A characterization of ruled hypersurfaces and homogeneous real hypersurfaces of type (A_0), (A_1), and (B) in non-flat complex space forms

Setsuo NAGAI

Dedicated to Professor U-Hang Ki and Professor Masafumi Okumura on the occasion of their 80th birthdays

Abstract. The purpose of this paper is to give a characterization of ruled hypersurfaces and homogeneous real hypersurfaces of type (A_0), (A_1), and (B) in non-flat complex space forms by using the second covariant derivatives of the shape operators.

1. Introduction

Let $\overline{M}_n(c)$ be an n-dimensional complex space form with constant holomorphic sectional curvature $c \neq 0$, and let J and g be its complex structure and Kähler metric, respectively. Complete and simply connected complex space forms are isometric to either a complex projective space $\mathbb{C}P_n(c)$ or a complex hyperbolic space $\mathbb{C}H_n(c)$ decided from $c > 0$ or $c < 0$, respectively.

Let M be a connected submanifold of $\overline{M}_n(c)$ with real codimension 1. We refer to this simply as a real hypersurface below. The induced metric of M is denoted by g. For a local unit normal vector field ν of M, we define the structure vector ξ of M by $\xi = -J\nu$. Further, the structure tensor field ϕ is defined by $JX = \phi X + g(X,\xi)\nu$. The structure vector ξ is said to be

---

principal if $A\xi = \alpha\xi$ is satisfied for some function α, where A is the shape operator of M.

A real hypersurface M is said to be a Hopf hypersurface if the structure vector ξ of M is principal. In a complex projective space $\mathbb{C}P_n(c)$, Hopf hypersurfaces with constant principal curvatures are simply the homogeneous real hypersurfaces (for details, see §2 Theorem K). They were classified by R. Takagi [14] (for details, see §2 Theorem T), and their principal curvatures were also calculated by R. Takagi [15] (for details, see §2). In a complex hyperbolic space $\mathbb{C}H_n(c)$, Hopf hypersurfaces with constant principal curvatures were classified by J. Berndt [2] (for details, see §2 Theorem B).

On the other hand, there are important examples of non-Hopf hypersurfaces in $\overline{M}_n(c)$, such as ruled hypersurfaces (for definition, see §2), with a totally geodesic ruling along a curve in $\overline{M}_n(c)$.

For a real hypersurface M in $\overline{M}_n(c)$, let T_0 be a distribution defined by $T_0 = \{X \in TM \,|\, g(X, \xi) = 0\}$, where TM denotes the tangent bundle of M. The distribution T_0 is called a holomorphic distribution (see [9]). T_0 is not integrable on any Hopf homogeneous real hypersurface of $\overline{M}_n(c)$ but is integrable on any ruled hypersurface of $\overline{M}_n(c)$. M. Kimura [5] obtained some properties of a ruled hypersurface of $\mathbb{C}P_n(c)$ and gave an example of minimal ruled hypersurface of $\mathbb{C}P_n(c)$. There are many characterization theorems of ruled hypersurfaces in $\overline{M}_n(c)$ (e.g., [1], [4], [6], [7], [9]). We also know many characterization theorems of homogeneous real hypersurfaces by using some formulas of the first covariant derivatives of the shape operators (e.g., [3], [7], [8], [9]).

In [12], we characterized homogeneous real hypersurfaces of type (A_0) and (B) simultaneously by using the second covariant derivatives of the shape operators under the conditions that the structure vector fields ξ are principal as follows:

Theorem N.[12][Theorem 4.1.] *Let M be a connected real hypersurface in a complex space form $\overline{M}_n(c)$ $(n \geq 2,\ c \neq 0)$ on which the structure vector ξ is principal with principal curvature $\alpha \neq 0$. Then the shape operator A*

satisfies

$$g((\nabla^2_{X,Y} A)Z, W) = \frac{\alpha}{4} \left\{ g(\phi AX, Y) g((4\phi A + \frac{2c}{\alpha}\phi)Z, W) \right.$$
$$+ g(\phi AX, Z) g((4\phi A + \frac{c}{\alpha}\phi)Y, W)$$
$$\left. + g(\phi AX, W) g((4\phi A + \frac{c}{\alpha}\phi)Y, Z) \right\},$$
$$X, Y, Z, W \in T_0,$$

if and only if M is locally congruent to a homogeneous real hypersurface of type (A_0) or (B).

The purpose of this paper is to generalize the above result. To that end, we prove the following theorem:

Theorem 4.1. *Let M be a connected real hypersurface in a complex space form $\overline{M}_n(c)$ ($n \geq 3$, $c \neq 0$). Then the shape operator A satisfies the following equation for some nonzero constant a:*

$$g((\nabla^2_{X,Y} A)Z, W) = \frac{c}{4a} \left\{ 2g(\phi AX, Y) g((2\phi A + a\phi)Z, W) \right.$$
$$+ g(\phi AX, Z) g((4\phi A + a\phi)Y, W)$$
$$\left. + g(\phi AX, W) g((4\phi A + a\phi)Y, Z) \right\}, \tag{4.1}$$
$$X, Y, Z, W \in T_0,$$

if and only if M is locally congruent to one of the following:

(A_0) *a horosphere in $\mathbb{C}H_n(c)$;*

(A_1) *a geodesic hypersphere in $\overline{M}_n(c)$ or a tube over a complex hyperbolic hyperplane $\mathbb{C}H_{n-1}(c)$ in $\mathbb{C}H_n(c)$;*

(B) *a tube over a totally geodesic and totally real space form of real dimension n in $\overline{M}_n(c)$;*

(R) *a ruled hypersurface in $\overline{M}_n(c)$.*

In the following all manifolds are assumed to be C^∞ and connected.

