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Polynomials and pseudoconvexity for Riemann domains over $\mathbb{C}^n$

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Abstract. We prove that a Riemann domain (G, π) over $\mathbb{C}^n$ is pseudoconvex if and only if for any continuous mapping $\varphi : \overline{D} \times [0, \delta] \to \overline{\overline{G}}$ of the form $(\overline{\overline{\pi}} \circ \varphi)_j(w, t) = p_j(w) + a_j t$ $(j = 1, 2, \ldots, n)$, where $(\overline{\overline{G}}, \overline{\overline{\pi}})$ is abstract closure of (G, π), $D = \{w \in \mathbb{C}\ ;\ |w| < \varepsilon\}$, $\varepsilon > 0$, $\delta > 0$, $a_j \in \mathbb{C}$ and $p_j(w)$ is a polynomial of w of degree at most 2, with $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset G$, it follows that $\varphi(\overline{D} \times [0, \delta]) \subset G$.

1. Introduction

A pair (G, π) is called a *Riemann domain over* $\mathbb{C}^n$ if G is a connected Hausdorff space and $\pi : G \to \mathbb{C}^n$ is a local homeomorphism. There are several definition of pseudoconvexity for Riemann domains over $\mathbb{C}^n$. Among others, a Riemann domain (G, π) is *pseudoconvex* if it satisfies the continuity principle, that is, for any continuous mapping $\varphi : \overline{D} \times [0, \delta] \to \overline{\overline{G}}$, where $D = \{w \in \mathbb{C}\ ;\ |w| < \varepsilon\}$, $\varepsilon > 0$ and $\delta > 0$, such that $(\overline{\overline{\pi}} \circ \varphi)_j(w, t)$ is a holomorphic function of w in D for any $t \in [0, \delta]$ and for any $j \in \{1, 2, \ldots, n\}$ with $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset G$, it follows that $\varphi(\overline{D} \times [0, \delta]) \subset G$. Here $(\overline{\overline{G}}, \overline{\overline{\pi}})$ is abstract closure of (G, π)(see Section 2). Yasuoka [3] proved that a domain Ω in $\mathbb{C}^n$ is pseudoconvex if and only if for any continuous mapping $\varphi : \overline{D} \times [0, \delta] \to \mathbb{C}^n$, where $D = \{w \in \mathbb{C}\ ;\ |w| < \varepsilon\}$, $\varepsilon > 0$ and $\delta > 0$, such that $\varphi_j(w, t) = p_j(w) + a_j t$ $(j = 1, 2, \ldots, n)$, $a_j \in \mathbb{C}$, $p_j(w)$ is a polynomial of w of degree at most 2 and $(\partial\varphi_1/\partial w, \partial\varphi_2/\partial w, \ldots, \partial\varphi_n/\partial w) \neq (0, 0, \ldots, 0)$

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for any $t \in [0,\delta]$ with $\varphi(\overline{D} \times (0,\delta]) \cup \varphi(\partial D \times \{0\}) \subset \Omega$, it follows that $\varphi(\overline{D} \times [0,\delta]) \subset \Omega$.

In this paper, we show that a Riemann domain (G,π) over $\mathbb{C}^n$ is pseudoconvex if and only if for any continuous mapping $\varphi : \overline{D} \times [0,\delta] \to \overline{\overline{G}}$, where $D = \{w \in \mathbb{C} \ ; \ |w| < \varepsilon\}$, $\varepsilon > 0$ and $\delta > 0$, such that $(\overline{\overline{\pi}} \circ \varphi)_j(w,t) = p_j(w) + a_j t$ $(j = 1, 2, \ldots, n)$, $a_j \in \mathbb{C}$, $p_j(w)$ is a polynomial of w of degree at most 2 with $\varphi(\overline{D}\times(0,\delta])\cup\varphi(\partial D\times\{0\}) \subset G$, it follows that $\varphi(\overline{D}\times[0,\delta]) \subset G$.

2. Riemann domains and abstract boundary points

Let (G,π) be a Riemann domain over $\mathbb{C}^n$ and let $\overline{\overline{\partial}}G$ be the set of all filter bases α that satisfies the following four conditions.

(1) There exists a point $z_0 \in \mathbb{C}^n$ such that $\lim \pi(\alpha) = z_0$.

(2) For any $V \in \beta_c(z_0)$, there exists exactly one connected component U of $\pi^{-1}(V)$ such that $U \in \alpha$.

(3) For any $U \in \alpha$, there exists $V \in \beta_c(z_0)$ such that U is a connected component of $\pi^{-1}(V)$.

(4) α has no accumulation point in G.

Here $\beta_c(z_0)$ is the set of all connected open neighborhoods of z_0 in $\mathbb{C}^n$. The set $\overline{\overline{\partial}}G$ is called the *abstract boundary* of G. We put $\overline{\overline{G}} = G \cup \overline{\overline{\partial}}G$ and define $\overline{\overline{\pi}} : \overline{\overline{G}} \to \mathbb{C}^n$ by

$$\overline{\overline{\pi}}(x) = \begin{cases} \pi(x) & (x \in G), \\ \lim \pi(x) & (x \in \overline{\overline{\partial}}G). \end{cases}$$

The topology of $\overline{\overline{G}}$ is as follows.
For every $\alpha \in \overline{\overline{\partial}}G$ and for every $U \in \alpha$ we put

$$\widehat{U}_\alpha = U \cup \{\beta \in \overline{\overline{\partial}}G \ ; \ \text{there exists } W \in \beta \text{ such that } W \subset U\}.$$

Then $\widehat{U}_\alpha$ is a fundamental neighborhood of α and $\overline{\overline{\pi}}$ is continuous. $(\overline{\overline{G}}, \overline{\overline{\pi}})$ is said to be *abstract closure* of (G,π) (see Jarnicki–Pflug [1, p. 33]).

