

Existence and stability of singularly perturbed standing pulse solutions of a three-component FitzHugh-Nagumo system

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Abstract. In this article, a singularly perturbed three-component FitzHugh-Nagumo system, which is proposed in [2], is considered. As a simple localized pattern, the existence of standing pulse solutions with high accurate approximations for a small parameter and their stability are shown by using an analytic singular perturbation technique.

1. Introduction

Various localized patterns are observed in many reaction-diffusion systems. Here we focus our attention on the type of FitzHugh-Nagumo systems. The two-component FitzHugh-Nagumo system (1.1), which describes the conduction of nerve impulse along nerve axons originally ([14], [5]), is very famous and has been studied energetically.

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + u - u^3 - v, \\ \tau v_t = D v_{xx} + u - v, \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R}. \quad (1.1)$$

Here we assume $0 < \varepsilon \ll 1$, $\tau > 0$, $D > 0$ (Originally $D = 0$ in the FitzHugh-Nagumo model). Let us consider standing pulse solutions

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$(u, v)(x; \varepsilon)$ of (1.1), which satisfy

$$\begin{cases} 0 = \varepsilon^2 u_{xx} + u - u^3 - v, \\ 0 = Dv_{xx} + u - v, \\ u(-\infty) = 0 = u(+\infty), \quad v(-\infty) = 0 = v(+\infty). \end{cases} \quad x \in \mathbb{R},$$

In [12], it was shown that there exist two types of destabilization of standing pulse solutions when τ increases. One is the appearance of traveling pulse solutions via the drift bifurcation at $\tau = \tau_D$ and the other is that of standing breathers via the Hopf bifurcation at $\tau = \tau_H$. For (1.1), it is believed that $0 < \tau_H < \tau_D$ (see [12], [17] for example). That is, for $0 < \tau < \tau_H$, $(u, v)(x; \varepsilon)$ are stable and at $\tau = \tau_H$, stable standing breathers bifurcate and then, $(u, v)(x; \varepsilon)$ become unstable for $\tau > \tau_H$. Though traveling pulses bifurcate at $\tau = \tau_D$ ($> \tau_H$), these bifurcated solutions are unstable because $(u, v)(x; \varepsilon)$ become still unstable for $\tau_H < \tau < \tau_D$. This observation implies that stable traveling pulse solutions never bifurcate from the branch of stable standing pulse solutions.

But in the papers [6] and [1], they introduced a three-component system as a phenomenological model of gas-discharge patterns and showed that the additional third component can stabilize standing pulse solutions and yield stable traveling pulse solutions. By using this three-component system, [15] and [16] showed rich dynamics numerically, which include pulse collision, pulse scattering, pulse annihilation among others.

In 2009, motivated by these works, Doelman et al [2] and [7] proposed the following three-component FitzHugh-Nagumo system with special scaling for small $\varepsilon > 0$:

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \beta w + \gamma), \\ \tau v_t = v_{xx} + u - v, \\ \theta w_t = D^2 w_{xx} + u - w, \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R} \quad (1.2)$$

where $0 < \varepsilon \ll 1$, $\tau, \theta > 0$, $D > 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$. This system seems to be the natural extension of (1.1) with the small inhibitors v and w in the first equation of (1.2). Furthermore they gave two cases numerically, the bifurcation of a stable traveling pulse from a stable standing pulse in the left panel and that of a stable standing breather from a stable standing pulse

in the right panel of Figure 1. In [2] and [7], they showed the existence and the stability of standing pulse solutions of (1.2) by using geometric singular perturbation method.

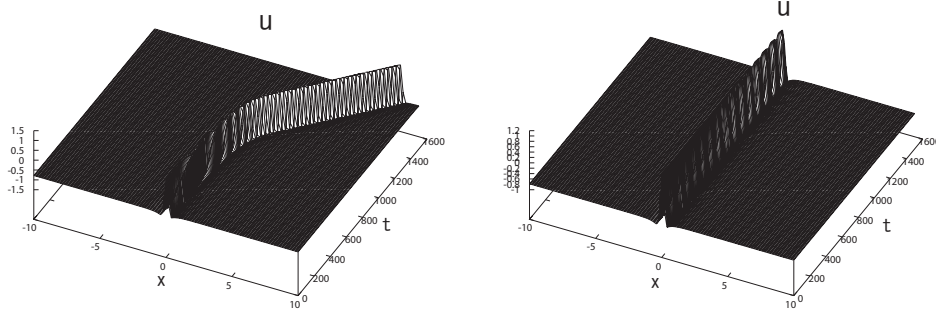


Figure 1: Numerical solutions of (1.2) for $(\alpha, \beta, \gamma, D, \theta, \varepsilon, \tau) = (6, 3, 4, 2, 10, 0.1, 120)$ in the left panel and for $(\alpha, \beta, \gamma, D, \theta, \varepsilon, \tau) = (6, 3, 4, 2, 23, 0.1, 80)$ in the right panel, respectively.

The aim of this paper is to show the same results as that in [2] and [7] by using analytic singular perturbation method. In Section 2, we construct standing pulse solutions of (1.2) with approximate solutions up to $O(\varepsilon^2)$, in which outer and inner approximations are included, by using analytic singular perturbation method. This information is of very importance for analyzing linearized eigenvalue problems. In Section 3, we solve two linearized eigenvalue problems depending on two types of destabilization, the out-of-phase and the in-phase modes. Based on the analytic singular perturbation method, we construct the Evans functions (algebraic equations with respect to unknown eigenvalues). In Section 4, we give the proof of lemmas. Finally in Section 5, we give a few comments on our results.

2. Existence of standing pulse solutions

Let us consider the three-component FitzHugh-Nagumo system

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \beta w + \gamma), \\ \tau v_t = v_{xx} + u - v, \\ \theta w_t = D^2 w_{xx} + u - w. \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R} \quad (2.1)$$

The constant solutions of (2.1) are given by the relation

$$\begin{cases} u - u^3 - \varepsilon(\alpha v + \beta w + \gamma) = 0, \\ u - v = 0, \\ u - w = 0, \end{cases}$$

which is reduced to $u^3 + (-1 + \varepsilon\alpha + \varepsilon\beta)u + \varepsilon\gamma = 0$. This equation has three roots and we write them as $u_0(\varepsilon), u_{\pm}(\varepsilon)$, where $u_0(\varepsilon) = O(\varepsilon)$, $u_{\pm}(\varepsilon) = \pm 1 + O(\varepsilon)$ ($\varepsilon \rightarrow 0$). We can find that $(u, v, w) = (u_{\pm}(\varepsilon), u_{\pm}(\varepsilon), u_{\pm}(\varepsilon))$ are stable and $(u, v, w) = (u_0(\varepsilon), u_0(\varepsilon), u_0(\varepsilon))$ is an unstable solution of (2.1). Then, the system (2.1) is called a bistable one. For example, $(u, v, w) = (u_-(\varepsilon), u_-(\varepsilon), u_-(\varepsilon))$ is asymptotically stable. If we give a suitable large local perturbation to this constant state, we can find that this state is destabilized and develops into a standing pulse solution (see Figure 2), which satisfies the stationary problem

$$\begin{cases} \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \beta w + \gamma) = 0, \\ v_{xx} + u - v = 0, \\ D^2 w_{xx} + u - w = 0, \\ (u, v, w)(\pm\infty) = (u_-(\varepsilon), u_-(\varepsilon), u_-(\varepsilon)). \end{cases} \quad x \in \mathbb{R} \quad (2.2)$$

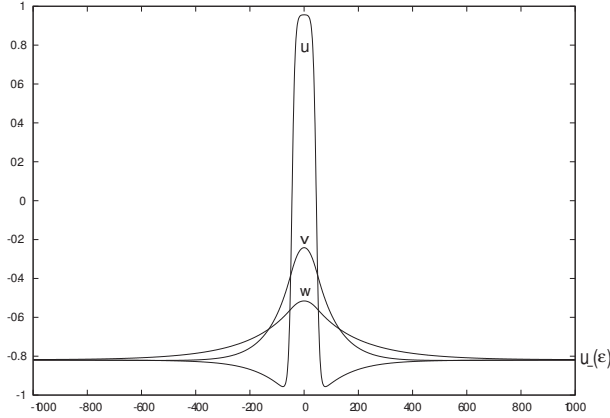


Figure 2: A standing pulse solution of (2.1)

Since this solution has a symmetric property at $x = 0$, it suffices for us

to consider the following problem on the half interval $[0, \infty)$:

$$\begin{cases} \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \beta w + \gamma) = 0, \\ v_{xx} + u - v = 0, \\ D^2 w_{xx} + u - w = 0, \\ (u_x, v_x, w_x)(0) = (0, 0, 0), \\ (u, v, w)(\infty) = (u_-(\varepsilon), u_-(\varepsilon), u_-(\varepsilon)). \end{cases} \quad x \in (0, \infty) \quad (2.3)$$

Since the highest derivative of the u component contains a small parameter ε in (2.3), we can expect that the component of u has a sharp transition layer. Then we define a position of the layer $x = l(\varepsilon)$ by $u(l(\varepsilon)) = 0$ and values of $a(\varepsilon)$ and $b(\varepsilon)$ by $v(l(\varepsilon)) = a(\varepsilon)$ and $w(l(\varepsilon)) = b(\varepsilon)$, respectively (see Figure 3). Moreover we divide $[0, \infty)$ into two parts $I_1 = [0, l(\varepsilon)]$ and

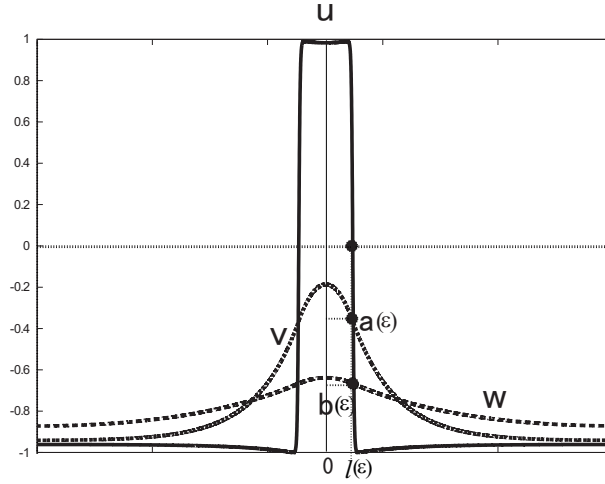


Figure 3: The layer position $l(\varepsilon)$ of a standing pulse solution for sufficiently small $\varepsilon > 0$

$I_2 = [l(\varepsilon), \infty)$ and write (2.3) as

$$\begin{cases} \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \beta w + \gamma) = 0, \\ v_{xx} + u - v = 0, \\ D^2 w_{xx} + u - w = 0, \\ (u_x, v_x, w_x)(0) = (0, 0, 0), \\ (u, v, w)(l(\varepsilon)) = (0, a(\varepsilon), b(\varepsilon)) \end{cases} \quad x \in (0, l(\varepsilon)) \quad (2.4)$$

and

$$\begin{cases} \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \beta w + \gamma) = 0, \\ v_{xx} + u - v = 0, \\ D^2 w_{xx} + u - w = 0, \\ (u, v, w)(l(\varepsilon)) = (0, a(\varepsilon), b(\varepsilon)), \\ (u, v, w)(\infty) = (u_-(\varepsilon), u_-(\varepsilon), u_-(\varepsilon)). \end{cases} \quad x \in (l(\varepsilon), \infty) \quad (2.5)$$

Here we note that $\ell(\varepsilon)$, $a(\varepsilon)$ and $b(\varepsilon)$ are unknown constants.

The standard classical singular perturbation process is as follows: First, to solve (2.4) and (2.5) independently under the assumption that

$$\begin{cases} l(\varepsilon) = l_0 + \varepsilon l_1 + \varepsilon^2 l_2 + \cdots, & (l_0 > 0) \\ a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots, \\ b(\varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \cdots \end{cases} \quad (2.6)$$

are known. Second, to match these solutions smoothly at $x = l(\varepsilon)$ and then determine these parameters $l(\varepsilon)$, $a(\varepsilon)$ and $b(\varepsilon)$. Here we emphasize that we need approximate solutions of (2.3) at least up to $O(\varepsilon^2)$ to examine the stability property of a standing pulse solution (see §3).

2.1. Solutions of (2.4) on the interval $I_1 = [0, \ell(\varepsilon)]$

Using the transformation $y = x/l(\varepsilon)$, we have

$$\begin{cases} \varepsilon^2 u_{yy} + l(\varepsilon)^2(u - u^3 - \varepsilon(\alpha v + \beta w + \gamma)) = 0, \\ v_{yy} + l(\varepsilon)^2(u - v) = 0, \\ D^2 w_{yy} + l(\varepsilon)^2(u - w) = 0, \\ (u_y, v_y, w_y)(0) = (0, 0, 0), \\ (u, v, w)(1) = (0, a(\varepsilon), b(\varepsilon)). \end{cases} \quad y \in (0, 1) \quad (2.7)$$

2.1.1. Construction of outer approximations of (2.7)

We begin with constructing outer approximations $(u, v, w)(y; \varepsilon)$ of the form

$$\begin{cases} u(y; \varepsilon) = U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y) + \cdots, \\ v(y; \varepsilon) = V_0(y) + \varepsilon V_1(y) + \varepsilon^2 V_2(y) + \cdots, \\ w(y; \varepsilon) = W_0(y) + \varepsilon W_1(y) + \varepsilon^2 W_2(y) + \cdots. \end{cases} \quad (2.8)$$

Substituting (2.6) and (2.8) into (2.7), we equate the coefficients of the same powers of ε .

$O(\varepsilon^0)$:

$$\begin{cases} U_0 - U_0^3 = 0, \\ V_0'' + l_0^2(U_0 - V_0) = 0, & y \in (0, 1) \\ D^2W_0'' + l_0^2(U_0 - W_0) = 0, \\ V_0'(0) = 0, \quad V_0(1) = a_0, \\ W_0'(0) = 0, \quad W_0(1) = b_0. \end{cases}$$

From Figure 3, we take $U_0 = 1$. Then V_0 satisfies

$$\begin{cases} V_0'' - l_0^2V_0 = -l_0^2, & y \in (0, 1) \\ V_0'(0) = 0, \quad V_0(1) = a_0 \end{cases}$$

and we easily have $V_0(y) = 1 + (a_0 - 1) \cosh(l_0 y) / \cosh(l_0)$. Similarly we also have $W_0(y) = 1 + (b_0 - 1) \cosh(l_0 y / D) / \cosh(l_0 / D)$.

$O(\varepsilon^1)$:

$$\begin{cases} (1 - 3U_0^2)U_1 - (\alpha V_0 + \beta W_0 + \gamma) = 0, \\ V_1'' + l_0^2(U_1 - V_1) + 2l_0l_1(U_0 - V_0) = 0, & y \in (0, 1) \\ D^2W_1'' + l_0^2(U_1 - W_1) + 2l_0l_1(U_0 - W_0) = 0, \\ V_1'(0) = 0, \quad V_1(1) = a_1, \\ W_1'(0) = 0, \quad W_1(1) = b_1. \end{cases}$$

By the first equation, we obtain $U_1(y) = -(\alpha V_0(y) + \beta W_0(y) + \gamma)/2$. The next lemma directly follows from the constant variation method.

Lemma 2.1. *A solution of the following boundary value problem:*

$$\begin{cases} V'' - l_0^2V = f(y), & y \in (0, 1) \\ V'(0) = 0, \quad V(1) = a \end{cases}$$

is given uniquely by $V(y) =$

$$\begin{aligned} & e^{l_0 y} \left\{ \frac{1}{2l_0} \int_1^y e^{-l_0 s} f(s) ds + \frac{e^{-l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) f(s) ds + \frac{a}{2 \cosh(l_0)} \right\} \\ & + e^{-l_0 y} \left\{ -\frac{1}{2l_0} \int_1^y e^{l_0 s} f(s) ds - \frac{e^{l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) f(s) ds + \frac{a}{2 \cosh(l_0)} \right\}. \end{aligned}$$

Using this lemma, we have

$$\begin{aligned}
V_1(y) &= e^{l_0 y} \left\{ \frac{1}{2l_0} \int_1^y e^{-l_0 s} f_1(s) ds + \frac{e^{-l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) f_1(s) ds \right. \\
&\quad \left. + \frac{a_1}{2 \cosh(l_0)} \right\} + e^{-l_0 y} \left\{ -\frac{1}{2l_0} \int_1^y e^{l_0 s} f_1(s) ds \right. \\
&\quad \left. - \frac{e^{l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) f_1(s) ds + \frac{a_1}{2 \cosh(l_0)} \right\}, \\
W_1(y) &= e^{l_0 y/D} \left\{ \frac{D}{2l_0} \int_1^y e^{-l_0 s/D} g_1(s) ds \right. \\
&\quad \left. + \frac{De^{-l_0/D}}{2l_0 \cosh(l_0/D)} \int_0^1 \cosh(l_0 s/D) g_1(s) ds + \frac{b_1}{2 \cosh(l_0/D)} \right\} \\
&\quad + e^{-l_0 y/D} \left\{ -\frac{D}{2l_0} \int_1^y e^{l_0 s/D} g_1(s) ds \right. \\
&\quad \left. - \frac{De^{l_0/D}}{2l_0 \cosh(l_0/D)} \int_0^1 \cosh(l_0 s/D) g_1(s) ds + \frac{b_1}{2 \cosh(l_0/D)} \right\},
\end{aligned}$$

where $f_1(y) = -l_0^2 U_1(y) - 2l_0 l_1(1 - V_0(y)) = l_0^2(\alpha V_0(y) + \beta W_0(y) + \gamma)/2 - 2l_0 l_1(1 - V_0(y))$ and $g_1(y) = -\{l_0^2 U_1(y) + 2l_0 l_1(1 - W_0(y))\}/D^2 = l_0^2(\alpha V_0(y) + \beta W_0(y) + \gamma)/(2D^2) - 2l_0 l_1(1 - W_0(y))/D^2$.

$O(\varepsilon^2)$:

$$\begin{cases} (1 - 3U_0^2)U_2 - 3U_0U_1^2 - (\alpha V_1 + \beta W_1) = 0, \\ V_2'' + l_0^2(U_2 - V_2) + 2l_0 l_1(U_1 - V_1) + (l_1^2 + 2l_0 l_2)(U_0 - V_0) = 0, \quad y \in (0, 1) \\ D^2 W_2'' + l_0^2(U_2 - W_2) + 2l_0 l_1(U_1 - W_1) \\ \quad + (l_1^2 + 2l_0 l_2)(U_0 - W_0) = 0, \\ V_2'(0) = 0, \quad V_2(1) = a_2 - \psi_0(0), \\ W_2'(0) = 0, \quad W_2(1) = b_2 - \rho_0(0), \end{cases}$$

where $\psi_0(0)$ and $\rho_0(0)$ will be determined later (see §2.1.2). Similarly to the case of $O(\varepsilon^1)$, we have

$$\begin{aligned}
U_2(y) &= -(3U_1^2(y) + \alpha V_1(y) + \beta W_1(y))/2, \\
V_2(y) &= e^{l_0 y} \left\{ \frac{1}{2l_0} \int_1^y e^{-l_0 s} f_2(s) ds + \frac{e^{-l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) f_2(s) ds \right. \\
&\quad \left. + \frac{a_2 - \psi_0(0)}{2 \cosh(l_0)} \right\} + e^{-l_0 y} \left\{ -\frac{1}{2l_0} \int_1^y e^{l_0 s} f_2(s) ds \right. \\
&\quad \left. - \frac{e^{l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) f_2(s) ds + \frac{a_2 - \psi_0(0)}{2 \cosh(l_0)} \right\}, \\
W_2(y) &= e^{l_0 y/D} \left\{ \frac{D}{2l_0} \int_1^y e^{-l_0 s/D} g_2(s) ds \right. \\
&\quad \left. + \frac{De^{-l_0/D}}{2l_0 \cosh(l_0/D)} \int_0^1 \cosh(l_0 s/D) g_2(s) ds + \frac{b_2 - \rho_0(0)}{2 \cosh(l_0/D)} \right\} \\
&\quad + e^{-l_0 y/D} \left\{ -\frac{D}{2l_0} \int_1^y e^{l_0 s/D} g_2(s) ds \right. \\
&\quad \left. - \frac{De^{l_0/D}}{2l_0 \cosh(l_0/D)} \int_0^1 \cosh(l_0 s/D) g_2(s) ds + \frac{b_2 - \rho_0(0)}{2 \cosh(l_0/D)} \right\},
\end{aligned}$$

where $f_2(y) = -l_0^2 U_2(y) - 2l_0 l_1 (U_1(y) - V_1(y)) - (l_1^2 + 2l_0 l_2)(1 - V_0(y)) = l_0^2(3U_1^2 + \alpha V_1 + \beta W_1)/2 - 2l_0 l_1 (U_1(y) - V_1(y)) - (l_1^2 + 2l_0 l_2)(1 - V_0(y))$ and $g_2(y) = -\{l_0^2 U_2(y) - 2l_0 l_1 (U_1(y) - W_1(y)) - (l_1^2 + 2l_0 l_2)(1 - W_0(y))\} / D^2 = l_0^2(3U_1^2 + \alpha V_1 + \beta W_1)/(2D^2) - \{2l_0 l_1 (U_1(y) - W_1(y)) + (l_1^2 + 2l_0 l_2)(1 - W_0(y))\} / D^2$.

2.1.2. Construction of inner approximations of (2.7)

Since the u component of the outer approximations constructed in §2.1.1 does not satisfy the boundary condition at $y = 1$, we have to modify this defect. Hence, we introduce the stretched variable $\xi = (y - 1)/\varepsilon$ and look for inner approximations ϕ_i, ψ_i, ρ_i ($i = 0, 1, 2$) of the following form in a neighborhood of $y = 1$:

$$\begin{cases} u(y) &= U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y) + \cdots \\ &\quad + \phi_0(\frac{y-1}{\varepsilon}) + \varepsilon \phi_1(\frac{y-1}{\varepsilon}) + \varepsilon^2 \phi_2(\frac{y-1}{\varepsilon}) + \cdots, \\ v(y) &= V_0(y) + \varepsilon V_1(y) + \varepsilon^2 V_2(y) + \cdots \\ &\quad + \varepsilon^2 \psi_0(\frac{y-1}{\varepsilon}) + \varepsilon^3 \psi_1(\frac{y-1}{\varepsilon}) + \varepsilon^4 \psi_2(\frac{y-1}{\varepsilon}) + \cdots, \\ w(y) &= W_0(y) + \varepsilon W_1(y) + \varepsilon^2 W_2(y) + \cdots \\ &\quad + \varepsilon^2 \rho_0(\frac{y-1}{\varepsilon}) + \varepsilon^3 \rho_1(\frac{y-1}{\varepsilon}) + \varepsilon^4 \rho_2(\frac{y-1}{\varepsilon}) + \cdots, \end{cases} \quad (2.9)$$

so that $(u, v, w)(y)$ satisfies the boundary condition at $y = 1$. Substituting (2.9) into (2.7) and using $\xi = (y - 1)/\varepsilon$, we equate the coefficients of the same power of ε .

$O(\varepsilon^0)$:

$$\begin{cases} \ddot{\phi}_0 - l_0^2 \phi_0(\phi_0 + 1)(\phi_0 + 2) = 0, \\ \ddot{\psi}_0 + l_0^2 \phi_0 = 0, \\ D^2 \ddot{\rho}_0 + l_0^2 \phi_0 = 0, \\ \phi_0(-\infty) = 0, \phi_0(0) = -1, \\ \psi_0(-\infty) = 0, \dot{\psi}_0(-\infty) = 0, \\ \rho_0(-\infty) = 0, \dot{\rho}_0(-\infty) = 0. \end{cases} \quad \xi \in (-\infty, 0) \quad (2.10)$$

Here we use the notations $\dot{\phi} = \frac{d\phi}{d\xi}$ and $\ddot{\phi} = \frac{d^2\phi}{d\xi^2}$. From the first and fourth equations, we have $\phi_0(\xi) = -1 - \tanh(l_0 \xi / \sqrt{2})$. And then $\psi_0(\xi) = -l_0^2 \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \phi_0(\zeta) d\zeta d\eta$ and $\rho_0(\xi) = -l_0^2 \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \phi_0(\zeta) d\zeta d\eta / D^2$.

$O(\varepsilon^1) :$

$$\left\{ \begin{array}{l} \ddot{\phi}_1 + l_0^2 \{ U_1(1) + \phi_1 - 3(U_0(1) + \phi_0)^2(U_1(1) + \phi_1) \\ \quad - (\alpha V_0(1) + \beta W_0(1) + \gamma) \} \\ \quad + 2l_0 l_1 \{ U_0(1) + \phi_0 - (U_0(1) + \phi_0)^3 \} = 0, \\ \ddot{\psi}_1 + l_0^2 \phi_1 + 2l_0 l_1 \phi_0 = 0, \quad \xi \in (-\infty, 0) \\ D^2 \ddot{\rho}_1 + l_0^2 \phi_1 + 2l_0 l_1 \phi_0 = 0, \\ \phi_1(-\infty) = 0, \quad \phi_1(0) = -U_1(1), \\ \psi_1(-\infty) = 0, \quad \dot{\psi}_1(-\infty) = 0, \\ \rho_1(-\infty) = 0, \quad \dot{\rho}_1(-\infty) = 0. \end{array} \right. \quad (2.11)$$

Since ϕ_1 satisfies

$$\left\{ \begin{array}{l} \ddot{\phi}_1 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\phi_1 = -3l_0^2(\alpha a_0 + \beta b_0 + \gamma)\phi_0(\phi_0 + 2)/2 \\ \quad + 2l_0 l_1 \phi_0(\phi_0 + 1)(\phi_0 + 2), \quad \xi \in (-\infty, 0) \\ \phi_1(-\infty) = 0, \quad \phi_1(0) = \frac{1}{2}(\alpha a_0 + \beta b_0 + \gamma), \end{array} \right. \quad (2.12)$$

applying the method of constant variation to (2.12) we obtain

$$\begin{aligned} \phi_1(\xi) &= \frac{1}{2}(\alpha a_0 + \beta b_0 + \gamma)\dot{\phi}_0(\xi)/\dot{\phi}_0(0) + \dot{\phi}_0(\xi) \int_{\xi}^0 (\dot{\phi}_0(\eta))^{-2} \int_{-\infty}^{\eta} \dot{\phi}_0(\zeta) \\ &\quad \times \{ 3l_0^2(\alpha a_0 + \beta b_0 + \gamma)\phi_0(\phi_0 + 2)/2 - 2l_0 l_1 \phi_0(\phi_0 + 1)(\phi_0 + 2) \} d\zeta d\eta. \end{aligned}$$

Furthermore we have

$$\begin{aligned} \psi_1(\xi) &= - \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \{ l_0^2 \phi_1(\zeta) + 2l_0 l_1 \phi_0(\zeta) \} d\zeta d\eta, \\ \rho_1(\xi) &= - \frac{1}{D^2} \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \{ l_0^2 \phi_1(\zeta) + 2l_0 l_1 \phi_0(\zeta) \} d\zeta d\eta. \end{aligned}$$

$O(\varepsilon^2)$:

$$\left\{ \begin{array}{l} \ddot{\phi}_2 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\phi_2 = -l_0^2 \{ (U_1'(1)\xi + U_2(1) \\ - 3(1 + \phi_0)^2(U_1'(1)\xi + U_2(1)) - 3(1 + \phi_0)(U_1(1) + \phi_1)^2 \\ - \alpha V_0'(1)\xi - \alpha V_1(1) - \beta W_0'(1)\xi - \beta W_1(1) \} - 2l_0l_1 \{ U_1(1) + \phi_1 \\ - 3(1 + \phi_0)^2(U_1(1) + \phi_1) - (\alpha V_0(1) + \beta W_0(1) + \gamma) \} \\ + (2l_0l_2 + l_1^2)\phi_0(\phi_0 + 1)(\phi_0 + 2), \\ \ddot{\psi}_2 + l_0^2(\phi_2 - \psi_0) + 2l_0l_1\phi_1 + (l_1^2 + 2l_0l_2)\phi_0 = 0, \quad \xi \in (-\infty, 0) \\ D^2\ddot{\rho}_2 + l_0^2(\phi_2 - \rho_0) + 2l_0l_1\phi_1 + (l_1^2 + 2l_0l_2)\phi_0 = 0, \\ \phi_2(-\infty) = 0, \quad \phi_2(0) = (3U_1^2(1) + \alpha a_1 + \beta b_1)/2, \\ \psi_2(-\infty) = 0, \quad \dot{\psi}_2(-\infty) = 0, \\ \rho_2(-\infty) = 0, \quad \dot{\rho}_2(-\infty) = 0. \end{array} \right. \quad (2.13)$$