2. Preliminaries

In this section, we present some preliminary results of real hypersurfaces in a complex space form.

Let $\overline{M}_n(c)$ $(c \neq 0)$ be an n-dimensional complex space form with constant holomorphic sectional curvature c and let J and g be its complex structure and Kähler metric, respectively.

M is a real hypersurface of $\overline{M}_n(c)$. Further, we denote by g the induced Riemannian metric on M and by ν a local unit normal vector field along M in $\overline{M}_n(c)$.

The Gauss and Weingarten formulas are:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)\nu, \tag{2.1}$$

$$\overline{\nabla}_X \nu = -AX, \tag{2.2}$$

where $\overline{\nabla}$ and ∇ respectively denote the Levi–Civita connection on $\overline{M}_n(c)$ and M, and A is the shape operator of M in $\overline{M}_n(c)$.

We define an almost contact metric structure (ϕ, ξ, η, g) on M as in the usual way:

$$\xi = -J\nu, \ \eta(X) = g(X, \xi), \ \phi X = (JX)^T, \ X \in TM, \tag{2.3}$$

where $(\ )^T$ denotes the tangential component of a vector. These structure tensors satisfy the following equations:

$$\phi^2 = -I + \eta \otimes \xi, \ \phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(\xi) = 1, \tag{2.4}$$

where I denotes the identity mapping of TM.

From (2.1) and (2.3), we easily have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX, \tag{2.6}$$

for tangent vectors $X, Y \in TM$.

In our case the Gauss and Codazzi equations of M become

$$\begin{aligned} R(X, Y)Z = &\frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\left\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\right\}. \qquad (2.8)$$

The Ricci formula of the tensor field T on M is:

$$\nabla^2_{X,Y}T - \nabla^2_{Y,X}T = R(X,Y)\cdot T, \quad X, Y \in TM, \qquad (2.9)$$

where the second covariant derivative $\nabla^2_{X,Y}T$ of T is defined by

$$\nabla^2_{X,Y}T = \nabla_X(\nabla_Y T) - \nabla_{\nabla_X Y}T$$

and $R(X,Y)$ acts on T as a derivation.

A real hypersurface M of $\mathbb{C}P_n(c)$ is said to be a homogeneous real hypersurface if it is an orbit of an analytic subgroup of the isometry group of $\mathbb{C}P_n(c)$. We know the complete classification of homogeneous real hypersurfaces of $\mathbb{C}P_n(c)$.

Theorem T ([14]). *Let M be a homogeneous real hypersurface of $\mathbb{C}P_n(c)$. Then M is locally congruent to one of the following spaces*:

(A_1) *a geodesic hypersphere*;

(A_2) *a tube of radius r $(0 < r < \frac{\pi}{\sqrt{c}})$ over a totally geodesic $\mathbb{C}P_k(c)$ $(1 \leq k \leq n-2)$*;

(B) *a tube of radius r $(0 < r < \frac{\pi}{2\sqrt{c}})$ over a complex quadric Q_{n-1}*;

(C) *a tube of radius r $(0 < r < \frac{\pi}{2\sqrt{c}})$ over $\mathbb{C}P_1 \times \mathbb{C}P_{\frac{n-1}{2}}$ where n (≥ 5) is odd*;

(D) *a tube of radius r $(0 < r < \frac{\pi}{2\sqrt{c}})$ over a complex Grassmann $G_{2,5}$ and $n = 9$*;

(E) *a tube of radius r $(0 < r < \frac{\pi}{2\sqrt{c}})$ over a Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$.*

For homogeneity of a real hypersurface in $\mathbb{C}P_n(c)$, there is a criterion obtained by M. Kimura [5]. His theorem is

Theorem K ([5]). *Let M be a connected real hypersurface in $\mathbb{C}P_n(c)$. Then M has constant principal curvatures and the structure vector ξ is*

principal if and only if M is congruent to an open subset of a homogeneous real hypersurface.

In $\mathbb{C}H_n(c)$ J. Berndt [2] obtained the complete classification of Hopf hypersurfaces with constant principal curvatures. His theorem is the following:

Theorem B ([2]). *Let M be a connected real hypersurface of $\mathbb{C}H_n(c)$ ($n \geq 2$) with constant principal curvatures. Further, assume that the structure vector ξ is principal. Then M is orientable and holomorphic congruent to an open part of one of the following hypersurfaces:*

(A_0) *a horosphere in $\mathbb{C}H_n(c)$;*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $\mathbb{C}H_{n-1}(c)$ in $\mathbb{C}H_n(c)$;*

(A_2) *a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic $\mathbb{C}H_k(c)$ ($1 \leq k \leq n-2$);*

(B) *a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic totally real submanifold $\mathbb{R}H_n$.*

Concerning the principal curvatures α, λ_1, λ_2, λ_3, λ_4 and their multiplicities m_α, m_{λ_1}, m_{λ_2}, m_{λ_3}, m_{λ_4} of homogeneous real hypersurfaces in $\mathbb{C}P_n(c)$, we have the following Table 1 (see [15]): Here, α is the principal curvature corresponding to the principal direction ξ.

