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On exchange π-UU unital rings

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Abstract. We prove that a ring R is exchange 2-UU if, and only if, $J(R)$ is nil and $R/J(R) \cong B \times C$, where B is a Boolean ring and C is a ring with $C \subseteq \prod_{\mu} \mathbb{Z}_3$ for some ordinal μ. We thus somewhat improve on a result due to Abdolyousefi-Chen (J. Algebra Appl., 2018) by showing that it is a simple consequence of already well-known results of Danchev-Lam (Publ. Math. Debrecen, 2016) and Danchev (Commun. Korean Math. Soc., 2017).

1. Introduction and Background

Everywhere in the text of the current article, all our rings are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are mainly in agreement with the stated in [7]. For instance, for an arbitrary ring R, $U(R)$ will always denote the unit group with n-th power $U^n(R) = \{u^n \mid u \in U(R)\}$, where $n \in \mathbb{N}$, $J(R)$ the Jacobson radical, and $Nil(R)$ the set of all nilpotents. Recall also that a ring R is said to be *tripotent* provided that the equality $x^3 = x$ holds for all $x \in R$.

We also need some other fundamentals as follows:

Definition 1.1. *([6]) A ring R is said to be UU if $U(R) = 1 + Nil(R)$.*

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Definition 1.2. *A ring R is said to be exchange if, for each $r \in R$, there is an idempotent $e \in rR$ such that $1 - e \in (1 - r)R$.*

It was proved in [6] that a ring R is an exchange UU ring if, and only if, $J(R)$ is nil and $R/J(R)$ is Boolean.

Before proceed by proving our chief result, we need a few more technicalities, mainly developed by the current author in the papers cited in the reference list. And so, generalizing Definition 1.1, one can state the following.

Definition 1.3. *Let $n \in \mathbb{N}$ be fixed. A ring R is called n-UU if, for any $u \in U(R)$, $u^n \in 1 + Nil(R)$, that is, the inclusion $U^n(R) \subseteq 1 + Nil(R)$ holds. If n is the minimal natural with this property, R is just said to be strongly n-UU.*

Clearly, UU rings just coincide with (strongly) 1-UU rings.

This can be freely expanded to the following:

Definition 1.4. *A ring R is called π-UU if, for any $u \in U(R)$, there exists $i \in \mathbb{N}$ depending on u such that $u^i \in 1 + Nil(R)$.*

The leitmotif of the present paper is to study exchange n-UU rings in the cases $n = 2$ and $n = 3$. Our results will considerably strengthen those from [1] and will also provide the interested reader with new simpler proofs. In closing we state a question which remains unanswered.

2. Main Results

The next statement considerably supersedes [1, Lemma 4.4] by dropping off the unnecessary limitation on the ring to be "exchange". The used technique was developed in [4] and [5].

Proposition 2.1. *Let R be a 2-UU ring. Then $J(R)$ is nil.*

Proof. Given $x \in J(R)$, it follows that $(1+x)^2 = 1+2x+x^2 \in 1+Nil(R)$ which amounts to $2x + x^2 \in Nil(R)$. Similarly, replacing x by $-x$, we derive that $-2x + x^2 \in Nil(R)$. Since these two sums commute, it follows immediately that $2x^2 \in Nil(R)$. Finally, using the above trick for x^2, we deduce that $2x^2 + x^4 \in Nil(R)$. Since $2x^2 \in Nil(R)$, we conclude that $x^4 \in Nil(R)$, i.e., $x \in Nil(R)$, as required. □

Corollary 2.2. *A ring R is 2-UU if, and only if, $J(R)$ is nil and $R/J(R)$ is 2-UU.*

Proof. According to Proposition 2.1, the argument follows in the same manner as [6, Theorem 2.4 (2)]. □

Lemma 2.3. *Let R be a ring. Then the following two points hold:*

(i) If R is n-UU for some $n \in \mathbb{N}$, then eRe is also n-UU for any $e \in Id(R)$.

(ii) If R is π-UU, then eRe is also π-UU for any $e \in Id(R)$.

