f(a)) = 0$, and

$$-\ln d + \Re(f) \leq -K \sum_{\nu=1}^{q} |\pi_\nu - \pi_\nu(a)|^2$$

on $\overline{B^{(Z)}(a, r)}$. where $B^{(Z)}(a, r)$ denotes the open ball in Z of radius r with center a. By a translation of coordinates, we may further assume that $\pi(a) = 0$ in $\mathbb{C}^n$. Let

$$\ell : \mathbb{C}^q \to \mathbb{C}^n, \quad \ell(z_1, z_2, \dots, z_q) := (z_1, z_2, \dots, z_q, 0, \dots, 0).$$

Since $\|\ell(z)\| = \|z\|$ for every $z \in \mathbb{C}^q$, we have that $\ell(\mathbf{B}_q(0, r)) = \mathbf{B}_n(0, r) \cap L$. Let $B := B(a, d(a))$, $Q := \mathbf{B}_n(0, d(a))$, and $\sigma := (\pi|_B)^{-1} : Q \to B$. Then, we have that $B^{(Z)}(a, r) = \sigma(\mathbf{B}_n(0, r) \cap L)$. Let $\tilde{f}(z) := f(\sigma(\ell(z)))$ for every $z \in \mathbf{B}_q(0, r)$. Take an arbitrary $\varepsilon \in (0, \mathrm{e}^{-\Re(\tilde{f}(0))})$. By Lemma 3.4, there exist $\rho_1 \in (0, r)$, $\delta > 0$, and $M > 0$ such that

$$\ln |P(z) - t| \leq \Re(\tilde{f}(z)) - \varepsilon t + M \|z\|^3$$

for every $(z,t) \in \overline{\mathbf{B}_q(0,\rho_1)} \times [0,\delta]$, where

$$P(z) := \sum_{|\alpha|\le 2} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}\,\mathrm{e}^{\tilde f}}{\partial z^\alpha}(0)z^\alpha, \quad \alpha = (\alpha_1,\alpha_2,\dots,\alpha_q),$$

on $\mathbb{C}^q$. Take an arbitrary $\rho \in (0, \min\{\rho_1, K/M\})$. Let $E := \overline{\mathbf{B}_q(0,\rho)} \times [0,\delta]$. For every $(z,t) \in E \setminus \{(0,0)\}$, we have that

$$\Re(\tilde f(z)) - \varepsilon t + M\|z\|^3 < \Re(\tilde f(z)) + K\|z\|^2$$

and therefore we have that

$$|P(z) - t| < \mathrm{e}^{\Re(\tilde f(z)) + K\|z\|^2} \le \mathrm{e}^{\ln d(\sigma(\ell(z)))} = d(\sigma(\ell(z))).$$

On the other hand, we have that

$$|P(0) - 0| = \left|\mathrm{e}^{\tilde f(0)}\right| = \mathrm{e}^{\Re(f(a))} = \mathrm{e}^{\ln d(a)} = d(a).$$

By Lemma 3.3, there exist $b^{(0)} \in \partial Q$ and $\xi^{(0)} \in \breve{\partial} D$ such that $\lim_{z\to b^{(0)}} \sigma(z) = \xi^{(0)}$ in $\breve{D}$. Let

$$\varphi : \mathbb{C}^q \times \mathbb{C} \to \mathbb{C}^n, \quad \varphi(z,t) := (P(z) - t)\,u^{(0)} + \ell(z),$$

where $u^{(0)} = (u_1^{(0)}, u_2^{(0)}, \dots, u_n^{(0)}) := b^{(0)}/P(0)$. Then, for every $(z,t) \in E \setminus \{(0,0)\}$, we have that

$$\|\varphi(z,t) - \ell(z)\| = |P(z) - t| < d(\sigma(\ell(z)))$$

and therefore

$$\varphi(z,t) \in \mathbf{B}_n(\ell(z), d(\sigma(\ell(z)))) = \pi(B(\sigma(\ell(z)), d(\sigma(\ell(z))))) \subset \pi(W),$$

where $W := \bigcup_{x\in B} B(x, d(x))$. By Proposition 2.2, the map $\pi|_W : W \to \pi(W)$ is biholomorphic. Let

$$\breve\varphi(z,t) := \begin{cases} (\pi|_W)^{-1}(\varphi(z,t)) & \text{if } (z,t) \in E \setminus \{(0,0)\}, \\ \xi^{(0)} & \text{if } (z,t) = (0,0). \end{cases}$$