Concerning the principal curvatures α, λ_1, λ_2 and their multiplicities m_α, m_{λ_1}, m_{λ_2} of Hopf homogeneous real hypersurfaces in $\mathbb{C}H_n(c)$, we have the following Table 2 (see [10]): Here, α is the principal curvature corresponding to the principal direction ξ.

Next, we explain some fundamental facts of ruled hypersurfaces in complex space forms (for details, see [1], [4], [5], [6]). Let $\gamma : I \to \overline{M}_n(c)$ be an arbitrary (regular) curve in $\overline{M}_n(c)$. Then for every t ($\in I$) there exists a totally geodesic hyperplane $\overline{M}_{n-1}^{(t)}(c)$ (in $\overline{M}_n(c)$) through the point $\gamma(t)$ which is orthogonal to the holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Let $M = \bigcup_{t \in I} \overline{M}_{n-1}^{(t)}(c)$. Then M obtains a real hypersurface of $\overline{M}_n(c)$, which is called a ruled hypersurface.

Type	Principal curvatures	Multiplicities
(A_1)	$\alpha = \sqrt{c}\cot\sqrt{c}r$	$m_\alpha = 1$
	$\lambda_1 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r$	$m_{\lambda_1} = 2(n-1)$
(A_2)	$\alpha = \sqrt{c}\cot\sqrt{c}r$	$m_\alpha = 1$
	$\lambda_1 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r$	$m_{\lambda_1} = 2(n-k-1)$
	$\lambda_2 = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}}{2}r$	$m_{\lambda_2} = 2k$
(B)	$\alpha = \sqrt{c}\cot\sqrt{c}r$	$m_\alpha = 1$
	$\lambda_1 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}(r-\frac{\pi}{2\sqrt{c}})$	$m_{\lambda_1} = n-2$
	$\lambda_2 = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}}{2}(r-\frac{\pi}{2\sqrt{c}})$	$m_{\lambda_2} = n-2$
(C)	$\alpha = \sqrt{c}\cot\sqrt{c}r$	$m_\alpha = 1$
	$\lambda_i = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}(r-\frac{\pi i}{2\sqrt{c}})$	$m_{\lambda_i} = n-3\ (i=2,4)$
	$(i = 1, 2, 3, 4)$	$m_{\lambda_i} = 2\ (i=1,3)$
(D)	$\alpha = \sqrt{c}\cot\sqrt{c}r$	$m_\alpha = 1$
	$\lambda_i = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}(r-\frac{\pi i}{2\sqrt{c}})$	$m_{\lambda_i} = 4\ (i=1,2,3,4)$
	$(i = 1, 2, 3, 4)$	
(E)	$\alpha = \sqrt{c}\cot\sqrt{c}r$	$m_\alpha = 1$
	$\lambda_i = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}(r-\frac{\pi i}{2\sqrt{c}})$	$m_{\lambda_i} = 8\ (i=2,4)$
	$(i = 1, 2, 3, 4)$	$m_{\lambda_i} = 6\ (i=1,3)$

Table 1: Principal curvatures in $\mathbb{C}P_n(c)$

Type	Principal curvatures	Multipricities
(A_0)	$\alpha = \sqrt{-c}$	1
	$\lambda_1 = \frac{\sqrt{-c}}{2}$	$2n-2$
(A)	$\alpha = \sqrt{-c}\coth\sqrt{-c}r$	1
	$\lambda_1 = \frac{\sqrt{-c}}{2}\coth\frac{\sqrt{-c}}{2}r$	$2(n-k-1)$
	$\lambda_2 = \frac{\sqrt{-c}}{2}\tanh\frac{\sqrt{-c}}{2}r$	$2k$
(B)	$\alpha = \sqrt{-c}\tanh\sqrt{-c}r$	1
	$\lambda_1 = \frac{\sqrt{-c}}{2}\coth\frac{\sqrt{-c}}{2}r$	$n-1$
	$\lambda_2 = \frac{\sqrt{-c}}{2}\tanh\frac{\sqrt{-c}}{2}r$	$n-1$

Table 2: Principal curvatures in $\mathbb{C}H_n(c)$

Let $T_0 = \{X \in TM | \ X \perp \xi\}$ be the holomorphic distribution of M. Then the distribution T_0 is integrable on a ruled hypersurface M. Fur-

thermore, the shape operator A of M satisfies the following:

$$
\begin{aligned}
A\xi &= \alpha\xi + U \quad (U \neq 0), \\
AU &= g(U,U)\xi, \\
AX &= 0, \quad X \perp \xi,\, U,
\end{aligned}
\tag{2.10}
$$

where U is the T_0-component of $A\xi$ (for details, see [6]).

Now, we define the concept of η-parallel second fundamental form as follows.