Let $\mathcal{F}(G)$ be the set of all filter bases of G satisfying the above three conditions (1), (2) and (3). We define $\sigma^G : \overline{\overline{G}} \to \mathcal{F}(G)$ by

$$\sigma^G(x) = \begin{cases} \alpha^x & (x \in G), \\ x & (x \in \overline{\overline{\partial}}G), \end{cases}$$

where

$$\alpha^x = \{U_x \ ; \text{there exists } V \in \beta_c(\pi(x)) \text{ such that } U_x \text{ is a connected component of } \pi^{-1}(V) \text{ and } x \in U_x\}.$$

Then σ^G is well-defined. Moreover for every $\alpha \in \mathcal{F}(G)$ and for every $U \in \alpha$, we put

$$U_\alpha = \{\beta \in \mathcal{F}(G) \ ; \ \text{there exists } W \in \beta \text{ such that } W \subset U\}.$$

Then the family $\{U_\alpha \ ; \ U \in \alpha\}$ satisfies the axiom of fundamental system of neighborhoods. Therefore $\mathcal{F}(G)$ is a topological space.

Proposition 2.1. *σ^G is homeomorphic.*

Proof. It is obvious that α^x is a filter base that satisfies the above three conditions (1), (2), (3) and $\lim \alpha^x = x$. We show that σ^G is bijective. Obviously σ^G is injective. To see that σ^G is surjective, we put

$$\mathcal{F}_0(G) = \{\alpha \in \mathcal{F}(G) \ ; \ \alpha \text{ has an accumulation point in } G\}.$$

Then we have $\mathcal{F}(G) = \mathcal{F}_0(G) \cup \overline{\overline{\partial}}G$ and $\mathcal{F}_0(G) \cap \overline{\overline{\partial}}G = \emptyset$. Let $\alpha \in \mathcal{F}(G)$. If $\alpha \in \mathcal{F}_0(G)$, we can put $\lim \alpha = x \in G$(see Jarnicki–Pflug [1, p. 30]) and see that $\sigma^G(x) = \alpha^x = \alpha$. If $\alpha \in \overline{\overline{\partial}}G$, it is clear that $\sigma^G(\alpha) = \alpha$. Hence σ^G is surjective.

According to the definition of topology, we see that σ^G is homeomorphic. □

Therefore we can regard $\overline{\overline{G}}$ as $\mathcal{F}(G)$ by σ^G. Since $\mathcal{F}(G)$ and $\mathcal{F}_0(G)$ is useful, we sometimes use these symbols.

Next we consider a subdomain of a Riemann domain over $\mathbb{C}^n$. Let (G, π) be a Riemann domain over $\mathbb{C}^n$ and let G_0 be a subdomain of G. Then

$(G_0, \pi|_{G_0})$ is a Riemann domain over $\mathbb{C}^n$. We define the mapping which allows us to regard $\overline{\overline{G_0}}$ as a subset of $\overline{\overline{G}}$. Let $\mathcal{F}(G_0)$ be the set of all filter bases of G_0 that satisfies the above three conditions (1), (2) and (3). For every $\alpha \in \mathcal{F}(G_0)$, we put

$$\begin{aligned}\widehat{\alpha} = \{C_U \subset G\ ;\ &\text{there exist } U \in \alpha \text{ and } V \in \beta_c(\lim \pi(\alpha)) \\ &\text{such that } U \subset C_U \subset \pi^{-1}(V) \\ &\text{and } C_U \text{ is a connected component of } \pi^{-1}(V)\}\end{aligned}$$

and let $\wedge : \mathcal{F}(G_0) \to \mathcal{F}(G)$, $\alpha \mapsto \widehat{\alpha}$. Then the mapping $\wedge$ is well-defined and continuous. We put $\psi_{G_0} = (\sigma^G)^{-1} \circ \wedge \circ \sigma^{G_0}$.

Remark 2.1. Let $\alpha \in \overline{\overline{\partial}} G_0$. If α has an accumulation point x_α in G, then $\widehat{\alpha} \in \mathcal{F}_0(G)$. Especially, $\lim \widehat{\alpha} = \lim \alpha = x_\alpha$. Since π is continuous, it follows that $\pi(x_\alpha) = \pi(\lim \widehat{\alpha}) = \pi(\lim \alpha) = \lim \pi(\alpha)$. Then x_α is unique, because G is a Hausdorff space. If α has no accumulation point in G, then it is clear that $\widehat{\alpha} \in \overline{\overline{\partial}} G$.

An open subset G_1 of G is said to be *univalent* if $\pi|_{G_1} : G_1 \to \pi(G_1)$ is homeomorphic.

Lemma 2.1. *Let (G, π) be a Riemann domain over $\mathbb{C}^n$, let G_0 be a univalent subdomain of G. Assume that G_0 satisfies the following condition.*

For every $z \in \partial\pi(G_0)$ and for every $V \in \beta_c(z)$, there exists $V_0 \in \beta_c(z)$ such that $V_0 \subset V$ and $(\pi|_{G_0})^{-1}(V_0)$ is connected.

Then the following two statements hold.

(1) $\overline{\overline{\pi|_{G_0}}} : \overline{\overline{G_0}} \to \overline{\pi(G_0)}$ *is homeomorphic.*

(2) $\psi_{G_0} : \overline{\overline{G_0}} \to \psi_{G_0}(\overline{\overline{G_0}})$ *is homeomorphic.*

Proof. (1)We define $f : \partial\pi(G_0) \to \overline{\overline{\partial}} G_0$, $z \mapsto \alpha_z$. Here α_z is an abstract boundary point of G_0 with $\lim \pi(\alpha_z) = z$. Then f is well-defined. In fact, we put

$$\begin{aligned}\alpha_z = \{C\ ;\ &\text{there exist } V \in \beta_c(z) \text{ and } \{x_\nu\}_{\nu \in \mathbb{N}} \subset G \text{ such that} \\ &C \text{ is a connected component of } \pi^{-1}(V) \cap G_0 \text{ and} \\ &\text{almost all } x_\nu \text{ lie in } C \text{ and } \lim_{\nu \to +\infty} \pi(x_\nu) = z\}.\end{aligned}$$

We shall show $\alpha_z \in \overline{\overline{\partial}} G_0$, $\lim \pi(\alpha_z) = z$ and $\alpha_z \neq \emptyset$. It is clear that $\emptyset \notin \alpha_z$ and $\alpha_z \neq \emptyset$.