Similarly to the case of $O(\varepsilon^1)$, we have

$$\begin{aligned} \phi_2(\xi) &= \frac{1}{2}(3U_1^2(1) + \alpha a_1 + \beta b_1)\dot{\phi}_0(\xi)/\dot{\phi}_0(0) - \dot{\phi}_0(\xi) \int_{\xi}^0 \left(\dot{\phi}_0(\eta) \right)^{-2} \int_{-\infty}^{\eta} \dot{\phi}_0(\zeta) \\ &\times [l_0^2 \{ (U_1'(1)\zeta + U_2(1))(2 + 6\phi_0 + 3\phi_0^2) + 3(1 + \phi_0)(U_1(1) + \phi_1)^2 + \alpha V_0'(1)\zeta \\ &+ \beta W_0'(1)\zeta + \alpha a_1 + \beta b_1 \} - 2l_0l_1 \{ U_1(1) + \phi_1 - 3(1 + \phi_0)^2(U_1(1) + \phi_1) \\ &- (\alpha a_0 + \beta b_0 + \gamma) \} + (2l_0l_2 + l_1^2)\phi_0(\phi_0 + 1)(\phi_0 + 2)] d\zeta d\eta, \\ \psi_2(\xi) &= - \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \{ l_0^2(\phi_2 - \psi_0) + 2l_0l_1\phi_1 + (l_1^2 + 2l_0l_2)\phi_0 \} d\zeta d\eta, \\ \rho_2(\xi) &= - \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \{ l_0^2(\phi_2 - \rho_0) + 2l_0l_1\phi_1 + (l_1^2 + 2l_0l_2)\phi_0 \} d\zeta d\eta / D^2. \end{aligned}$$

2.1.3. Exact solutions of (2.7)

Using the above outer and inner approximations, we can construct uniform approximations of (2.7) up to order $O(\varepsilon^2)$, which take the form

$$\left\{ \begin{array}{l} U(y; \varepsilon) = U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y) \\ \quad + \theta(y) \left(\phi_0\left(\frac{y-1}{\varepsilon}\right) + \varepsilon \phi_1\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^2 \phi_2\left(\frac{y-1}{\varepsilon}\right) \right), \\ V(y; \varepsilon) = V_0(y) + \varepsilon V_1(y) + \varepsilon^2 V_2(y) \\ \quad + \theta(y) \left(\varepsilon^2 \psi_0\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^3 \psi_1\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^4 \psi_2\left(\frac{y-1}{\varepsilon}\right) \right), \\ W(y; \varepsilon) = W_0(y) + \varepsilon W_1(y) + \varepsilon^2 W_2(y) \\ \quad + \theta(y) \left(\varepsilon^2 \rho_0\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^3 \rho_1\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^4 \rho_2\left(\frac{y-1}{\varepsilon}\right) \right), \end{array} \right.$$

where $\theta(y) \in C^\infty[0, 1]$ satisfies

$$\theta(y) = 0, \quad 0 \leq y \leq \frac{1}{2}; \quad 0 \leq \theta(y) \leq 1, \quad \frac{1}{2} \leq y \leq \frac{3}{4}; \quad \theta(y) = 1, \quad \frac{3}{4} \leq y \leq 1.$$

Moreover we assume that $l(\varepsilon)$, $a(\varepsilon)$ and $b(\varepsilon)$ are given in the following form:

$$\begin{cases} l(\varepsilon) = l_0 + \varepsilon l_1 + \varepsilon^2 l_2 + \varepsilon^3 \tilde{l}(\varepsilon), \\ a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 \tilde{a}(\varepsilon), \\ b(\varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 \tilde{b}(\varepsilon). \end{cases}$$

We can easily find that $(U, V, W)(y; \varepsilon)$ satisfies the boundary condition of (2.7) at $y = 0$ exactly, but at $y = 1$ it becomes

$$\begin{cases} U(1; \varepsilon) = 0, \\ V(1; \varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 \psi_1(0) + \varepsilon^4 \psi_2(0), \\ W(1; \varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 \rho_1(0) + \varepsilon^4 \rho_2(0). \end{cases}$$

So we modify $(U, V, W)(y; \varepsilon)$ a little to satisfy the boundary condition at $y = 1$ exactly and add the remainder term $(\varepsilon^2 \tilde{U}, \varepsilon^2 \tilde{V}, \varepsilon^2 \tilde{W})$, we seek exact solutions of (2.7) of the form

$$\begin{cases} u(y; \varepsilon) = U(y; \varepsilon) + \varepsilon^2 \tilde{U}(y; \varepsilon), \\ v(y; \varepsilon) = V(y; \varepsilon) + a^*(\varepsilon) + \varepsilon^2 \tilde{V}(y; \varepsilon), \\ w(y; \varepsilon) = W(y; \varepsilon) + b^*(\varepsilon) + \varepsilon^2 \tilde{W}(y; \varepsilon), \end{cases} \quad (2.14)$$

where $a^*(\varepsilon) = \varepsilon^3(\tilde{a}(\varepsilon) - \psi_1(0) - \varepsilon \psi_2(0))$ and $b^*(\varepsilon) = \varepsilon^3(\tilde{b}(\varepsilon) - \rho_1(0) - \varepsilon \rho_2(0))$. Substituting (2.14) into (2.7), we obtain

$$\begin{cases} \varepsilon^2(U_{yy} + \varepsilon^2 \tilde{U}_{yy}) + l(\varepsilon)^2 \left[U + \varepsilon^2 \tilde{U} - (U + \varepsilon^2 \tilde{U})^3 \right. \\ \quad \left. - \varepsilon \left(\alpha(V + a^* + \varepsilon^2 \tilde{V}) + \beta(W + b^* + \varepsilon^2 \tilde{W}) + \gamma \right) \right] = 0, \\ V_{yy} + \varepsilon^2 \tilde{V}_{yy} + l(\varepsilon)^2 \left(U + \varepsilon^2 \tilde{U} - (V + a^* + \varepsilon^2 \tilde{V}) \right) = 0, \\ D^2(W_{yy} + \varepsilon^2 \tilde{W}_{yy}) + l(\varepsilon)^2 \left(U + \varepsilon^2 \tilde{U} - (W + b^* + \varepsilon^2 \tilde{W}) \right) = 0, \\ (\tilde{U}_y, \tilde{V}_y, \tilde{W}_y)(0; \varepsilon) = (0, 0, 0), \quad (\tilde{U}, \tilde{V}, \tilde{W})(1; \varepsilon) = (0, 0, 0). \end{cases} \quad (2.15)$$

Then, we define the following operator $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = (T_1, T_2, T_3)(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon)$:

$$\begin{cases} T_1(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) \equiv \varepsilon^2 \tilde{U}_{yy} + l(\varepsilon)^2 (\tilde{U} - 3U^2 \tilde{U} - 3\varepsilon^2 U \tilde{U}^2 - \varepsilon^4 \tilde{U}^3 \\ \quad - \varepsilon \alpha \tilde{V} - \varepsilon \beta \tilde{W}) + \frac{1}{\varepsilon^2} U_{yy} \\ \quad + l(\varepsilon)^2 \{ U - U^3 - \varepsilon (\alpha(V + a^*) - \beta(W + b^*) + \gamma) \}, \\ T_2(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) \equiv \tilde{V}_{yy} + l(\varepsilon)^2 (\tilde{U} - \tilde{V}) + \frac{1}{\varepsilon^2} [V_{yy} + l(\varepsilon)^2 (U - V - a^*)], \\ T_3(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) \equiv D^2 \tilde{W}_{yy} + l(\varepsilon)^2 (\tilde{U} - \tilde{W}) \\ \quad + \frac{1}{\varepsilon^2} [D^2 W_{yy} + l(\varepsilon)^2 (U - W - b^*)] \end{cases}$$

from $X \equiv A_\varepsilon \times B \times B$ to $Y \equiv C[0, 1] \times C[0, 1] \times C[0, 1]$, where $A_\varepsilon \equiv \{\tilde{U} \in C_\varepsilon^2[0, 1] \mid \tilde{U}_y(0) = 0, \tilde{U}(1) = 0\}$, $B \equiv \{\tilde{V} \in C^2[0, 1] \mid \tilde{V}_y(0) = 0, \tilde{V}(1) = 0\}$ and

$$C_\varepsilon^2[0, 1] \equiv \left\{ u \in C^2[0, 1] \mid \|u\|_{C_\varepsilon^2[0, 1]} \equiv \sup_{y \in [0, 1]} \sum_{i=0}^2 \left| \left(\varepsilon \frac{d}{dy} \right)^i u(y) \right| < +\infty \right\}.$$

We find that $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon)$ is the continuously differentiable operator and (2.15) is equivalent to $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = 0$.

Lemma 2.2. *There exist $\varepsilon_0 > 0$ and a positive constant C such that for any $\varepsilon \in (0, \varepsilon_0)$ the followings hold:*

- (i) $\|T_t(t_1; \varepsilon) - T_t(t_2; \varepsilon)\|_{X \rightarrow Y} \leq C \|t_1 - t_2\|_X$ for any $t_1, t_2 \in X$,
- (ii) $\|T(0; \varepsilon)\|_Y \leq C\varepsilon$,
- (iii) $\|T_t^{-1}(0; \varepsilon)\|_{Y \rightarrow X} \leq C$.

The proof is given in §4.

By this lemma, we can apply the Implicit Function Theorem to $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = 0$ and find that $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = 0$ has solutions $(\tilde{U}(\varepsilon), \tilde{V}(\varepsilon), \tilde{W}(\varepsilon)) \in X$ satisfying $\|(\tilde{U}(\varepsilon), \tilde{V}(\varepsilon), \tilde{W}(\varepsilon))\|_X = o(1)$ ($\varepsilon \rightarrow 0$). Thus we have the exact solutions of (2.7) on $[0, 1]$

$$\begin{cases} u(y; \varepsilon) = U(y; \varepsilon) + \varepsilon^2 \tilde{U}(y; \varepsilon), \\ v(y; \varepsilon) = V(y; \varepsilon) + a^*(\varepsilon) + \varepsilon^2 \tilde{V}(y; \varepsilon), \\ w(y; \varepsilon) = W(y; \varepsilon) + b^*(\varepsilon) + \varepsilon^2 \tilde{W}(y; \varepsilon), \end{cases} \quad (2.16)$$

which implies that (2.4) has the following exact solutions on $I_1 = [0, l(\varepsilon)]$:

$$\begin{cases} u(x; \varepsilon) = U(x/l(\varepsilon); \varepsilon) + \varepsilon^2 \tilde{U}(x/l(\varepsilon); \varepsilon), \\ v(x; \varepsilon) = V(x/l(\varepsilon); \varepsilon) + a^*(\varepsilon) + \varepsilon^2 \tilde{V}(x/l(\varepsilon); \varepsilon), \\ w(x; \varepsilon) = W(x/l(\varepsilon); \varepsilon) + b^*(\varepsilon) + \varepsilon^2 \tilde{W}(x/l(\varepsilon); \varepsilon). \end{cases} \quad (2.17)$$

2.2. Solutions of (2.5) on the interval $I_2 = [l(\varepsilon), \infty)$

Using the transformation $y = x - l(\varepsilon)$ in (2.5), we have

$$\begin{cases} \varepsilon^2 u_{yy} + u - u^3 - \varepsilon(\alpha v + \beta w + \gamma) = 0, \\ v_{yy} + u - v = 0, \\ D^2 w_{yy} + u - w = 0, \\ (u, v, w)(0) = (0, a(\varepsilon), b(\varepsilon)), \\ (u, v, w)(\infty) = (u_-(\varepsilon), u_-(\varepsilon), u_-(\varepsilon)), \end{cases} \quad y \in (0, \infty) \quad (2.18)$$

where $u_-(\varepsilon)$ has the following asymptotic expansion:

$$\begin{aligned} u_-(\varepsilon) &= u_0^* + u_1^* \varepsilon + u_2^* \varepsilon^2 + \cdots \\ &\equiv -1 + \frac{1}{2}(\alpha + \beta - \gamma)\varepsilon + \frac{1}{8}(\alpha + \beta - \gamma)(\alpha + \beta - 3\gamma)\varepsilon^2 + \cdots. \end{aligned}$$

2.2.1. Construction of outer approximations of (2.18)

We begin with constructing outer approximations $(u, v, w)(y; \varepsilon)$ of the form

$$\begin{cases} u(y) = U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y) + \cdots, \\ v(y) = V_0(y) + \varepsilon V_1(y) + \varepsilon^2 V_2(y) + \cdots, \\ w(y) = W_0(y) + \varepsilon W_1(y) + \varepsilon^2 W_2(y) + \cdots. \end{cases} \quad (2.19)$$

Substituting (2.19) into (2.18), we equate the coefficients of the same powers of ε .

$O(\varepsilon^0)$:

$$\begin{cases} U_0 - U_0^3 = 0, \\ V_0'' + U_0 - V_0 = 0, & y \in (0, \infty) \\ D^2 W_0'' + U_0 - W_0 = 0, \\ V_0(0) = a_0, \quad V_0(\infty) = u_0^*, \\ W_0(0) = b_0, \quad W_0(\infty) = u_0^*. \end{cases}$$

From Figure 3, we take $U_0 = -1$. Then V_0 satisfies

$$\begin{cases} V_0'' - V_0 = 1, & y \in (0, \infty) \\ V_0(0) = a_0, \quad V_0(\infty) = u_0^* = -1. \end{cases}$$

We easily find $V_0(y) = -1 + (a_0 + 1)e^{-y}$. Similarly we also have $W_0(y) = -1 + (b_0 + 1)e^{-y/D}$.

$O(\varepsilon^1)$:

$$\begin{cases} (1 - 3U_0^2)U_1 - (\alpha V_0 + \beta W_0 + \gamma) = 0, \\ V_1'' + U_1 - V_1 = 0, & y \in (0, \infty) \\ D^2 W_1'' + U_1 - W_1 = 0, \\ V_1(0) = a_1, \quad V_1(\infty) = u_1^*, \\ W_1(0) = b_1, \quad W_1(\infty) = u_1^*. \end{cases}$$

By the first equation, we have $U_1(y) = -(\alpha V_0(y) + \beta W_0(y) + \gamma)/2$. Applying the method of constant variation, we have

$$\begin{aligned} V_1(y) &= u_1^* + e^{-y}(a_1 - u_1^*) - e^{-y} \int_0^y e^{2x} \int_x^\infty e^{-s} f_3(s) ds dx, \\ W_1(y) &= u_1^* + e^{-y/D}(b_1 - u_1^*) - \frac{1}{D^2} e^{-y/D} \int_0^y e^{2x/D} \int_x^\infty e^{-s/D} f_3(s) ds dx, \end{aligned}$$

where $f_3(y) = -U_1(y) + u_1^* = (\alpha(V_0(y) + 1) + \beta(W_0(y) + 1))/2$.

$O(\varepsilon^2)$:

$$\begin{cases} (1 - 3U_0^2)U_2 - 3U_0U_1^2 - (\alpha V_1 + \beta W_1) = 0, \\ V_2'' + U_2 - V_2 = 0, \\ D^2W_2'' + U_2 - W_2 = 0, \\ V_2(0) = a_2 - \psi_0(0), \quad V_2(\infty) = u_2^*, \\ W_2(0) = b_2 - \rho_0(0), \quad W_2(\infty) = u_2^*, \end{cases} \quad y \in (0, \infty)$$

where $\psi_0(0)$ and $\rho_0(0)$ will be determined later (see §2.2.2). Similarly to the case of $O(\varepsilon^1)$, we have $U_2(y) = -(-3U_1^2(y) + \alpha V_1(y) + \beta W_1(y))/2$,

$$\begin{aligned} V_2(y) &= u_2^* + e^{-y}(a_2 - \psi_0(0) - u_2^*) - e^{-y} \int_0^y e^{2x} \int_x^\infty e^{-s} f_4(s) ds dx, \\ W_2(y) &= u_2^* + e^{-y/D}(b_2 - \rho_0(0) - u_2^*) \\ &\quad - \frac{1}{D^2} e^{-y/D} \int_0^y e^{2x/D} \int_x^\infty e^{-s/D} f_4(s) ds dx, \end{aligned}$$

where $f_4(y) = -U_2(y) + u_2^* = (-3U_1^2(y) + \alpha V_1(y) + \beta W_1(y))/2 + u_2^*$.

2.2.2. Construction of inner approximations of (2.18)

Since the u component of the outer approximations constructed in §2.2.1 does not satisfy the boundary condition at $y = 0$, we have to modify this defect. Hence, we introduce the stretched variable $\xi = y/\varepsilon$ and look for inner approximations ϕ_i, ψ_i, ρ_i ($i = 0, 1, 2$) of the following form in a neigh-

borhood of $y = 0$:

$$\left\{ \begin{array}{lcl} u(y) & = & U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y) + \cdots \\ & & + \phi_0(\frac{y}{\varepsilon}) + \varepsilon \phi_1(\frac{y}{\varepsilon}) + \varepsilon^2 \phi_2(\frac{y}{\varepsilon}) + \cdots, \\ v(y) & = & V_0(y) + \varepsilon V_1(y) + \varepsilon^2 V_2(y) + \cdots \\ & & + \varepsilon^2 \psi_0(\frac{y}{\varepsilon}) + \varepsilon^3 \psi_1(\frac{y}{\varepsilon}) + \varepsilon^4 \psi_2(\frac{y}{\varepsilon}) + \cdots, \\ w(y) & = & W_0(y) + \varepsilon W_1(y) + \varepsilon^2 W_2(y) + \cdots \\ & & + \varepsilon^2 \rho_0(\frac{y}{\varepsilon}) + \varepsilon^3 \rho_1(\frac{y}{\varepsilon}) + \varepsilon^4 \rho_2(\frac{y}{\varepsilon}) + \cdots, \end{array} \right. \quad (2.20)$$

so that $(u, v, w)(y)$ satisfies the boundary condition at $y = 0$. Substituting (2.20) into (2.18) and using $\xi = y/\varepsilon$, we equate the coefficients of the same powers of ε .

$O(\varepsilon^0)$:

$$\left\{ \begin{array}{l} \ddot{\phi}_0 - \phi_0(\phi_0 - 1)(\phi_0 - 2) = 0, \\ \ddot{\psi}_0 + \phi_0 = 0, \\ D^2 \ddot{\rho}_0 + \phi_0 = 0, \\ \phi_0(0) = 1, \phi_0(\infty) = 0, \\ \psi_0(\infty) = 0, \dot{\psi}_0(\infty) = 0, \\ \rho_0(\infty) = 0, \dot{\rho}_0(\infty) = 0. \end{array} \right. \quad \xi \in (0, \infty)$$

From the first and fourth equations, we obtain $\phi_0(\xi) = 1 - \tanh(\xi/\sqrt{2})$. And then we have $\psi_0(\xi) = -\int_\xi^\infty \int_\eta^\infty \phi_0(\zeta) d\zeta d\eta$, $\rho_0(\xi) = -\int_\xi^\infty \int_\eta^\infty \phi_0(\zeta) d\zeta d\eta / D^2$.

$O(\varepsilon^1)$:

$$\left\{ \begin{array}{l} \ddot{\phi}_1 + U_1(0) + \phi_1 - 3(U_0(0) + \phi_0)^2(U_1(0) + \phi_1) \\ \quad - (\alpha V_0(0) + \beta W_0(0) + \gamma) = 0, \\ \ddot{\psi}_1 + \phi_1 = 0, \\ D^2 \ddot{\rho}_1 + \phi_1 = 0, \\ \phi_1(0) = -U_1(0), \phi_1(\infty) = 0, \\ \psi_1(\infty) = 0, \dot{\psi}_1(\infty) = 0, \\ \rho_1(\infty) = 0, \dot{\rho}_1(\infty) = 0. \end{array} \right. \quad \xi \in (0, \infty)$$

Since ϕ_1 satisfies

$$\left\{ \begin{array}{l} \ddot{\phi}_1 - (2 - 6\phi_0 + 3\phi_0^2)\phi_1 = -3(\alpha a_0 + \beta b_0 + \gamma)\phi_0(\phi_0 - 2)/2, \\ \phi_1(0) = \frac{1}{2}(\alpha a_0 + \beta b_0 + \gamma), \phi_1(\infty) = 0, \end{array} \right. \quad \xi \in (0, \infty)$$

applying the method of constant variation, we obtain

$$\begin{aligned}\phi_1(\xi) &= \frac{1}{2}(\alpha a_0 + \beta b_0 + \gamma)\dot{\phi}_0(\xi)/\dot{\phi}_0(0) \\ &+ \frac{3}{2}(\alpha a_0 + \beta b_0 + \gamma)\dot{\phi}_0(\xi) \int_0^\xi (\dot{\phi}_0(\eta))^{-2} \int_\eta^\infty \dot{\phi}_0(\zeta)\phi_0(\phi_0 - 2)d\zeta d\eta.\end{aligned}$$

Then we have $\psi_1(\xi) = -\int_\xi^\infty \int_\eta^\infty \phi_1(\zeta)d\zeta d\eta$, $\rho_1(\xi) = -\int_\xi^\infty \int_\eta^\infty \phi_1(\zeta)d\zeta d\eta/D^2$.
 $O(\varepsilon^2)$:

$$\left\{ \begin{array}{l} \ddot{\phi}_2 - (2 - 6\phi_0 + 3\phi_0^2)\phi_2 = -\left\{ \frac{3}{2}(\alpha a_1 + \beta b_1)\phi_0(\phi_0 - 2) \right. \\ \quad - \xi U_1'(0)(2 - 6\phi_0 + 3\phi_0^2) - \frac{3}{2}U_1^2(0)(2 - 6\phi_0 + 3\phi_0^2) \\ \quad \left. - 3(\phi_0 - 1)(U_1(0) + \phi_1)^2 - \alpha \xi V_0'(0) - \beta \xi W_0'(0) \right\}, \quad \xi \in (0, \infty) \\ \ddot{\psi}_2 + \phi_2 - \psi_0 = 0, \\ D^2\ddot{\rho}_2 + \phi_2 - \rho_0 = 0, \\ \phi_2(0) = (-3U_1^2(0) + \alpha a_1 + \beta b_1)/2, \quad \phi_2(\infty) = 0, \\ \psi_2(\infty) = 0, \quad \dot{\psi}_2(\infty) = 0, \\ \rho_2(\infty) = 0, \quad \dot{\rho}_2(\infty) = 0. \end{array} \right.$$

Similarly to the case of $O(\varepsilon^1)$, we have

$$\begin{aligned}\phi_2(\xi) &= \frac{1}{2}(-3U_1^2(0) + \alpha a_1 + \beta b_1)\dot{\phi}_0(\xi)/\dot{\phi}_0(0) \\ &+ \dot{\phi}_0(\xi) \int_0^\xi (\dot{\phi}_0(\eta))^{-2} \int_\eta^\infty \dot{\phi}_0(\zeta) \left\{ \frac{3}{2}(\alpha a_1 + \beta b_1) \right. \\ &\times \phi_0(\phi_0 - 2) - \zeta U_1'(0)(2 - 6\phi_0 + 3\phi_0^2) \\ &- \frac{3}{2}U_1^2(0)(2 - 6\phi_0 + 3\phi_0^2) - 3(\phi_0 - 1)(U_1(0) + \phi_1)^2 \\ &\left. - \alpha \zeta V_0'(0) - \beta \zeta W_0'(0) \right\} d\zeta d\eta, \\ \psi_2(\xi) &= \int_\xi^\infty \int_\eta^\infty (\psi_0(\zeta) - \phi_2(\zeta))d\zeta d\eta, \\ \rho_2(\xi) &= \int_\xi^\infty \int_\eta^\infty (\rho_0(\zeta) - \phi_2(\zeta))d\zeta d\eta/D^2.\end{aligned}$$

2.2.3. Exact solutions of (2.18)

Using the above outer and inner approximations, we can construct uniform approximations of (2.18), which take the form

$$\begin{cases} U(y; \varepsilon) = U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y) + \phi_0(\frac{y}{\varepsilon}) + \varepsilon \phi_1(\frac{y}{\varepsilon}) + \varepsilon^2 \phi_2(\frac{y}{\varepsilon}), \\ V(y; \varepsilon) = V_0(y) + \varepsilon V_1(y) + \varepsilon^2 V_2(y) + \varepsilon^2 \psi_0(\frac{y}{\varepsilon}) + \varepsilon^3 \psi_1(\frac{y}{\varepsilon}) + \varepsilon^4 \psi_2(\frac{y}{\varepsilon}), \\ W(y; \varepsilon) = W_0(y) + \varepsilon W_1(y) + \varepsilon^2 W_2(y) + \varepsilon^2 \rho_0(\frac{y}{\varepsilon}) + \varepsilon^3 \rho_1(\frac{y}{\varepsilon}) + \varepsilon^4 \rho_2(\frac{y}{\varepsilon}). \end{cases}$$

Here it is assumed that $u_-(\varepsilon) = u_0^* + u_1^* \varepsilon + u_2^* \varepsilon^2 + \tilde{u}_-(\varepsilon) \varepsilon^3$ and

$$\begin{cases} l(\varepsilon) = l_0 + \varepsilon l_1 + \varepsilon^2 l_2 + \varepsilon^3 \tilde{l}(\varepsilon), \\ a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 \tilde{a}(\varepsilon), \\ b(\varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 \tilde{b}(\varepsilon) \end{cases}$$

are given. $(U, V, W)(y; \varepsilon)$ satisfy the following conditions at $y = 0$,

$$\begin{cases} U(0; \varepsilon) = 0, \\ V(0; \varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 \psi_1(0) + \varepsilon^4 \psi_2(0), \\ W(0; \varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 \rho_1(0) + \varepsilon^4 \rho_2(0) \end{cases}$$

and at $y = \infty$, $U(\infty) = V(\infty) = W(\infty) = u_0^* + u_1^* \varepsilon + u_2^* \varepsilon^2$.

Here we modify these a little to satisfy the both boundary conditions at $y = 0$ and $y = \infty$ exactly and add the remainder term $(\varepsilon^2 \tilde{U}, \varepsilon^2 \tilde{V}, \varepsilon^2 \tilde{W})$, we seek exact solutions of (2.18) of the form

$$\begin{cases} u(y; \varepsilon) = U(y; \varepsilon) + u_-^*(y; \varepsilon) + \varepsilon^2 \tilde{U}(y; \varepsilon), \\ v(y; \varepsilon) = V(y; \varepsilon) + a^*(\varepsilon) + u_-^*(y; \varepsilon) + \varepsilon^2 \tilde{V}(y; \varepsilon), \\ w(y; \varepsilon) = W(y; \varepsilon) + b^*(\varepsilon) + u_-^*(y; \varepsilon) + \varepsilon^2 \tilde{W}(y; \varepsilon), \end{cases} \quad (2.21)$$

where $a^*(\varepsilon) = \varepsilon^3(\tilde{a}(\varepsilon) - \psi_1(0) - \varepsilon \psi_2(0))$, $b^*(\varepsilon) = \varepsilon^3(\tilde{b}(\varepsilon) - \rho_1(0) - \varepsilon \rho_2(0))$, $u_-^*(y; \varepsilon) = \varepsilon^3 \tilde{u}_-(\varepsilon)(1 - e^{-y})$. Substituting (2.21) into (2.18), we obtain

$$\begin{cases} \varepsilon^2(U_{yy} + u_{-yy}^* + \varepsilon^2 \tilde{U}_{yy}) + U + u_-^* + \varepsilon^2 \tilde{U} \\ \quad - (U + u_-^* + \varepsilon^2 \tilde{U})^3 - \varepsilon \left\{ \alpha(V + a^* + u_-^* + \varepsilon^2 \tilde{V}) \right. \\ \quad \quad \left. + \beta(W + b^* + u_-^* + \varepsilon^2 \tilde{W}) + \gamma \right\} = 0, \\ V_{yy} + u_{-yy}^* + \varepsilon^2 \tilde{V}_{yy} + U + u_-^* + \varepsilon^2 \tilde{U} \\ \quad - (V + a^* + u_-^* + \varepsilon^2 \tilde{V}) = 0, \\ D^2(W_{yy} + u_{-yy}^* + \varepsilon^2 \tilde{W}_{yy}) + U + u_-^* + \varepsilon^2 \tilde{U} \\ \quad - (W + b^* + u_-^* + \varepsilon^2 \tilde{W}) = 0, \\ (\tilde{U}, \tilde{V}, \tilde{W})(0; \varepsilon) = (0, 0, 0), \quad (\tilde{U}, \tilde{V}, \tilde{W})(\infty; \varepsilon) = (0, 0, 0). \end{cases} \quad (2.22)$$

Then, we define the following operator $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = (T_1, T_2, T_3)(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon)$:

$$\begin{cases} T_1(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) \equiv \varepsilon^2 \tilde{U}_{yy} + \tilde{U} - 3U^2 \tilde{U} - 3\varepsilon^2 U \tilde{U}^2 - \varepsilon^4 \tilde{U}^3 - \varepsilon \alpha \tilde{V} - \varepsilon \beta \tilde{W} \\ \quad + \frac{1}{\varepsilon^2} [U_{yy} + u_{-yy}^* + U - U^3 \\ \quad - \varepsilon \{ \alpha(V + a^* + u_-^*) + \beta(W + b^* + u_-^*) + \gamma \}], \\ T_2(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) \equiv \tilde{V}_{yy} + (\tilde{U} - \tilde{V}) \\ \quad + \frac{1}{\varepsilon^2} [V_{yy} + u_{-yy}^* + (U - V - a)], \\ T_3(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) \equiv D^2 \tilde{W}_{yy} + (\tilde{U} - \tilde{W}) \\ \quad + \frac{1}{\varepsilon^2} [D^2 W_{yy} + D^2 u_{-yy}^* + (U - W - b^*)] \end{cases}$$

from $\bar{X} \equiv \bar{A}_\varepsilon \times \bar{B} \times \bar{B}$ to $\bar{Y} \equiv X_\kappa[0, \infty) \times X_\kappa[0, \infty) \times X_\kappa[0, \infty)$, where $\bar{A}_\varepsilon \equiv \{\tilde{U} \in X_{\kappa, \varepsilon}^2[0, \infty) \mid \tilde{U}(0) = 0, \tilde{U}(\infty) = 0\}$, $\bar{B} \equiv \{\tilde{V} \in X_\kappa^2[0, \infty) \mid \tilde{V}(0) = 0, \tilde{V}(\infty) = 0\}$, $X_\kappa[0, \infty) \equiv \left\{ u \in C^2[0, \infty) \mid \|u\|_{X_\kappa[0, \infty)} \equiv \sup_{y \in [0, \infty)} e^{\kappa y} |u(y)| < +\infty \right\}$, $X_\kappa^2[0, \infty) \equiv \left\{ u \in C^2[0, \infty) \mid \|u\|_{X_\kappa^2[0, \infty)} \equiv \sup_{y \in [0, \infty)} \sum_{i=0}^2 e^{\kappa y} \left| \left(\frac{d}{dy} \right)^i u(y) \right| < +\infty \right\}$, $X_{\kappa, \varepsilon}^2[0, \infty) \equiv \left\{ u \in C^2[0, \infty) \mid \|u\|_{X_{\kappa, \varepsilon}^2[0, \infty)} \equiv \sup_{y \in [0, \infty)} \sum_{i=0}^2 e^{\kappa y} \left| \left(\varepsilon \frac{d}{dy} \right)^i u(y) \right| < +\infty \right\}$ and $\kappa > 0$. We find that $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon)$ is the continuously differentiable operator and (2.22) is equivalent to $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = 0$.