Then, we have that $\breve\pi \circ \breve\varphi = \varphi$ on E. Suppose that there exists a sequence $\{(z^{(k)}, t^{(k)})\} \subset E \setminus \{(0,0)\}$ which converges to $(0,0)$ in $\mathbb{C}^q \times \mathbb{C}$ such that the sequence $\{\breve{\varphi}(z^{(k)}, t^{(k)})\}$ converges to some point η in D. Then, we have that $W \cap B(\eta, d(\eta)) \neq \emptyset$ and therefore there exists $x \in B$ such that $B(x, d(x)) \cap B(\eta, d(\eta)) \neq \emptyset$. Then, by Proposition 2.1, there exists a path $\beta : [0,1] \to D$ which joins x to η such that $\pi \circ \beta$ is the line segment which joins $\pi(x)$ to $b^{(0)}$. Since $(\pi \circ \beta)([0,1)) \subset Q$ and $\beta(0) = x = \sigma(\pi(x))$, we have that $\beta = \sigma \circ (\pi \circ \beta)$ on $[0,1)$ by the identity principle for liftings. It follows that

$$\eta = \lim_{t \to 1-0} \beta(t) = \lim_{t \to 1-0} \sigma((1-t)\,\pi(x) + tb^{(0)}) = \xi^{(0)} \in \breve{\partial} D,$$

which is a contradiction. Therefore, for every sequence $\{(z^{(k)}, t^{(k)})\} \subset E \setminus \{(0,0)\}$ which converges to $(0,0)$ in $\mathbb{C}^q \times \mathbb{C}$, the sequence $\{\breve{\varphi}(z^{(k)}, t^{(k)})\}$ has no convergent subsequence in D. For every $s > 0$, there exist $\rho' \in (0, \rho)$ and $\delta' \in (0, \delta)$ such that $\varphi(\mathbf{B}_q(0, \rho') \times (-\delta', \delta')) \subset \mathbf{B}_n(b^{(0)}, s)$. There exists $k_0 \in \mathbb{N}$ such that $(z^{(k)}, t^{(k)}) \in \mathbf{B}_q(0, \rho') \times [0, \delta')$ and $1/k < \delta'$ for every $k \geq k_0$. Since the set $\mathbf{B}_q(0, \rho') \times [0, \delta') \setminus \{(0,0)\}$ is arcwise connected, there exists a path $\gamma : [0,1] \to D$ which joins $\breve{\varphi}(z^{(k)}, t^{(k)})$ to $\breve{\varphi}(0, 1/l)$ and $(\pi \circ \gamma)([0,1]) \subset W \cap \mathbf{B}_n(b^{(0)}, s)$ for every $k, l \geq k_0$. Therefore, the sequence $\{\breve{\varphi}(z^{(k)}, t^{(k)})\}$ is equivalent to $\{\breve{\varphi}(0, 1/k)\}$ and we have that

$$\lim_{k \to \infty} \breve{\varphi}(x^{(k)}, t^{(k)}) = \lim_{k \to \infty} \breve{\varphi}(0, 1/k) = \lim_{k \to \infty} \sigma((P(0) - 1/k)\, u^{(0)}) = \xi^{(0)}$$

in $\breve{D}$. It follows that $\lim_{(z,t) \to (0,0)} \breve{\varphi}(z, t) = \xi^{(0)}$ in $\breve{D}$. Thus, we proved that the map $\breve{\varphi} : E \to \breve{D}$ is continuous. Let $F := \overline{\mathbf{B}_q(0,1)} \times [0,1]$, let $\breve{\lambda} : F \to \breve{D}$, $\breve{\lambda}(\zeta, \tau) := \breve{\varphi}(\rho\zeta, \delta\,(1-\tau))$, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) : \mathbb{C}^q \times \mathbb{C} \to \mathbb{C}^n$, $\lambda(\zeta, \tau) := \varphi(\rho\zeta, \delta\,(1-\tau))$. Then, $\breve{\lambda}$ is continuous, $\breve{\lambda}(F \setminus \{(0,1)\}) = \breve{\varphi}(E \setminus \{(0,0)\}) \subset D$, $\breve{\pi} \circ \breve{\lambda} = \lambda$ on F, and $\breve{\lambda}(0,1) = \breve{\varphi}(0,0) = \xi^{(0)} \in \breve{\partial} D$. We have that