Let C_1 and C_2 be elements of α_z. Then there exist $\{x_\nu^{(1)}\}_{\nu \in \mathbb{N}} \subset G$ and $\{x_\nu^{(2)}\}_{\nu \in \mathbb{N}} \subset G$ such that $\lim_{\nu \to +\infty} \pi(x_\nu^{(1)}) = z$ and $\lim_{\nu \to +\infty} \pi(x_\nu^{(2)}) = z$. And there exist $V_1 \in \beta_c(z)$, $V_2 \in \beta_c(z)$, $\nu_1 \in \mathbb{N}$ and $\nu_2 \in \mathbb{N}$ such that $\{x_\nu^{(1)}\}_{\nu \geq \nu_1} \subset C_1 \subset \pi^{-1}(V_1) \cap G_0$ and $\{x_\nu^{(2)}\}_{\nu \geq \nu_2} \subset C_2 \subset \pi^{-1}(V_2) \cap G_0$, where C_1 is a connected component of $\pi^{-1}(V_1) \cap G_0$ and C_2 is a connected component of $\pi^{-1}(V_2) \cap G_0$. Since $V_1 \cap V_2$ is an open neighborhood of z, there is $V_0 \in \beta_c(z)$ such that $\pi^{-1}(V_0) \cap G_0$ is connected, $V_0 \subset V_1 \cap V_2$ and there is $N \in \mathbb{N}$ such that for every $\nu > N$, we get $x_\nu^{(1)}$, $x_\nu^{(2)} \in \pi^{-1}(V_0)$. We obtain $(\pi^{-1}(V_0) \cap G_0) \cap C_1 \neq \emptyset$ and $(\pi^{-1}(V_0) \cap G_0) \cap C_2 \neq \emptyset$. Hence $\pi^{-1}(V_0) \cap G_0 \subset C_1 \cap C_2$ and $\pi^{-1}(V_0) \cap G_0 \in \alpha_z$. Therefore α_z is a filter base of G_1.

For any $V \in \beta_c(z)$, there is $V_0 \in \beta_c(z)$ such that $V_0 \subset V$ and $\pi^{-1}(V_0) \cap G_0$ is connected. Then we obtain $\pi^{-1}(V_0) \cap G_0 \in \alpha_z$ and $\pi(\pi^{-1}(V_0) \cap G_0) \subset V_0 \cap \pi(G_0) \subset V_0 \subset V$. Hence $\lim \pi_1(\alpha_z) = z$.

We show that for any $V \in \beta_c(z)$, there exists exactly one connected component C of $\pi^{-1}(V) \cap G_0$ such that $C \in \alpha_z$. For any $V \in \beta_c(z)$, let C_1 and C_2 be connected components of $\pi^{-1}(V) \cap G_0$ that satisfy the following condition. There exist $\{x_\nu^{(i)}\}_{\nu \in \mathbb{N}} \subset G$ $(i = 1, 2)$ and $\nu_i \in \mathbb{N}$ $(i = 1, 2)$ such that $\{x_\nu^{(i)}\}_{\nu \geq \nu_i} \subset C_i$ $(i = 1, 2)$ and $\lim_{\nu \to +\infty} \pi_1(x_\nu^{(i)}) = z$ $(i = 1, 2)$. By the assumption, there exists $V_0 \in \beta_c(z)$ such that $V_0 \subset V$ and $\pi^{-1}(V_0) \cap G_0$ is connected. Now V_0 contains almost all $\{\pi(x_\nu^{(1)})\}_{\nu \in \mathbb{N}}$ and $\{\pi(x_\nu^{(2)})\}_{\nu \in \mathbb{N}}$. Thus $\pi^{-1}(V_0) \cap G_0 \cap C_1 \neq \emptyset$ and $\pi^{-1}(V_0) \cap G_0 \cap C_2 \neq \emptyset$. Since C_1 and C_2 are connected components of $\pi^{-1}(V_0) \cap G_0$, we have $\pi^{-1}(V_0) \cap G_0 \subset C_1$ and $\pi^{-1}(V_0) \cap G_0 \subset C_2$. It follows that $C_1 = C_2$.

Obviously α_z satisfies that for any $U \in \alpha_z$, there exists $V \in \beta_c(z)$ such that U is a connected component of $\pi^{-1}(V)$. It is clear that α_z has no accumulation point. Therefore α_z is an abstract boundary point with $\lim \pi(\alpha_z) = z$.

Then this α_z is unique. In fact, suppose that α' is an abstract boundary point of G_0 with $\lim \pi(\alpha') = z$. Assume that $\alpha' \neq \alpha_z$. Then there exist $U' \in \alpha'$ and $U \in \alpha_z$ such that $U \cap U' = \emptyset$. Moreover there exist $V \in \beta_c(z)$ and $V' \in \beta_c(z)$ such that U is a connected component of $\pi^{-1}(V) \cap G_0$ and

U' is a connected component of $\pi^{-1}(V') \cap G_0$. Now $V \cap V'$ is an open neighborhood of z. Thus there is $V_0 \in \beta_c(z)$ such that $V_0 \subset V \cap V'$ and $\pi^{-1}(V_0) \cap G_0$ is connected by the assumption of G_0. Then it follows that $U \cap \pi^{-1}(V_0) \subset \pi^{-1}(V_0) \cap G_0$ and $U' \cap \pi^{-1}(V_0) \subset \pi^{-1}(V_0) \cap G_0$.