Lemma 2.3. *There exist $\varepsilon_0 > 0$ and positive constants κ and C such that for any $\varepsilon \in (0, \varepsilon_0)$ the followings hold:*

- (i) $\|T_t(t_1; \varepsilon) - T_t(t_2; \varepsilon)\|_{\bar{X} \rightarrow \bar{Y}} \leq C \|t_1 - t_2\|_{\bar{X}}$ for any $t_1, t_2 \in \bar{X}$,
- (ii) $\|T(0; \varepsilon)\|_{\bar{Y}} \leq C\varepsilon$,
- (iii) $\|T_t^{-1}(0; \varepsilon)\|_{\bar{Y} \rightarrow \bar{X}} \leq C$.

The proof is similar to that of Lemma 2.2, so we omit it.

By this lemma, we can apply the Implicit Function Theorem to $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = 0$ and find that $T(\tilde{U}, \tilde{V}, \tilde{W}; \varepsilon) = 0$ has solutions $(\tilde{U}(\varepsilon), \tilde{V}(\varepsilon), \tilde{W}(\varepsilon)) \in$

\bar{X} satisfying $\|(\tilde{U}(\varepsilon), \tilde{V}(\varepsilon), \tilde{W}(\varepsilon))\|_{\bar{X}} = o(1)$ ($\varepsilon \rightarrow 0$). Thus we have the solutions of (2.18) on $[0, \infty)$

$$\begin{cases} u(y; \varepsilon) = U(y; \varepsilon) + u_-^*(y) + \varepsilon^2 \tilde{U}(y; \varepsilon), \\ v(y; \varepsilon) = V(y; \varepsilon) + a^*(\varepsilon) + u_-^*(y) + \varepsilon^2 \tilde{V}(y; \varepsilon), \\ w(y; \varepsilon) = W(y; \varepsilon) + b^*(\varepsilon) + u_-^*(y) + \varepsilon^2 \tilde{W}(y; \varepsilon), \end{cases} \quad (2.23)$$

which implies that (2.5) has the following exact solutions on $I_2 = [l(\varepsilon), \infty)$:

$$\begin{cases} u(x; \varepsilon) = U(x - l(\varepsilon); \varepsilon) + u_-^*(x - l(\varepsilon); \varepsilon) + \varepsilon^2 \tilde{U}(x - l(\varepsilon); \varepsilon), \\ v(x; \varepsilon) = V(x - l(\varepsilon); \varepsilon) + a^*(\varepsilon) + u_-^*(x - l(\varepsilon); \varepsilon) \\ \quad + \varepsilon^2 \tilde{V}(x - l(\varepsilon); \varepsilon), \\ w(x; \varepsilon) = W(x - l(\varepsilon); \varepsilon) + b^*(\varepsilon) + u_-^*(x - l(\varepsilon); \varepsilon) \\ \quad + \varepsilon^2 \tilde{W}(x - l(\varepsilon); \varepsilon). \end{cases} \quad (2.24)$$

2.3. Solutions of (2.3) on the interval $[0, \infty)$

Finally, we construct solutions of (2.3) on the interval $[0, \infty)$, matching the solutions constructed in §2.1 and §2.2 in C^1 -sense. To distinguish the solutions in each interval $I_1 = [0, l(\varepsilon)]$ and $I_2 = [l(\varepsilon), \infty)$, we write $u, v, w, U_0, V_0, W_0, \phi_0, \psi_0, \rho_0 \dots$ on the interval I_i as $u^{(i)}, v^{(i)}, w^{(i)}, U_0^{(i)}, V_0^{(i)}, W_0^{(i)}, \phi_0^{(i)}, \psi_0^{(i)}, \rho_0^{(i)} \dots$ for $i = 1, 2$. We have already constructed the solutions to be continuous at $x = l(\varepsilon)$. Then, for our purpose, we impose the following three conditions:

$$\begin{cases} \Phi(\varepsilon) \equiv l(\varepsilon) \left(u_x^{(1)}(l(\varepsilon)) - u_x^{(2)}(l(\varepsilon)) \right) = 0, \\ \Psi(\varepsilon) \equiv l(\varepsilon) \left(v_x^{(1)}(l(\varepsilon)) - v_x^{(2)}(l(\varepsilon)) \right) = 0, \\ \Pi(\varepsilon) \equiv l(\varepsilon) \left(w_x^{(1)}(l(\varepsilon)) - w_x^{(2)}(l(\varepsilon)) \right) = 0, \end{cases} \quad (2.25)$$

which enable us to show that $l(\varepsilon)$, $a(\varepsilon)$, $b(\varepsilon)$ are uniquely determined.

Substitute each solutions on I_i ($i = 1, 2$) to (2.25), we easily find that

the following relations hold for any small $\varepsilon > 0$:

$$\left\{ \begin{array}{l} \Phi(\varepsilon) = \frac{1}{\varepsilon}(\dot{\phi}_0^{(1)}(0) - l_0 \dot{\phi}_0^{(2)}(0)) + \{U_0'^{(1)}(1) + \dot{\phi}_1^{(1)}(0) \\ - l_0(U_0'^{(2)}(0) + \dot{\phi}_1^{(2)}(0)) - l_1 \dot{\phi}_0^{(2)}(0)\} \\ + \varepsilon\{U_1'^{(1)}(1) + \dot{\phi}_2^{(1)}(0) - l_0(U_1'^{(2)}(0) + \dot{\phi}_2^{(2)}(0)) \\ - l_1(U_0'^{(2)}(0) + \dot{\phi}_1^{(2)}(0)) - l_2 \dot{\phi}_0^{(2)}(0)\} + O(\varepsilon^2) \\ \equiv \frac{1}{\varepsilon}\Phi_{-1}(\varepsilon) + \Phi_0(\varepsilon) + \varepsilon\Phi_1(\varepsilon) + O(\varepsilon^2) = 0, \\ \Psi(\varepsilon) = (V_0'^{(1)}(1) - l_0 V_0'^{(2)}(0)) + \varepsilon\{V_1'^{(1)}(1) + \dot{\psi}_0^{(1)}(0) \\ - l_0(V_1'^{(2)}(0) + \dot{\psi}_0^{(2)}(0)) - l_1 V_0'^{(2)}(0)\} + O(\varepsilon^2) \\ \equiv \Psi_0(\varepsilon) + \varepsilon\Psi_1(\varepsilon) + O(\varepsilon^2) = 0, \\ \Pi(\varepsilon) = (W_0'^{(1)}(1) - l_0 W_0'^{(2)}(0)) + \varepsilon\{W_1'^{(1)}(1) + \dot{\rho}_0^{(1)}(0) \\ - l_0(W_1'^{(2)}(0) + \dot{\rho}_0^{(2)}(0)) - l_1 W_0'^{(2)}(0)\} + O(\varepsilon^2) \\ \equiv \Pi_0(\varepsilon) + \varepsilon\Pi_1(\varepsilon) + O(\varepsilon^2) = 0. \end{array} \right.$$

$O(\varepsilon^{-1})$:

$$\Phi_{-1} = \dot{\phi}_0^{(1)}(0) - l_0 \dot{\phi}_0^{(2)}(0) = 0.$$

Note that this is always true.

$O(\varepsilon^0)$:

$$\left\{ \begin{array}{l} \Phi_0 = U_0'^{(1)}(1) + \dot{\phi}_1^{(1)}(0) - l_0(U_0'^{(2)}(0) + \dot{\phi}_1^{(2)}(0)) - l_1 \dot{\phi}_0^{(2)}(0) \\ = \sqrt{2}l_0(\alpha a_0 + \beta b_0 + \gamma) - \frac{1}{\sqrt{2}}l_1 + l_0\sqrt{2}(\alpha a_0 + \beta b_0 + \gamma) + \frac{1}{\sqrt{2}}l_1 \\ = 2\sqrt{2}l_0(\alpha a_0 + \beta b_0 + \gamma) = 0, \\ \Psi_0 = V_0'^{(1)}(1) - l_0 V_0'^{(2)}(0) = l_0(a_0 - 1) \tanh l_0 + l_0(a_0 + 1) = 0, \\ \Pi_0 = W_0'^{(1)}(1) - l_0 W_0'^{(2)}(0) = \frac{l_0}{D}(b_0 - 1) \tanh \frac{l_0}{D} + \frac{l_0}{D}(b_0 + 1) = 0. \end{array} \right.$$

By the relations $\Psi_0 = 0$ and $\Pi_0 = 0$, we have $a_0 = -e^{-2l_0}$ and $b_0 = -e^{-2l_0/D}$, respectively. Substitute these into $\Phi_0 = 0$, we have

$$\alpha e^{-2l_0} + \beta e^{-2l_0/D} = \gamma. \quad (2.26)$$

If we find $l_0 > 0$ satisfying (2.26) for given α, β, γ, D , we can get a_0 and b_0 uniquely from $\Psi_0 = 0$ and $\Pi_0 = 0$, respectively.

$O(\varepsilon^1)$:

$$\begin{cases} \Phi_1 = U_1'^{(1)}(1) + \dot{\phi}_2^{(1)}(0) - l_0(U_1'^{(2)}(0) + \dot{\phi}_2^{(2)}(0)) \\ \quad - l_1(U_0'^{(2)}(0) + \dot{\phi}_1^{(2)}(0)) - l_2\dot{\phi}_0^{(2)}(0) = 0, \\ \Psi_1 = V_1'^{(1)}(1) + \dot{\psi}_0^{(1)}(0) - l_0(V_1'^{(2)}(0) + \dot{\psi}_0^{(2)}(0)) - l_1V_0'^{(2)}(0) = 0, \\ \Pi_1 = W_1'^{(1)}(1) + \dot{\rho}_0^{(1)}(0) - l_0(W_1'^{(2)}(0) + \dot{\rho}_0^{(2)}(0)) - l_1W_0'^{(2)}(0) = 0. \end{cases}$$

Though it seems that Φ_1 contains the term l_2 , it is independent of l_2 . Indeed, we can calculate as follows.

The terms including l_2 in Φ_1

$$\begin{aligned} &= 2l_0l_2 \int_{-\infty}^0 \dot{\phi}_0^{(1)} \phi_0^{(1)} (\phi_0^{(1)} + 1)(\phi_0^{(1)} + 2) d\xi / \dot{\phi}_0^{(1)}(0) - l_2\dot{\phi}_0^{(2)}(0) \\ &= -\frac{2\sqrt{2}}{4}l_2 + \frac{1}{\sqrt{2}}l_2 = 0. \end{aligned}$$

Furthermore we can show that Φ_1 is also independent of l_1 . To show this, we consider Φ_1 as a function of l_1 and write it as $\Phi_1 = \Phi_1(l_1)$. We have the following lemma:

Lemma 2.4. $\frac{\partial}{\partial l_1}\Phi_1(l_1) = 0$.

The proof will be given in §4.

The terms including a_1 or b_1 in Φ_1

$$\begin{aligned} &= \int_{-\infty}^0 (\alpha a_1 + \beta b_1) \dot{\phi}_0^{(1)} \left\{ -\frac{1}{2}l_0^2(2 + 6\phi_0^{(1)} + 3(\phi_0^{(1)})^2 + l_0^2) \right\} d\xi / \dot{\phi}_0^{(1)}(0) \\ &\quad + l_0 \int_0^\infty (\alpha a_1 + \beta b_1) \dot{\phi}_0^{(2)} \left\{ \frac{1}{2}l_0^2(2 - 6\phi_0^{(2)} + 3(\phi_0^{(2)})^2 + 2) \right\} d\xi \\ &= (\alpha a_1 + \beta b_1)(\sqrt{2}l_0 + \sqrt{2}l_0) = 2\sqrt{2}l_0(\alpha a_1 + \beta b_1). \end{aligned}$$

By $\Phi_1 = 0$, we have $2\sqrt{2}l_0(\alpha a_1 + \beta b_1) = \text{constant}$. On the other hand,

the terms including l_1 or a_1 in Ψ_1

$$\begin{aligned} &= l_1 \left\{ -l_0 e^{-l_0} \int_0^1 e^{l_0 s} (1 - V_0^{(1)}) ds - 2l_0 e^{-l_0} \tanh l_0 \int_0^1 \cosh(l_0 s) (1 - V_0^{(1)}) ds \right. \\ &\quad \left. - l_0 e^{-l_0} \int_0^1 e^{-l_0 s} (1 - V_0^{(1)}) ds + a_0 + 1 \right\} + a_1(l_0 + l_0 \tanh l_0) \\ &= -l_0 \frac{2e^{-l_0}}{e^{l_0} + e^{-l_0}} l_1 + a_1(l_0 \frac{2e^{l_0}}{e^{l_0} + e^{-l_0}}). \end{aligned}$$

By $\Psi_1 = 0$, we have $a_1 = e^{-2l_0}l_1 + \text{constant}$. Similarly we have $b_1 = e^{-2\frac{l_0}{D}}l_1/D + \text{constant}$ by $\Pi_1 = 0$. Then, substituting these a_1 and b_1 into $2\sqrt{2}l_0(\alpha a_1 + \beta b_1) = \text{constant}$, we obtain

$$\left(\alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}/D}\right) l_1 = \text{constant}.$$

If $\alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}/D} \neq 0$, l_1 is uniquely determined and then a_1 and b_1 are also uniquely determined. Similarly higher order terms l_k , a_k , b_k ($k = 2, 3, \dots$) are also successively determined.

Now we justify the above process. Put $l(\varepsilon) = l_0 + \varepsilon\bar{l}$, $a(\varepsilon) = a_0 + \varepsilon\bar{a}$, $b(\varepsilon) = b_0 + \varepsilon\bar{b}$ and define $\Phi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$, $\Psi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$, $\Pi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$ by $\Phi(\varepsilon) = \varepsilon\Phi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$, $\Psi(\varepsilon) = \varepsilon\Psi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$, $\Pi(\varepsilon) = \varepsilon\Pi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$, respectively. We find that there exist two small positive constants δ and ε_0 such that $\Phi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$, $\Psi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$, $\Pi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon)$ are continuous for $\bar{l} \in (l_1 - \delta, l_1 + \delta)$, $\bar{a} \in (a_1 - \delta, a_1 + \delta)$ and $\bar{b} \in (b_1 - \delta, b_1 + \delta)$, $\varepsilon \in [0, \varepsilon_0]$ and are C^1 -class functions for \bar{l} , \bar{a} , \bar{b} . Furthermore, we can easily find that

$$\begin{cases} \Phi^*(l_1, a_1, b_1; 0) = 0, & \Psi^*(l_1, a_1, b_1; 0) = 0, \\ \Pi^*(l_1, a_1, b_1; 0) = 0, & \frac{\partial \Phi^*}{\partial l}(l_1, a_1, b_1; 0) = 0, \\ \frac{\partial \Phi^*}{\partial \bar{a}}(l_1, a_1, b_1; 0) = 2\sqrt{2}l_0\alpha, & \frac{\partial \Phi^*}{\partial b}(l_1, a_1, b_1; 0) = 2\sqrt{2}l_0\beta, \\ \frac{\partial \Psi^*}{\partial l}(l_1, a_1, b_1; 0) = -l_0 \frac{2e^{-l_0}}{e^{l_0} + e^{-l_0}}, & \frac{\partial \Psi^*}{\partial \bar{a}}(l_1, a_1, b_1; 0) = l_0 \frac{2e^{l_0}}{e^{l_0/D} + e^{-l_0/D}}, \\ \frac{\partial \Psi^*}{\partial b}(l_1, a_1, b_1; 0) = 0, & \frac{\partial \Pi^*}{\partial l}(l_1, a_1, b_1; 0) = -\frac{l_0}{D^2} \frac{2e^{-l_0/D}}{e^{l_0/D} + e^{-l_0/D}}, \\ \frac{\partial \Pi^*}{\partial \bar{a}}(l_1, a_1, b_1; 0) = 0, & \frac{\partial \Pi^*}{\partial b}(l_1, a_1, b_1; 0) = \frac{l_0}{D} \frac{2e^{l_0/D}}{e^{l_0/D} + e^{-l_0/D}}. \end{cases}$$

This implies that

$$\begin{aligned} & \frac{\partial(\Phi^*, \Psi^*, \Pi^*)}{\partial(\bar{l}, \bar{a}, \bar{b})}(l_1, a_1, b_1; 0) \\ &= \begin{vmatrix} 0 & 2\sqrt{2}l_0\alpha & 2\sqrt{2}l_0\beta \\ -l_0 \frac{2e^{-l_0}}{e^{l_0} + e^{-l_0}} & l_0 \frac{2e^{l_0}}{e^{l_0/D} + e^{-l_0/D}} & 0 \\ -\frac{l_0}{D^2} \frac{2e^{-l_0/D}}{e^{l_0/D} + e^{-l_0/D}} & 0 & \frac{l_0}{D} \frac{2e^{l_0/D}}{e^{l_0/D} + e^{-l_0/D}} \end{vmatrix} \\ &= 8\sqrt{2}\frac{l_0^3}{D}e^{l_0}e^{l_0/D} \frac{\alpha e^{-2l_0} + \frac{\beta}{D}e^{-2l_0/D}}{(e^{l_0} + e^{-l_0})(e^{l_0/D} + e^{-l_0/D})}. \end{aligned}$$

Then if $\alpha e^{-2l_0} + \frac{\beta}{D}e^{-2\frac{l_0}{D}} \neq 0$, we can apply the Implicit Function Theorem to $\Phi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon) = 0$, $\Psi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon) = 0$ and $\Pi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon) = 0$ and find that

there exist $\bar{l} = \bar{l}(\varepsilon)$, $\bar{a} = \bar{a}(\varepsilon)$, $\bar{b} = \bar{b}(\varepsilon)$ ($\varepsilon \in [0, \varepsilon_0]$) satisfying $\bar{l}(0) = l_1$, $\bar{a}(0) = a_1$, $\bar{b}(0) = b_1$ such that

$$\Phi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon) = 0, \quad \Psi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon) = 0, \quad \Pi^*(\bar{l}, \bar{a}, \bar{b}; \varepsilon) = 0.$$

Then we have the following existence theorem:

Theorem 2.1. *For given α, β, γ, D , if there exists $l_0 > 0$ satisfying $\alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}} = \gamma$ and $\alpha e^{-2l_0} + \frac{\beta}{D} e^{-2\frac{l_0}{D}} \neq 0$, then (2.3) has a standing pulse solution $(u, v, w)(x; \varepsilon)$ for a sufficiently small $\varepsilon > 0$, which is explicitly represented by (2.17) on I_1 and (2.24) on I_2 , respectively.*

Corollary 2.1. ([2]) *Assume that α, β, D satisfy $|\alpha D| > |\beta|$. For the number K of standing pulse solutions of (2.3), which have the asymptotic forms stated in Theorem 2.1 for a sufficiently small $\varepsilon > 0$, we have*

- (a1) $\text{sgn}(\alpha) = \text{sgn}(\beta) = \text{sgn}(\gamma)$, and $|\gamma| < |\alpha + \beta|$, then $K=1$.
 - (a2) $\text{sgn}(\alpha) = \text{sgn}(\beta) = \text{sgn}(\gamma)$, and $|\gamma| > |\alpha + \beta|$, then $K=0$.
 - (a3) $\text{sgn}(\alpha) = \text{sgn}(\beta)$ and $\text{sgn}(\alpha) \neq \text{sgn}(\gamma)$, then $K=0$.
 - (b1) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $\text{sgn}(\gamma) = -1$, then $K=0$.
 - (b2) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $0 < \gamma < \alpha + \beta$, then $K=1$.
 - (b3) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $\alpha + \beta < \gamma < \gamma_{c1}$, then $K=2$.
 - (b4) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $\gamma > \gamma_{c1}$, then $K=0$.
 - (c1) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $\gamma < \alpha + \beta$, then $K=0$.
 - (c2) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $\alpha + \beta < \gamma < 0$, then $K=1$.
 - (c3) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $0 < \gamma < \gamma_{c1}$, then $K=2$.
 - (c4) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $\gamma > \gamma_{c1}$, then $K=0$.
 - (d1) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $\gamma < \gamma_{c2}$, then $K=0$.
 - (d2) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $\gamma_{c2} < \gamma < 0$, then $K=2$.
 - (d3) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $0 < \gamma < \alpha + \beta$, then $K=1$.
 - (d4) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$, and $\gamma > \alpha + \beta$, then $K=0$.
 - (e1) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $\gamma < \gamma_{c2}$, then $K=0$.
 - (e2) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $\gamma_{c2} < \gamma < \alpha + \beta$, then $K=2$.
 - (e3) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $\alpha + \beta < \gamma < 0$, then $K=1$.
 - (e4) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$, and $\gamma > 0$, then $K=0$,
- where $\gamma_{c1} = (-\alpha)^{-\frac{1}{D-1}} \beta^{-\frac{D}{D-1}} (D^{-\frac{1}{D-1}} - D^{-\frac{D}{D-1}})$ and $\gamma_{c2} = \alpha^{-\frac{1}{D-1}} (-\beta)^{-\frac{D}{D-1}} (D^{-\frac{1}{D-1}} - D^{-\frac{D}{D-1}})$.

Proof. We put $g(l_0) = \alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}}$. The left panel of Figure 4 shows the curve $\gamma = g(l_0)$ with $\alpha < 0, \beta > 0$ and the right one does the curve $\gamma = g(l_0)$ with $\alpha > 0, \beta < 0$. Here γ_{c_1} (resp. γ_{c_2}) is the maximal (resp. minimal) of the curve $\gamma = g(l_0)$ for $l_0 > 0$ in the left (resp. right) panel at $l_0 = l_c$. Clearly l_c is determined as $l_c = \frac{D}{2(D-1)} \log\left(-\frac{\alpha D}{\beta}\right)$. Then, the number of standing pulse solutions corresponds to the number of intersection points of $\gamma = g(l_0)$ and $\gamma = l_0$ except for the critical point $l_0 = l_c$, at which $\alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}}/D = 0$. The case (b2), (b3) and (d2), (d3) follow from Figure 4 directly. The other cases will be obtained similarly. \square

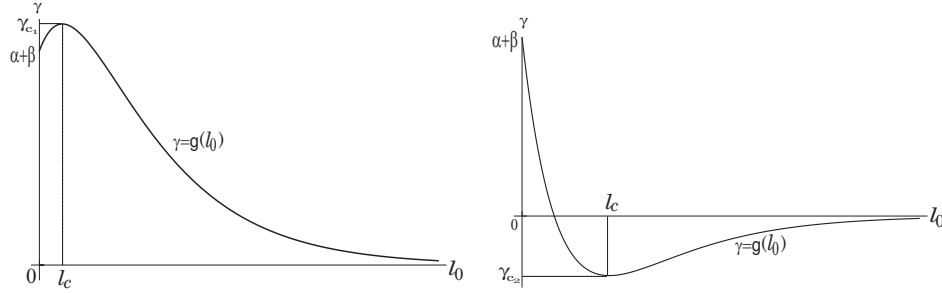


Figure 4: (b) $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$ (d) $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$

3. Stability of standing pulse solutions

Here we will study the stability of the standing pulse solutions $(u, v, w) = (u, v, w)(x; \varepsilon)$. Van Heijster et al [7] showed that the stability of the standing pulse solutions does not depend on the parameters τ and θ when τ and θ are $O(1)$ with respect to ε , on the other hand, when $\tau = O(1/\varepsilon^2)$ and/or $\theta = O(1/\varepsilon^2)$, there may appear the two types of bifurcation, one is a drift bifurcation and the other is a Hopf bifurcation. Hence, we restrict our attention to the latter case, so that we set $\tau = \hat{\tau}/\varepsilon^2, \theta = \hat{\theta}/\varepsilon^2$ with $\hat{\tau} > 0, \hat{\theta} > 0$ in the subsequent analysis. In order to discuss the stability, we consider the following linearized eigenvalue problems of (2.1) around

$(u, v, w)(x; \varepsilon):$

$$L^\varepsilon \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \hat{\lambda} \begin{bmatrix} \varepsilon^2 & 0 \\ 0 & \hat{\tau} & 0 \\ 0 & 0 & \hat{\theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad (3.1)$$

where

$$L^\varepsilon \equiv \begin{bmatrix} \varepsilon^2 \frac{d^2}{dx^2} + 1 - 3u^2 & -\varepsilon\alpha & -\varepsilon\beta \\ 1 & \frac{d^2}{dx^2} - 1 & 0 \\ 1 & 0 & D^2 \frac{d^2}{dx^2} - 1 \end{bmatrix}$$

and $(p, q, r)(x; \varepsilon; \hat{\lambda}) \in BC(\mathbb{R}) \times BC(\mathbb{R}) \times BC(\mathbb{R})$. Here we put an eigenvalue $\lambda = \varepsilon^2 \hat{\lambda}$. The operator L^ε with the usual domain becomes a sectorial operator for $\varepsilon > 0$ and the spectral analysis of (3.1) derives the nonlinear stability or instability (for instance, see Henry [8]). Therefor our problem consists of the following two parts: (i) distribution of the essential spectrum, (ii) distribution of isolated eigenvalues. For the problem (i), noting that $(u_-(\varepsilon), u_-(\varepsilon), u_-(\varepsilon))$ is the stable constant solution of (2.1), we can conclude the following lemma:

Lemma 3.1. ([7], [8]) *There exists a positive constant d such that for sufficiently small positive ε , the essential spectrum of (3.1) satisfies*

$$\operatorname{Re}\{\text{essential spectrum of (3.1)}\} \leq -d,$$

where $d \equiv \min\{1/(2\hat{\tau}), 1/(2\hat{\theta})\} > 0$.

Next, we consider the distribution of eigenvalues. A complex number $\hat{\lambda}$ is called an eigenvalue of (3.1) if this equation has a nontrivial solution $(p, q, r)(x; \varepsilon; \hat{\lambda})$ which belongs to $BC(\mathbb{R}) \times BC(\mathbb{R}) \times BC(\mathbb{R})$. Clearly $\left(\frac{du}{dx}, \frac{dv}{dx}, \frac{dw}{dx}\right)(x; \varepsilon)$ satisfies the equation (3.1) with $\hat{\lambda} = 0$. This implies that $\hat{\lambda} = 0$ is an eigenvalue of (3.1), which corresponds to a translation invariance of the standing pulse solutions.