$$\begin{aligned} \lambda(\zeta, \tau) &= (P(\rho\zeta) - \delta\,(1-\tau))\, u^{(0)} + \ell(\rho\zeta) \\ &= \Big\{(P(\rho\zeta) - \delta)\, u^{(0)} + \ell(\rho\zeta)\Big\} + \tau\delta u^{(0)} \end{aligned}$$

and every component of $(P(\rho\zeta) - \delta)\, u^{(0)} + \ell(\rho\zeta)$ is a polynomial of $\zeta_1, \zeta_2, \ldots, \zeta_n$ of degree at most 2. Let $H := \mathbb{C}\, u^{(0)} + L$, which is a complex linear subspace of $\mathbb{C}^n$. For every $(z, t) \in \mathbb{C}^q \times \mathbb{C}$, we have that $\varphi(z, t) = (P(z) - t)\, u^{(0)} + \ell(z) \in H$. Suppose that $u^{(0)} \in L$. Then, we have that $H = L$ and $\check{\lambda}(F \setminus \{(0,1)\}) = \check{\varphi}(E \setminus \{(0,1)\}) \subset \pi^{-1}(L)$. Since $\pi^{-1}(L)$ is weakly q-pseudoconvex, we have that $\xi^{(0)} = \lim_{(z,t)\to(0,1)} \check{\lambda}(z,t) \in D$ by Lemma 3.2, which is a contradiction. It follows that $u_0 \notin L$ and $\dim H = q+1$. Therefore, there exists $m \in \{q+1, q+2, \ldots, n\}$ such that $u_m^{(0)} \neq 0$. Let

$$\psi : \mathbb{C}^n \to \mathbb{C}^q \times \mathbb{C}, \quad \psi(w_1, w_2, \ldots, w_n) = (z_1, z_2, \ldots, z_q, t),$$

be the holomorphic map defined by

$$\begin{cases} z_\nu = w_\nu - \left(u_\nu^{(0)}/u_m^{(0)}\right) \cdot w_m \quad (\nu = 1, 2, \ldots, q), \\ t = P(\left(w_\nu - \left(u_\nu^{(0)}/u_m^{(0)}\right) \cdot w_m\right)_{\nu=1}^{q}) - \left(1/u_m^{(0)}\right) \cdot w_m. \end{cases}$$

Then, by direct computations, we can verify that $(\psi|_H) \circ \varphi = \mathsf{id}_{\mathbb{C}^q \times \mathbb{C}}$ and $\varphi \circ (\psi|_H) = \mathsf{id}_H$. It follows that $\varphi : \mathbb{C}^q \times \mathbb{C} \to \varphi(\mathbb{C}^q \times \mathbb{C}) = H$ is biholomorphic. Since the map $\mathbb{C}^q \times \mathbb{C} \to \mathbb{C}^q \times \mathbb{C}$, $(\zeta, \tau) \to (\rho\zeta, \delta\,(1-\tau))$, is a complex affine automorphism, the map $\lambda : \mathbb{C}^q \times \mathbb{C} \to H$ is also biholomorphic. Consequently, $\check{\lambda}$ satisfies the supposition of condition (2) but does not satisfies the conclusion of it, which is a contradiction. □

5. Corollaries

As a corollary to Theorem 4.1, we have the following characterization of a q-pseudoconvex unramified domain over $\mathbb{C}^n$ by the continuity property (cf. Słodkowski [16, Theorem 4.3] and Vâjâitu [19, Corollary 1]).