Definition 2.1. *The shape operator A of a real hypersurface M is said to be η-parallel if A satisfies the following*:

$$
g((\nabla_X A)Y, Z) = 0, \quad X,\, Y,\, Z \in T_0.
\tag{2.11}
$$

There are many characterization theorems for ruled hypersurfaces in a complex space form (see, for example, [1], [4], [5], [6], [7]). The following proposition can be easily deduced:

Proposition 2.2. *Let M be a real hypersurface of a complex space form $\overline{M}_n(c)$. If the shape operator A of M satisfies*

$$
g(AX, Y) = 0, \quad X,\, Y \in T_0,
\tag{2.12}
$$

then M is locally congruent to a ruled hypersurface of $\overline{M}_n(c)$.

Proof Let X and Y be vector fields of M whose values belong to T_0. Then, accoding to (2.6) and (2.12), we have

$$
g(\nabla_X Y, \xi) = -g(Y, \phi AX) = g(AX, \phi Y) = 0.
$$

This means that

$$
\nabla_X Y \in T_0, \quad X,\, Y \in T_0.
\tag{2.13}
$$

Furthermore, for vector fields X and Y whose values belong to T_0, we have

$$
\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)\nu = \nabla_X Y.
\tag{2.14}
$$

From (2.13) and (2.14), we conclude that the leaves of the distribution T_0 are totally geodesic in M and in $\overline{M}_n(c)$. Because T_0 is J-invariant, all the leaves of the distribution T_0 are totally geodesic complex hyperplanes $\overline{M}_{n-1}(c)$ of $\overline{M}_n(c)$. This proves the proposition. $\square$

Remark 2.3. *For a real hypersurface M of $\overline{M}_n(c)$, the following two conditions are equivalent (see Proposition 5 in [7]):*
(i) The holomorphic distribution T_0 is integrable.
(ii) $g((\phi A + A\phi)X, Y) = 0$ for any $X, Y \in T_0$.

Remark 2.4. *Under the condition of Proposition 2.2, the shape operator A of M is η-parallel (see [7] Proposition 4).*

For ruled real hypersurfaces in a complex space form, M. Kimura and S. Maeda obtained the following characterization theorem:

Theorem 2.5 (M. Kimura and S. Maeda [7]) *Let M be a real hypersurface of $\mathbb{C}P_n(c)$. Then the shape operator A of M is η-parallel and the holomorphic distribution T_0 is integrable if and only if M is locally congruent to a ruled hypersurface of $\mathbb{C}P_n(c)$.*

For a ruled hypersurface of $\overline{M}_n(c)$, S-S. Ahn, S-B. Lee, and Y-J. Suh obtained the following:

Theorem 2.6 (S-S. Ahn, S-B. Lee, and Y-J. Suh [1]) *Let M be a connected real hypersurface of $\overline{M}_n(c)$, $c \neq 0$ and $n \geq 3$. If the shape operator A of M satisfies*

$$g((\nabla_X A)Y, Z) = 0$$

and

$$g((A\phi - \phi A)X, Y) = 0$$

for any vector fields X, Y, and Z in T_0, and the structure vector field ξ is not principal, then M is locally congruent to a ruled hypersurface.

3. Lemmas

In this section, we present some lemmas that we will use to prove our main theorem.

For the shape operators of real hypersurfaces in a non-flat complex space form $\overline{M}_n(c)$ $(c \neq 0)$, we know the following:

Lemma 3.1 ([11], [13]) *Let M be a homogeneous real hypersurface of type (A) in a non-flat complex space form $\overline{M}_n(c)$ $(c \neq 0)$. Then the shape operator A of M satisfies the following equations:*

$$\phi A - A\phi = 0, \tag{3.1}$$

$$A^2 - \alpha A - \frac{c}{4}I = -\frac{c}{4}\eta \otimes \xi, \tag{3.2}$$

$$(\nabla_X A)Y = -\frac{c}{4}\left\{\eta(Y)\phi X + g(\phi X, Y)\xi\right\}, \tag{3.3}$$

where α and I denote the principal curvature in the direction of the structure vector ξ and the identity mapping of TM, respectively.

Lemma 3.2 ([3]) *Let M be a homogeneous real hypersurface of type (A_0) or (B) in a non-flat complex space form $\overline{M}_n(c)$ $(c \neq 0)$. Then the shape operator A of M satisfies the following equations:*

$$\phi A + A\phi + \frac{c}{\alpha}\phi = 0, \tag{3.4}$$

$$A^2 + \frac{c}{\alpha}A - \frac{c}{4}I = (\alpha^2 + \frac{3}{4}c)\eta \otimes \xi, \tag{3.5}$$

$$
(\nabla_X A)Y = -\frac{\alpha}{4}\left\{2\eta(X)(A\phi - \phi A)Y \right. \\
\left. +\eta(Y)(A\phi - 3\phi A)X + g((A\phi - 3\phi A)X, Y)\xi\right\}. \tag{3.6}
$$