The eigenvalue problem (3.1) can be rewritten equivalently as

$$\frac{d}{dx} \bar{V} = A(x; \varepsilon; \hat{\lambda}) \bar{V}, \quad (3.2)$$

where $\bar{V} = \bar{V}(x; \varepsilon; \hat{\lambda}) = \left(p, \varepsilon \frac{dp}{dx}, q, \frac{dq}{dx}, r, \frac{dr}{dx} \right)$ and $A(x; \varepsilon; \hat{\lambda})$ is defined by

$$A(x; \varepsilon; \hat{\lambda}) \equiv \begin{bmatrix} 0 & \frac{1}{\varepsilon} & 0 & 0 & 0 & 0 \\ \frac{1}{\varepsilon}(-1 + 3u^2 + \varepsilon^2 \hat{\lambda}) & 0 & \alpha & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 + \hat{\tau} \hat{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{D^2} & 0 & 0 & 0 & \frac{1}{D^2}(1 + \hat{\theta} \hat{\lambda}) & 0 \end{bmatrix}.$$

Since $V(x; \varepsilon) \equiv \left(u, \varepsilon \frac{du}{dx}, v, \frac{dv}{dx}, w, \frac{dw}{dx} \right) \rightarrow (u_-(\varepsilon), 0, u_-(\varepsilon), 0, u_-(\varepsilon), 0)$ as $x \rightarrow \pm\infty$, $\bar{V}(x; \varepsilon; \hat{\lambda})$ obeys the following linearized equation:

$$\frac{d}{dx} \bar{V} = A(\infty; \varepsilon; \hat{\lambda}) \bar{V} \quad (3.3)$$

when $x \rightarrow \pm\infty$. Let $\mu_i(\varepsilon; \hat{\lambda}) (i = 1, 2, \dots, 6)$ denote eigenvalues of the matrix $A(\infty; \varepsilon; \hat{\lambda})$, where we suppose $\text{Re}\{\mu_1\} \leq \text{Re}\{\mu_2\} \leq \text{Re}\{\mu_3\} \leq \text{Re}\{\mu_4\} \leq \text{Re}\{\mu_5\} \leq \text{Re}\{\mu_6\}$. By the standard argument, we may assume that $\mu_i(\varepsilon; \hat{\lambda}) (i = 1, 2, \dots, 6)$ depend analytically on $\hat{\lambda}$.

Lemma 3.2. *There exists a positive constant d independent of ε such that*

$$\text{Re}\{\mu_1\} \leq \text{Re}\{\mu_2\} \leq \text{Re}\{\mu_3\} < 0 < \text{Re}\{\mu_4\} \leq \text{Re}\{\mu_5\} \leq \text{Re}\{\mu_6\}$$

hold for any $\hat{\lambda} \in \mathbb{C}_d \equiv \{\lambda \in \mathbb{C} \mid \text{Re}\{\lambda\} \geq -d\}$.

The proof is given in §4.

Then a nontrivial solution $\bar{V}(x; \varepsilon; \hat{\lambda})$ of (3.2) corresponding to an eigenvalue $\hat{\lambda} \in \mathbb{C}_d$ must satisfy

$$\bar{V}(x; \varepsilon; \hat{\lambda}) \rightarrow \mathbf{0} \quad (x \rightarrow \pm\infty). \quad (3.4)$$

Thus we consider the following eigenvalue problem:

$$\begin{cases} \varepsilon^2 \hat{\lambda} p = \varepsilon^2 p_{xx} + (1 - 3u^2)p - \varepsilon \alpha q - \varepsilon \beta r, \\ \hat{\tau} \hat{\lambda} q = q_{xx} + p - q, \\ \hat{\theta} \hat{\lambda} r = D^2 r_{xx} + p - r, \\ (p, q, r)(\pm\infty) = (0, 0, 0). \end{cases} \quad x \in \mathbb{R} \quad (3.5)$$

Furthermore, by virtue of the symmetry of the standing pulse solutions, the eigenvalue problem (3.1) on \mathbb{R} is decomposed into a pair of the following eigenvalue problems on \mathbb{R}_+ (see [9]):

$$\begin{cases} \varepsilon^2 \hat{\lambda} p = \varepsilon^2 p_{xx} + (1 - 3u^2)p - \varepsilon \alpha q - \varepsilon \beta r, \\ \hat{\tau} \hat{\lambda} q = q_{xx} + p - q, \\ \hat{\theta} \hat{\lambda} r = D^2 r_{xx} + p - r, \\ (p_x, q_x, r_x)(0) = (0, 0, 0), \\ (p, q, r)(\infty) = (0, 0, 0) \end{cases} \quad x \in (0, \infty) \quad (3.6)$$

and

$$\begin{cases} \varepsilon^2 \hat{\lambda} p = \varepsilon^2 p_{xx} + (1 - 3u^2)p - \varepsilon \alpha q - \varepsilon \beta r, \\ \hat{\tau} \hat{\lambda} q = q_{xx} + p - q, \\ \hat{\theta} \hat{\lambda} r = D^2 r_{xx} + p - r, \\ (p, q, r)(0) = (0, 0, 0), \\ (p, q, r)(\infty) = (0, 0, 0). \end{cases} \quad x \in (0, \infty) \quad (3.7)$$

3.1. Eigenvalue problem (3.6)

Similarly to the construction of standing pulse solutions, let us consider the following problems with suitable boundary conditions:

$$\begin{cases} \varepsilon^2 \hat{\lambda} p = \varepsilon^2 p_{xx} + (1 - 3u^2)p - \varepsilon \alpha q - \varepsilon \beta r, \\ \hat{\tau} \hat{\lambda} q = q_{xx} + p - q, \\ \hat{\theta} \hat{\lambda} r = D^2 r_{xx} + p - r, \\ (p_x, q_x, r_x)(0) = (0, 0, 0), \\ (p, q, r)(l(\varepsilon)) = \left(\frac{a}{\varepsilon}, b, c\right), \end{cases} \quad x \in (0, l(\varepsilon)) = I_1 \quad (3.8)$$

and

$$\begin{cases} \varepsilon^2 \hat{\lambda} p = \varepsilon^2 p_{xx} + (1 - 3u^2)p - \varepsilon \alpha q - \varepsilon \beta r, \\ \hat{\tau} \hat{\lambda} q = q_{xx} + p - q, \\ \hat{\theta} \hat{\lambda} r = D^2 r_{xx} + p - r, \\ (p, q, r)(l(\varepsilon)) = \left(\frac{a}{\varepsilon}, b, c\right), \\ (p, q, r)(\infty) = (0, 0, 0), \end{cases} \quad x \in (l(\varepsilon), \infty) = I_2 \quad (3.9)$$

where a, b, c are given real numbers. For any $\hat{\lambda} \in \mathbb{C}_d$, let $(p^{(1)}, q^{(1)}, r^{(1)})(x; \varepsilon; \hat{\lambda}; a, b, c)$ and $(p^{(2)}, q^{(2)}, r^{(2)})(x; \varepsilon; \hat{\lambda}; a, b, c)$ be solutions of (3.8) and (3.9), respectively. Then, any solution $\bar{V}(x; \varepsilon; \lambda)$ of (3.8) satisfying $(p_x, q_x, r_x)(0) = (0, 0, 0)$ is represented as a linear combination of three independent solutions

$$\begin{aligned} \bar{V}_1(x; \varepsilon; \hat{\lambda}) &= \begin{bmatrix} p^{(1)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ \varepsilon p_x^{(1)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ q^{(1)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ q_x^{(1)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ r^{(1)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ r_x^{(1)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \end{bmatrix}, \quad \bar{V}_2(x; \varepsilon; \hat{\lambda}) = \begin{bmatrix} p^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ \varepsilon p_x^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ q^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ q_x^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ r^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ r_x^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \end{bmatrix}, \\ \bar{V}_3(x; \varepsilon; \hat{\lambda}) &= \begin{bmatrix} p^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ \varepsilon p_x^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ q^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ q_x^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ r^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ r_x^{(1)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \end{bmatrix}. \end{aligned}$$

By virtue of Lemma 3.2, any solution of (3.9) satisfying $(p, q, r)(\infty) = (0, 0, 0)$ is represented as a linear combination of three independent solutions

$$\begin{aligned} \bar{V}_4(x; \varepsilon; \hat{\lambda}) &= \begin{bmatrix} p^{(2)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ \varepsilon p_x^{(2)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ q^{(2)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ q_x^{(2)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ r^{(2)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \\ r_x^{(2)}(x; \varepsilon; \hat{\lambda}; 1, 0, 0) \end{bmatrix}, \quad \bar{V}_5(x; \varepsilon; \hat{\lambda}) = \begin{bmatrix} p^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ \varepsilon p_x^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ q^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ q_x^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ r^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \\ r_x^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 1, 0) \end{bmatrix}, \\ \bar{V}_6(x; \varepsilon; \hat{\lambda}) &= \begin{bmatrix} p^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ \varepsilon p_x^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ q^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ q_x^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ r^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \\ r_x^{(2)}(x; \varepsilon; \hat{\lambda}; 0, 0, 1) \end{bmatrix}. \end{aligned}$$

The coefficient matrix $A(x; \varepsilon; \hat{\lambda})$ of (3.3) depends analytically on $\hat{\lambda}$. Then, we may assume, without loss of generality, that $\bar{V}_i(x; \varepsilon; \hat{\lambda})$ ($i = 1, 2, \dots, 6$) also depend analytically on $\hat{\lambda}$.

Let $\bar{V}(x; \varepsilon; \hat{\lambda})$ be a nontrivial solutions of (3.6) corresponding to an eigenvalue $\hat{\lambda}$. We know that there exist constants α_i ($i = 1, 2, \dots, 6$) satisfying $\sum_{i=1}^6 |\alpha_i| \neq 0$ such that $\bar{V}(x; \varepsilon; \hat{\lambda})$ must be represented as

$$\bar{V}(x; \varepsilon; \hat{\lambda}) = \begin{cases} \alpha_1 \bar{V}_1(x; \varepsilon; \hat{\lambda}) + \alpha_2 \bar{V}_2(x; \varepsilon; \hat{\lambda}) + \alpha_3 \bar{V}_3(x; \varepsilon; \hat{\lambda}), & x \in I_1 \\ \alpha_4 \bar{V}_4(x; \varepsilon; \hat{\lambda}) + \alpha_5 \bar{V}_5(x; \varepsilon; \hat{\lambda}) + \alpha_6 \bar{V}_6(x; \varepsilon; \hat{\lambda}), & x \in I_2. \end{cases} \quad (3.10)$$

At $x = l(\varepsilon)$, we have the relation

$$\begin{aligned} & \alpha_1 \bar{V}_1(l(\varepsilon); \varepsilon; \hat{\lambda}) + \alpha_2 \bar{V}_2(l(\varepsilon); \varepsilon; \hat{\lambda}) + \alpha_3 \bar{V}_3(l(\varepsilon); \varepsilon; \hat{\lambda}) \\ &= \alpha_4 \bar{V}_4(l(\varepsilon); \varepsilon; \hat{\lambda}) + \alpha_5 \bar{V}_5(l(\varepsilon); \varepsilon; \hat{\lambda}) + \alpha_6 \bar{V}_6(l(\varepsilon); \varepsilon; \hat{\lambda}). \end{aligned} \quad (3.11)$$

This implies that $\hat{\lambda}$ is an eigenvalue of (3.6) if and only if six vectors $\bar{V}_i(l(\varepsilon); \varepsilon; \hat{\lambda})$ ($i = 1, 2, \dots, 6$) are linearly dependent. Setting

$$\begin{aligned} g_N(\varepsilon; \hat{\lambda}) &= \det[\bar{V}_1(l(\varepsilon); \varepsilon; \hat{\lambda}), \bar{V}_2(l(\varepsilon); \varepsilon; \hat{\lambda}), \\ &\quad \bar{V}_3(l(\varepsilon); \varepsilon; \hat{\lambda}), \bar{V}_4(l(\varepsilon); \varepsilon; \hat{\lambda}), \bar{V}_5(l(\varepsilon); \varepsilon; \hat{\lambda}), \bar{V}_6(l(\varepsilon); \varepsilon; \hat{\lambda})], \end{aligned}$$

we find that $g_N(\varepsilon; \hat{\lambda})$ is an analytic function of $\hat{\lambda} \in \mathbb{C}_d$ and have the next lemma.

Lemma 3.3. *The number $\hat{\lambda} \in \mathbb{C}_d$ is an eigenvalue of (3.6) if and only if $g_N(\varepsilon; \hat{\lambda}) = 0$.*

We call $g_N(\varepsilon; \hat{\lambda})$ the *Evans function* of the standing pulse solutions corresponding to (3.6). Therefore, making use of the equation $g_N(\varepsilon; \hat{\lambda}) = 0$ we can examine the distribution of eigenvalues $\hat{\lambda} \in \mathbb{C}_d$. For this purpose, we have to construct functions $\bar{V}_i(x; \varepsilon; \hat{\lambda})$ ($i = 1, 2, \dots, 6$) as we constructed standing pulse solutions in §2. According to the dependency of $\hat{\lambda} \in \mathbb{C}_d$ on ε , we must consider the following two cases:

- (I) $\hat{\lambda} = \hat{\lambda}(\varepsilon) = O(1)$ in \mathbb{C}_d ($\varepsilon \rightarrow 0$),
- (II) $\hat{\lambda} = \hat{\lambda}(\varepsilon) \rightarrow \infty$ in \mathbb{C}_d ($\varepsilon \rightarrow 0$).

We analyze here only the case (I) because we can treat the case (II) similarly and we conclude that $g_N(\varepsilon; \hat{\lambda}) \neq 0$ for any $\hat{\lambda}$ satisfying the case (II). Then, we suppose that $\hat{\lambda} = O(1)$ as $\varepsilon \rightarrow 0$.

3.1.1. Construction of $\bar{V}_1, \bar{V}_2, \bar{V}_3$

First we solve the problem (3.8) as we have done in §2.2.1. Using the transformation $y = x/l(\varepsilon)$ in (3.8), we have

$$\begin{cases} \varepsilon^2 p_{yy} + l(\varepsilon)^2((1 - 3u^2 - \varepsilon^2 \hat{\lambda})p - \varepsilon \alpha q - \varepsilon \beta r) = 0, \\ q_{yy} + l(\varepsilon)^2(p - (1 + \hat{\tau} \hat{\lambda})q) = 0, \\ D^2 r_{yy} + l(\varepsilon)^2(p - (1 + \hat{\theta} \hat{\lambda})r) = 0, \\ (p_y, q_y, r_y)(0) = (0, 0, 0), \\ (p, q, r)(1) = \left(\frac{a}{\varepsilon}, b, c\right). \end{cases} \quad y \in (0, 1) \quad (3.12)$$

We first consider outer approximations of the form

$$\begin{cases} p(y) = P_0(y) + \varepsilon P_1(y) + \varepsilon^2 P_2(y) + \cdots, \\ q(y) = Q_0(y) + \varepsilon Q_1(y) + \varepsilon^2 Q_2(y) + \cdots, \\ r(y) = R_0(y) + \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \cdots. \end{cases} \quad (3.13)$$

Here we note that in an outer region, the part of the inner approximations of the standing pulse solutions $(u, v, w)(x; \varepsilon)$ decays exponentially to 0 when $\varepsilon \rightarrow 0$. Substituting (3.13) into (3.12), we equate the coefficients of the same powers of ε .

$O(\varepsilon^0)$:

$$\begin{cases} -2P_0 = 0, \\ Q_0'' + l_0^2(P_0 - (1 + \hat{\tau} \hat{\lambda})Q_0) = 0, \\ D^2 R_0'' + l_0^2(P_0 - (1 + \hat{\theta} \hat{\lambda})R_0) = 0, \\ Q_0'(0) = 0, \quad Q_0(1) = b, \\ R_0'(0) = 0, \quad R_0(1) = c. \end{cases} \quad y \in (0, 1)$$

$P_0 = 0$ and then Q_0 satisfies

$$\begin{cases} Q_0'' - l_0^2(1 + \hat{\tau} \hat{\lambda})Q_0 = 0, \\ Q_0'(0) = 0, \quad Q_0(1) = b. \end{cases} \quad y \in (0, 1)$$

We easily have $Q_0(y; \hat{\lambda}; b) = b \cosh(l_0 \mu(\hat{\lambda})y) / \cosh(l_0 \mu(\hat{\lambda}))$ and similarly to Q_0 , $R_0(y; \hat{\lambda}; c) = c \cosh(l_0 \nu(\hat{\lambda})y) / \cosh(l_0 \nu(\hat{\lambda}))$, where $\mu(\hat{\lambda}) = \sqrt{1 + \hat{\tau} \hat{\lambda}}$ and $\nu(\hat{\lambda}) = \sqrt{1 + \hat{\theta} \hat{\lambda}}/D$.

$O(\varepsilon^1) :$

$$\begin{cases} -2P_1 - \alpha Q_0 - \beta R_0 = 0, \\ Q_1'' + l_0^2(P_1 - (1 + \hat{\tau}\hat{\lambda})Q_1) + 2l_0l_1(P_0 - (1 + \hat{\tau}\hat{\lambda})Q_0) = 0, & y \in (0, 1) \\ D^2R_1'' + l_0^2(P_1 - (1 + \hat{\theta}\hat{\lambda})R_1) + 2l_0l_1(P_0 - (1 + \hat{\theta}\hat{\lambda})R_0) = 0, \\ Q_1'(0) = 0, \quad Q_1(1) = -q_0(0; a), \\ R_1'(0) = 0, \quad R_1(1) = -r_0(0; a), \end{cases}$$

where $q_0(0; a)$ and $r_0(0; a)$ will be determined later. We have $P_1(y; \hat{\lambda}; b, c) = -(\alpha Q_0(y; \hat{\lambda}; b) + \beta R_0(y; \hat{\lambda}; c))/2$ and then from Lemma 2.1,

$$\begin{aligned} Q_1(y; \hat{\lambda}; a, b, c) &= e^{l_0\mu(\hat{\lambda})y} \left\{ \frac{1}{2l_0\mu(\hat{\lambda})} \int_1^y e^{-l_0\mu(\hat{\lambda})s} \bar{f}_1(s; \hat{\lambda}; b, c) ds \right. \\ &\quad + \frac{e^{-l_0\mu(\hat{\lambda})}}{2l_0\mu(\hat{\lambda}) \cosh(l_0\mu(\hat{\lambda}))} \int_0^1 \cosh(l_0\mu(\hat{\lambda})s) \bar{f}_1(s; \hat{\lambda}; b, c) ds \\ &\quad \left. - \frac{q_0(0; a)}{2 \cosh(l_0\mu(\hat{\lambda}))} \right\} + e^{-l_0\mu(\hat{\lambda})y} \left\{ -\frac{1}{2l_0\mu(\hat{\lambda})} \int_1^y e^{l_0\mu(\hat{\lambda})s} \bar{f}_1(s; \hat{\lambda}; b, c) ds \right. \\ &\quad \left. - \frac{e^{l_0\mu(\hat{\lambda})}}{2l_0\mu(\hat{\lambda}) \cosh(l_0\mu(\hat{\lambda}))} \int_0^1 \cosh(l_0\mu(\hat{\lambda})s) \bar{f}_1(s; \hat{\lambda}; b, c) ds - \frac{q_0(0; a)}{2 \cosh(l_0\mu(\hat{\lambda}))} \right\}, \\ R_1(y; \hat{\lambda}; a, b, c) &= e^{l_0\nu(\hat{\lambda})y} \left\{ \frac{1}{2l_0\nu(\hat{\lambda})} \int_1^y e^{-l_0\nu(\hat{\lambda})s} \bar{g}_1(s; \hat{\lambda}; b, c) ds \right. \\ &\quad + \frac{e^{-l_0\nu(\hat{\lambda})}}{2l_0\nu(\hat{\lambda}) \cosh(l_0\nu(\hat{\lambda}))} \int_0^1 \cosh(l_0\nu(\hat{\lambda})s) \bar{g}_1(s; \hat{\lambda}; b, c) ds \\ &\quad \left. - \frac{r_0(0; a)}{2 \cosh(l_0\nu(\hat{\lambda}))} \right\} + e^{-l_0\nu(\hat{\lambda})y} \left\{ -\frac{1}{2l_0\nu(\hat{\lambda})} \int_1^y e^{l_0\nu(\hat{\lambda})s} \bar{g}_1(s; \hat{\lambda}; b, c) ds \right. \\ &\quad \left. - \frac{e^{l_0\nu(\hat{\lambda})}}{2l_0\nu(\hat{\lambda}) \cosh(l_0\nu(\hat{\lambda}))} \int_0^1 \cosh(l_0\nu(\hat{\lambda})s) \bar{g}_1(s; \hat{\lambda}; b, c) ds - \frac{r_0(0; a)}{2 \cosh(l_0\nu(\hat{\lambda}))} \right\}, \end{aligned}$$

where $\bar{f}_1(y; \hat{\lambda}; b, c) = -l_0^2 P_1(y; \hat{\lambda}; b, c) + 2l_0l_1(1 + \hat{\tau}\hat{\lambda})Q_0(y; \hat{\lambda}; b)$ and $\bar{g}_1(y; \hat{\lambda}; b, c) = -(l_0^2 P_1(y; \hat{\lambda}; b, c) - 2l_0l_1(1 + \hat{\theta}\hat{\lambda})R_0(y; \hat{\lambda}; c))/D^2$.

$O(\varepsilon^2)$:

$$\left\{ \begin{array}{l} l_0^2(-6U_0U_1P_1 - 2P_2) + 2l_0l_1(-2P_1) - l_0^2(\alpha Q_1 + \beta R_1) \\ \quad - 2l_0l_1(\alpha Q_0 + \beta R_0) = 0, \\ Q_2'' + l_0^2(P_2 - (1 + \hat{\tau}\hat{\lambda})Q_2) + 2l_0l_1(P_1 - (1 + \hat{\tau}\hat{\lambda})Q_1) \\ \quad + (l_1^2 + 2l_0l_2)(P_0 - (1 + \hat{\tau}\hat{\lambda})Q_0) = 0, \quad y \in (0, 1) \\ D^2R_2'' + l_0^2(P_2 - (1 + \hat{\theta}\hat{\lambda})R_2) + 2l_0l_1(P_1 - (1 + \hat{\theta}\hat{\lambda})R_1) \\ \quad + (l_1^2 + 2l_0l_2)(P_0 - (1 + \hat{\theta}\hat{\lambda})R_0) = 0, \\ Q_2'(0) = 0, \quad Q_2(1) = -q_1(0; a), \\ R_2'(0) = 0, \quad R_2(1) = -r_1(0; a), \end{array} \right.$$

where $q_1(0; a)$ and $r_1(0; a)$ will be determined later. Similarly to the case of $O(\varepsilon^1)$, we have

$$\begin{aligned} P_2(y; \hat{\lambda}; a, b, c) &= -\frac{1}{2l_0} \left\{ (6l_0U_1(y) + 4l_1)P_1(y; \hat{\lambda}; b, c) \right. \\ &\quad + l_0(\alpha Q_1(y; \hat{\lambda}; a, b, c) + \beta R_1(y; \hat{\lambda}; a, b, c)) \\ &\quad \left. + 2l_1(\alpha Q_0(y; \hat{\lambda}; b) + \beta R_0(y; \hat{\lambda}; c)) \right\}, \\ Q_2(y; \hat{\lambda}; a, b, c) &= e^{l_0\mu(\hat{\lambda})y} \left\{ \frac{1}{2l_0\mu(\hat{\lambda})} \int_1^y e^{-l_0\mu(\hat{\lambda})s} \bar{f}_2(s; \hat{\lambda}; a, b, c) ds \right. \\ &\quad + \frac{e^{-l_0\mu(\hat{\lambda})}}{2l_0\mu(\hat{\lambda}) \cosh(l_0\mu(\hat{\lambda}))} \int_0^1 \cosh(l_0\mu(\hat{\lambda})s) \bar{f}_2(s; \hat{\lambda}; a, b, c) ds \\ &\quad \left. - \frac{q_1(0; a)}{2 \cosh(l_0\mu(\hat{\lambda}))} \right\} + e^{-l_0\mu(\hat{\lambda})y} \\ &\quad \times \left\{ -\frac{1}{2l_0\mu(\hat{\lambda})} \int_1^y e^{l_0\mu(\hat{\lambda})s} \bar{f}_2(s; \hat{\lambda}; a, b, c) ds - \frac{e^{l_0\mu(\hat{\lambda})}}{2l_0\mu(\hat{\lambda}) \cosh(l_0\mu(\hat{\lambda}))} \right. \\ &\quad \left. \times \int_0^1 \cosh(l_0\mu(\hat{\lambda})s) \bar{f}_2(s; \hat{\lambda}; a, b, c) ds - \frac{q_1(0; a)}{2 \cosh(l_0\mu(\hat{\lambda}))} \right\}, \\ R_2(y; \hat{\lambda}; a, b, c) &= e^{l_0\nu(\hat{\lambda})y} \left\{ \frac{1}{2l_0\nu(\hat{\lambda})} \int_1^y e^{-l_0\nu(\hat{\lambda})s} \bar{g}_2(s; \hat{\lambda}; a, b, c) ds \right. \\ &\quad + \frac{e^{-l_0\nu(\hat{\lambda})}}{2l_0\nu(\hat{\lambda}) \cosh(l_0\nu(\hat{\lambda}))} \int_0^1 \cosh(l_0\nu(\hat{\lambda})s) \bar{g}_2(s; \hat{\lambda}; a, b, c) ds \\ &\quad \left. - \frac{r_1(0; a)}{2 \cosh(l_0\nu(\hat{\lambda}))} \right\} + e^{-l_0\nu(\hat{\lambda})y} \end{aligned}$$

$$\times \left\{ -\frac{1}{2l_0\nu(\hat{\lambda})} \int_1^y e^{l_0\nu(\hat{\lambda})s} \bar{g}_2(s; \hat{\lambda}; a, b, c) ds - \frac{e^{l_0\nu(\hat{\lambda})}}{2l_0\nu(\hat{\lambda}) \cosh(l_0\nu(\hat{\lambda}))} \right. \\ \left. \times \int_0^1 \cosh(l_0\nu(\hat{\lambda})s) \bar{g}_2(s; \hat{\lambda}; a, b, c) ds - \frac{r_1(0; a)}{2 \cosh(l_0\nu(\hat{\lambda}))} \right\},$$

where $\bar{f}_2(y; \hat{\lambda}; a, b, c) = -l_0^2 P_2(y; \hat{\lambda}; b, c) - 2l_0 l_1 (P_1(y; \hat{\lambda}; b, c) - (1 + \hat{\tau}\hat{\lambda})Q_1(y; \hat{\lambda}; a, b, c)) - (l_1^2 + 2l_0 l_2)(P_0(y) - (1 + \hat{\tau}\hat{\lambda})Q_0(y; \hat{\lambda}; c))$ and $\bar{g}_2(y; \hat{\lambda}; a, b, c) = \{-l_0^2 P_2(y; \hat{\lambda}; b, c) - 2l_0 l_1 \times (P_1(y; \hat{\lambda}; b, c) - (1 + \hat{\theta}\hat{\lambda})R_1(y; \hat{\lambda}; a, b, c)) - (l_1^2 + 2l_0 l_2)(P_0(y) - (1 + \hat{\theta}\hat{\lambda})R_0(y; \hat{\lambda}; c))\}/D^2$.

Since the p component dose not satisfy the boundary condition at $y = 1$, we have to modify this defect. Hence, we introduce the stretched variable $\xi = (y - 1)/\varepsilon$ and look for inner approximations p_i, q_i, r_i ($i = 0, 1, 2$) of the form

$$\begin{cases} p(y) &= P_0(y) + \varepsilon P_1(y; \hat{\lambda}; b, c) + \varepsilon^2 P_2(y; \hat{\lambda}; a, b, c) + \cdots \\ &\quad + \frac{1}{\varepsilon} p_0\left(\frac{y-1}{\varepsilon}\right) + p_1\left(\frac{y-1}{\varepsilon}\right) + \varepsilon p_2\left(\frac{y-1}{\varepsilon}\right) + \cdots, \\ q(y) &= Q_0(y; \hat{\lambda}; b) + \varepsilon Q_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 Q_2(y; \hat{\lambda}; a, b, c) + \cdots \\ &\quad + \varepsilon q_0\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^2 q_1\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^3 q_2\left(\frac{y-1}{\varepsilon}\right) + \cdots, \\ r(y) &= R_0(y; \hat{\lambda}; c) + \varepsilon R_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 R_2(y; \hat{\lambda}; a, b, c) + \cdots \\ &\quad + \varepsilon r_0\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^2 r_1\left(\frac{y-1}{\varepsilon}\right) + \varepsilon^3 r_2\left(\frac{y-1}{\varepsilon}\right) + \cdots, \end{cases} \quad (3.14)$$

so that $(p, q, r)(y)$ satisfies the boundary condition at $y = 1$. Substituting this into (3.12) and using $\xi = (y - 1)/\varepsilon$, we equate the coefficients of the same powers of ε .