Let $\{x_\nu\}_{\nu\in\mathbb{N}}$ be determined by U and let $\{x'_\nu\}_{\nu\in\mathbb{N}}$ be determined by U'. Then there is $N \in \mathbb{N}$ such that for every $\nu > N$, we have $\pi(x_\nu) \in V_0$ and $\pi(x'_\nu) \in V_0$. Hence $U \cap \pi^{-1}(V_0) \neq \emptyset$ and $U' \cap \pi^{-1}(V_0) \neq \emptyset$.
Therefore we have $U \cap \pi^{-1}(V_0) \supset \pi^{-1}(V_0) \cap G_0$ and $U' \cap \pi^{-1}(V_0) \supset \pi^{-1}(V_0) \cap G_0$. It follows that $U \cap \pi^{-1}(V_0) = \pi^{-1}(V_0) \cap G_0$ and $U' \cap \pi^{-1}(V_0) = \pi^{-1}(V_0) \cap G_0$. This is a contradiction. Hence f is well-defined. It is easy to see that f is bijective.

Define $F : \overline{\pi(G_0)} \to \overline{\overline{G_0}}$ as $F|_{\partial\pi(G_0)} = f$ and $F|_{\pi(G_0)} = (\pi|_{G_0})^{-1}$. Then F is homeomorphic. In fact,we put $z \in \partial\pi(G_0)$ and $f(z) = \alpha_z$. Let $\widehat{U}_{\alpha_z} = U \cup \{\beta \in \overline{\overline{\partial}}G_0$; there exists $W \in \beta$ such that $W \subset U\}$ be a neighborhood of α_z. Then there exists $V \in \beta_c(z)$ such that U is a connected component of $\pi^{-1}(V) \cap G_0$. By the assumption of G_0, there is $V_0 \in \beta_c(z)$ such that $V_0 \subset V$ and $\pi^{-1}(V_0) \cap G_0$ is connected. We shall prove that $F(V_0 \cap \overline{\pi(G_0)}) \subset \widehat{U}_{\alpha_z}$. It is clear that $F(V_0 \cap \pi(G_0)) = \pi^{-1}(V_0 \cap \pi(G_0)) = \pi^{-1}(V_0) \cap G_0 \subset U$. Hence we only have to show that $F(V_0 \cap \partial\pi(G_0)) \subset \{\beta \in \overline{\overline{\partial}}G_0$; there exists $W \in \beta$ such that $W \subset U\}$. Let $z_0 \in V_0 \cap \partial\pi(G_0)$ and $F(z_0) = \alpha_0$, since $V_0 \in \beta_c(z_0)$, there is $C_0 \in \alpha_0$ such that $C_0 \subset \pi^{-1}(V_0) \cap G_0$. Since $\pi^{-1}(V_0) \cap G_1$ is connected, then $C_0 = \pi^{-1}(V_0) \cap G_1 \subset U$. Thus $\alpha_0 \in \{\beta \in \overline{\overline{\partial}}G_0$; there exists $W \in \beta$ such that $W \subset U\}$. Consequently, F is continuous. Since $F^{-1} = \overline{\overline{\pi}}_1$, F is homeomorphic. Therefore $\overline{\overline{\pi|_{G_0}}}$ is homeomorphic.

Next we shall show (2). Put $\psi_{G_0} = \psi$. Then $\mathrm{id}_{\partial\pi(G_0)} = \overline{\overline{\pi}} \circ \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1}\Big|_{\partial\pi(G_0)}$ and $\mathrm{id}_{\psi(\overline{\overline{\partial}}G_0)} = \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1} \circ \overline{\overline{\pi}}|_{\psi(\overline{\overline{\partial}}G_0)}$ hold.

In fact, let $z \in \partial\pi(G_0)$ and let $\overline{\overline{\pi|_{G_0}}}^{-1}(z) = \alpha$, where α is an abstract boundary point of G_0 with $\lim \pi(\alpha) = z$.
Case 1 : $\widehat{\alpha} \in \mathcal{F}_0(G)$.

Then $\psi \circ \overline{\overline{\pi|_{G_0}}}^{-1}(z) = \psi(\alpha) = (\sigma^G)^{-1}(\widehat{\alpha}) = \lim \widehat{\alpha} = \lim \alpha \in G$, it follows that $\overline{\overline{\pi}} \circ \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1}(z) = \overline{\overline{\pi}}(\lim \widehat{\alpha}) = \overline{\overline{\pi}}(\lim \alpha) = \pi(\lim \alpha) = \lim \pi(\alpha) = z$.
Case 2 : $\widehat{\alpha} \in \overline{\overline{\partial}}G$.

Then $\overline{\overline{\pi}}(\widehat{\alpha}) = \lim \pi(\widehat{\alpha}) = \lim \pi(\alpha) = z$.
Thus we obtain $\mathrm{id}_{\partial\pi(G_0)} = \overline{\overline{\pi}} \circ \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1}\Big|_{\partial\pi(G_0)}$.
It remains to show that $\mathrm{id}_{\psi(\overline{\overline{\partial}}G_0)} = \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1} \circ \overline{\overline{\pi}}|_{\psi(\overline{\overline{\partial}}G_0)}$. Let $x \in \psi(\overline{\overline{\partial}}G_0)$.
Case 1 : $x \in G$.

Then $\psi \circ \overline{\overline{\pi|_{G_0}}}^{-1} \circ \overline{\overline{\pi}}(x) = \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1} \circ \pi(x) = \psi(\alpha)$, where α is an abstract boundary point of G_0 with $\lim \pi(\alpha) = \pi(x)$. There is $\beta \in \overline{\overline{\partial}}G_0$ such that $\lim \widehat{\beta} = \lim \beta = x \in G$. Thus $\lim \pi(\beta) = \pi(x)$. It follows from bijectivity of $\overline{\overline{\pi|_{G_0}}}^{-1}$ that $\beta = \alpha$. Therefore, $\psi(\alpha) = \lim \alpha = \lim \beta = x$.
Case 2 : $x \in \overline{\overline{\partial}}G$.