Corollary 5.1. *Let q and n be integers such that $1 \leq q \leq n$. Let (D, π) be an unramified domain over $\mathbb{C}^n$. Then, the following two conditions are equivalent.*

(1) (D, π) *is q-pseudoconvex.*

(2) *Let $\check{\lambda} : \overline{\mathbf{B}_q(0,1)} \times [0,1] \to \check{D}$ be a continuous map which satisfies the following two conditions:*

- $\check{\lambda}((\overline{\mathbf{B}_q(0,1)} \times [0,1)) \cup (\partial \mathbf{B}_q(0,1) \times \{1\})) \subset D$.
- *The map $\check{\lambda}(\,\cdot\,, t) : \mathbf{B}_q(0,1) \to D$ is holomorphic for every $t \in [0,1)$.*

Then, we have that $\breve{\lambda}(\overline{\mathbf{B}_q(0,1)} \times \{1\}) \subset D$.

We have the following characterization of a q-pseudoconvex unramified domain over $\mathbb{C}^n$, which generalizes Lelong [11, p. 201], Hitotumatu [8, Proposition 14], Alessandrini–Silva [2, p. 86], Słodkowski [16, Corollary 4.8], and Pawlaschyk–Zeron [14, Proposition 3.14].

Corollary 5.2. *Let q and n be integers such that $1 \leq q \leq n$. Let (D, π) be an unramified domain over $\mathbb{C}^n$. Then, the following two conditions are equivalent.*

(1) *(D, π) is q-pseudoconvex.*

(2) *For every $(q+1)$-dimensional complex affine subspace H of $\mathbb{C}^n$, the closed complex submanifold $\pi^{-1}(H)$ of D is weakly q-pseudoconvex.*

Proof.

(1) $\to$ (2). The assertion is a direct consequence of Propositions 2.6 and 2.7.

(2) $\to$ (1). Seeking a contradiction, suppose that (D, π) is not q-pseudoconvex. Let $F := \overline{\mathbf{B}_q(0,1)} \times [0,1]$. By Theorem 4.1, there exists a continuous map $\breve{\lambda} : F \to \breve{D}$ which satisfies the following three conditions:

- $\breve{\lambda}(F \setminus \{(0,1)\}) \subset D$.
- There exists a holomorphic map $\lambda : \mathbb{C}^{q+1} \to \mathbb{C}^n$ such that the image $H := \lambda(\mathbb{C}^{q+1})$ is a $(q+1)$-dimensional complex affine subspace of $\mathbb{C}^n$, the induced map $\lambda : \mathbb{C}^{q+1} \to H$ is biholomorphic, and $\breve{\pi} \circ \breve{\lambda} = \lambda$ on F.
- $\breve{\lambda}(0,1) \in \breve{\partial} D$.

Then, by assumption, $\pi^{-1}(H)$ is weakly q-pseudoconvex. Since $\breve{\lambda}(F \setminus \{(0,1)\}) \subset D \cap \pi^{-1}(H)$, we have that $\breve{\lambda}(0,1) = \lim_{(z,t) \to (0,1)} \breve{\lambda}(z,t) \in D$ by Lemma 3.2, which is a contradiction. □

In the case where $q = 1$, we have the following characterization of a pseudoconvex unramified domain over $\mathbb{C}^n$, which generalizes Yasuoka [21, Theorem 2] and refines Sugiyama [17, Theorem 3.1].

Corollary 5.3. *Let (D, π) be an unramified domain over $\mathbb{C}^n$. Then, the following two conditions are equivalent.*

(1) *(D, π) is pseudoconvex.*

(2) *Let $\check{\lambda} : \overline{B_1(0,1)} \times [0,1] \to \check{D}$ be a continuous map which satisfies the following two conditions:*

- *$\check{\lambda}(\overline{B_1(0,1)} \times [0,1] \setminus \{(0,1)\}) \subset D$.*
- *There exists a holomorphic map $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) : \mathbb{C}^2 \to \mathbb{C}^n$ of the form*
$$\lambda_\nu(z,t) = P_\nu(z) + c_\nu t,$$
where $P_\nu(z)$ is a polynomial of z of degree at most 2 and $c_\nu \in \mathbb{C}$ for every $\nu = 1, 2, \ldots, n$, such that the image $H := \lambda(\mathbb{C}^2)$ is a complex affine subspace of dimension 2, the induced map $\lambda : \mathbb{C}^2 \to H$ is biholomorphic, and $\check{\pi} \circ \check{\lambda} = \lambda$ on $\overline{B_1(0,1)} \times [0,1]$.

Then, we have that $\check{\lambda}(0,1) \in D$.