$O(\varepsilon^{-1})$:

$$\begin{cases} \ddot{p}_0 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)p_0 = 0, \\ \ddot{q}_0 + l_0^2 p_0 = 0, \\ D^2 \ddot{r}_0 + l_0^2 p_0 = 0, \\ p_0(-\infty) = 0, \quad p_0(0) = a, \\ q_0(-\infty) = 0, \quad \dot{q}_0(-\infty) = 0, \\ r_0(-\infty) = 0, \quad \dot{r}_0(-\infty) = 0. \end{cases} \quad \xi \in (-\infty, 0)$$

By the first and fourth equations, we have $p_0(\xi; a) = a\dot{\phi}_0(\xi)/\dot{\phi}_0(0)$, and then $q_0(\xi; a) = -l_0^2 \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} p_0(\zeta) d\zeta d\eta$ and $r_0(\xi; a) = -l_0^2 \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} p_0(\zeta) d\zeta d\eta / D^2$.

$O(\varepsilon^0)$:

$$\left\{ \begin{array}{l} \ddot{p}_1 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)p_1 - 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)p_0 \\ \quad - 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1)p_0 = 0, \\ \ddot{q}_1 + l_0^2p_1 + 2l_0l_1p_0 = 0, \quad \xi \in (-\infty, 0) \\ D^2\ddot{r}_1 + l_0^2p_1 + 2l_0l_1p_0 = 0, \\ p_1(-\infty) = 0, \quad p_1(0) = -P_0(1) = 0, \\ q_1(-\infty) = 0, \quad \dot{q}_1(-\infty) = 0, \\ r_1(-\infty) = 0, \quad \dot{r}_1(-\infty) = 0. \end{array} \right. \quad (3.15)$$

Since p_1 satisfies the following equations:

$$\left\{ \begin{array}{l} \ddot{p}_1 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)p_1 = 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)p_0 \\ \quad + 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1)p_0, \quad \xi \in (-\infty, 0) \\ p_1(-\infty) = 0, \quad p_1(0) = 0, \end{array} \right.$$

we obtain

$$\begin{aligned} p_1(\xi; a) = & -\dot{\phi}_0(\xi) \int_{\xi}^0 (\dot{\phi}_0(\eta))^{-2} \int_{-\infty}^{\eta} \dot{\phi}_0(\zeta) \{2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)p_0 \\ & + 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1)p_0\} d\zeta d\eta \end{aligned}$$

and then

$$\begin{aligned} q_1(\xi; a) &= - \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \{l_0^2p_1(\zeta) + 2l_0l_1p_0(\zeta)\} d\zeta d\eta, \\ r_1(\xi; a) &= -\frac{1}{D^2} \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \{l_0^2p_1(\zeta) + 2l_0l_1p_0(\zeta)\} d\zeta d\eta. \end{aligned}$$

$O(\varepsilon^1)$:

$$\left\{ \begin{array}{l} \ddot{p}_2 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)(P_1(1; \hat{\lambda}; b, c) + p_2) - 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1)p_1 \\ \quad - 3l_0^2\{(U_1(1) + \phi_1)^2 + 2(1 + \phi_0)(U_1'(1)\xi + U_2(1) + \phi_2)\}p_0 \\ \quad - 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)p_1 - 12l_0l_1(1 + \phi_0)(U_1(1) + \phi_1)p_0 \\ \quad - (l_1^2 + 2l_0l_2)(2 + 6\phi_0 + 3\phi_0^2)p_0 - l_0^2(\hat{\lambda}p_0 + \alpha b + \beta c) = 0, \\ \ddot{q}_2 + l_0^2(p_2 - (1 + \hat{\tau}\hat{\lambda})q_0) + 2l_0l_1p_1 + (l_1^2 + 2l_0l_2)p_0 = 0, \quad \xi \in (-\infty, 0) \\ D^2\ddot{r}_2 + l_0^2(p_2 - (1 + \hat{\theta}\hat{\lambda})r_0) + 2l_0l_1p_1 + (l_1^2 + 2l_0l_2)p_0 = 0, \\ p_2(-\infty) = 0, \quad p_2(0) = -P_1(1; \hat{\lambda}; b, c) = (\alpha b + \beta c)/2, \\ q_2(-\infty) = 0, \quad \dot{q}_2(-\infty) = 0, \\ r_2(-\infty) = 0, \quad \dot{r}_2(-\infty) = 0. \end{array} \right. \quad (3.16)$$

Similarly to the case of $O(\varepsilon^0)$, we have

$$\begin{aligned}
p_2(\xi; \hat{\lambda}; a, b, c) &= \frac{1}{2}(\alpha b + \beta c)\dot{\phi}_0(\xi)/\dot{\phi}_0(0) - \dot{\phi}_0(\xi) \int_{\xi}^0 \left(\dot{\phi}_0(\eta)\right)^{-2} \int_{-\infty}^{\eta} \dot{\phi}_0(\zeta) \\
&\quad \times \left[l_0^2(2 + 6\phi_0 + 3\phi_0^2)P_1(1; \hat{\lambda}; b, c) + 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1)p_1 \right. \\
&\quad + 3l_0^2\{(U_1(1) + \phi_1)^2 + 2(1 + \phi_0)(U_1'(1)\zeta + U_2(1) + \phi_2)\}p_0 \\
&\quad + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)p_1 + 12l_0l_1(1 + \phi_0)(U_1(1) + \phi_1)p_0 \\
&\quad \left. + (l_1^2 + 2l_0l_2)(2 + 6\phi_0 + 3\phi_0^2)p_0 + l_0^2(\hat{\lambda}p_0 + \alpha b + \beta c) \right] d\zeta d\eta, \\
q_2(\xi; \hat{\lambda}; a, b, c) &= - \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \left\{ l_0^2(p_2 - (1 + \hat{\tau}\hat{\lambda})q_0) + 2l_0l_1p_1 \right. \\
&\quad \left. + (l_1^2 + 2l_0l_2)p_0 \right\} d\zeta d\eta, \\
r_2(\xi; \hat{\lambda}; a, b, c) &= - \frac{1}{D^2} \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \left\{ l_0^2(p_2 - (1 + \hat{\theta}\hat{\lambda})r_0) + 2l_0l_1p_1 \right. \\
&\quad \left. + (l_1^2 + 2l_0l_2)p_0 \right\} d\zeta d\eta.
\end{aligned}$$

For any fixed $\hat{\lambda}^* \in \mathbb{C}_d$, let us define $\Delta_{\nu} \equiv \{\hat{\lambda} \in \mathbb{C}_d \mid |\hat{\lambda} - \hat{\lambda}^*| \leq \nu\}$ ($\nu > 0$). Using the above approximate solutions, we can construct uniform approximations up to $O(\varepsilon^2)$ of (3.12) for any $\hat{\lambda} \in \Delta_{\nu}$, which take the form

$$\left\{ \begin{array}{l} P(y; \varepsilon; \hat{\lambda}; a, b, c) = P_0(y) + \varepsilon P_1(y; \hat{\lambda}; b, c) + \varepsilon^2 P_2(y; \hat{\lambda}; a, b, c) \\ \quad + \theta(y) \left(\frac{1}{\varepsilon} p_0\left(\frac{y-1}{\varepsilon}; a\right) + p_1\left(\frac{y-1}{\varepsilon}; a\right) + \varepsilon^2 p_2\left(\frac{y-1}{\varepsilon}; \hat{\lambda}; a, b, c\right) \right), \\ Q(y; \varepsilon; \hat{\lambda}; a, b, c) = Q_0(y; \hat{\lambda}; b) + \varepsilon Q_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 Q_2(y; \hat{\lambda}; a, b, c) \\ \quad + \theta(y) \left(\varepsilon q_0\left(\frac{y-1}{\varepsilon}; a\right) + \varepsilon^2 q_1\left(\frac{y-1}{\varepsilon}; a\right) + \varepsilon^3 q_2\left(\frac{y-1}{\varepsilon}; \hat{\lambda}; a, b, c\right) \right), \\ R(y; \varepsilon; \hat{\lambda}; a, b, c) = R_0(y; \hat{\lambda}; c) + \varepsilon R_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 R_2(y; \hat{\lambda}; a, b, c) \\ \quad + \theta(y) \left(\varepsilon r_0\left(\frac{y-1}{\varepsilon}; a\right) + \varepsilon^2 r_1\left(\frac{y-1}{\varepsilon}; a\right) + \varepsilon^3 r_2\left(\frac{y-1}{\varepsilon}; \hat{\lambda}; a, b, c\right) \right), \end{array} \right.$$

where $\theta(y)$ is the same function as is defined in §2.1.3. Obviously $(P, Q, R)(y; \varepsilon; \hat{\lambda}; a, b, c)$ satisfies the boundary condition at $y = 0$, but it does not satisfy that at $y = 1$, because it becomes

$$\left\{ \begin{array}{l} P(1; \varepsilon; \hat{\lambda}; a, b, c) = \frac{a}{\varepsilon}, \\ Q(1; \varepsilon; \hat{\lambda}; a, b, c) = b + \varepsilon^3 q_2(0; \hat{\lambda}; a, b, c), \\ W(1; \varepsilon; \hat{\lambda}; a, b, c) = c + \varepsilon^3 r_2(0; \hat{\lambda}; a, b, c). \end{array} \right.$$

So we modify it a little to satisfy the boundary condition at $y = 1$ exactly and add the remainder term $(\varepsilon^2 \tilde{P}, \varepsilon^2 \tilde{Q}, \varepsilon^2 \tilde{R})$ to it and look for exact

solutions of (3.12) of the form

$$\begin{cases} p(y; \varepsilon; \hat{\lambda}; a, b, c) &= P(y; \varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{P}(y; \varepsilon; \hat{\lambda}; a, b, c), \\ q(y; \varepsilon; \hat{\lambda}; a, b, c) &= Q(y; \varepsilon; \hat{\lambda}; a, b, c) + q^*(\varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{Q}(y; \varepsilon; \hat{\lambda}; a, b, c), \\ r(y; \varepsilon; \hat{\lambda}; a, b, c) &= R(y; \varepsilon; \hat{\lambda}; a, b, c) + r^*(\varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{R}(y; \varepsilon; \hat{\lambda}; a, b, c), \end{cases}$$

where $q^*(\varepsilon; \hat{\lambda}; a, b, c) = -\varepsilon^3 q_2(0; \hat{\lambda}; a, b, c)$ and $r^*(\varepsilon; \hat{\lambda}; a, b, c) = -\varepsilon^3 r_2(0; \hat{\lambda}; a, b, c)$.

Substituting this into (3.12), we have

$$\begin{cases} \varepsilon^2(P_{yy} + \varepsilon^2 \tilde{P}_{yy}) + l(\varepsilon)^2 \left[(1 - 3u^2 - \varepsilon^2 \tilde{\lambda})(P + \varepsilon^2 \tilde{P}) \right. \\ \quad \left. - \varepsilon \left(\alpha(Q + q^* + \varepsilon^2 \tilde{Q}) + \beta(R + r^* + \varepsilon^2 \tilde{R}) \right) \right] = 0, \\ Q_{yy} + \varepsilon^2 \tilde{Q}_{yy} + l(\varepsilon)^2 \left(P + \varepsilon^2 \tilde{P} - (1 + \tilde{\tau} \tilde{\lambda})(Q + q^* + \varepsilon^2 \tilde{Q}) \right) = 0, \\ D^2(R_{yy} + \varepsilon^2 \tilde{R}_{yy}) + l(\varepsilon)^2 \left(P + \varepsilon^2 \tilde{P} - (1 + \tilde{\theta} \tilde{\lambda})(R + r^* + \varepsilon^2 \tilde{R}) \right) = 0, \\ (\tilde{P}_y, \tilde{Q}_y, \tilde{R}_y)(0; \varepsilon) = (0, 0, 0), \quad (\tilde{P}, \tilde{Q}, \tilde{R})(1; \varepsilon) = (0, 0, 0). \end{cases} \quad (3.17)$$

Then, for $t = (\tilde{P}, \tilde{Q}, \tilde{R})$, we define the following operator $T(t; \varepsilon; \hat{\lambda}; a, b, c) = (T_1, T_2, T_3)(t; \varepsilon; \hat{\lambda}; a, b, c)$:

$$\begin{cases} T_1(t; \varepsilon; \hat{\lambda}; a, b, c) \equiv \varepsilon^2 \tilde{P}_{yy} + l(\varepsilon)^2 ((1 - 3u^2 - \varepsilon^2 \hat{\lambda})\tilde{P} - \varepsilon \alpha \tilde{Q} - \varepsilon \beta \tilde{R}) \\ \quad + \frac{1}{\varepsilon^2} \left[P_{yy} + l(\varepsilon)^2 \left\{ (1 - 3u^2 - \varepsilon^2 \hat{\lambda})P \right. \right. \\ \quad \left. \left. - \varepsilon (\alpha(Q + q^*) - \beta(R + r^*)) \right\} \right], \\ T_2(t; \varepsilon; \hat{\lambda}; a, b, c) \equiv \tilde{Q}_{yy} + l(\varepsilon)^2 (\tilde{P} - (1 + \hat{\tau} \hat{\lambda})\tilde{Q}) \\ \quad + \frac{1}{\varepsilon^2} \left[Q_{yy} + l(\varepsilon)^2 \left(P - (1 + \hat{\tau} \hat{\lambda})(Q + q^*) \right) \right], \\ T_3(t; \varepsilon; \hat{\lambda}; a, b, c) \equiv D^2 \tilde{R}_{yy} + l(\varepsilon)^2 (\tilde{P} - (1 + \hat{\theta} \hat{\lambda})\tilde{R}) \\ \quad + \frac{1}{\varepsilon^2} \left[D^2 R_{yy} + l(\varepsilon)^2 \left(P - (1 + \hat{\theta} \hat{\lambda})(R + r^*) \right) \right] \end{cases}$$

from $X \times (0, \varepsilon_0) \times \Delta_\nu$ to Y , where X and Y are defined in §2.1.3. We find that $T(t; \varepsilon; \hat{\lambda})$ is the continuously differentiable operator and (3.17) is equivalent to $T(t; \varepsilon; \hat{\lambda}; a, b, c) = 0$.

Lemma 3.4. *For any given $\hat{\lambda}^* \in \mathbb{C}_d$, there exist positive constants ε_0, ν_0 and C such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\hat{\lambda} \in \Delta_{\nu_0}$,*

$$(i) \quad \|T_t(t_1; \varepsilon; \hat{\lambda}; a, b, c) - T_t(t_2; \varepsilon; \hat{\lambda}; a, b, c)\|_{X \rightarrow Y} \leq C \|t_1 - t_2\|_X$$

for any $t_1, t_2 \in X$,

- (ii) $\|T(0; \varepsilon; \hat{\lambda}; a, b, c)\|_Y \leq C\varepsilon,$
- (iii) $\|T_t^{-1}(0; \varepsilon; \hat{\lambda}; a, b, c)\|_{Y \rightarrow X} \leq C.$

Moreover the results (i) - (iii) hold for $\partial T / \partial \hat{\lambda}$ in place of T .

By this lemma, we can apply the Implicit Function Theorem to $T(t; \varepsilon; \hat{\lambda}; a, b, c) = 0$. Thus, under the same assumption of Lemma 3.4, there exists $t(\varepsilon; \hat{\lambda}; a, b, c) \in X$ satisfying $T(t; \varepsilon; \hat{\lambda}; a, b, c) = 0$. $t(\varepsilon; \hat{\lambda}; a, b, c)$ and $\partial t / \partial \hat{\lambda}(\varepsilon; \hat{\lambda}; a, b, c)$ are uniformly continuous with respect to $(\varepsilon, \hat{\lambda}) \in (0, \varepsilon_0) \times \Delta_{\nu_0}$ in the X -topology and satisfy

$$\begin{aligned} \|t(\varepsilon; \hat{\lambda}; a, b, c)\|_X, \quad \|\partial t / \partial \hat{\lambda}(\varepsilon; \hat{\lambda}; a, b, c)\|_X &= o(1) \\ &\text{as } \varepsilon \rightarrow 0 \text{ uniformly in } \hat{\lambda} \in \Delta_{\nu_0}. \end{aligned}$$

Thus, we have the exact solutions of (3.12) on $[0, 1]$ of the form

$$\begin{cases} p(y; \varepsilon; \hat{\lambda}; a, b, c) = P(y; \varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{P}(y; \varepsilon; \hat{\lambda}; a, b, c), \\ q(y; \varepsilon; \hat{\lambda}; a, b, c) = Q(y; \varepsilon; \hat{\lambda}; a, b, c) + q^*(\varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{Q}(y; \varepsilon; \hat{\lambda}; a, b, c), \\ r(y; \varepsilon; \hat{\lambda}; a, b, c) = R(y; \varepsilon; \hat{\lambda}; a, b, c) + r^*(\varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{R}(y; \varepsilon; \hat{\lambda}; a, b, c), \end{cases}$$

which implies that (3.8) has the following exact solutions on $[0, l(\varepsilon)]$:

$$\begin{cases} p^{(1)}(x; \varepsilon; \hat{\lambda}; a, b, c) &= P(\frac{x}{l(\varepsilon)}; \varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{P}(\frac{x}{l(\varepsilon)}; \varepsilon; \hat{\lambda}; a, b, c), \\ q^{(1)}(x; \varepsilon; \hat{\lambda}; a, b, c) &= Q(\frac{x}{l(\varepsilon)}; \varepsilon; \hat{\lambda}; a, b, c) + q^*(\varepsilon; \hat{\lambda}; a, b, c) \\ &\quad + \varepsilon^2 \tilde{Q}(\frac{x}{l(\varepsilon)}; \varepsilon; \hat{\lambda}; a, b, c), \\ r^{(1)}(x; \varepsilon; \hat{\lambda}; a, b, c) &= R(\frac{x}{l(\varepsilon)}; \varepsilon; \hat{\lambda}; a, b, c) + r^*(\varepsilon; \hat{\lambda}; a, b, c) \\ &\quad + \varepsilon^2 \tilde{R}(\frac{x}{l(\varepsilon)}; \varepsilon; \hat{\lambda}; a, b, c). \end{cases} \quad (3.18)$$

3.1.2. Construction of $\bar{V}_4, \bar{V}_5, \bar{V}_6$

Next, we consider the problem (3.9). By using the transformation $y = x - l(\varepsilon)$, we have

$$\begin{cases} \varepsilon^2 p_{yy} + (1 - 3u^2 - \varepsilon^2 \hat{\lambda})p - \varepsilon \alpha q - \varepsilon \beta r = 0, \\ q_{yy} + p - (1 + \hat{\tau} \hat{\lambda})q = 0, \\ D^2 r_{yy} + p - (1 + \hat{\theta} \hat{\lambda})r = 0, \\ (p, q, r)(0) = \left(\frac{a}{\varepsilon}, b, c\right), \\ (p, q, r)(\infty) = (0, 0, 0). \end{cases} \quad y \in (0, \infty) \quad (3.19)$$

First, we consider outer approximations of the form

$$\begin{cases} p(y) = P_0(y) + \varepsilon P_1(y) + \varepsilon^2 P_2(y) + \cdots, \\ q(y) = Q_0(y) + \varepsilon Q_1(y) + \varepsilon^2 Q_2(y) + \cdots, \\ r(y) = R_0(y) + \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \cdots. \end{cases} \quad (3.20)$$

Substituting this into (3.19), we equate the coefficients of the same powers of ε

$O(\varepsilon^0)$:

$$\begin{cases} -2P_0 = 0, \\ Q_0'' + P_0 - (1 + \hat{\tau}\hat{\lambda})Q_0 = 0, \\ D^2 R_0'' + P_0 - (1 + \hat{\theta}\hat{\lambda})R_0 = 0, \\ Q_0(0) = b, \quad Q_0(\infty) = 0, \\ R_0(0) = c, \quad R_0(\infty) = 0. \end{cases} \quad y \in (0, \infty)$$

$P_0 = 0$ and Q_0 satisfies

$$\begin{cases} Q_0'' - (1 + \hat{\tau}\hat{\lambda})Q_0 = 0, \\ Q_0(0) = b, \quad Q_0(\infty) = 0. \end{cases} \quad y \in (0, \infty)$$

We easily find that $Q_0(y; \hat{\lambda}; b) = be^{-\mu(\hat{\lambda})y}$ and similarly to Q_0 , $R_0(y; \hat{\lambda}; c) = ce^{-\nu(\hat{\lambda})y}$.

$O(\varepsilon^1)$:

$$\begin{cases} -2P_1 - (\alpha Q_0 + \beta R_0) = 0, \\ Q_1'' + P_1 - (1 + \hat{\tau}\hat{\lambda})Q_1 = 0, \\ D^2 R_1'' + P_1 - (1 + \hat{\theta}\hat{\lambda})R_1 = 0, \\ Q_1(0) = -q_0(0; a), \quad Q_1(\infty) = 0, \\ R_1(0) = -r_0(0; a), \quad R_1(\infty) = 0, \end{cases} \quad y \in (0, \infty)$$

where $q_0(0; a)$ and $r_0(0; a)$ will be determined later. We have $P_1(y; \hat{\lambda}; b, c) = -(\alpha Q_0(y; \hat{\lambda}; b) + \beta R_0(y; \hat{\lambda}; c))/2$ and Q_1 satisfies

$$\begin{cases} Q_1'' - (1 + \hat{\tau}\hat{\lambda})Q_1 = -P_1, \\ Q_1(0) = -q_0(0; a), \quad Q_1(\infty) = 0. \end{cases} \quad y \in (0, \infty)$$

By using the method of constant variation, we get

$$\begin{aligned}
Q_1(y; \hat{\lambda}; a, b, c) &= -q_0(0; a)e^{-\mu(\hat{\lambda})y} - e^{-\mu(\hat{\lambda})y} \\
&\quad \times \int_0^y e^{2\mu(\hat{\lambda})x} \int_x^\infty e^{-\mu(\hat{\lambda})s} \bar{f}_3(s; \hat{\lambda}; b, c) ds dx, \\
R_1(y; \hat{\lambda}; a, b, c) &= -r_0(0; a)e^{-\nu(\hat{\lambda})y} - e^{-\nu(\hat{\lambda})y} \\
&\quad \times \frac{1}{D^2} \int_0^y e^{2\nu(\hat{\lambda})x} \int_x^\infty e^{-\nu(\hat{\lambda})s} \bar{f}_3(s; \hat{\lambda}; b, c) ds dx,
\end{aligned}$$

where $\bar{f}_3(y; \hat{\lambda}; b, c) = -P_1(y; \hat{\lambda}; b, c)$.

$O(\varepsilon^2)$:

$$\begin{cases} -2P_2 - 6U_0U_1P_1 - (\alpha Q_1 + \beta R_1) = 0, \\ Q_2'' + P_2 - (1 + \hat{\tau}\hat{\lambda})Q_2 = 0, \\ D^2R_2'' + P_2 - (1 + \hat{\theta}\hat{\lambda})R_2 = 0, \\ Q_2(0) = -q_1(0; a), \quad Q_2(\infty) = 0, \\ R_2(0) = -r_1(0; a), \quad R_2(\infty) = 0. \end{cases} \quad y \in (0, \infty)$$

where $q_1(0; a)$ and $r_1(0; a)$ will be determined later. Similarly to the case of $O(\varepsilon^1)$, we have

$$\begin{aligned}
P_2(y; \hat{\lambda}; a, b, c) &= -\frac{1}{2} \{ -6U_1(y)P_1(y; \hat{\lambda}; b, c) + \alpha Q_1(y; \hat{\lambda}; a, b, c) \\
&\quad + \beta R_1(y; \hat{\lambda}; a, b, c) \}, \\
Q_2(y; \hat{\lambda}; a, b, c) &= -q_1(0; a)e^{-\mu(\hat{\lambda})y} - e^{-\mu(\hat{\lambda})y} \\
&\quad \times \int_0^y e^{2\mu(\hat{\lambda})x} \int_x^\infty e^{-\mu(\hat{\lambda})s} \bar{f}_4(s; \hat{\lambda}; b, c) ds dx, \\
R_2(y; \hat{\lambda}; a, b, c) &= -r_1(0; a)e^{-\nu(\hat{\lambda})y} - e^{-\nu(\hat{\lambda})y} \\
&\quad \times \frac{1}{D^2} \int_0^y e^{2\nu(\hat{\lambda})x} \int_x^\infty e^{-\nu(\hat{\lambda})s} \bar{f}_4(s; \hat{\lambda}; b, c) ds dx,
\end{aligned}$$

where $\bar{f}_4(y; \hat{\lambda}; a, b, c) = -P_2(y; \hat{\lambda}; a, b, c)$.

Since the p component does not satisfy the boundary condition at $y = 0$, we have to modify this defect. Hence, we introduce the stretched variable

$\xi = y/\varepsilon$ and look for inner approximations p_i, q_i, r_i ($i = 0, 1, 2$) of the form

$$\begin{cases} p(y) &= P_0(y) + \varepsilon P_1(y; \hat{\lambda}; b, c) + \varepsilon^2 P_2(y; \hat{\lambda}; a, b, c) + \cdots \\ &\quad + \frac{1}{\varepsilon} p_0(\frac{y}{\varepsilon}) + p_1(\frac{y}{\varepsilon}) + \varepsilon p_2(\frac{y}{\varepsilon}) + \cdots, \\ q(y) &= Q_0(y; \hat{\lambda}; b) + \varepsilon Q_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 Q_2(y; \hat{\lambda}; a, b, c) + \cdots \\ &\quad + \varepsilon q_0(\frac{y}{\varepsilon}) + \varepsilon^2 q_1(\frac{y}{\varepsilon}) + \varepsilon^3 q_2(\frac{y}{\varepsilon}) + \cdots, \\ r(y) &= R_0(y; \hat{\lambda}; c) + \varepsilon R_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 R_2(y; \hat{\lambda}; a, b, c) + \cdots \\ &\quad + \varepsilon r_0(\frac{y}{\varepsilon}) + \varepsilon^2 r_1(\frac{y}{\varepsilon}) + \varepsilon^3 r_2(\frac{y}{\varepsilon}) + \cdots, \end{cases} \quad (3.21)$$

so that $(p, q, r)(y)$ satisfies the boundary condition at $y = 0$. Substituting this into (3.19) and using $\xi = y/\varepsilon$, we equate the coefficient of the same powers of ε .

$O(\varepsilon^{-1})$:

$$\begin{cases} \ddot{p}_0 - (2 - 6\phi_0 + 3\phi_0^2)p_0 = 0, \\ \ddot{q}_0 + p_0 = 0, \\ D^2 \ddot{r}_0 + p_0 = 0, \\ p_0(0) = a, \quad p_0(\infty) = 0, \\ q_0(\infty) = 0, \quad \dot{q}_0(\infty) = 0, \\ r_0(\infty) = 0, \quad \dot{r}_0(\infty) = 0. \end{cases} \quad \xi \in (0, \infty)$$

By the first and fourth equations, we have $p_0(\xi; a) = a\dot{\phi}_0(\xi)/\dot{\phi}_0(0)$, and then $q_0(\xi; a) = -\int_{\xi}^{\infty} \int_{\eta}^{\infty} p_0(\zeta) d\zeta d\eta$ and $r_0(\xi; a) = -\int_{\xi}^{\infty} \int_{\eta}^{\infty} p_0(\zeta) d\zeta d\eta / D^2$.

$O(\varepsilon^0)$:

$$\begin{cases} \ddot{p}_1 - (2 - 6\phi_0 + 3\phi_0^2)p_1 - 6(-1 + \phi_0)(U_1(0) + \phi_1)p_0 = 0, \\ \ddot{q}_1 + p_1 = 0, \\ D^2 \ddot{r}_1 + p_1 = 0, \\ p_1(0) = -P_0(0), \quad p_1(\infty) = 0, \\ q_1(\infty) = 0, \quad \dot{q}_1(\infty) = 0, \\ r_1(\infty) = 0, \quad \dot{r}_1(\infty) = 0. \end{cases} \quad \xi \in (0, \infty)$$

Since p_1 satisfies the following equations:

$$\begin{cases} \ddot{p}_1 - (2 - 6\phi_0 + 3\phi_0^2)p_1 = 6(-1 + \phi_0)(U_1(0) + \phi_1)p_0, \\ p_1(0) = -P_0(0) = 0, \quad p_1(\infty) = 0, \end{cases} \quad \xi \in (0, \infty)$$

we obtain

$$p_1(\xi; a) = -\dot{\phi}_0(\xi) \int_0^\xi (\dot{\phi}_0(\eta))^{-2} \int_\eta^\infty \dot{\phi}_0(\zeta) 6(-1 + \phi_0)(U_1(0) + \phi_1) p_0 d\zeta d\eta$$

and then $q_1(\xi; a) = -\int_\xi^\infty \int_\eta^\infty p_1(\zeta) d\zeta d\eta$ and $r_1(\xi; a) = -\int_\xi^\infty \int_\eta^\infty p_1(\zeta) d\zeta d\eta / D^2$.