Then there exists $\beta \in \overline{\overline{\partial}}G_0$ such that $\psi(\beta) = \widehat{\beta} = x$. We get $\psi \circ \overline{\overline{\pi|_{G_0}}}^{-1} \circ \overline{\overline{\pi}}(x) = \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1} \circ \overline{\overline{\pi}}(\widehat{\beta}) = \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1}(\lim \pi(\widehat{\beta})) = \psi(\alpha)$.
Here α is an abstract boundary point of G_0 with $\lim \pi(\alpha) = \lim \pi(\widehat{\beta}) = \lim \pi(\beta)$. It follows from bijectivity of $\overline{\overline{\pi|_{G_0}}}^{-1}$ that $\beta = \alpha$. Thus $\psi(\alpha) = \psi(\beta) = \widehat{\beta} = x$. It follows that $\mathrm{id}_{\psi(\overline{\overline{\partial}}G_0)} = \psi \circ \overline{\overline{\pi|_{G_0}}}^{-1} \circ \overline{\overline{\pi}}|_{\psi(\overline{\overline{\partial}}G_0)}$.
We obtain that $\psi|^{-1}_{\psi(\overline{\overline{\partial}}G_0)} = \overline{\overline{\pi|_{G_0}}}^{-1} \circ \overline{\overline{\pi}}|_{\psi(\overline{\overline{\partial}}G_0)}$. It follows that $\psi : \overline{\overline{G_0}} \to \psi(\overline{\overline{G_0}})$ is homeomorphic. ☐

Let $G_0 \subset G$ be an open univalent neighborhood of $x \in G$, let $|\cdot|$ be the maximum norm in $\mathbb{C}^n$ and let $r \in (0, +\infty]$. If

$$\pi(G_0) = P(\pi(x), r) = \{z \in \mathbb{C}^n \ ; \ |\pi(x) - z| < r\},$$

then G_0 is called a *polydisk* with radius r and center x and is denoted by $\widehat{P}(x, r)$. We define

$$\delta_G(x) = \sup\{r \in (0, +\infty] \ ; \ \widehat{P}(x, r) \text{ exists}\},$$

which is called the *boundary distance function.* The set $\widehat{P}(x, \delta_G(x))$ is called the *maximal polydisk* with center x.

Corollary 2.1. *Let (G, π) be a Riemann domain over $\mathbb{C}^n$ and let $\widehat{P}(x, \delta_G(x))$ be a maximal polydisk. Then both $\overline{\overline{\pi|_{\widehat{P}(x,\delta_G(x))}}} : \overline{\overline{\widehat{P}(x, \delta_G(x))}} \to \overline{P(x, \delta_G(x))}$ and $\psi_{\widehat{P}(x,\delta_G(x))} : \overline{\overline{\widehat{P}(x, \delta_G(x))}} \to \psi_{\widehat{P}(x,\delta_G(x))}(\overline{\overline{\widehat{P}(x, \delta_G(x))}})$ are homeomorphic.*

3. O_2-pseudoconvex domains and pseudoconvex domains

After defining the O_m-pseudoconvexity ($m \in \mathbb{N}$), we show that the pseudoconvexity is equivalent to the O_2-pseudoconvexity. Let (G, π) be a Riemann domain over $\mathbb{C}^n$.

Definition 3.1. Let $m \in \mathbb{N}$ and let $\varphi : \overline{D} \times [0, \delta] \to \overline{\overline{G}}$ be a continuous map, where $D = \{w \in \mathbb{C} \ ; \ |w| < \varepsilon\}$, $\varepsilon > 0$ and $\delta > 0$. If $(\overline{\overline{\pi}} \circ \varphi)_j(w, t) = p_j(w) + a_j t$ $(j = 1, 2, \ldots, n)$, $a_j \in \mathbb{C}$, $p_j(w)$ is a polynomial of w of degree at most m, then φ is called a *family of analytic disks of degree* m.

Definition 3.2. We say that G is *O_m-pseudoconvex* if for any family φ of analytic disks of degree m with $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset G$, we have $\varphi(\overline{D} \times [0, \delta]) \subset G$.

Remark 3.1. The O_m-pseudoconvexity is invariant under affine transformations.

For any $a \in \mathbb{C}$ and for any $\varepsilon \in (0, +\infty]$, the set $\{z \in \mathbb{C} \ ; \ |z - a| < \varepsilon\}$ is denoted by $D(a, \varepsilon)$.

Lemma 3.1 (Yasuoka [3, Lemma 1])**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to [-\infty, +\infty)$ be an upper semi-continuous function. If f is not subharmonic on Ω, then there exist $a \in \Omega$, $D = D(a, \varepsilon) \Subset \Omega$, $h \in C^\infty(\overline{D})$ and $C > 0$ such that*

$$\begin{cases} h_{z\bar{z}}(z) = -C & \textit{for } z \in D, \\ h(a) = f(a), & \\ h(z) \geq f(z) & \textit{for } z \in D. \end{cases}$$

Theorem 3.1. *Let (G, π) be a Riemann domain over $\mathbb{C}^n$. Then the following two statements are equivalent.*

(1) (G, π) *is pseudoconvex.*

(2) (G, π) *is O_2-pseudoconvex.*

Proof. The implication (1)$\Rightarrow$(2) is trivial. We show another implication. We can assume that G and $\mathbb{C}^n$ are not homeomorphic. Seeking a contradiction, suppose that (G, π) is not pseudoconvex. Then $-\log \delta_G(x)$ is not plurisubharmonic on G (see Jarnicki–Pflug[1, p. 143]). By an affine transformation which conserves the distance, we can assume that $-\log \delta_G(\pi_x^{-1}(w\xi_0))$ is not subharmonic on $D(0,\varepsilon) \subset \{w \in \mathbb{C} \ ; \ |w\xi_0| < \delta_G(x)\}$. Here $x \in G$, $\varepsilon > 0$, $\pi_x = \pi|_{\widehat{P}(x,\delta_G(x))}$ and $\xi_0 = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. It follows from Lemma 3.1 that there exist $a_0 \in D(0,\varepsilon)$, $D_0 = D(a_0, \varepsilon_0) \Subset D(0,\varepsilon)$, $h \in C^\infty(\overline{D}_0)$ and $C > 0$ such that

$$\begin{cases} -h_{z\bar{z}}(w) = C & \text{for } w \in D_0, \\ -h(a_0) = \log \delta_G(\pi_x^{-1}(a_0\xi_0)), & \\ -h(w) \leq \log \delta_G(\pi_x^{-1}(w\xi_0)) & \text{for } w \in D_0. \end{cases}$$