Acknowledgment. The first author is partially supported by JSPS KAKENHI Grant Number JP17K05301.

References

[1] M. Abe and S. Sugiyama, A characterization of subpluriharmonicity for a function of several complex variables, *Bulletin of the Graduate School of Integrated Arts and Sciences, Hiroshima Univ., II: Studies in Environmental Sciences* (to appear).

[2] L. Alessandrini and A. Silva, On a theorem of Lelong, *Math. Nachr.*, **147** (1990), 83–88.

[3] J.-P. Demailly, Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-desssus d'une variété kählérienne complète, *Ann. Sci. École Norm. Sup. (4)*, **15** (1982), 457–511.

[4] K. Fritzsche and H. Grauert, *From holomorphic functions to complex manifolds*, *Grad. Texts in Math.*, vol. 213, Springer, New York, 2002.

[5] O. Fujita, Domaines pseudoconvexes d'ordre général et fonctions pseudoconvexes d'ordre général, *J. Math. Kyoto Univ.*, **30** (1990), 637–649.

[6] O. Fujita, On the equivalence of the q-plurisubharmonic functions and the pseudoconvex functions of general order, *Annual Reports of Graduate School of Human Culture, Nara Women's Univ.*, **7** (1991), 77–81.

[7] R. E. Greene and H. Wu, Embedding of open Riemannian manifolds by harmonic functions, *Ann. Inst. Fourier (Grenoble)*, **25** (1975), 215–235.

[8] S. Hitotumatu, On some conjectures concerning pseudo-convex domains, *J. Math. Soc. Japan*, **6** (1954), 177–195.

[9] L. R. Hunt and J. J. Murray, q-plurisubharmonic functions and a generalized Dirichlet problem, *Michigan Math. J.*, **25** (1978), 299–316.

[10] M. Jarnicki and P. Pflug, *Extension of holomorphic functions*, *De Gruyter Expositions in Mathematics*, vol. 34, Walter de Gruyter, Berlin, 2000.

[11] P. Lelong, Domaines convexes par rapport aux fonctions plurisousharmoniques. *J. Analyse Math.*, **2** (1952), 178–208.

[12] K. Matsumoto, Pseudoconvex Riemann domains of general order over Stein manifolds, *Mem. Fac. Sci. Kyushu Univ. Ser. A. Math.*, **44** (1990), 95–109.

[13] T. Ohsawa, Completeness of noncompact analytic spaces, *Publ. Res. Inst. Math. Sci.*, **20** (1984), 683–692.

[14] T. Pawlaschyk and E. S. Zeron, On convex hulls and pseudoconvex domains generated by q-plurisubharmonic functions, part I, *J. Math. Anal. Appl.*, **408** (2013), 394–408.

[15] A. Popa-Fischer, A generalization to the q-convex case of a theorem of Fornæss and Narasimhan, *Michigan Math. J.*, **50** (2002), 483–492.

[16] Z. Słodkowski, Local maximum property and q-plurisubharmonic functions in uniform algebras, *J. Math. Anal. Appl.*, **115** (1986), 105–130.

[17] S. Sugiyama, Polynomials and pseudoconvexity for Riemann domains over $\mathbb{C}^n$, *Toyama Math. J.*, **38** (2016), 101–114.

[18] S. Sugiyama, Generalized Cartan-Behnke-Stein's theorem and q-pseudoconvexity in a Stein manifold, *Tohoku Math. J. (2)* (to appear).

[19] V. Vâjâitu, On increasing unions of q-complete manifolds. *Rev. Roumaine Math. Pures Appl.*, **43** (1998), 235–244.

[20] V. Vâjâitu, A Levi problem for continuous strongly q-plurisubharmonic functions, *C. R. Acad. Sci. Paris Sér. I Math.*, **328** (1999), 573–578.

[21] T. Yasuoka, Polynomials and pseudoconvexity, *Math. Sem. Notes Kobe Univ.*, **11** (1983), 139–148.

Makoto Abe
School of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
e-mail: abem@hiroshima-u.ac.jp

Tadashi Shima
School of Informatics and Data Science
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
e-mail: tadashi@hiroshima-u.ac.jp

Shun Sugiyama
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan
e-mail: d171772@hiroshima-u.ac.jp

(Received February 4, 2019)