$O(\varepsilon^1)$:

$$\left\{ \begin{array}{l} \ddot{p}_2 - (2 - 6\phi_0 + 3\phi_0^2)(P_1(0; \hat{\lambda}; b, c) + p_2) - 6(-1 + \phi_0)(U_1(0) + \phi_1)p_1 \\ - 3\{(U_1(0) + \phi_1)^2 + 2(-1 + \phi_0)(U_1'(0)\xi + U_2(0) + \phi_2)\} p_0 \\ - \hat{\lambda}p_0 - \alpha Q_0(0) - \beta R_0(0) = 0, \\ \ddot{q}_2 + p_2 - (1 + \hat{\tau}\hat{\lambda})q_0 = 0, \\ D^2\ddot{r}_2 + p_2 - (1 + \hat{\theta}\hat{\lambda})r_0 = 0, \\ p_2(0) = -P_1(0; \hat{\lambda}; b, c) = \frac{1}{2}(\alpha b + \beta c), \quad p_2(\infty) = 0, \\ q_2(\infty) = 0, \quad \dot{q}_2(\infty) = 0, \\ r_2(\infty) = 0, \quad \dot{r}_2(\infty) = 0. \end{array} \right. \quad \xi \in (0, \infty)$$

Similarly to the case of $O(\varepsilon^0)$, we have

$$\begin{aligned} p_2(\xi; \hat{\lambda}; a, b, c) &= \frac{1}{2}(\alpha b + \beta c) \dot{\phi}_0(\xi) / \dot{\phi}_0(0) \\ &\quad - \dot{\phi}_0(\xi) \int_0^\xi (\dot{\phi}_0(\eta))^{-2} \int_\eta^\infty \dot{\phi}_0(\zeta) [(2 - 6\phi_0 + 3\phi_0^2)P_1(0) \\ &\quad + 6(-1 + \phi_0)(U_1(0) + \phi_1)p_1 + 3\{(U_1(0) + \phi_1)^2 \\ &\quad + 2(-1 + \phi_0)(U_1'(0)\zeta + U_2(0) + \phi_2)\} p_0 + \hat{\lambda}p_0 + \alpha b + \beta c] d\zeta d\eta, \\ q_2(\xi; \hat{\lambda}; a, b, c) &= -\int_\xi^\infty \int_\eta^\infty (p_2(\zeta) - (1 + \hat{\tau}\hat{\lambda})q_0(\zeta)) d\zeta d\eta, \\ r_2(\xi; \hat{\lambda}; a, b, c) &= -\int_\xi^\infty \int_\eta^\infty (p_2(\zeta) - (1 + \hat{\theta}\hat{\lambda})r_0(\zeta)) d\zeta d\eta / D^2. \end{aligned}$$

Using the above approximate solutions, we can construct uniform approximations up to $O(\varepsilon^2)$ of (3.19) for any fixed $\hat{\lambda} \in \Delta_\nu$, which take the form

$$\left\{ \begin{array}{l} P(y; \varepsilon; \hat{\lambda}; a, b, c) = P_0(y) + \varepsilon P_1(y; \hat{\lambda}; b, c) + \varepsilon^2 P_2(y; \hat{\lambda}; a, b, c) \\ \quad + \frac{1}{\varepsilon} p_0(\frac{y}{\varepsilon}; a) + p_1(\frac{y}{\varepsilon}; a) + \varepsilon p_2(\frac{y}{\varepsilon}; \hat{\lambda}; a, b, c), \\ Q(y; \varepsilon; \hat{\lambda}; a, b, c) = Q_0(y; \hat{\lambda}; b) + \varepsilon Q_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 Q_2(y; \hat{\lambda}; a, b, c) \\ \quad + \varepsilon q_0(\frac{y}{\varepsilon}; a) + \varepsilon^2 q_1(\frac{y}{\varepsilon}; a) + \varepsilon^3 q_2(\frac{y}{\varepsilon}; \hat{\lambda}; a, b, c), \\ R(y; \varepsilon; \hat{\lambda}; a, b, c) = R_0(y) + \varepsilon R_1(y; \hat{\lambda}; a, b, c) + \varepsilon^2 R_2(y; \hat{\lambda}; a, b, c) \\ \quad + \varepsilon r_0(\frac{y}{\varepsilon}; a) + \varepsilon^2 r_1(\frac{y}{\varepsilon}; a) + \varepsilon^3 r_2(\frac{y}{\varepsilon}; \hat{\lambda}; a, b, c). \end{array} \right.$$

Obviously $(P, Q, R)(y; \varepsilon; \hat{\lambda}; a, b, c)$ satisfies at $y = 0$

$$\begin{cases} P(0; \varepsilon; \hat{\lambda}; a, b, c) = \frac{a}{\varepsilon}, \\ Q(0; \varepsilon; \hat{\lambda}; a, b, c) = b + \varepsilon^3 q_2(0; \hat{\lambda}; a, b, c), \\ R(0; \varepsilon; \hat{\lambda}; a, b, c) = c + \varepsilon^3 r_2(0; \hat{\lambda}; a, b, c) \end{cases}$$

and at $y = \infty$ $(P, Q, R)(\infty; \varepsilon; \hat{\lambda}; a, b, c) = (0, 0, 0)$. So we modify this defect to satisfy both boundary conditions exactly and add the remainder term $(\varepsilon^2 \tilde{P}, \varepsilon^2 \tilde{Q}, \varepsilon^2 \tilde{R})$ to it and look for exact solutions of (3.19), which take the form

$$\begin{cases} p(y; \varepsilon; \hat{\lambda}; a, b, c) &= P(y; \varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{P}(y; \varepsilon; \hat{\lambda}; a, b, c) \\ q(y; \varepsilon; \hat{\lambda}; a, b, c) &= Q(y; \varepsilon; \hat{\lambda}; a, b, c) + q^*(y; \varepsilon; \hat{\lambda}; a, b, c) \\ &+ \varepsilon^2 \tilde{Q}(y; \varepsilon; \hat{\lambda}; a, b, c), \\ r(y; \varepsilon; \hat{\lambda}; a, b, c) &= R(y; \varepsilon; \hat{\lambda}; a, b, c) + r^*(y; \varepsilon; \hat{\lambda}; a, b, c) \\ &+ \varepsilon^2 \tilde{R}(y; \varepsilon; \hat{\lambda}; a, b, c), \end{cases} \quad (3.22)$$

where $q^*(y; \varepsilon; \hat{\lambda}; a, b, c) = -\varepsilon^3 q_2(0; \hat{\lambda}; a, b, c)e^{-y}$ and $r^*(y; \varepsilon; \hat{\lambda}; a, b, c) = -\varepsilon^3 \times r_2(0; \hat{\lambda}; a, b, c)e^{-y}$. Substituting this into (3.19), we have

$$\begin{cases} \varepsilon^2(P_{yy} + \varepsilon^2 \tilde{P}_{yy}) + (1 - 3u^2 - \varepsilon^2 \tilde{\lambda})(P + \varepsilon^2 \tilde{P}) \\ \quad - \varepsilon \left(\alpha(Q + q^* + \varepsilon^2 \tilde{Q}) + \beta(R + r^* + \varepsilon^2 \tilde{R}) \right) = 0, \\ Q_{yy} + q_{yy}^* + \varepsilon^2 \tilde{Q}_{yy} + P + \varepsilon^2 \tilde{P} - (1 + \hat{\tau} \hat{\lambda})(Q + q^* + \varepsilon^2 \tilde{Q}) = 0, \\ D^2(R_{yy} + r_{yy}^* + \varepsilon^2 \tilde{R}_{yy}) + P + \varepsilon^2 \tilde{P} - (1 + \hat{\theta} \hat{\lambda})(R + r^* + \varepsilon^2 \tilde{R}) = 0, \\ (\tilde{P}, \tilde{Q}, \tilde{R})(0; \varepsilon) = (0, 0, 0), \quad (\tilde{P}, \tilde{Q}, \tilde{R})(\infty; \varepsilon) = (0, 0, 0). \end{cases} \quad (3.23)$$

Then, for $t = (\tilde{P}, \tilde{Q}, \tilde{R})$, we define the following operator $T(t; \varepsilon; \hat{\lambda}; a, b, c) = (T_1, T_2, T_3)(t; \varepsilon; \hat{\lambda}; a, b, c)$:

$$\begin{cases} T_1(t; \varepsilon; \hat{\lambda}; a, b, c) \equiv \varepsilon^2 \tilde{P}_{yy} + (1 - 3u^2 - \varepsilon^2 \hat{\lambda})\tilde{P} - \varepsilon \alpha \tilde{Q} - \varepsilon \beta \tilde{R} \\ \quad + \frac{1}{\varepsilon^2} \left[P_{yy} + (1 - 3u^2 - \varepsilon^2 \hat{\lambda})P - \varepsilon \{ \alpha(Q + q^*) - \beta(R + r^*) \} \right], \\ T_2(t; \varepsilon; \hat{\lambda}; a, b, c) \equiv \tilde{Q}_{yy} + \tilde{P} - (1 + \hat{\tau} \hat{\lambda})\tilde{Q} + \frac{1}{\varepsilon^2} \left[Q_{yy} + q_{yy}^* + P \right. \\ \quad \left. - (1 + \hat{\tau} \hat{\lambda})(Q + q^*) \right], \\ T_3(t; \varepsilon; \hat{\lambda}; a, b, c) \equiv D^2 \tilde{R}_{yy} + \tilde{P} - (1 + \hat{\theta} \hat{\lambda})\tilde{R} + \frac{1}{\varepsilon^2} \left[D^2(R_{yy} + r_{yy}^*) + P \right. \\ \quad \left. - (1 + \hat{\theta} \hat{\lambda})(R + r^*) \right] \end{cases}$$

from $\bar{X} \times (0, \varepsilon_0) \times \Delta_\nu$ to \bar{Y} , where \bar{X} and \bar{Y} are defined in §2.2.3. We find that $T(t; \varepsilon; \hat{\lambda})$ is the continuously differentiable operator and (3.23) is equivalent to $T(t; \varepsilon; \hat{\lambda}; a, b, c) = 0$.

Lemma 3.5. *For any given $\hat{\lambda}^* \in \mathbb{C}_d$, there exist positive constants ε_0, ν_0 and C such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\hat{\lambda} \in \Delta_{\nu_0}$,*

- (i) $\|T_t(t_1; \varepsilon; \hat{\lambda}; a, b, c) - T_t(t_2; \varepsilon; \hat{\lambda}; a, b, c)\|_{\bar{X} \rightarrow \bar{Y}} \leq C\|t_1 - t_2\|_{\bar{X}}$
for any $t_1, t_2 \in \bar{X}$,
- (ii) $\|T(0; \varepsilon; \hat{\lambda}; a, b, c)\|_{\bar{Y}} \leq C\varepsilon$,
- (iii) $T_t^{-1}(0; \varepsilon; \hat{\lambda}; a, b, c)\|_{\bar{Y} \rightarrow \bar{X}} \leq C$.

Moreover the results (i) - (iii) hold for $\partial T / \partial \hat{\lambda}$ in place of T .

By this lemma, we can apply the Implicit Function Theorem to $T(t; \varepsilon; \hat{\lambda}; a, b, c) = 0$. Thus, under the same assumption of Lemma 3.5, there exists $t(\varepsilon; \hat{\lambda}; a, b, c) \in \bar{X}$ satisfying $T(t; \varepsilon; \hat{\lambda}; a, b, c) = 0$. $t(\varepsilon; \hat{\lambda}; a, b, c)$ and $\partial t / \partial \hat{\lambda}(\varepsilon; \hat{\lambda}; a, b, c)$ are uniformly continuous with respect to $(\varepsilon, \hat{\lambda}) \in (0, \varepsilon_0) \times \Delta_{\nu_0}$ in the \bar{X} -topology and satisfy

$$\|t(\varepsilon; \hat{\lambda}; a, b, c)\|_{\bar{X}}, \|\partial t / \partial \hat{\lambda}(\varepsilon; \hat{\lambda}; a, b, c)\|_{\bar{X}} = o(1) \text{ as } \varepsilon \rightarrow 0 \text{ uniformly in } \hat{\lambda} \in \Delta_{\nu_0}.$$

Thus, we have the exact solutions of (3.19) on $[0, \infty)$ of the form

$$\begin{cases} p(y; \varepsilon; \hat{\lambda}; a, b, c) &= P(y; \varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{P}(y; \varepsilon), \\ q(y; \varepsilon; \hat{\lambda}; a, b, c) &= Q(y; \varepsilon; \hat{\lambda}; a, b, c) + q^*(y; \varepsilon; \hat{\lambda}; a, b, c) \\ &\quad + \varepsilon^2 \tilde{Q}(y; \varepsilon; \hat{\lambda}; a, b, c), \\ r(y; \varepsilon; \hat{\lambda}; a, b, c) &= R(y; \varepsilon; \hat{\lambda}; a, b, c) + r^*(y; \varepsilon; \hat{\lambda}; a, b, c) \\ &\quad + \varepsilon^2 \tilde{R}(y; \varepsilon; \hat{\lambda}; a, b, c), \end{cases} \quad (3.24)$$

which implies that (3.9) has the following exact solutions on $[l(\varepsilon), \infty)$:

$$\begin{cases} p^{(2)}(x; \varepsilon; \lambda; a, b, c) &= P(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c) \\ &\quad + \varepsilon^2 \tilde{P}(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c), \\ q^{(2)}(x; \varepsilon; \lambda; a, b, c) &= Q(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c) \\ &\quad + q^*(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{Q}(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c), \\ r^{(2)}(x; \varepsilon; \lambda; a, b, c) &= R(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c) \\ &\quad + r^*(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c) + \varepsilon^2 \tilde{R}(x - l(\varepsilon); \varepsilon; \hat{\lambda}; a, b, c). \end{cases} \quad (3.25)$$

3.1.3. Evans function $g_N(\varepsilon; \hat{\lambda})$ corresponding to (3.6)

By using these $(p^{(1)}, q^{(1)}, r^{(1)})(x; \varepsilon; \hat{\lambda}; a, b, c)$ and $(p^{(2)}, q^{(2)}, r^{(2)})(x; \varepsilon; \hat{\lambda}; a, b, c)$, we can calculate $\bar{V}^{(i)}(\varepsilon; \hat{\lambda}) \equiv \bar{V}_i(l(\varepsilon); \varepsilon; \hat{\lambda})$ ($i = 1, 2, \dots, 6$) as follows:

$$\bar{V}^{(1)}(\varepsilon; \hat{\lambda}) = \begin{bmatrix} \frac{1}{\varepsilon} \\ a_{10} + \varepsilon a_{11} + O(\varepsilon^2) \\ 0 \\ a_{20} + \varepsilon a_{21} + O(\varepsilon^2) \\ 0 \\ a_{30} + \varepsilon a_{31} + O(\varepsilon^2) \end{bmatrix}, \bar{V}^{(2)}(\varepsilon; \hat{\lambda}) = \begin{bmatrix} 0 \\ \varepsilon b_{11} + O(\varepsilon^2) \\ 1 \\ b_{20} + \varepsilon b_{21} + O(\varepsilon^2) \\ 0 \\ \varepsilon b_{31} + O(\varepsilon^2) \end{bmatrix},$$

$$\bar{V}^{(3)}(\varepsilon; \hat{\lambda}) = \begin{bmatrix} 0 \\ \varepsilon c_{11} + O(\varepsilon^2) \\ 0 \\ \varepsilon c_{21} + O(\varepsilon^2) \\ 1 \\ c_{30} + \varepsilon c_{31} + O(\varepsilon^2) \end{bmatrix}, \bar{V}^{(4)}(\varepsilon; \hat{\lambda}) = \begin{bmatrix} \frac{1}{\varepsilon} \\ d_{10} + \varepsilon d_{11} + O(\varepsilon^2) \\ 0 \\ d_{20} + \varepsilon d_{21} + O(\varepsilon^2) \\ 0 \\ d_{30} + \varepsilon d_{31} + O(\varepsilon^2) \end{bmatrix},$$

$$\bar{V}^{(5)}(\varepsilon; \hat{\lambda}) = \begin{bmatrix} 0 \\ \varepsilon e_{11} + O(\varepsilon^2) \\ 1 \\ e_{20} + \varepsilon e_{21} + O(\varepsilon^2) \\ 0 \\ \varepsilon e_{31} + O(\varepsilon^2) \end{bmatrix}, \bar{V}^{(6)}(\varepsilon; \hat{\lambda}) = \begin{bmatrix} 0 \\ \varepsilon f_{11} + O(\varepsilon^2) \\ 0 \\ \varepsilon f_{21} + O(\varepsilon^2) \\ 1 \\ f_{30} + \varepsilon f_{31} + O(\varepsilon^2) \end{bmatrix},$$

where $a_{10}, a_{11}, \dots, f_{31}$ are defined as follows:

$$\begin{aligned}
a_{10} &= \frac{1}{l_0} \dot{p}_1^{(1)}(0; 1), & a_{11} &= -\frac{l_1}{l_0^2} \dot{p}_1^{(1)}(0; 1) + \frac{1}{l_0} \dot{p}_2^{(1)}(0; \hat{\lambda}; 1, 0, 0), \\
a_{20} &= \frac{1}{l_0} \dot{q}_0^{(1)}(0; 1), & a_{21} &= -\frac{l_1}{l_0^2} \dot{q}_0^{(1)}(0; 1) + \frac{1}{l_0} \{Q_1'^{(1)}(1; \hat{\lambda}; 1, 0, 0) + \dot{q}_1^{(1)}(0; 1)\}, \\
a_{30} &= \frac{1}{l_0} \dot{r}_0^{(1)}(0; 1), & a_{31} &= -\frac{l_1}{l_0^2} \dot{r}_0^{(1)}(0; 1) + \frac{1}{l_0} \{R_1'^{(1)}(1; \hat{\lambda}; 1, 0, 0) + \dot{r}_1^{(1)}(0; 1)\}, \\
b_{11} &= \frac{1}{l_0} \dot{p}_2^{(1)}(0; \hat{\lambda}; 0, 1, 0), & b_{20} &= \frac{1}{l_0} Q_0'^{(1)}(1; \hat{\lambda}; 1), \\
b_{21} &= -\frac{l_1}{l_0^2} Q_0'^{(1)}(1; \hat{\lambda}; 1) + \frac{1}{l_0} Q_1'^{(1)}(1; \hat{\lambda}; 0, 1, 0), & b_{31} &= \frac{1}{l_0} R_1'^{(1)}(1; \hat{\lambda}; 0, 1, 0), \\
c_{11} &= \frac{1}{l_0} \dot{p}_2^{(1)}(0; \hat{\lambda}; 0, 0, 1), & c_{21} &= \frac{1}{l_0} Q_1'^{(1)}(1; \hat{\lambda}; 0, 0, 1), \\
c_{30} &= \frac{1}{l_0} R_0'^{(1)}(1; \hat{\lambda}; 1), & c_{31} &= -\frac{l_1}{l_0^2} R_0'^{(1)}(1; \hat{\lambda}; 1) + \frac{1}{l_0} R_1'^{(1)}(1; \hat{\lambda}; 0, 0, 1), \\
d_{10} &= \dot{p}_1^{(2)}(0; 1), & d_{11} &= \dot{p}_2^{(2)}(0; \hat{\lambda}; 1, 0, 0), & d_{20} &= \dot{q}_0^{(2)}(0; 1), \\
d_{21} &= Q_1'^{(2)}(0; \hat{\lambda}; 1, 0, 0) + \dot{q}_1^{(2)}(0; 1), \\
d_{30} &= \dot{r}_0^{(2)}(0; 1), & d_{31} &= R_1'^{(2)}(0; \hat{\lambda}; 1, 0, 0) + \dot{r}_1^{(2)}(0; 1), \\
e_{11} &= \dot{p}_2^{(2)}(0; \hat{\lambda}; 0, 1, 0), & e_{20} &= Q_0'^{(2)}(0; \hat{\lambda}; 1), & e_{21} &= Q_1'^{(2)}(0; \hat{\lambda}; 0, 1, 0), \\
e_{31} &= R_1'^{(2)}(0; \hat{\lambda}; 0, 1, 0), & f_{11} &= \dot{p}_2^{(2)}(0; \hat{\lambda}; 0, 0, 1), \\
f_{21} &= Q_1'^{(2)}(0; \hat{\lambda}; 0, 0, 1), & f_{30} &= R_0'^{(2)}(0; \hat{\lambda}; 1), & f_{31} &= R_1'^{(2)}(0; \hat{\lambda}; 0, 0, 1).
\end{aligned}$$

First, we can show

Lemma 3.6. $a_{10} = d_{10} = 0.$

The proof will be given in §4.

Using this lemma, we find that $g_N(\varepsilon; \hat{\lambda})$ is represented as follows:

$$\begin{aligned}
g_N(\varepsilon; \hat{\lambda}) &= \{(a_{11} - d_{11})(b_{20} - e_{20})(c_{30} - f_{30}) \\
&\quad - (a_{20} - d_{20})(b_{11} - e_{11})(c_{30} - f_{30}) \\
&\quad - (a_{30} - d_{30})(b_{20} - e_{20})(c_{11} - f_{11})\} + O(\varepsilon).
\end{aligned}$$

Next, we easily find that

$$\left\{ \begin{array}{ll} a_{20} - d_{20} = -2\sqrt{2}, & a_{30} - d_{30} = \frac{2\sqrt{2}}{D^2}, \\ b_{20} - e_{20} = \sqrt{1 + \hat{\tau}\hat{\lambda}} \frac{2e^{l_0\sqrt{1+\hat{\tau}\hat{\lambda}}}}{e^{l_0\sqrt{1+\hat{\tau}\hat{\lambda}}} + e^{-l_0\sqrt{1+\hat{\tau}\hat{\lambda}}}}, & \\ b_{11} - e_{11} = 2\sqrt{2}\alpha, & c_{11} - f_{11} = 2\sqrt{2}\beta, \\ c_{30} - f_{30} = \frac{1}{D} \sqrt{1 + \hat{\theta}\hat{\lambda}} \frac{2e^{l_0\sqrt{1+\hat{\theta}\hat{\lambda}}/D}}{e^{l_0\sqrt{1+\hat{\theta}\hat{\lambda}}/D} + e^{-l_0\sqrt{1+\hat{\theta}\hat{\lambda}}/D}}. & \end{array} \right. \quad (3.26)$$

Furthermore we obtain

Lemma 3.7. $a_{11} - d_{11} = \frac{4}{3}\sqrt{2}\hat{\lambda} - 4\left\{\alpha(1 - e^{-2l_0}) + \frac{\beta}{D}(1 - e^{-2l_0/D})\right\}.$

The proof will be given in §4.

We look for a root $\hat{\lambda}$ of $g_N(0; \hat{\lambda}) = 0$. Then, this root $\hat{\lambda}$ should satisfy

$$\begin{aligned} a_{11} - d_{11} &= (a_{20} - d_{20})(b_{11} - e_{11})/(b_{20} - e_{20}) \\ &\quad + (a_{30} - d_{30})(c_{11} - f_{11})/(c_{30} - f_{30}). \end{aligned} \quad (3.27)$$

Applying Lemma 3.7 and substituting (3.26) into (3.27), we have

$$\begin{aligned} \frac{\sqrt{2}}{3}\hat{\lambda} + \alpha \left(\frac{1}{\sqrt{1+\hat{\tau}\hat{\lambda}}} - 1 + e^{-2l_0} \right) + \frac{\beta}{D} \left(\frac{1}{\sqrt{1+\hat{\theta}\hat{\lambda}}} - 1 + e^{-2l_0/D} \right) \\ = - \left(\frac{\alpha}{\sqrt{1+\hat{\tau}\hat{\lambda}}} e^{-2l_0\sqrt{1+\hat{\tau}\hat{\lambda}}} + \frac{\beta}{D\sqrt{1+\hat{\theta}\hat{\lambda}}} e^{-2l_0\sqrt{1+\hat{\theta}\hat{\lambda}/D}} \right). \end{aligned} \quad (3.28)$$

Here we remark that by the standard argument, if we get a *single* root $\hat{\lambda}$ of $g_N(0; \hat{\lambda}) = 0$ (i.e., $\frac{d}{d\hat{\lambda}}g_N(0; \hat{\lambda}) \neq 0$), a root $\hat{\lambda}(\varepsilon) = \hat{\lambda} + O(\varepsilon)$ of $g_N(\varepsilon; \hat{\lambda}) = 0$ is uniquely determined. Then, the relation (3.28) plays an important role to determine roots of $g_N(\varepsilon; \hat{\lambda}) = 0$.

3.2. Evans function $g_D(\varepsilon; \hat{\lambda})$ corresponding to (3.7)

Noting the boundary condition of (3.7) at $x = 0$, we consider the following problems with suitable boundary conditions:

$$\left\{ \begin{array}{l} \varepsilon^2 \hat{\lambda} p = \varepsilon^2 p_{xx} + (1 - 3u^2)p - \varepsilon \alpha q - \varepsilon \beta r, \\ \hat{\tau} \hat{\lambda} q = q_{xx} + p - q, \\ \hat{\theta} \hat{\lambda} r = D^2 r_{xx} + p - r, \\ (p, q, r)(0) = (0, 0, 0), \\ (p, q, r)(l(\varepsilon)) = \left(\frac{a}{\varepsilon}, b, c \right), \end{array} \right. \quad x \in (0, l(\varepsilon)) = I_1 \quad (3.29)$$

and

$$\left\{ \begin{array}{l} \varepsilon^2 \hat{\lambda} p = \varepsilon^2 p_{xx} + (1 - 3u^2)p - \varepsilon \alpha q - \varepsilon \beta r, \\ \hat{\tau} \hat{\lambda} q = q_{xx} + p - q, \\ \hat{\theta} \hat{\lambda} r = D^2 r_{xx} + p - r, \\ (p, q, r)(l(\varepsilon)) = \left(\frac{a}{\varepsilon}, b, c \right), \\ (p, q, r)(\infty) = (0, 0, 0), \end{array} \right. \quad x \in (l(\varepsilon), \infty) = I_2 \quad (3.30)$$

where a, b, c are given real numbers. For any $\hat{\lambda} \in \mathbb{C}_d$, let $(p^{(1)}, q^{(1)}, r^{(1)})(x; \varepsilon; \hat{\lambda}; a, b, c)$ and $(p^{(2)}, q^{(2)}, r^{(2)})(x; \varepsilon; \hat{\lambda}; a, b, c)$ be solutions of (3.29) and (3.30), respectively. Then, any solution $\bar{V}(x; \varepsilon; \lambda)$ of (3.29) satisfying $(p, q, r)(0) = (0, 0, 0)$ is represented as a linear combination of three independent solutions $\bar{V}_1, \bar{V}_2, \bar{V}_3$. By virtue of Lemma 3.2, any solution of (3.30) satisfying $(p, q, r)(\infty) = (0, 0, 0)$ is represented as a linear combination of three independent solutions $\bar{V}_4, \bar{V}_5, \bar{V}_6$. Here $\bar{V}_i (i = 1, 2, \dots, 6)$ have the same definition as those in §3.1.

By the same argument in §3.1, we have the next lemma.