From Lemma 3.1, Lemma 3.2, Table 1 and Table 2, we have

Lemma 3.3. *Let M be a homogeneous real hypersurface of type (A_0), (A_1), or (B) in $\overline{M}_n(c)$ $(c \neq 0)$. Then, the shape operator A of M satisfies the following equations for some nonzero constant a:*

$$\phi A + A\phi + a\phi = 0, \tag{3.7}$$

$$
(\nabla_X A)Y = \frac{c}{4a}\left\{2\eta(X)(2\phi A + a\phi)Y + \eta(Y)(4\phi A + a\phi)X \right. \\
\left. +g((4\phi A + a\phi)X, Y)\xi\right\}. \tag{3.8}
$$

So, taking the covariant derivative of (3.8), we have the following:

Lemma 3.4. *Let M be a homogeneous real hypersurface of type (A_0), (A_1), or (B) in $\overline{M}_n(c)$ $(c \neq 0)$. Then, the shape operator A of M satisfies the following equation for some nonzero constant a:*

$$
\begin{aligned}
g((\nabla^2_{X,Y} A)Z, W) = \frac{c}{4a} &\{ 2g(\phi AX, Y)g((2\phi A + a\phi)Z, W) \\
&+ g(\phi AX, Z)g((4\phi A + a\phi)Y, W) \\
&+ g(\phi AX, W)g((4\phi A + a\phi)Y, Z) \}, \quad X, Y, Z, W \in T_0.
\end{aligned}
$$
(3.9)

For a ruled hypersurface in $\overline{M}_n(c)$ $(c \neq 0)$, we have

Lemma 3.5. *Let M be a ruled hypersurface in $\overline{M}_n(c)$ $(c \neq 0)$. Then, both-sides of (3.9) vanish.*

Proof For a ruled hypersurface M, the leaves of the distribution T_0 are totally geodesic both in M and $\overline{M}_n(c)$. Furthermore, the shape operator A of M is η-parallel (see [6]). From this it follows that

$$
\begin{aligned}
&g((\nabla^2_{X,Y} A)Z, W) \\
&= g(\nabla_X((\nabla_Y A)Z) - (\nabla_Y A)(\nabla_X Z), W) - g((\nabla_{\nabla_X Y} A)Z, W) \\
&= 0, \quad X, Y, Z, W \in T_0.
\end{aligned}
$$

This means that the left side of (3.9) vanishes.

According to (2.10) and Remark 2.3, the right side of (3.9) also vanishes. This proves the lemma. □

The following is known about the principal curvatures of a Hopf hypersurface:

Lemma 3.6 ([8], [13]) *Let M be a Hopf hypersurface of $\overline{M}_n(c)$ $(n \geq 2, c \neq 0)$. Then the principal curvature α corresponding to the structure vector ξ is locally constant, and for any principal curvature vector $X \in T_0$ with $AX = \lambda X$, we have*

$$
(2\lambda - \alpha)A\phi X = (\alpha\lambda + \frac{c}{2})\phi X.
$$

4.   Proof of Theorem

In this section we shall prove our main theorem.

Theorem 4.1. *Let M be a connected real hypersurface in a complex space form $\overline{M}_n(c)$ ($n \geq 3$, $c \neq 0$). Then the shape operator A satisfies the following equation for some nonzero constant a:*

$$g((\nabla^2_{X,Y}A)Z, W) = \frac{c}{4a} \left\{ 2g(\phi AX, Y)g((2\phi A + a\phi)Z, W) \right.$$
$$+ g(\phi AX, Z)g((4\phi A + a\phi)Y, W)$$
$$\left. + g(\phi AX, W)g((4\phi A + a\phi)Y, Z) \right\}, \tag{4.1}$$
$$X,\, Y,\, Z,\, W \in T_0,$$

if and only if M is locally congruent to one of the following:

(A_0) *a horosphere in $\mathbb{C}H_n(c)$;*

(A_1) *a geodesic hypersphere in $\overline{M}_n(c)$ or a tube over a complex hyperbolic hyperplane $\mathbb{C}H_{n-1}(c)$ in $\mathbb{C}H_n(c)$;*

(B) *a tube over a totally geodesic and totally real space form of real dimension n in $\overline{M}_n(c)$;*

(R) *a ruled hypersurface in $\overline{M}_n(c)$.*

By making use of the Codazzi equation (2.8), we find the following general formula:

$$g((\nabla^2_{X,Y}A)Z - (\nabla^2_{X,Z}A)Y, W)$$
$$= \frac{c}{4} \left\{ g(\phi AX, Y)g(\phi Z, W) - g(\phi AX, Z)g(\phi Y, W) \right. \tag{4.2}$$
$$\left. - 2g(\phi AX, W)g(\phi Y, Z) \right\},$$

on arbitrary tangent vectors X, Y, Z, $W \in T_0$. Therefore the condition (4.1) and (4.2) implies that

$$g(\phi AX, W)g((\phi A + A\phi + a\phi)Y, Z) = 0. \tag{4.3}$$

We assume here that $a \neq 0$.