By translation, we may let $a_0 = 0$. Put

$$\begin{aligned} &\widehat{P}(w) = \widehat{P}(\pi_x^{-1}(w\xi_0), \delta_G(\pi_x^{-1}(w\xi_0))) \text{ and} \\ &P(w) = P(w\xi_0, \delta_G(\pi_x^{-1}(w\xi_0))). \end{aligned}$$

Then we consider the maximal polydisk $\widehat{P}(0)$. By Corollary 2.1, $\partial P(0)$ and $\overline{\overline{\partial}}\widehat{P}(0)$ are homeomorphic. Moreover $\overline{\overline{\partial}}G \cap \psi_{\widehat{P}(0)}(\overline{\overline{\partial}}\widehat{P}(0)) \neq \emptyset$. Then there is $u \in \overline{\overline{\partial}}G \cap \psi_{\widehat{P}(0)}(\overline{\overline{\partial}}\widehat{P}(0))$ such that $\overline{\overline{\pi}}(u) \in \partial P(0)$. We can assume that there exist $z_{k+1}^{(0)}, \dots, z_n^{(0)} \in D(0, \delta_G(\pi_x^{-1}(0)))$ such that

$$\overline{\overline{\pi}}(u) = (\delta_G(\pi_x^{-1}(0)), \dots, \delta_G(\pi_x^{-1}(0)), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \in \partial P(0).$$

Define $h_1(w) = -h(w) - C|w|^2$. Then h_1 is harmonic on D_0. Since D_0 is simply connected, there exists exactly one conjugate harmonic function h_2 on D_0 with $h_2(0) = 0$.

Let $p(w) +$ (terms of order ≥ 3) be the power series expansion of the holomorphic function $\exp(h_1(w) + \mathrm{i}h_2(w))$ at $w = 0$.

For any $\delta \in (0, +\infty]$, we define the family $\chi : \overline{D_0} \times [0, \delta] \to \mathbb{C}^n$ of analytic disks of degree 2 in $\mathbb{C}^n$ by

$$\chi(w,t) = \begin{cases} \chi_1(w,t) = p(w) - t + w\xi_1, \\ \chi_2(w,t) = p(w) - t + w\xi_2, \\ \vdots \\ \chi_k(w,t) = p(w) - t + w\xi_k, \\ \chi_{k+1}(w,t) = z_{k+1}^{(0)} + w\xi_{k+1}, \\ \vdots \\ \chi_n(w,t) = z_n^{(0)} + w\xi_n. \end{cases}$$

We can choose D_0 so that

$$|p(w) - t| \leq |\exp(h_1(w) + \mathrm{i}h_2(w)) - t| + L_1|w|^3$$

for all $(w,t) \in D_0 \times [0,\delta]$, where L_1 is a positive constant.
Moreover we can assume that $0 < \delta_G(\pi_x^{-1}(0)) < 1$ by Remark 3.1. Let ε_0 and δ be sufficiently small. Then we obtain

$$|\exp(h_1(w) + \mathrm{i}h_2(w) - t)| \geq |\exp(h_1(w) + \mathrm{i}h_2(w)) - t|$$

for all $(w,t) \in D_0 \times [0,\delta]$. Thus

$$|p(w) - t| \leq \exp(h_1(w) - t) + L_1|w|^3 \tag{1}$$

for all $(w,t) \in D_0 \times [0,\delta]$. Since $h_1(0) > 0$, we can easily prove that

$$\log|p(w) - t| \leq h_1(w) - t + L_2|w|^3$$

for all $(w,t) \in D_0 \times [0,\delta]$, where L_2 is a positive constant.
For any $\varepsilon_1 \in (0, \min\{\varepsilon_0, \frac{C}{L_2}\})$ and put $D = D(0,\varepsilon_1)$. Then we have

$$h_1(w) - t + L_2|w|^3 \leq h_1(w) + C|w|^2 - t = -h(w) - t \tag{2}$$

for all $(w,t) \in \overline{D} \times [0,\delta]$. Then we consider $|\chi(w,t) - w\xi_0|$.
Case 1 : $|\chi(w,t) - w\xi_0| = |p(w) - t|$.

Inequality (2) implies

$$\begin{aligned} \log|\chi(w,t) - w\xi_0| &= \log|p(w) - t| \\ &\leq -h(w) - t \leq \log\delta_G(\pi_x^{-1}(w\xi_0)) - t \end{aligned}$$

for all $(w,t) \in \overline{D} \times [0,\delta]$. Therefore we have

$$|\chi(w,t) - w\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0))$$

for all $(w,t) \in \overline{D} \times (0,\delta]$.

Case 2 : There is $l \in \{k+1, k+2, \dots, n\}$ such that $|\chi(w,t) - w\xi_0| = |z_l^{(0)}|$.

We can choose $\overline{D}$ so that

$$|z_l^{(0)}| < \delta_G(\pi_x^{-1}(w\xi_0))$$

for all $w \in \overline{D}$ by continuity of $\delta_G(\pi_x^{-1}(w\xi_0))$.