Lemma 3.8. *The number $\hat{\lambda} \in \mathbb{C}_d$ is an eigenvalue of (3.7) if and only if $g_D(\varepsilon; \hat{\lambda}) = 0$, where $g_D(\varepsilon; \hat{\lambda}) = \{(a_{11} - d_{11})(b_{20} - e_{20})(c_{30} - f_{30}) - (a_{20} - d_{20})(b_{11} - e_{11})(c_{30} - f_{30}) - (a_{30} - d_{30})(b_{20} - e_{20})(c_{11} - f_{11})\} + O(\varepsilon)$.*

$g_D(\varepsilon; \hat{\lambda})$ also is called the *Evans function* of the standing pulse solutions corresponding to (3.7). To calculate $g_D(\varepsilon; \hat{\lambda})$, only two terms $b_{20} = \frac{1}{l_0} Q_0'^{(1)}(1; \hat{\lambda}; 1)$ and $c_{30} = \frac{1}{l_0} R_0'^{(1)}(1; \hat{\lambda}; 1)$ are different from those calculated in §3.1. Other terms are the same as those of (3.26) and Lemma 3.7. For these two terms, we have

$$b_{20} = \sqrt{1 + \hat{\tau}\hat{\lambda}} \frac{\cosh(l_0 \sqrt{1 + \hat{\tau}\hat{\lambda}})}{\sinh(l_0 \sqrt{1 + \hat{\tau}\hat{\lambda}})}, \quad c_{30} = \frac{1}{D} \sqrt{1 + \hat{\theta}\hat{\lambda}} \frac{\cosh(l_0 \sqrt{1 + \hat{\theta}\hat{\lambda}}/D)}{\sinh(l_0 \sqrt{1 + \hat{\theta}\hat{\lambda}}/D)}.$$

depending on the Dirichlet boundary condition at $x = 0$. Using these relations, we get

$$b_{20} - e_{20} = \sqrt{1 + \hat{\tau}\hat{\lambda}} \frac{2e^{l_0 \sqrt{1 + \hat{\tau}\hat{\lambda}}}}{e^{l_0 \sqrt{1 + \hat{\tau}\hat{\lambda}}} - e^{-l_0 \sqrt{1 + \hat{\tau}\hat{\lambda}}}},$$

$$c_{30} - f_{30} = \frac{1}{D} \sqrt{1 + \hat{\theta}\hat{\lambda}} \frac{2e^{l_0 \sqrt{1 + \hat{\theta}\hat{\lambda}}/D}}{e^{l_0 \sqrt{1 + \hat{\theta}\hat{\lambda}}/D} - e^{-l_0 \sqrt{1 + \hat{\theta}\hat{\lambda}}/D}}.$$

Then, substitute these into $g_D(0; \hat{\lambda}) = 0$. We find that

$$\begin{aligned} \frac{\sqrt{2}}{3} \hat{\lambda} + \alpha \left(\frac{1}{\sqrt{1 + \hat{\tau}\hat{\lambda}}} - 1 + e^{-2l_0} \right) + \frac{\beta}{D} \left(\frac{1}{\sqrt{1 + \hat{\theta}\hat{\lambda}}} - 1 + e^{-2l_0/D} \right) \\ = \frac{\alpha}{\sqrt{1 + \hat{\tau}\hat{\lambda}}} e^{-2l_0 \sqrt{1 + \hat{\tau}\hat{\lambda}}} + \frac{\beta}{D \sqrt{1 + \hat{\theta}\hat{\lambda}}} e^{-2l_0 \sqrt{1 + \hat{\theta}\hat{\lambda}}/D}. \end{aligned} \quad (3.31)$$

If we get a *single* root $\hat{\lambda}$ of $g_D(0; \hat{\lambda}) = 0$ (i.e., $\frac{d}{d\hat{\lambda}} g_D(0; \hat{\lambda}) \neq 0$), a root $\hat{\lambda}(\varepsilon) = \hat{\lambda} + O(\varepsilon)$ of $g_D(\varepsilon; \hat{\lambda}) = 0$ is uniquely determined. Then, the relation (3.31) plays the important role to determine roots of $g_D(\varepsilon; \hat{\lambda}) = 0$.

3.3. Distribution of eigenvalues of (3.1)

In §3.1 and §3.2, we find that eigenvalues $\hat{\lambda} \in \mathbb{C}_d$ of the eigenvalue problem (3.1) are determined by the relations (3.28) or (3.31). We easily know that (3.28) does not have the zero root, on the other hand, (3.31) has the zero root, which corresponds to the translation free of the standing pulse solution. Then, we give the following result on the stability of the standing pulse solution of (2.1):

Theorem 3.1. *Suppose that there exists $l_0 > 0$ satisfying $\alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}} = \gamma$ and $\alpha e^{-2l_0} + \frac{\beta}{D} e^{-2\frac{l_0}{D}} \neq 0$ for given α, β, γ, D . Then the standing pulse solution $(u, v, w)(x; \varepsilon)$ of (2.1) is stable if any $\hat{\lambda}$ satisfying (3.28) or (3.31) has a negative real part for given $\hat{\tau}$ and $\hat{\theta}$ except for the simple 0 eigenvalue, where $\tau = \varepsilon^2 \hat{\tau}$ and $\theta = \varepsilon^2 \hat{\theta}$.*

Especially, when $\varepsilon^2 \tau, \varepsilon^2 \theta = o(1)$ for small $\varepsilon > 0$, it corresponds to the case $\hat{\tau} = 0$ and $\hat{\theta} = 0$ in our eigenvalue problem (3.1). Then, for these case (3.28) and (3.31) deduce to

$$\hat{\lambda} = -3\sqrt{2}(\alpha e^{-2l_0} + \frac{\beta}{D} e^{-2l_0/D}) \text{ and } \hat{\lambda} = 0, \quad (3.32)$$

respectively. $\lambda = \varepsilon^2 \hat{\lambda} = 0$ comes from the translation free of the standing pulse solutions and the other eigenvalue $\lambda = \varepsilon^2 \hat{\lambda} = -\varepsilon^2 3\sqrt{2}(\alpha e^{-2l_0} + \beta e^{-2l_0/D}/D)$ is essential to determine their stability. Therefore we have

Theorem 3.2. ([7]) *Suppose that there exists $l_0 > 0$ satisfying $\alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}} = \gamma$ and $\alpha e^{-2l_0} + \frac{\beta}{D} e^{-2\frac{l_0}{D}} \neq 0$ for given α, β, γ, D . And assume that $\varepsilon^2 \tau, \varepsilon^2 \theta = o(1)$ for small positive ε . Then the standing pulse solution of (2.1) is stable if and only if $\alpha e^{-2l_0} + \beta e^{-2l_0/D}/D > 0$.*

Then, we can determine the stability of the standing pulse solutions (2.1) given by Corollary 2.1.

Corollary 3.1. *Suppose that $\varepsilon^2 \tau, \varepsilon^2 \theta = o(1)$ for small positive ε . The stability of the standing pulse solutions of (2.1) given in Corollary 2.1 are*

classified as follows for the case $|\alpha D| > |\beta|$:

- (a1a) when $\text{sgn}(\alpha) = \text{sgn}(\beta) = \text{sgn}(\gamma) = 1$ and $\gamma < \alpha + \beta$, the solution (u, v, w) is stable,
- (a1b) when $\text{sgn}(\alpha) = \text{sgn}(\beta) = \text{sgn}(\gamma) = -1$ and $\gamma > \alpha + \beta$, the solution (u, v, w) is unstable,
- (b2) when $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$ and $0 < \gamma < \alpha + \beta$, the solution (u, v, w) is stable.
- (b3) when $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$ and $\alpha + \beta < \gamma < \gamma_{c1}$, the solution (u, v, w) is stable for $0 < l_0 < l_c$, unstable for $l_0 > l_c$.
- (c2) when $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$ and $\alpha + \beta < \gamma < 0$, the solution (u, v, w) is unstable,
- (c3) when $\text{sgn}(\alpha) = -1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$ and $0 < \gamma < \gamma_{c1}$, the solution (u, v, w) is stable for $0 < l_0 < l_c$, unstable for $l_0 > l_c$,
- (d2) when $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$ and $\gamma_{c2} < \gamma < 0$, the solution (u, v, w) is unstable for $0 < l_0 < l_c$, stable for $l_0 > l_c$,
- (d3) when $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta > 0$ and $0 < \gamma < \alpha + \beta$, the solution (u, v, w) is stable,
- (e2) when $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$ and $\gamma_{c2} < \gamma < \alpha + \beta$, the solution (u, v, w) is unstable for $0 < l_0 < l_c$, stable for $l_0 > l_c$,
- (e3) when $\text{sgn}(\alpha) = 1 = -\text{sgn}(\beta)$, $\alpha + \beta < 0$ and $\alpha + \beta < \gamma < 0$, the solution (u, v, w) is unstable,

where $\gamma_{c1} = (-\alpha)^{-\frac{1}{D-1}} \beta^{-\frac{D}{D-1}} (D^{-\frac{1}{D-1}} - D^{-\frac{D}{D-1}})$, $\gamma_{c2} = \alpha^{-\frac{1}{D-1}} (-\beta)^{-\frac{D}{D-1}} (D^{-\frac{1}{D-1}} - D^{-\frac{D}{D-1}})$, $l_c = \frac{D}{2(D-1)} \log \left(-\frac{\alpha D}{\beta} \right)$.

Proof. When we differentiate $g(l_0) = \alpha e^{-2l_0} + \beta e^{-2\frac{l_0}{D}}$ with respect to l_0 , we have $dg/dl_0(l_0) = -2\alpha e^{-2l_0} - 2\beta e^{-2\frac{l_0}{D}}/D$, from which we obtain the relation $\tilde{\lambda} = -3\sqrt{2}(dg/dl_0)(l_0)/2$. Then, the stability of the standing pulse solution (u, v, w) is determined by the slop of the curve $\gamma = g(l_0)$, that is, if the slop of $\gamma = g(l_0)$ is negative (resp. positive), (u, v, w) is stable (resp. unstable). For example, Figure 4 corresponds to the case (b2), (b3) (the left) and (d2), (d3) (the right). \square

4. Proof of lemmas

4.1. Proof of Lemma 2.2

First, we show (i).

$$\begin{aligned}
T_1(\tilde{U} + k; \varepsilon) - T_1(\tilde{U}; \varepsilon) &= \varepsilon^2(\tilde{U} + k)_{yy} + l(\varepsilon)^2 \\
&\times \left\{ \tilde{U} + k - 3U^2(\tilde{U} + k) - 3\varepsilon^2 U(\tilde{U} + k)^2 - \varepsilon^4(\tilde{U} + k)^3 \right\} \\
&- \left[\varepsilon^2 \tilde{U}_{yy} + l(\varepsilon)^2 \left\{ \tilde{U} - 3U^2 \tilde{U} - 3\varepsilon^2 U \tilde{U}^2 - \varepsilon^4 \tilde{U}^3 \right\} \right] \\
&= \varepsilon^2 k_{yy} + l(\varepsilon)^2 \left\{ 1 - 3U^2 - 6\varepsilon^2 U \tilde{U} - 3\varepsilon^4 \tilde{U}^2 \right\} k + O(|k|^2),
\end{aligned}$$

which implies $\frac{\partial T_1}{\partial \tilde{U}} = \varepsilon^2 \frac{d^2}{dy^2} + l(\varepsilon)^2 \left\{ 1 - 3U^2 - 6\varepsilon^2 U \tilde{U} - 3\varepsilon^4 \tilde{U}^2 \right\}$. Similarly we have

$$\left\{ \begin{array}{lll} \frac{\partial T_1}{\partial \tilde{V}} = -l(\varepsilon)^2 \varepsilon \alpha, & \frac{\partial T_1}{\partial \tilde{W}} = -l(\varepsilon)^2 \varepsilon \beta, \\ \frac{\partial T_2}{\partial \tilde{U}} = l(\varepsilon)^2, & \frac{\partial T_2}{\partial \tilde{V}} = \frac{d^2}{dy^2} - l(\varepsilon)^2, & \frac{\partial T_2}{\partial \tilde{W}} = 0, \\ \frac{\partial T_3}{\partial \tilde{U}} = l(\varepsilon)^2, & \frac{\partial T_3}{\partial \tilde{V}} = 0, & \frac{\partial T_3}{\partial \tilde{W}} = D^2 \frac{d^2}{dy^2} - l(\varepsilon)^2. \end{array} \right.$$

Using these results, we get for $t_i = (\tilde{U}_i, \tilde{V}_i, \tilde{W}_i) \in X$ ($i = 1, 2$), $T_t(t_i; \varepsilon)$

$$= \begin{bmatrix} \varepsilon^2 \frac{d^2}{dy^2} + l(\varepsilon)^2 \left\{ 1 - 3U^2 - 6\varepsilon^2 U \tilde{U}_i - 3\varepsilon^4 \tilde{U}_i^2 \right\} & -l(\varepsilon)^2 \varepsilon \alpha & -l(\varepsilon)^2 \varepsilon \beta \\ l(\varepsilon)^2 & \frac{d^2}{dy^2} - l(\varepsilon)^2 & 0 \\ l(\varepsilon)^2 & 0 & D^2 \frac{d^2}{dy^2} - l(\varepsilon)^2 \end{bmatrix}.$$

Thus, we know

$$T_t(t_1; \varepsilon) - T_t(t_2; \varepsilon) = \begin{bmatrix} l(\varepsilon)^2 \left\{ -6\varepsilon^2 U(\tilde{U}_1 - \tilde{U}_2) - 3\varepsilon^4(\tilde{U}_1^2 - \tilde{U}_2^2) \right\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which shows that for $h = (h_U, h_V, h_W) \in X$,

$$\begin{aligned}
\|T_t(t_1; \varepsilon) - T_t(t_2; \varepsilon)\|_{X \rightarrow Y} &= \sup_{h \neq 0} \frac{\|(T_t(t_1; \varepsilon) - T_t(t_2; \varepsilon))h\|_Y}{\|h\|_X} \\
&= \sup_{h \neq 0} \frac{\|l(\varepsilon)^2 \{-6\varepsilon^2 U(\tilde{U}_1 - \tilde{U}_2) - 3\varepsilon^4(\tilde{U}_1^2 - \tilde{U}_2^2)\}h_U\|_{C[0,1]}}{\|h_U\|_{A_\varepsilon} + \|h_V\|_B + \|h_W\|_B} \\
&\leq \sup_{h \neq 0} \frac{3\varepsilon^2 l(\varepsilon)^2 \|\{2U + \varepsilon^2(\tilde{U}_1 + \tilde{U}_2)\}(\tilde{U}_1 - \tilde{U}_2)h_U\|_{C[0,1]}}{\|h_U\|_{C[0,1]}} \\
&\leq 3\varepsilon^2 l(\varepsilon)^2 \|2U + \varepsilon^2(\tilde{U}_1 + \tilde{U}_2)\|_{C[0,1]} \|\tilde{U}_1 - \tilde{U}_2\|_{C[0,1]}.
\end{aligned}$$

If we take ε being small, it holds true that

$$\|T_t(t_1; \varepsilon) - T_t(t_2; \varepsilon)\|_{X \rightarrow Y} \leq 1 \cdot \|(\tilde{U}_1 - \tilde{U}_2)\|_{C[0,1]} \leq \|(t_1 - t_2)\|_X,$$

which implies the assertion (i) with $C = 1$.

From the constructions of approximate solutions, we easily find (ii) is true.

Finally, we show (iii). Put

$$\begin{aligned}
T_t(0; \varepsilon) &= \begin{bmatrix} \varepsilon^2 \frac{d^2}{dy^2} + l(\varepsilon)^2(1 - 3U^2) & -l(\varepsilon)^2 \varepsilon \alpha & -l(\varepsilon)^2 \varepsilon \beta \\ l(\varepsilon)^2 & \frac{d^2}{dy^2} - l(\varepsilon)^2 & 0 \\ l(\varepsilon)^2 & 0 & D^2 \frac{d^2}{dy^2} - l(\varepsilon)^2 \end{bmatrix}, \\
&\equiv \begin{bmatrix} L_\varepsilon & S_1 & S_2 \\ S_3 & M_\varepsilon & 0 \\ S_4 & 0 & N_\varepsilon \end{bmatrix}.
\end{aligned}$$

By using the same method as that in [4], $L_\varepsilon : A_\varepsilon \rightarrow C[0, 1]$ has an inverse operator L_ε^{-1} such that there exists a positive constant K_1 which is independent of ε satisfying $\|L_\varepsilon^{-1}\|_{C[0,1] \rightarrow A_\varepsilon} \leq K_1$ for any small $\varepsilon > 0$. Applying the constant variation method to $M_\varepsilon p = q$, we find that for any $q \in C[0, 1]$, there exists a unique $p \in B$ satisfying $M_\varepsilon p = q$. Thus, there exists an inverse operator M_ε^{-1} of $M_\varepsilon : B \rightarrow C[0, 1]$, which satisfies

$$\|M_\varepsilon^{-1}q\|_B = \|p\|_B = \sup_{y \in [0,1]} |p(y)| + \sup_{y \in [0,1]} |p'(y)| + \sup_{y \in [0,1]} |p''(y)|$$

and for any $y \in [0, 1]$, the following inequalities hold

$$\begin{aligned}
|p(y)| &\leq e^{l_0 y} \left\{ \frac{1}{2l_0} \int_0^1 e^{-l_0 s} |q(s)| ds + \frac{e^{-l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) |q(s)| ds \right\} \\
&\quad + e^{-l_0 y} \left\{ \frac{1}{2l_0} \int_0^1 e^{l_0 s} |q(s)| ds + \frac{e^{l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) |q(s)| ds \right\} \\
&\leq e^{l_0} \left\{ \frac{1}{2l_0} \sup_{y \in [0,1]} |q(s)| + \frac{1}{2l_0} \sup_{y \in [0,1]} |q(s)| \right\} \\
&\quad + \frac{1}{2l_0} e^{l_0} \sup_{y \in [0,1]} |q(s)| + \frac{e^{l_0}}{2l_0} \sup_{y \in [0,1]} |q(s)| \\
&\leq K_2 \|q\|_{C[0,1]}, \\
|p'(y)| &\leq l_0 e^{l_0 y} \left\{ \frac{1}{2l_0} \int_0^1 e^{-l_0 s} |q(s)| ds + \frac{e^{-l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) |q(s)| ds \right\} \\
&\quad + l_0 e^{-l_0 y} \left\{ \frac{1}{2l_0} \int_0^1 e^{l_0 s} |q(s)| ds + \frac{e^{l_0}}{2l_0 \cosh(l_0)} \int_0^1 \cosh(l_0 s) |q(s)| ds \right\} \\
&\leq K_3 \|q\|_{C[0,1]}, \\
|p''(y)| &= |l_0^2 p(y) + q(y)| \leq l_0^2 |p(y)| + |q(y)| \leq K_4 \|q\|_{C[0,1]},
\end{aligned}$$

where K_2, K_3 and K_4 are positive constants independent of ε . Therefore we find that there exists a positive constant K_5 , which independent of ε , satisfying

$$\|M_\varepsilon^{-1} q\|_B \leq K_5 \|q\|_{C[0,1]},$$

which implies that

$$\|M_\varepsilon^{-1}\|_{C[0,1] \rightarrow B} = \sup_{q \neq 0} \frac{\|M_\varepsilon^{-1} q\|_B}{\|q\|_{C[0,1]}} \leq K_5.$$

Similarly to the above, we know that there is a positive constant K_6 , which independent of ε , satisfying $\|N_\varepsilon^{-1}\|_{C[0,1] \rightarrow B} \leq K_6$.

Next, we consider the following problem::

$$T_t(0; \varepsilon)P = \begin{bmatrix} L_\varepsilon P_1 & S_1 & S_2 \\ S_3 & M_\varepsilon & 0 \\ S_4 & 0 & N_\varepsilon \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}.$$

where $P = (P_1, P_2, P_3) \in X$ and $F = (F_1, F_2, F_3) \in Y$, which is equivalent to

$$\begin{cases} L_\varepsilon P_1 + S_1 P_2 + S_2 P_3 = F_1, \\ S_3 P_1 + M_\varepsilon P_2 = F_2, \\ S_4 P_1 + N_\varepsilon P_3 = F_3. \end{cases} \quad (4.1)$$

From the second and third equations of (4.1), we have $P_2 = M_\varepsilon^{-1}(F_2 - S_3 P_1)$ and $P_3 = N_\varepsilon^{-1}(F_3 - S_4 P_1)$, respectively. Substitute these into the first of (4.1), we know that

$$L_\varepsilon P_1 + S_1 M_\varepsilon^{-1}(F_2 - S_3 P_1) + S_2 N_\varepsilon^{-1}(F_3 - S_4 P_1) = F_1.$$

And opetating L_ε^{-1} from the left, we have

$$\begin{aligned} \{I_{A_\varepsilon} - L_\varepsilon^{-1}(S_1 M_\varepsilon^{-1} S_3 + S_2 N_\varepsilon^{-1} S_4)\} P_1 \\ = L_\varepsilon^{-1}(F_1 - S_1 M_\varepsilon^{-1} F_2 - S_2 N_\varepsilon^{-1} F_3). \end{aligned}$$

Here we show $\|L_\varepsilon^{-1}(S_1 M_\varepsilon^{-1} S_3 + S_2 N_\varepsilon^{-1} S_4)\|_{A_\varepsilon \rightarrow A_\varepsilon} < 1$ for small $\varepsilon > 0$.

$$\begin{aligned} & \|L_\varepsilon^{-1}(S_1 M_\varepsilon^{-1} S_3 + S_2 N_\varepsilon^{-1} S_4)\|_{A_\varepsilon \rightarrow A_\varepsilon} \\ & \leq \|L_\varepsilon^{-1}\|_{C[0,1] \rightarrow A_\varepsilon} \|S_1 M_\varepsilon^{-1} S_3 + S_2 N_\varepsilon^{-1} S_4\|_{A_\varepsilon \rightarrow C[0,1]} \\ & \leq K_1 \left(\|\ell(\varepsilon)^2 \varepsilon \alpha M_\varepsilon^{-1} \ell(\varepsilon)^2\|_{A_\varepsilon \rightarrow C[0,1]} + \|\ell(\varepsilon)^2 \varepsilon \beta N_\varepsilon^{-1} \ell(\varepsilon)^2\|_{A_\varepsilon \rightarrow C[0,1]} \right) \\ & \leq K_1 \left(\varepsilon |\alpha| \ell(\varepsilon)^4 \|M_\varepsilon^{-1}\|_{C[0,1] \rightarrow B} + \varepsilon |\beta| \ell(\varepsilon)^4 \|N_\varepsilon^{-1}\|_{C[0,1] \rightarrow B} \right) \\ & \leq \varepsilon \ell(\varepsilon)^4 K_1 (|\alpha| K_5 + |\beta| K_6) < 1, \end{aligned}$$

which means the existence of an inverse operator of $\{I_{A_\varepsilon} - L_\varepsilon^{-1}(S_1 M_\varepsilon^{-1} S_3 + S_2 N_\varepsilon^{-1} S_4)\}$. Thus when we multiply this inverse operator to (4.1) from the left, we know

$$\begin{aligned} P_1 = \{I_{A_\varepsilon} - L_\varepsilon^{-1}(S_1 M_\varepsilon^{-1} S_3 + S_2 N_\varepsilon^{-1} S_4)\}^{-1} L_\varepsilon^{-1} (F_1 \\ - S_1 M_\varepsilon^{-1} F_2 - S_2 N_\varepsilon^{-1} F_3). \end{aligned}$$

Therefore we can estimate

$$\begin{aligned} \|P_1\|_{A_\varepsilon} & < (\|I_{A_\varepsilon}\|_{A_\varepsilon \rightarrow A_\varepsilon} + 1) \|L_\varepsilon^{-1}\|_{C[0,1] \rightarrow A_\varepsilon} \|F_1 - S_1 M_\varepsilon^{-1} F_2 \\ & \quad - S_2 N_\varepsilon^{-1} F_3\|_{C[0,1]} \\ & \leq 2K_1 (\|F_1\|_{C[0,1]} + \|S_1 M_\varepsilon^{-1} F_2\|_{C[0,1]} + \|S_2 N_\varepsilon^{-1} F_3\|_{C[0,1]}) \\ & \leq K_7 (\|F_1\|_{C[0,1]} + \|F_2\|_{C[0,1]} + \|F_3\|_{C[0,1]}). \end{aligned}$$

Using this, we have

$$\begin{aligned}\|P_2\|_B &\leq \|M_\varepsilon^{-1}(F_2 - S_3 P_1)\|_B \leq K_8 (\|F_1\|_{C[0,1]} + \|F_2\|_{C[0,1]} + \|F_3\|_{C[0,1]}), \\ \|P_3\|_B &\leq \|M_\varepsilon^{-1}(F_3 - S_4 P_1)\|_B \leq K_9 (\|F_1\|_{C[0,1]} + \|F_2\|_{C[0,1]} + \|F_3\|_{C[0,1]}),\end{aligned}$$

where K_7, K_8, K_9 are positive constants independent of ε . Using these, we obtain

$$\|T_t(0; \varepsilon)^{-1} F\|_X = \|P\|_X \leq K_{10} \|F\|_Y$$

for some positive constant K_{10} independent of ε , which implies

$$\|T_t(0; \varepsilon)^{-1}\|_{Y \rightarrow X} = \sup_{F \neq 0} \frac{\|T_t(0; \varepsilon)^{-1} F\|_X}{\|F\|_Y} \leq K_{10}.$$

This completes the proof of (iii). \square

4.2. Proof of Lemma 2.4

Let us consider the l_1 -dependency of

$$\begin{aligned}\Phi_1 &= U_1'^{(1)}(1) + \dot{\phi}_2^{(1)}(0) - l_0(U_1'^{(2)}(0) + \dot{\phi}_2^{(2)}(0)) \\ &\quad - l_1(U_0'^{(2)}(0) + \dot{\phi}_1^{(2)}(0)) - l_2 \dot{\phi}_0^{(2)}(0).\end{aligned}$$

Noting that $U_0'^{(2)}(0) = 0$ and $\dot{\phi}_1^{(2)}(0) = 0$, we find that the terms including l_1 in Φ_1 is the only term $\dot{\phi}_2^{(1)}(0)$. So we omit the upper index (1) in each term here after with no confusion. Then, we have

the terms including l_1 in Φ_1

$$\begin{aligned}&= \left(\dot{\phi}_0(0)\right)^{-1} \int_{-\infty}^0 \dot{\phi}_0(\zeta) [3l_0^2(1 + \phi_0)\phi_1^2 + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)\phi_1 \\ &\quad + l_1^2\phi_0(\phi_0 + 1)(\phi_0 + 2)] d\zeta.\end{aligned}$$

Noting that ϕ_1 also depends on l_1 , we find that

$$\begin{aligned}\frac{\partial}{\partial l_1} \Phi_1(l_1) &= \left(\dot{\phi}_0(0)\right)^{-1} \int_{-\infty}^0 \dot{\phi}_0(\zeta) \left[6l_0^2(1 + \phi_0)\phi_1 \frac{\partial \phi_1}{\partial l_1} \right. \\ &\quad + 2l_0(2 + 6\phi_0 + 3\phi_0^2)\phi_1 \\ &\quad \left. + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2) \frac{\partial \phi_1}{\partial l_1} + 2l_1\phi_0(\phi_0 + 1)(\phi_0 + 2) \right] d\zeta.\end{aligned}$$

Since ϕ_1 satisfies

$$\begin{cases} \ddot{\phi}_1 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\phi_1 = 2l_0l_1\phi_0(\phi_0 + 1)(\phi_0 + 2), & \xi \in (-\infty, 0) \\ \phi_1(-\infty) = 0, \quad \phi_1(0) = -U_1(1) = 0, \end{cases} \quad (4.2)$$

we know that $R(\xi) \equiv \frac{\partial}{\partial l_1} \phi_1(\xi)$ satisfies

$$\begin{cases} \ddot{R} - l_0^2(2 + 6\phi_0 + 3\phi_0^2)R = 2l_0\phi_0(\phi_0 + 1)(\phi_0 + 2), & \xi \in (-\infty, 0) \\ R(-\infty) = 0, & R(0) = 0. \end{cases} \quad (4.3)$$

On the other hand, since ϕ_0 satisfies

$$\begin{cases} \ddot{\phi}_0 - l_0^2\phi_0(\phi_0 + 1)(\phi_0 + 2) = 0, & \xi \in (-\infty, 0) \\ \phi_0(-\infty) = 0, & \phi_0(0) = -U_0(1) = -1, \end{cases}$$

$P(\xi) = \frac{\partial}{\partial l_0} \phi_0(\xi)$ does

$$\begin{cases} \ddot{P} - l_0^2(2 + 6\phi_0 + 3\phi_0^2)P = 2l_0\phi_0(\phi_0 + 1)(\phi_0 + 2), & \xi \in (-\infty, 0) \\ P(-\infty) = 0, & P(0) = 0. \end{cases} \quad (4.4)$$