In what follows $e_1, \ldots, e_{2n-2}$ denotes an orthonormal basis of T_0 at a point in M, and the index j runs from 1 to $2n - 2$. Letting $X = e_j$ and $W = \phi e_j$ in (4.3), and taking the summation on j, we have

$$(h - \alpha)g((\phi A + A\phi + a\phi)Y, Z) = 0, \tag{4.4}$$

where h and α denote the mean curvature of M and the function defined by $\alpha = g(A\xi, \xi)$, respectively. Furthermore, letting $Y = e_j$ and $Z = \phi e_j$ in (4.4) and taking the summation on j, we obtain

$$(h - \alpha)\left\{(h - \alpha) + (n - 1)a\right\} = 0. \tag{4.5}$$

From (4.5) either $h - \alpha = 0$ or $(h - \alpha) + (n - 1)a = 0$ holds on M, since M is connected and $a \neq 0$. We therefore devide the discussion into the following two cases:

Case 1. The equation $h - \alpha = 0$ holds on M.

Case 2. The equation $(h - \alpha) + (n - 1)a = 0$ holds on M.

We first consider Case 1, in which we have $(h-\alpha)+(n-1)a = (n-1)a \neq 0$. So, from (4.3), the following holds on M:

$$g(\phi AX, Y) = 0, \quad X, Y \in T_0. \tag{4.6}$$

This leads to

$$A\xi = \alpha\xi + U,$$
$$AU = g(U, U)\xi,$$
$$AX = 0, \quad X \perp \xi, U,$$

where the vector field U is defined by $U = A\xi - \alpha\xi$. This means that M is locally congruent to a ruled hypersurface of $\overline{M}_n(c)$ by Proposition 2.2 (see also [7]).

On the other hand, let M be a ruled hypersurface of $\overline{M}_n(c)$. Then, (4.1) holds by Lemma 3.5.

We next consider Case 2. In this case we have $h - \alpha \neq 0$. So, from (4.4),

$$(\phi A + A\phi + a\phi)X = -g(\phi U, X)\xi, \quad X \in T_0. \tag{4.7}$$

On the other hand, using (4.1), (2.7), and (2.9), we have

$$\frac{c}{4}\{g((\phi A + A\phi)Y, Z)g(\phi X, W) - g((\phi A + A\phi)X, Z)g(\phi Y, W)$$

$$+ g((\phi A + A\phi)Y, W)g(\phi X, Z) - g((\phi A + A\phi)X, W)g(\phi Y, Z)\}$$

$$- \frac{c}{2}\{g((\phi A - A\phi)Z, W)g(\phi X, Y) + g((\phi A + A\phi)X, Y)g(\phi Z, W)\}$$

$$- \frac{c}{a}g((\phi A + A\phi)X, Y)g(\phi A Z, W) + g(AY, Z)g(\frac{c}{4}X - A^2 X, W) \qquad (4.8)$$

$$- g(AX, Z)g(\frac{c}{4}Y - A^2 Y, W) - g(AX, W)g(\frac{c}{4}Y - A^2 Y, Z)$$

$$+ g(AY, W)g(\frac{c}{4}X - A^2 X, Z) = 0, \quad X, Y, Z, W \in T_0$$

(for details, see [12]).

Combining (4.7) with (4.8), we are led to

$$g(AY, Z)g(\frac{c}{4}X - A^2 X, W) - g(AX, Z)g(\frac{c}{4}Y - A^2 Y, W)$$

$$- g(AX, W)g(\frac{c}{4}Y - A^2 Y, Z) + g(AY, W)g(\frac{c}{4}X - A^2 X, Z) = 0. \qquad (4.9)$$

We assert that ξ is principal. We shall prove our assertion by reductio ad absurdum.

Let Ω be the open subset of M defined by $\Omega = \{p \in M \,|\, U(p) \neq 0\}$. In the following we assume that the set Ω is not empty and all discussion concerns the set Ω unless otherwise stated.

We restate the orthonormal frame field e_1, e_2, ..., e_{2n-1} in such a way that $e_1 = \xi$, $e_2 = U/\|U\|$, $e_3 = \phi e_2$, $e_{2i+1} = \phi e_{2i}$ $(i \geq 2)$.

We now define the local differentiable functions α, β, γ, δ, and h_{ij} as follows:

$$\alpha = g(A\xi, \xi), \ \beta = g(A\xi, e_2), \ \gamma = g(Ae_2, e_2), \ \delta = g(Ae_2, e_3),$$

$$h_{ij} = g(Ae_i, e_j), \ i, j = 1, 2, \ldots, 2n - 1.$$

Hereafter the index p runs 2, 4, ..., $2(n - 1)$ and $p*$ means $p* = p + 1$. We further assume that e_i $(i \geq 4)$ are chosen in such a manner that

$$h_{ij} = \lambda_i \delta_{ij}. \quad (i, j \geq 4).$$

This is in fact possible by (4.7). Furthermore, the following are satisfied by (4.7):

$$g(Ae_3, e_3) = -(\gamma + a), \ \lambda_{p*} = -(\lambda_p + a), \ p = 4, 6, \ldots, 2(n - 1).$$

Let us first demonstrate the following lemma:

Lemma 4.2. $h_{2p} = h_{2p*} = 0 \quad (p = 4, 6, \ldots, 2(n-1))$.