Thus Case 1 and Case 2 imply

$$|\chi(w,t) - w\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0)) \tag{3}$$

for all $(w,t) \in \overline{D} \times (0,\delta]$. By inequality (2), for every $w \in \partial D$, we get

$$h_1(w) + L_2|w|^3 < h_1(w) + C|w|^2 = -h(w).$$

Hence

$$\log|p(w)| \le h_1(w) + L_2|w|^3 < -h(w) \le \log \delta_G(\pi_x^{-1}(w\xi_0))$$

for all $w \in \partial D$. Consequently, we have $|p(w)| < \delta_G(\pi_x^{-1}(w\xi_0))$ for any $w \in \partial D$. It follows that

$$|\chi(w,0) - w\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0)) \tag{4}$$

for all $w \in \partial D$. We made preparations to define the family of analytic disks of degree 2 of $\overline{\overline{G}}$. Put $G_0 = \bigcup_{w \in \overline{D}} \widehat{P}(w)$ and $\pi_{G_0} = \pi|_{G_0}$. Then G_0 is connected and π_{G_0} is homeomorphic (cf. Narasimhan [2, p. 107]). Define $J : \overline{D} \times (0,\delta] \to G_0 \subset \overline{\overline{G}}$ by $J(w,t) = \pi_w^{-1} \circ \chi(w,t)$. We shall show that J is continuous. For any $(w,t) \in \overline{D} \times (0,\delta]$, let $U(J(w,t))$ be an open neighborhood of $J(w,t)$. Then we can assume that $U(J(w,t)) \subset \widehat{P}(w)$. Since π_w is open, $\pi_w(U(J(w,t)))$ is an open neighborhood of $\chi(w,t)$. Now $\chi(w,t)$ is continuous. Therefore there is a neighborhood $D' \times T'$ of (w,t) such that $\chi(D' \times T') \subset \pi_w(U(J(w,t))) \subset P(w)$. Then we have $\pi_w^{-1}(\chi(D' \times$

$T'))\subset U(J(w,t))\subset \widehat{P}(w)$. Moreover there is an open neighborhood D'' of w such that

$$|w\xi_0 - w'\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0))$$

for all $w'\in D''$. Therefore for every $w'\in D''$, we get $P(w)\cap P(w')\neq\emptyset$ and $\pi_w^{-1}=\pi_{w'}^{-1}$ on $P(w)\cap P(w')$. Put $D'\cap D''=D'''$. Then D''' is an open neighborhood of w and we obtain $\chi(D'''\times T')\subset P(w)$. Therefore $\pi_{w'}^{-1}(\chi(D'''\times T'))\subset \pi_w^{-1}(\chi(D'''\times T'))\subset U(J(w,t))$ for any $w'\in D'''$. It follows that $J(D'''\times T')\subset U(J(w,t))$. This means that J is continuous. Next we extend J. we define $\overline{\overline{J}}:\overline{D}\times[0,\delta]\to\overline{\overline{G}}$ by

$$\overline{\overline{J}}(w,t)=\begin{cases} J(w,t) & (t\neq 0),\\ \lim\limits_{\nu\to+\infty} J(w_\nu,t_\nu) & (t=0),\end{cases}$$

where $\{(w_\nu,t_\nu)\}_{\nu\in\mathbb{N}}$ satisfies $(w_\nu,t_\nu)\to(w,0)$ $(\nu\to+\infty)$ and $t_\nu\neq 0$ for any $\nu\in\mathbb{N}$. Then $\overline{\overline{J}}$ is well-defined and continuous. In fact, first we shall show that the sequence $\{x_\nu\}_{\nu\in\mathbb{N}}=\{J(w_\nu,t_\nu)\}_{\nu\in\mathbb{N}}=\{\pi_{w_\nu}^{-1}\circ\chi(w_\nu,t_\nu)\}_{\nu\in\mathbb{N}}$ has a limit point.

Case 1 : $\chi(w,0)\in P(w)$.

We have $\pi_w^{-1}\circ\chi(w,0)\in\widehat{P}(w)\subset G$. Then we show that $\lim\limits_{\nu\to+\infty}\pi_{w_\nu}^{-1}\circ\chi(w_\nu,t_\nu)=\pi_w^{-1}\circ\chi(w,0)$. Let $U=U(\pi_w^{-1}\circ\chi(w,0))\subset\widehat{P}(w)$ an open neighborhood of $\pi_w^{-1}\circ\chi(w,0)$, then $\pi(U)$ is an open neighborhood of $\chi(w,0)$. Since $\chi(w,t)$ is continuous, there exists $N\in\mathbb{N}$ such that for every $\nu>N$, we get $\chi(w_\nu,t_\nu)\in\pi(U)\subset P(w)$. Thus for any $\nu>N$, we have $P(w)\cap P(w_\nu)\neq\emptyset$ and $\pi_w^{-1}=\pi_{w_\nu}^{-1}$ on $P(w)\cap P(w_\nu)$. Therefore $\pi_w^{-1}\circ\chi(w_\nu,t_\nu)=\pi_{w_\nu}^{-1}\circ\chi(w_\nu,t_\nu)\in U$ for all $\nu>N$. We get $\lim\limits_{\nu\to+\infty}\pi_{w_\nu}^{-1}\circ\chi(w_\nu,t_\nu)=\pi_w^{-1}\circ\chi(w,0)$.

Case 2 : $\chi(w,0)\in\partial P(w)$.

We get $(\overline{\overline{\pi}}_w)^{-1}\circ\chi(w,0)\in\overline{\overline{\partial}}\widehat{P}(w)$ by Corollary 1. Then $\wedge\circ(\overline{\overline{\pi}}_w)^{-1}\circ\chi(w,0)\in\mathcal{F}(G)$.

Case 2.1 : $\wedge\circ(\overline{\overline{\pi}}_w)^{-1}\circ\chi(w,0)=\alpha\in\mathcal{F}_0(G)$.