Proof. We shall show the validity only of (ii). The proof of (i) is analogous and so it will be omitted. As in [6], letting $w \in U(eRe)$ with inverse v, it follows that $w + 1 - e \in U(R)$ with inverse $v + 1 - e$. Therefore, there exists $i \in \mathbb{N}$ such that $(w + 1 - e)^i = w^i + 1 - e \in 1 + Nil(R)$, that is, $w^i - e = q \in Nil(R)$. But $q \in Nil(R) \cap (eRe) = Nil(eRe)$ which leads to $w^i = e + q \in 1_{eRe} + Nil(eRe)$, as expected. □

Lemma 2.4. *For any $n \in \mathbb{N}$ and any non-zero ring R the full matrix ring $\mathbb{M}_n(R)$ is not 2-UU.*

Proof. Since $\mathbb{M}_2(R)$ is isomorphic to a corner ring of $\mathbb{M}_n(R)$ for $n \geq 2$, in view of Lemma 2.3 it suffices to establish the claim for $n = 2$. To that goal, as in [6], let us consider the invertible matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with the inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, we infer that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ which is the same invertible element with the inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, and thus it is certainly not a nilpotent, as wanted. □

We shall now restate and reproof the main result from [1] by giving a more convenient form and more transparent proof arising from well-known recent results in [6] and [5], respectively. Actually, a new substantial achievement, including new points with more strategic estimations, arises as follows:

Theorem 2.5. *Suppose that R is a ring. Then the following five items are equivalent:*

(a) R is exchange 2-UU.

(b) $J(R)$ is nil and $R/J(R)$ is commutative invo-clean.

(c) $J(R)$ is nil and $R/J(R) \cong B \times C$, where $B \subseteq \prod_\lambda \mathbb{Z}_2$ and $C \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

(d) $J(R)$ is nil and $R/J(R)$ is tripotent.

(e) $J(R)$ is nil and $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

Proof. The equivalence (b) $\iff$ (c) is exactly [5, Corollary 2.17], whereas the equivalence (d) $\iff$ (e) is obvious.

We shall show that (a) $\iff$ (b) is valid. To prove the left-to-right implication, we first consider the semi-primitive case when $J(R) = \{0\}$. Imitating the basic idea from the proof of [6, Theorem 4.1], we arrive at the case when $eRe \cong \mathbb{M}_2(T)$ for some idempotent $e \in R$ and some non-zero ring T depending on R, provided $Nil(R) \neq \{0\}$. However, with Lemma 2.3 at hand we deduce that eRe is 2-UU, while with the aid of Lemma 2.4 this property does not hold for $\mathbb{M}_2(T)$. This contradiction substantiates that R is reduced, i.e., $Nil(R) = \{0\}$ and thus abelian. Hence R is clean with $U^2(R) = \{1\}$ which allows us to conclude with an appeal to [5] that R is abelian invo-clean and so commutative invo-clean. Suppose now that $J(R) \neq \{0\}$. The fact that $J(R)$ is nil follows directly from Proposition 2.1. Owing to [8] and Corollary 2.2, one sees that $R/J(R)$ is exchange 2-UU, and so by what we have just already shown so far, the factor-ring $R/J(R)$ has to be commutative invo-clean, as asserted.

As for the right-to-left implication, it follows immediately by virtue of [8] that R is an exchange ring. That R is a 2-UU ring follows like this: Using the isomorphisms $U(R)/(1+J(R)) \cong U(R/J(R)) \cong U(B) \times \prod_\mu U(\mathbb{Z}_3)$, we so have $U^2(R/J(R)) = \{1\}$. Furthermore, for any $u \in U(R)$ it must be that $u+J(R) \in U(R/J(R))$ and hence $(u+J(R))^2 = u^2+J(R) = 1+J(R)$ which means that $u^2 - 1 \in J(R) \subseteq Nil(R)$, as required.

The implication (c) $\Rightarrow$ (d) is elementary. What remains to illustrate is the truthfulness of the implication (d) $\Rightarrow$ (a). Since tripotent rings are always exchange, the application of [8] shows that R is exchange. On the other side, since $U(R/J(R)) \cong U(R)/(1+J(R))$ and $U^2(R/J(R)) = \{1\}$, as shown above it follows that R is a 2-UU ring, thus completing the proof after all. $\square$

The next construction manifestly demonstrates that the theorem is no longer true for n-UU rings when $n > 2$.

Example 2.6. Consider the full matrix 2×2 ring $R = \mathbb{M}_2(\mathbb{Z}_2)$. It was proved in [2] that R is nil-clean and hence exchange. Moreover, R is a 3-UU ring. However, it is easily checked that $J(R) = \{0\}$ and that R is even not tripotent (whence it is not boolean). In fact, $U(R)$ has 6 elements satisfying the following identities:

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Therefore, in all cases, $U^3(R) - 1 \subseteq Nil(R)$ and besides there are $u \in U(R)$ such that $u^3 = 1 \neq u$. This concludes our claim and thus the example ends.

We finish off our work with the following question of some interest and importance. Recall that a ring R is termed *π-Boolean* if, for any $r \in R$, there is $i \in \mathbb{N}$, which depends on r, with $r^i = r^{2i}$.

Problem 2.7. *Does it follow that R is an exchange π-UU ring if, and only if, $J(R)$ is nil and $R/J(R)$ is π-Boolean?*

If yes, this will resolve the basic problem from [3] in the affirmative.

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