Proof Letting $X = e_2$, $Y = \phi e_2$, $Z = e_p$, $W = e_2$ in (4.9), we have

$$
\begin{aligned}
&-\delta(\lambda_p + a)h_{2p} \\
&+\left\{\gamma(\lambda_p - a) - \beta^2 - 2\gamma^2 - 2\delta^2 - \sum_p h_{2p}^2 - \sum_p h_{2p*}^2 + \frac{c}{4}\right\} h_{2p*} \quad (4.10) \\
&= 0.
\end{aligned}
$$

Letting $X = e_2$, $Y = \phi e_2$, $Z = e_{p*}$, $W = \phi e_2$ in (4.9), we have

$$
\begin{aligned}
&\delta \lambda_p h_{2p} \\
&+\left\{2\delta^2 + 2(\gamma + a)^2 + \sum_p h_{2p}^2 + \sum_p h_{2p*}^2 - \frac{c}{4} - (\gamma + a)(\lambda_p + 2a)\right\} h_{2p*} \\
&= 0.
\end{aligned}
$$

$$(4.11)$$

Letting $X = e_2$, $Y = \phi e_2$, $Z = e_{p*}$, $W = e_2$ in (4.9), we have

$$
\begin{aligned}
&\left\{\gamma(\lambda_p + 2a) + \beta^2 + 2\gamma^2 + 2\delta^2 + \sum_p h_{2p}^2 + \sum_p h_{2p*}^2 - \frac{c}{4}\right\} h_{2p} \\
&+ \delta \lambda_p h_{2p*}^2 = 0.
\end{aligned} \quad (4.12)
$$

Letting $X = e_2$, $Y = \phi e_2$, $Z = e_p$, $W = \phi e_2$ in (4.9), we have

$$
\begin{aligned}
&\left\{(\gamma + a)(\lambda_p - a) + 2\delta^2 + 2(\gamma + a)^2 + \sum_p h_{2p}^2 + \sum_p h_{2p*}^2 - \frac{c}{4}\right\} h_{2p} \\
&+ \delta(\lambda_p + a)h_{2p*} = 0.
\end{aligned} \quad (4.13)
$$

From (4.10) and (4.11), we have

$$
a\delta h_{2p} + \left\{a(\lambda_p - \gamma) + \beta^2\right\} h_{2p*} = 0. \quad (4.14)
$$

From (4.12) and (4.13), we have

$$
\left\{a(\lambda_p + \gamma + a) - \beta^2\right\} h_{2p} + a\delta h_{2p*} = 0. \quad (4.15)
$$

Letting $X = e_2$, $Y = e_p$, $Z = e_{p*}$, $W = \phi e_2$ in (4.9), we have

$$\{\delta h_{2p} + (\lambda_p - \gamma)h_{2p*}\} h_{2p*} = 0. \tag{4.16}$$

Letting $X = \phi e_2$, $Y = e_p$, $Z = e_{p*}$, $W = e_2$ in (4.9), we have

$$\{(\lambda_p + \gamma + a)h_{2p} + \delta h_{2p*}\} h_{2p} = 0. \tag{4.17}$$

From (4.15) and (4.17),

$$\beta^2 h_{2p} = 0,$$

and from (4.14) and (4.16),

$$\beta^2 h_{2p*} = 0.$$

Since $\beta \neq 0$, the lemma holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Secondly, we prove the following lemma:

Lemma 4.3. $\delta = 0$.

Proof We shall prove the lemma by reductio ad absurdum. We assume that $\delta \neq 0$ is satisfied. Then, letting $X = e_2$, $Y = e_p$, $Z = e_p$, $W = \phi e_2$ in (4.9) and using Lemma 4.2, we are led to

$$\lambda_p^2 + a\lambda_p - \frac{c}{4} = 0. \tag{4.18}$$

This leads to $\lambda_p \neq 0$ since $c \neq 0$. Letting $X = e_2$, $Y = e_p$, $Z = e_p$, $W = e_2$ in (4.9), we have

$$\gamma(\lambda_p^2 - \frac{c}{4}) - (\beta^2 + \gamma^2 + \delta^2)\lambda_p = 0. \tag{4.19}$$

Combining (4.18) with (4.19), we have

$$a\gamma + \beta^2 + \gamma^2 + \delta^2 = 0. \tag{4.20}$$

Letting $X = \phi e_2$, $Y = e_p$, $Z = e_p$, $W = \phi e_2$ in (4.9), we have

$$(\gamma + a)(\lambda_p^2 - \frac{c}{4}) + \lambda_p \left\{\delta^2 + (\gamma + a)^2 - \frac{c}{4}\right\} = 0. \tag{4.21}$$

Combining (4.18) with (4.21), we have

$$\delta^2 + (\gamma^2 + a\gamma - \frac{c}{4}) = 0. \tag{4.22}$$

Letting $X = e_2$, $Y = e_{p*}$, $Z = e_{p*}$, $W = \phi e_2$ in (4.9), we have

$$(\gamma + a)\left\{(\lambda_p + a)^2 - \frac{c}{4}\right\} - (\lambda_p + a)\left\{\delta^2 + (\gamma + a)^2 - \frac{c}{4}\right\} = 0. \quad (4.23)$$