We can put $\lim\alpha=x$. Then $\pi(x)=\chi(w,0)$. We shall show $\lim\limits_{\nu\to+\infty}\pi_{w_\nu}^{-1}\circ\chi(w_\nu,t_\nu)=x$. First we show for any $V\in\beta_c(\chi(w,0))$, there exists $N\in\mathbb{N}$ such that for any $\nu>N$, we get $x_\nu\in C_U\subset\pi^{-1}(V)$. Here C_U is an element of α with $U \subset C_U$ and U is connected component of $\pi_w^{-1}(V)$ with $U \in (\overline{\overline{\pi}}_w)^{-1} \circ \chi(w,0)$. Assume that there exists $V \in \beta_c(\chi(w,0))$ such that for any $N \in \mathbb{N}$, there exists $\nu > N$ such that $x_\nu \notin C_U \subset \pi^{-1}(V)$. It leads to a contradiction. We can assume that V is sufficiently small. Then we obtain a subsequence $\{x_{\nu(j)}\}_{j\in\mathbb{N}} \subset \{x_\nu\}_{\nu\in\mathbb{N}}$ with $x_{\nu(j)} \notin C_U$ for every $j \in \mathbb{N}$. Put $\pi(x_{\nu(j)}) = \chi(w_{\nu(j)}, t_{\nu(j)}) = \zeta_j$, then $\zeta_j \in P(w_{\nu(j)})$ for every $j \in \mathbb{N}$ and there is $N \in \mathbb{N}$ such that for every $j > N$, we have $\zeta_j \in V$. Moreover we obtain $P(w_{\nu(j)}) \cap P(w) = \emptyset$ for any $j > N$. In fact, suppose that there exists $j_0 \in \mathbb{N}$ such that $P(w_{\nu(j_0)}) \cap P(w) \neq \emptyset$. Since $V \cap P(w_{\nu(j)}) \neq \emptyset$ and $V \cap P(w) \neq \emptyset$, we can take a sufficiently small polydisk V such that $V \cap (P(w_{\nu(j)}) \cap P(w))$ is connected. Therefore $\pi_w^{-1}(V) = U \subset (\pi|_{\widehat{P}(w)\cup\widehat{P}(w_{\nu(j_0)})})^{-1}(V) \subset C_U \subset \pi^{-1}(V)$. Thus we get $x_{\nu(j_0)} \in C_U$. This is a contradiction. It follows that $P(w_{\nu(j)}) \cap P(w) = \emptyset$ for any $j > N$. However since $w_{\nu(j)}\xi_0 \to w\xi_0$ $(j \to +\infty)$, this is also a contradiction. Therefore $\lim_{\nu\to+\infty} x_\nu = x$.

Case 2.2 : $\wedge \circ (\overline{\overline{\pi}}_w)^{-1} \circ \chi(w,0) = \alpha \in \overline{\overline{\partial}}G$.

By the same as above, we get $\lim_{\nu\to+\infty} x_\nu = \alpha$. Then the limit value is independent of the choice of a sequence $\{(w_\nu, t_\nu)\}_{\nu\in\mathbb{N}}$ with $(w_\nu, t_\nu) \to (w,0)$ $(\nu \to +\infty)$ and $t_\nu \neq 0$ for any $\nu \in \mathbb{N}$. Hence $\overline{\overline{J}}$ is well-defined.

Next we show that $\overline{\overline{J}}$ is continuous. We consider a sequence $\{(w_\nu, t_\nu)\}_{\nu\in\mathbb{N}}$ which satisfies $t_i = 0$ for some $i \in \mathbb{N}$. In this case, since we need a sequence with $(w_\nu, t_\nu) \to (w_i, 0)$ $(\nu \to +\infty)$ and $t_\nu \neq 0$ for any $\nu \in \mathbb{N}$, we define a double sequence as follows. When $i_1 \in \mathbb{N}$ satisfies $(w_{i_1}, t_{i_1}) = (w_{i_1}, 0)$, we take a sequence $\{t_{i_1,j}\}_{j\in\mathbb{N}} \subset (0,\delta]$ with $\lim_{j\to+\infty} t_{i_1,j} = 0$. When $i_2 \in \mathbb{N}$ satisfies $(w_{i_2}, t_{i_2}) \neq (w_{i_2}, 0)$, we take sequence $\{t_{i_2,j}\}_{j\in\mathbb{N}} \subset (0,\delta]$ with $\lim_{j\to+\infty} t_{i_2,j} = t_{i_2}$. It follows from the argument in Case 2.1 and Case 2.2 that $\lim_{i\to+\infty}\lim_{j\to+\infty} J(w_i, t_{i,j}) = \overline{\overline{J}}(w,0)$.

Therefore $\overline{\overline{J}}$ is continuous. Inequalities (3) and (4) imply $\overline{\overline{J}}(\overline{D} \times (0,\delta]) \subset G$ and $\overline{\overline{J}}(\partial D \times \{0\}) \subset G$. Moreover $\overline{\overline{\pi}} \circ \overline{\overline{J}} = \chi$. Now (G,π) is O_2-pseudoconvex domain, therefore we obtain $\overline{\overline{J}}(\overline{D} \times [0, \delta]) \subset G$. However,

$$\begin{aligned}
\chi(0,0) &= (p(0), \dots, p(0), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\
&= (\exp(-h(0)), \dots, \exp(-h(0)), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\
&= (\exp(\log \delta_G(\pi_x^{-1}(0))), \dots, \exp(\log \delta_G(\pi_x^{-1}(0))), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\
&= (\delta_G(\pi_x^{-1}(0)), \dots, \delta_G(\pi_x^{-1}(0)), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\
&= \overline{\overline{\pi}}(u) \in \partial P^n(0).
\end{aligned}$$

It follows that $\overline{\overline{J}}(0,0) = \psi_{\widehat{P}(0)} \circ (\overline{\overline{\pi}}_0)^{-1} \circ \chi(0,0) = u \in \overline{\overline{\partial}}G$. This is a contradiction. Therefore G is pseudoconvex. □

References

[1] M. Jarnicki and P. Pflug, *Extension of Holomorphic Functions*, Walter de Gruyter, Berlin, 2000.

[2] R. Narasimhan, *Several Complex Variables*, Univ. Chicago Press, Chicago, 1971.

[3] T. Yasuoka, Polynomials and pseudoconvexity, *Math. Sem. Notes Kobe Univ.*, **11**(1983), 139–148.

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