By (4.3) and (4.4), we find that $P(\xi) = R(\xi)$ for $\xi \in (-\infty, 0)$. Next by (4.2), we know that $Q(\xi) = \frac{\partial}{\partial l_0} \phi_1(\xi)$ satisfies

$$\begin{cases} \ddot{Q} - l_0^2(2 + 6\phi_0 + 3\phi_0^2)Q = 2l_0(2 + 6\phi_0 + 3\phi_0^2)\phi_1 + 6l_0^2(1 + \phi_0)P\phi_1 \\ \quad + 2l_1\phi_0(\phi_0 + 1)(\phi_0 + 2) + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)P, & \xi \in (-\infty, 0) \\ Q(-\infty) = 0, & Q(0) = 0, \end{cases}$$

from which we have

$$\begin{aligned} Q(\xi) &= -\dot{\phi}_0(\xi) \int_{\xi}^0 \left(\dot{\phi}_0(\eta) \right)^{-2} \int_{-\infty}^{\eta} \dot{\phi}_0(\zeta) \left[2l_0(2 + 6\phi_0 + 3\phi_0^2)\phi_1 \right. \\ &\quad \left. + 6l_0^2(1 + \phi_0)P\phi_1 + 2l_1\phi_0(\phi_0 + 1)(\phi_0 + 2) + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)P \right] d\zeta d\eta. \end{aligned}$$

So we obtain

$$\begin{aligned} \dot{Q}(0) &= \left(\dot{\phi}_0(0) \right)^{-1} \int_{-\infty}^0 \dot{\phi}_0(\zeta) \left[2l_0(2 + 6\phi_0 + 3\phi_0^2)\phi_1 + 6l_0^2(1 + \phi_0)P\phi_1 \right. \\ &\quad \left. + 2l_1\phi_0(\phi_0 + 1)(\phi_0 + 2) + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)P \right] d\zeta. \end{aligned}$$

Thus we have $\frac{\partial}{\partial l_1} \Phi_1(l_1) = \dot{Q}(0)$. Finally, we differentiate

$$\begin{aligned} \Phi_0 &= U_0'^{(1)}(1) + \dot{\phi}_1^{(1)}(0) - l_0(U_0'^{(2)}(0) + \dot{\phi}_1^{(2)}(0)) - l_1\dot{\phi}_0^{(2)}(0) \\ &= \dot{\phi}_1^{(1)}(0) - l_1\dot{\phi}_0^{(2)}(0) = 0 \end{aligned}$$

by l_0 , we know that $\dot{Q}(0) = \frac{\partial}{\partial l_0} \dot{\phi}_1^{(1)}(0) = 0$, which implies $\frac{\partial}{\partial l_1} \Phi_1(l_1) = 0$. The proof is completed. \square

4.3. Proof of Lemma 3.1

Let $\omega \in \mathbb{C}$, $k \in \mathbb{R}$ and $P, Q, R \in \mathbb{C}$ and substitute $(U, V, W) = e^{\varepsilon^2 \omega t + i k x}$ (P, Q, R) in (2.1) with $\tau = \hat{\tau}/\varepsilon^2, \theta = \hat{\theta}/\varepsilon^2$. We have

$$\begin{cases} \varepsilon^2 \omega P = -k^2 \varepsilon^2 P + (1 - 3u^2)P - \varepsilon \alpha Q - \varepsilon \beta R, \\ \hat{\tau} \omega Q = -k^2 Q + P - Q, \\ \hat{\theta} \omega R = -k^2 D^2 R + P - R, \end{cases}$$

which is written in the following form

$$(-k^2 D + N(x) - \omega A) \mathbf{p} = \mathbf{0}, \quad -\infty < x < \infty,$$

where

$$D = \begin{bmatrix} \varepsilon^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{bmatrix}, N(x) = \begin{bmatrix} 1 - 3u^2 & -\varepsilon \alpha & -\varepsilon \beta \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

$$A = \begin{bmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \hat{\tau} & 0 \\ 0 & 0 & \hat{\theta} \end{bmatrix}, \mathbf{p} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

By [6], the location of the essential spectrum L^ε is determined by the following sets: $S_{\pm\infty} = \{\omega \in \mathbb{C} \mid \det(-k^2 D + \lim_{x \rightarrow \pm\infty} N(x) - \omega A) = 0, -\infty < k < \infty\}$.

$$\det(-k^2 D + \lim_{x \rightarrow \pm\infty} N(x) - \omega A)$$

$$= \begin{vmatrix} -\varepsilon^2 k^2 + 1 - 3(u_-(\varepsilon))^2 - \varepsilon^2 \omega & -\varepsilon \alpha & -\varepsilon \beta \\ 1 & -k^2 - 1 - \hat{\tau} \omega & 0 \\ 1 & 0 & -D^2 k^2 - 1 - \hat{\theta} \omega \end{vmatrix},$$

from which we have

$$\begin{aligned} & (-\varepsilon^2 k^2 + 1 - 3(u_-(\varepsilon))^2 - \varepsilon^2 \omega)(-k^2 - 1 - \hat{\tau} \omega)(-D^2 k^2 - 1 - \hat{\theta} \omega) \\ & - \varepsilon \alpha (D^2 k^2 + 1 + \hat{\theta} \omega) - \varepsilon \beta (k^2 + 1 + \hat{\tau} \omega) = 0. \end{aligned}$$

For sufficiently small $\varepsilon > 0$,

$$(-\varepsilon^2 k^2 + 1 - 3(u_-(\varepsilon))^2 - \varepsilon^2 \omega)(-k^2 - 1 - \hat{\tau} \omega)(-D^2 k^2 - 1 - \hat{\theta} \omega) = 0$$

holds approximately and noting $1 - 3(u_-(\varepsilon))^2 = -2 + O(\varepsilon)$, we find that

$$\begin{aligned}\omega &= (1 - 3(u_-(\varepsilon))^2 + O(\varepsilon))/\varepsilon^2 < -1/\varepsilon^2, \\ \omega &= -(k^2 + 1)/\hat{\tau} + O(\varepsilon) < -1/2\hat{\tau}, \quad \omega = -(D^2k^2 + 1)/\hat{\theta} + O(\varepsilon) < -1/2\hat{\theta}\end{aligned}$$

for any $k \in \mathbb{R}$. Then, putting $d = \min\{1/2\hat{\tau}, 1/2\hat{\theta}\} > 0$, we have our desired result. The proof is completed. \square

4.4. Proof of Lemma 3.2

By a simple calculation, we obtain

$$\begin{aligned}& \varepsilon^2 \det \left(A(\infty; \varepsilon; \hat{\lambda}) - \mu I \right) \\&= \begin{vmatrix} -\varepsilon\mu & 1 & 0 & 0 & 0 & 0 \\ -1 + 3(u_-(\varepsilon))^2 + \varepsilon^2\hat{\lambda} & -\varepsilon\mu & \varepsilon\alpha & 0 & \varepsilon\beta & 0 \\ 0 & 0 & -\mu & 1 & 0 & 0 \\ -1 & 0 & 1 + \hat{\tau}\hat{\lambda} & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 1 \\ -1/D^2 & 0 & 0 & 0 & (1 + \hat{\theta}\hat{\lambda})/D^2 & -\mu \end{vmatrix} \\&= (\varepsilon^2\mu^2 + 1 - 3(u_-(\varepsilon))^2 - \varepsilon^2\hat{\lambda})(\mu^2 - 1 - \hat{\tau}\hat{\lambda})(\mu^2 - (1 + \hat{\theta}\hat{\lambda})/D^2) \\&\quad + \varepsilon\alpha(\mu^2 - (1 + \hat{\theta}\hat{\lambda})/D^2) + \varepsilon\beta(\mu^2 - 1 - \hat{\tau}\hat{\lambda}) = 0.\end{aligned}$$

Thus, we know that

$$\begin{aligned}\varepsilon^2\mu^2 + 1 - 3(u_-(\varepsilon))^2 - \varepsilon^2\hat{\lambda} &= o(1), \quad \mu^2 - 1 - \hat{\tau}\hat{\lambda} = o(1), \\ \mu^2 - (1 + \hat{\theta}\hat{\lambda})/D^2 &= o(1) \text{ for small } \varepsilon > 0,\end{aligned}$$

which implies that

$$\mu^2 = (\varepsilon^2\hat{\lambda} + 2 + o(1))/\varepsilon^2, \quad \mu^2 = 1 + \hat{\tau}\hat{\lambda} + o(1), \quad \mu^2 = (1 + \hat{\theta}\hat{\lambda})/D^2 + o(1).$$

Here we set $d = \min\{1/2\hat{\tau}, 1/2\hat{\theta}\} > 0$. Then, for any $\lambda \in \mathbb{C}_d$ and small $\varepsilon > 0$, $A(\infty; \varepsilon; \hat{\lambda})$ has six eigenvalues, three of them have positive real parts and other three have negative real parts. The proof is completed. \square

4.5. Proof of Lemma 3.6

Let us calculate $a_{10} = \dot{p}_1^{(1)}(0; \varepsilon; \lambda; 1, 0, 0)/l_0$. In this proof, we omit the upper index (1). Noting the relation $\alpha a_0 + \beta b_0 + \gamma = 0$, (3.15) and (2.11)

can be rewritten

$$\begin{cases} \ddot{p}_1 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)p_1 \\ = 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)p_0 + 6l_0^2(1 + \phi_0)\phi_1p_0, & \xi \in (-\infty, 0) \\ p_1(-\infty) = 0, \quad p_1(0) = 0, \end{cases}$$

and

$$\begin{cases} \ddot{\phi}_1 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\phi_1 = 2l_0l_1\phi_0(\phi_0 + 1)(\phi_0 + 2), & \xi \in (-\infty, 0) \\ \phi_1(-\infty) = 0, \quad \phi_1(0) = 0 \end{cases} \quad (4.5)$$

respectively. When we differentiate the first equation of (4.5) by ξ , we have

$$\begin{aligned} (\dot{\phi}_1)'' - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_1 &= 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_0 \\ &\quad + 6l_0^2(1 + \phi_0)\dot{\phi}_0\phi_1, \quad \xi \in (-\infty, 0). \end{aligned}$$

Here putting $k(\xi) = \dot{\phi}_1(\xi)$, we have

$$\begin{cases} \ddot{k} - l_0^2(2 + 6\phi_0 + 3\phi_0^2)k \\ = \{2l_0l_1(2 + 6\phi_0 + 3\phi_0^2) + 6l_0^2(1 + \phi_0)\phi_1\} \dot{\phi}_0, & \xi \in (-\infty, 0) \\ k(-\infty) = 0, \quad k(0) = \dot{\phi}_1(0). \end{cases} \quad (4.6)$$

Furthermore putting $A(\xi) = p_1(\xi) - k(\xi)/\dot{\phi}_0(0)$ and noting $p_0(\xi; 1) = \dot{\phi}_0(\xi)/\dot{\phi}_0(0)$, we find that $A(\xi)$ satisfies the equations

$$\begin{cases} \ddot{A} - l_0^2(2 + 6\phi_0 + 3\phi_0^2)A = 0, & \xi \in (-\infty, 0) \\ A(-\infty) = 0, \quad A(0) = -\dot{\phi}_1(0)/\dot{\phi}_0(0) \end{cases} \quad (4.7)$$

from which $A(\xi) = \left(-\dot{\phi}_1(0)/\dot{\phi}_0(0)\right) \dot{\phi}_0(\xi)/\dot{\phi}_0(0)$ holds. Thus we have

$$p_1(\xi) = \dot{\phi}_1(\xi)/\dot{\phi}_0(0) - \dot{\phi}_1(0)\dot{\phi}_0(\xi)/(\dot{\phi}_0(0))^2.$$

Note that $\ddot{\phi}_1(0) = 0$ and $\ddot{\phi}_0(0) = 0$. We know that

$$\dot{p}_1(0) = \ddot{\phi}_1(0)/\dot{\phi}_0(0) - \dot{\phi}_1(0)\ddot{\phi}_0(0)/(\dot{\phi}_0(0))^2 = 0,$$

which shows $a_{10} = 0$. Similarly we have $d_{10} = 0$. \square

4.6. Proof of Lemma 3.7

Note that $a_{11} = -l_1 \dot{p}_1^{(1)}(0; \varepsilon; \lambda; 1, 0, 0)/l_0^2 + \dot{p}_2^{(1)}(0; \varepsilon; \lambda; 1, 0, 0)/l_0$. By the proof of Lemma 3.6, we already know $\dot{p}_1^{(1)}(0; \varepsilon; \lambda; 1, 0, 0) = 0$. Then, we calculate $\dot{p}_2^{(1)}(0; \varepsilon; \lambda; 1, 0, 0)$. First of all, we shall prepare the fundamental relations. For this purpose, we use the relation $u_{xx}^{(1)}(l(\varepsilon); \varepsilon) = u_{xx}^{(2)}(l(\varepsilon); \varepsilon)$ at $x = l(\varepsilon)$, where $u^{(1)}(x; \varepsilon)$ on $[0, l(\varepsilon)]$ and $u^{(2)}(x; \varepsilon)$ on $[l(\varepsilon), \infty)$ are the constructed solutions in §2.1 and §2.2, respectively. Substituting these constructed solutions into $u_{xx}^{(1)}(l(\varepsilon); \varepsilon) = u_{xx}^{(2)}(l(\varepsilon); \varepsilon)$ and equating the coefficients of the same powers of ε , we have

$$O(\varepsilon^{-2}) : \quad \ddot{\phi}_0^{(1)}(0) = l_0^2 \ddot{\phi}_0^{(2)}(0).$$

Indeed, we have $\ddot{\phi}_0^{(1)}(0) = \ddot{\phi}_0^{(2)}(0) = 0$.

$$O(\varepsilon^{-1}) : \quad \ddot{\phi}_1^{(1)}(0) = l_0^2 \ddot{\phi}_1^{(2)}(0) + 2l_0 l_1 \ddot{\phi}_0^{(2)}(0).$$

By using $\ddot{\phi}_1^{(2)}(0) = 0$, we have $\ddot{\phi}_1^{(1)}(0) = 0$.

$$O(\varepsilon^0) : \quad \ddot{\phi}_2^{(1)}(0) = l_0^2 \ddot{\phi}_2^{(2)}(0) + 2l_0 l_1 \ddot{\phi}_1^{(2)}(0) + (l_1^2 + 2l_0 l_2) \ddot{\phi}_0^{(2)}(0),$$

from which we have

$$\ddot{\phi}_2^{(1)}(0) = l_0^2 \ddot{\phi}_2^{(2)}(0). \quad (4.8)$$

Here after we omit the upper index (1) for simplicity. Differentiate the first equation of (2.10) and (2.11) with respect to ξ . We get

$$(\dot{\phi}_0)'' - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_0 = 0, \quad \xi \in (-\infty, 0) \quad (4.9)$$

and

$$\begin{aligned} (\dot{\phi}_1)'' + l_0^2 \left\{ -(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_1 - 6(1 + \phi_0)(U_1(1) + \phi_1)\dot{\phi}_0 \right\} \\ - 2l_0 l_1(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_0 = 0, \quad \xi \in (-\infty, 0), \end{aligned}$$

from which we have

$$\begin{aligned} (\dot{\phi}_1)'' - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_1 &= 6(1 + \phi_0)(U_1(1) + \phi_1)\dot{\phi}_0 \\ &+ 2l_0 l_1(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_0, \quad \xi \in (-\infty, 0). \end{aligned}$$

Since $\dot{\phi}_1(-\infty) = 0$, we obtain

$$\begin{aligned} \dot{\phi}_1(\xi) &= \dot{\phi}_1(0)\dot{\phi}_0(\xi)/\dot{\phi}_0(0) - \dot{\phi}_0(\xi) \int_{\xi}^0 \left(\dot{\phi}_0(\eta) \right)^{-2} \\ &\times \int_{-\infty}^{\eta} \dot{\phi}_0^2(\zeta) \{ 6(1 + \phi_0)(U_1(1) + \phi_1) + 2l_0 l_1(2 + 6\phi_0 + 3\phi_0^2) \} d\zeta d\eta. \end{aligned}$$

Thus we have

$$\begin{aligned} \ddot{\phi}_1(0) &= \dot{\phi}_1(0)\ddot{\phi}_0(0)/\dot{\phi}_0(0) + \left(\dot{\phi}_0(0)\right)^{-1} \\ &\times \int_{-\infty}^0 \dot{\phi}_0^2(\zeta) \{6(1+\phi_0)(U_1(1)+\phi_1) + 2l_0l_1(2+6\phi_0+3\phi_0^2)\} d\zeta \end{aligned}$$

Noting that $\ddot{\phi}_1(0) = 0$ and $\ddot{\phi}_0(0) = 0$, we have

$$\begin{aligned} \left(\dot{\phi}_0(0)\right)^{-1} \int_{-\infty}^0 \dot{\phi}_0^2(\zeta) \{6(1+\phi_0)(U_1(1)+\phi_1) \\ + 2l_0l_1(2+6\phi_0+3\phi_0^2)\} d\zeta = 0. \end{aligned} \quad (4.10)$$

Furthermore differentiating the first equation of (2.13) by ξ , we have

$$\begin{cases} (\dot{\phi}_2)'' + l_0^2 \left[-\{3(U_1(1)+\phi_1)^2 + 6(1+\phi_0)(U_1'(1)\xi + U_2(1)+\phi_2)\} \dot{\phi}_0 \right. \\ \quad -6(1+\phi_0)(U_1(1)+\phi_1)\dot{\phi}_1 - (2+6\phi_0+3\phi_0^2)(U_1'(1)+\dot{\phi}_2) \\ \quad \left. -\alpha V_0'(1) - \beta W_0'(1) \right] \\ \quad + 2l_0l_1 \left\{ -6(1+\phi_0)(U_1(1)+\phi_1)\dot{\phi}_0 - (2+6\phi_0+3\phi_0^2)\dot{\phi}_1 \right\} \\ \quad - (l_1^2 + 2l_0l_2)(2+6\phi_0+3\phi_0^2)\dot{\phi}_0 = 0 \quad , \xi \in (-\infty, 0). \end{cases} \quad (4.11)$$

Simply we put

$$\begin{aligned} S_1(\xi) &\equiv l_0^2 \left[\{3(U_1(1)+\phi_1)^2 + 6(1+\phi_0)(U_1'(1)\xi + U_2(1)+\phi_2)\} \dot{\phi}_0 \right. \\ &+ 6(1+\phi_0)(U_1(1)+\phi_1)\dot{\phi}_1 + (2+6\phi_0+3\phi_0^2)U_1'(1) + \alpha V_0'(1) + \beta W_0'(1) \Big] \\ &\quad + 2l_0l_1 \left\{ 6(1+\phi_0)(U_1(1)+\phi_1)\dot{\phi}_0 + (2+6\phi_0+3\phi_0^2)\dot{\phi}_1 \right\} \\ &\quad + (l_1^2 + 2l_0l_2)(2+6\phi_0+3\phi_0^2)\dot{\phi}_0. \end{aligned}$$

(4.11) can be rewritten as

$$(\dot{\phi}_2)'' - l_0^2(2+6\phi_0+3\phi_0^2)\dot{\phi}_2 = S_1(\xi), \quad \xi \in (-\infty, 0). \quad (4.12)$$

Multiplying (4.12) by $\dot{\phi}_0(\xi)$ and integrating it on $(-\infty, 0)$, we get

$$\int_{-\infty}^0 \dot{\phi}_0(\dot{\phi}_2)'' d\xi - l_0^2 \int_{-\infty}^0 (2+6\phi_0+3\phi_0^2)\dot{\phi}_0\dot{\phi}_2 d\xi = \int_{-\infty}^0 \dot{\phi}_0 S_1(\xi) d\xi.$$

Since the first term can be calculated as

$$\int_{-\infty}^0 \dot{\phi}_0(\dot{\phi}_2)'' d\xi = \left[\dot{\phi}_0\ddot{\phi}_2 \right]_{-\infty}^0 - \int_{-\infty}^0 \ddot{\phi}_0\ddot{\phi}_2 d\xi = \dot{\phi}_0(0)\ddot{\phi}_2(0) + \int_{-\infty}^0 (\dot{\phi}_0)''\dot{\phi}_2 d\xi,$$

we have

$$\dot{\phi}_0(0)\ddot{\phi}_2(0) + \int_{-\infty}^0 \left((\dot{\phi}_0)'' - l_0^2(2 + 6\phi_0 + 3\phi_0^2)\dot{\phi}_0 \right) \dot{\phi}_2 d\xi = \int_{-\infty}^0 \dot{\phi}_0 S_1(\xi) d\xi.$$

By the relation (4.9), we get

$$\dot{\phi}_0(0)\ddot{\phi}_2(0) = \int_{-\infty}^0 \dot{\phi}_0 S_1(\xi) d\xi. \quad (4.13)$$

Referring (3.16), p_2 satisfies

$$\begin{cases} \ddot{p}_2 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)p_2 = 6l_0^2(1 + \phi_0(U_1(1) + \phi_1))p_1 \\ + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2)p_1 + 3l_0^2 \{ (U_1(1) + \phi_1)^2 \\ + 2(1 + \phi_0)(U_1'(1)\xi + U_2(1) + \phi_2) \} p_0 \\ + 12l_0l_1(1 + \phi_0)(U_1(1) + \phi_1)p_0 \\ + (l_1^2 + 2l_0l_2)(2 + 6\phi_0 + 3\phi_0^2)p_0 + l_0^2\hat{\lambda}p_0, \\ p_2(-\infty) = 0, \quad p_2(0) = 0. \end{cases}$$

Substituting $p_0(\xi; 1) = \dot{\phi}_0(\xi)/\dot{\phi}_0(0)$ and $p_1(\xi) = \dot{\phi}_1(\xi)/\dot{\phi}_0(0) - \dot{\phi}_1(0)\dot{\phi}_0(\xi)/(\dot{\phi}_0(0))^2$, we get

$$\begin{aligned} & \ddot{p}_2 - l_0^2(2 + 6\phi_0 + 3\phi_0^2)p_2 \\ &= \{ 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1) + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2) \} \dot{\phi}_1/\dot{\phi}_0(0) \\ & - \{ 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1) + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2) \} \dot{\phi}_1(0)\dot{\phi}_0/(\dot{\phi}_0(0))^2 \\ & + 3l_0^2 \{ (U_1(1) + \phi_1)^2 + 2(1 + \phi_0)(U_1'(1)\xi + U_2(1) + \phi_2) \} \dot{\phi}_0/\dot{\phi}_0(0) \\ & + \left\{ 12l_0l_1(1 + \phi_0)(U_1(1) + \phi_1) + (l_1^2 + 2l_0l_2)(2 + 6\phi_0 + 3\phi_0^2) + l_0^2\tilde{\lambda} \right\} \dot{\phi}_0/\dot{\phi}_0(0), \\ &= \left[S_1(\xi) - l_0^2 \{ (2 + 6\phi_0 + 3\phi_0^2)U_1'(1) + \alpha V_0'(1) + \beta W_0'(1) \} + l_0^2\hat{\lambda}\dot{\phi}_0 \right] / \dot{\phi}_0(0) \\ & - \{ 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1) + 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2) \} \dot{\phi}_1(0)\dot{\phi}_0/(\dot{\phi}_0(0))^2 \\ &\equiv S_2(\xi). \end{aligned}$$

Solving this equation, we have

$$p_2(\xi) = -\dot{\phi}_0(\xi) \int_{\xi}^0 \left(\dot{\phi}_0(\eta) \right)^2 \int_{-\infty}^{\eta} \dot{\phi}_0(\zeta) S_2(\zeta) d\zeta,$$

which shows

$$\dot{p}_2(0) = \left(\dot{\phi}_0(0) \right)^{-1} \int_{-\infty}^0 \dot{\phi}_0(\zeta) S_2(\zeta) d\zeta$$

$$\begin{aligned}
&= \left(\dot{\phi}_0(0)\right)^{-2} \left[\int_{-\infty}^0 \dot{\phi}_0(\zeta) S_1(\zeta) d\zeta - l_0^2 \int_{-\infty}^0 \dot{\phi}_0(\zeta) \right. \\
&\times \left\{ (2 + 6\phi_0 + 3\phi_0^2) U_1'(1) + \alpha V_0'(1) + \beta W_0'(1) \right\} d\zeta \\
&- \dot{\phi}_1(0)/\dot{\phi}_0(0) \int_{-\infty}^0 \dot{\phi}_0^2(\zeta) \left\{ 6l_0^2(1 + \phi_0)(U_1(1) + \phi_1) \right. \\
&+ \left. 2l_0l_1(2 + 6\phi_0 + 3\phi_0^2) \right\} d\zeta + l_0^2 \hat{\lambda} \int_{-\infty}^0 \dot{\phi}_0^2(\zeta) d\zeta \Big].
\end{aligned}$$

Here note that $\int_{-\infty}^0 \dot{\phi}_0^2(\zeta) d\zeta = \frac{\sqrt{2}}{3} l_0$ and set $z = \phi_0(\zeta)$. It follows that

$$\begin{aligned}
&\int_{-\infty}^0 \dot{\phi}_0(\zeta) \left\{ (2 + 6\phi_0 + 3\phi_0^2) U_1'(1) + \alpha V_0'(1) + \beta W_0'(1) \right\} d\zeta \\
&= \int_0^{-1} \left\{ (2 + 6z + 3z^2) U_1'(1) + \alpha V_0'(1) + \beta W_0'(1) \right\} dz \\
&= -(\alpha V_0'(1) + \beta W_0'(1)).
\end{aligned}$$

By the relations (4.10) and (4.13), we know that

$$\dot{p}_2(0) = \left(\dot{\phi}_0(0)\right)^{-2} \left[\dot{\phi}_0(0) \ddot{\phi}_2(0) + l_0^2 (\alpha V_0'(1) + \beta W_0'(1)) + \frac{\sqrt{2}}{3} l_0^3 \hat{\lambda} \right].$$

Substitute $\dot{\phi}_0(0) = -l_0/\sqrt{2}$ and $\alpha V_0'(1) + \beta W_0'(1) = -l_0 \left\{ \alpha(1 - e^{-2l_0}) + \beta(1 - e^{-2l_0/D})/D \right\}$. We get finally

$$a_{11} = \frac{1}{l_0} \dot{p}_2(0) = -\frac{\sqrt{2}}{l_0^2} \ddot{\phi}_2(0) - 2 \left\{ \alpha(1 - e^{-2l_0}) + \frac{\beta}{D}(1 - e^{-2l_0/D}) \right\} + \frac{2\sqrt{2}}{3} \hat{\lambda}.$$

Similarly we have

$$d_{11} = \sqrt{2} \ddot{\phi}_2^{(2)}(0) + 2 \left\{ \alpha(1 - e^{-2l_0}) + \frac{\beta}{D}(1 - e^{-2l_0/D}) \right\} - \frac{2\sqrt{2}}{3} \hat{\lambda}.$$

Then,

$$\begin{aligned}
a_{11} - d_{11} &= -\frac{\sqrt{2}}{l_0^2} (\ddot{\phi}_2^{(1)}(0) - l_0^2 \ddot{\phi}_2^{(2)}(0)) \\
&+ 4 \left\{ \alpha(1 - e^{-2l_0}) + \frac{\beta}{D}(1 - e^{-2l_0/D}) \right\} + \frac{4\sqrt{2}}{3} \hat{\lambda}.
\end{aligned}$$

Therefore, by (4.8) we get

$$a_{11} - d_{11} = \frac{4\sqrt{2}}{3} \hat{\lambda} - 4 \left\{ \alpha(1 - e^{-2l_0}) + \frac{\beta}{D}(1 - e^{-2l_0/D}) \right\}.$$

□

5. Concluding remarks

In this paper, the existence of the standing pulse solutions of (1.2) with high accurate approximations for a small $\varepsilon > 0$ and their stability are shown. That is, the stability is determined by roots $\hat{\lambda}$ of (3.28) or (3.31). Each eigenvalue $\hat{\lambda}$ depends on the parameters $\hat{\tau}$ and $\hat{\theta}$. These facts imply that there is the possibility of the two types of bifurcations. One is a drift bifurcation when (3.31) has a double zero root. The other is a Hopf bifurcation when (3.28) has a pair of pure imaginary roots. See Figure 1. Though bifurcation phenomena from the standing pulse solution is a very interesting and important problem, we will discuss this problem in the forthcoming work since this article becomes too long.

Furthermore, when $\alpha\beta < 0$, there appears a saddle-node bifurcation point for some γ (see Figure 4). Thus, this three-component FitzHugh-Nagumo system (1.2) with $\tau = \hat{\tau}/\varepsilon^2$ and $\theta = \hat{\theta}/\varepsilon^2$ may have three types of bifurcation points, say a drift, a Hopf and a saddle-node bifurcation points, for suitable parameters $\alpha, \beta, \gamma, \hat{\tau}, \hat{\theta}$. Not only single bifurcation points but also double or triple bifurcation points may exist if we can choose the special parameters. The dynamics of the bifurcated solutions of (1.2) in a neighborhood of the above bifurcation points, moreover the dynamics of the interaction between heterogeneities and bifurcated traveling pulses are very interesting problems. Though we can apply the center manifold theory to these problems, for such purpose we have to construct eigenfunctions, with high accurate approximations for a small $\varepsilon > 0$, of the linearized problems and their adjoint problems (see [3] and [11]). In this sense, the result in this paper is the first key step for the above problems.

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