Combining (4.18) with (4.23), we have

$$\left\{\gamma(\gamma + a) + \delta^2\right\}\lambda_p + a(\delta^2 + \gamma^2 + a\gamma - \frac{c}{4}) = 0. \quad (4.24)$$

From (4.22), (4.24), and $\lambda_p \neq 0$, we are led to

$$c = 0,$$

which is a contradiction, so $\delta = 0$ must be satisfied. $\qquad\square$

Thirdly, we establish the following lemma:

Lemma 4.4.

$$\lambda_p = \lambda_{p*} = -\frac{a}{2}, \quad \gamma = -(\frac{c}{2a} + a).$$

Proof Letting $X = e_2$, $Y = \phi e_2$, $Z = \phi e_2$, $W = e_2$ in (4.9) and using Lemma 4.2 and Lemma 4.3, we have

$$(\gamma + a)\beta^2 + (2\gamma + a)(\gamma^2 + a\gamma - \frac{c}{4}) = 0. \quad (4.25)$$

Equation (4.21) is reduced to

$$(\lambda_p + \gamma + a)\left\{\lambda_p(\gamma + a) - \frac{c}{4}\right\} = 0. \quad (4.26)$$

Also, (4.23) is reduced to

$$(\lambda_p - \gamma)\left\{(\lambda_p + a)(\gamma + a) + \frac{c}{4}\right\} = 0. \quad (4.27)$$

Letting $X = e_p$, $Y = e_{p*}$, $Z = e_{p*}$, $W = e_p$ in (4.9), we have

$$(2\lambda_p + a)(\lambda_p^2 + a\lambda_p - \frac{c}{4}) = 0. \quad (4.28)$$

We assert that $\lambda_p \neq \gamma$. If $\lambda_p = \gamma$ were satisfied, we would have $(\gamma + a)\beta^2 = 0$ from (4.25) and (4.28). Since $\beta \neq 0$, we further would have $\gamma = -a$. Then, from (4.26) and (4.28) we would be led to $c = 0$ which is a contradiction, so $\lambda_p \neq \gamma$ must be satisfied.

We assert that $\lambda_p \neq -(\gamma + a)$. If this were not so, we would have $(\gamma + a)\beta^2 = 0$ from (4.25) and (4.27). Since $\beta \neq 0$, we further would have

$\gamma = -a$. Then, from (4.26) and (4.28) we would be led to $c = 0$ which is a contradiction, so $\lambda_p \neq -(\gamma + a)$ must be satisfied.

Therefore from (4.26) and (4.27) we get $\lambda_p = -\frac{a}{2}$, $\gamma = -(\frac{c}{2a} + a)$. This completes the proof.   □

We end this section with a proof of our main theorem. From Lemma 4.4, (4.25) is reduced to

$$2a^2\beta^2 + (a^2 + c)^2 = 0$$

which is a contradiction, so the structure vector ξ must be principal. This leads to our main theorem as follows. Since ξ is principal, (4.7) is reduced to

$$\phi A + A\phi + a\phi = 0. \tag{4.29}$$

Then, from Lemma 3.6, M has at most three distinct constant principal curvatures. So, owing to Theorem K and Theorem B, M is locally congruent to a homogeneous real hypersurface of type (A) or (B) in $\overline{M}_n(c)$ $(c \neq 0)$. Among these homogeneous real hypersurfaces, (4.29) is satisfied by type (A_0), (A_1), or (B). This concludes the proof of the theorem by the arguments above.

References

[1] S-S. Ahn, S-B. Lee and Y-J. Suh, *On Ruled Real Hypersurfaces in a Complex Space Form*, Tukuba J. Math. **17**, No. 2(1993), 311–322.

[2] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395**(1989), 132–141.

[3] U-H. Ki, H. S. Kim and H. Nakagawa, *A characterization of a real hypersurface of type B*, Tsukuba J. Math. **14**(1990), 9–26.

[4] U-H. Ki and N-G. Kim, *Ruled Real Hypersurfaces of a Complex Space Form*, Acta Math. Sinica, New Series **10**, No. 4(1994), 401–409.

[5] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296**(1986), 137–149.

[6] M. Kimura, *Sectional Curvatures of Holomorphic Planes on a Real Hypersurfaces in $P^n(\mathbb{C})$*, Math. Ann. **276**(1987), 487–497.

[7] M. Kimura and S. Maeda, *On Real Hypersurfaces of a Complex Projective Space*, Math. Z. **202**(1989), 299–311.

[8] Y. Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28**(1976), 529–540.

[9] S. Maeda and S. Udagawa, *Real hypersurfaces of a complex projective space in terms of holomorphic distribution*, Tsukuba J. Math. **14**, No.1(1990), 39–52.

[10] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan **37**(1985), 515–535.

[11] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geometriae Dedicata **20**(1986), 245–261.

[12] S. Nagai, *Some characterization of a real hypersurface of type (B) in complex space forms*, Saitama Math. J. **19**(2001), 37–46.

[13] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212**(1975), 355–364.

[14] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10**(1973), 495–506.

[15] R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures*, J. Math. Soc. Japan **27**(1975), 43–53.

Setsuo Nagai
Department of Mathematics
University of Toyama
Gofuku, Toyama 930-8555, Japan
e-mail: snagai@sci.u-toyama.ac.jp