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Rings with Jacobson units

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Abstract. We introduce and study the notion of *JU rings*, that are, rings having only *Jacobson units*. In parallel to the so-called UU rings, these rings also form a large class and have many interesting properties established in the present paper. For instance, it is proved that any exchange JU ring is semi-boolean, and vice versa. This somewhat extends a result due to Lee-Zhou (Glasg. Math. J., 2008) and Danchev-Lam (Publ. Math. Debrecen, 2016).

1. Introduction and Background

Everywhere in the text of the current paper, as it is customary, suppose that all rings are *associative* containing an *identity* element. For such a ring R, let $U(R)$ be the unit group of R, let $J(R)$ be the Jacobson radical of R, let $Nil(R)$ be the set of all nilpotents of R, and let $Idemp(R)$ be the set of all idempotents of R. Moreover, let R^+ denote the additive group of R. All other notions and notations are basically standard and follow those from [8].

The following useful equality holds: $J(R) = \{x \in R \mid 1 + RxR \subseteq U(R)\}$. Likewise, if R is commutative, then $Nil(R) = N(R)$ is the nil-radical of R.

It is well known that $1 + J(R) \subseteq U(R)$, that is, $J(R) \subseteq 1 + U(R)$. However, these inclusions could be strict, so that it is rather natural to state the following:

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Definition. A ring R is called a *JU ring* or a ring with *Jacobson units* if $U(R) = 1 + J(R)$.

This is tantamount to the equality $J(R) = 1 + U(R)$. Equivalently, since one can show that

$$U(R)/(1 + J(R)) \cong U(R/J(R)),$$

we observe that in the presence of this isomorphism all JU rings are just those rings R for which $U(R/J(R)) = 1$.

Moreover, note that nil ideals are always contained in the Jacobson radical, so, if $R/J(R)$ is commutative, then the commutator subgroup of $U(R)$ is contained in $1 + J(R)$. Furthermore, if $J(R)$ is nilpotent as an ideal, then $1 + J(R)$ is nilpotent as a group and thus $U(R)$ is solvable. In particular, if $(J(R))^2 = 0$, then $U(R)$ is a metabelian group.

The brief history of the best known achievements on that and certain other closely related themes is the following: In [1] was defined the class of *UU rings* as those rings for which $U(R) = 1 + Nil(R)$. These rings are next systematically studied in [4] by finding a necessary and sufficient condition when a ring is UU in terms of its unit group and its characteristic. Specifically, a ring R is a UU ring if and only if $U(R)$ is a 2-group and $2 \in Nil(R)$. Some other interesting things in this aspect the interested reader can see in [2] and [14].

The motivation of this article is to explore rings containing only JU units in parallel to the study in [4]. Doing that, we shall strengthen some well-known results in the subject.

The work is organized as follows: In the present section, we already provided the reader with the needed information concerning the principal known results. In the next two sections, we state some preliminary and main results, equipped with concrete examples. Further, we give some their applications to abelian groups and commutative group rings, and after that we finish off with several queries of interest.

2. Examples and Preliminaries

We start here with the following technical assertion.

Lemma 2.1. *For any ring R the following equality is true:*

$$U(R) + J(R) = U(R).$$

Proof. It is self-evident that the left hand-side contains the right hand-side. To treat the converse, given $x \in J(R)+U(R)$, we may write $x = a+u$, where $a \in J(R)$ and $u \in U(R)$. With the aid of the above characterization for $J(R) = \{a \in R \mid 1 + RaR \subseteq U(R)\}$, we easily check that $a + u = u(1 + u^{-1}a) \in U(R)$, as required. ∎

Imitating [13], recall that a ring is said to be *semi-boolean* if each its element is semi-boolean, i.e., it is the sum of an element from $J(R)$ and an element from $Idemp(R)$. This is equivalent to $R/J(R)$ is boolean and idempotents lift modulo $J(R)$. In the terminology of [2], semi-boolean rings are just called *J-clean* rings.

An other non-trivial example of JU rings is the next one which extends Corollary 2.5 from [2].

Example 2.2. *J-clean rings are JU.*

Proof. If u is a unit in the J-clean ring R, then $u = j + e$, where j is in $J(R)$ and e is in $Idemp(R)$. So, by Lemma 2.1, $e = u - j \in U(R)$ and hence $e = 1$. This means that $u = j + 1$, as required. ∎

Mimicking [1] or [4], a ring R is called a *UU* ring if $U(R) = 1 + Nil(R)$.

In accordance with [4] one may observe that a JU ring is a UU ring if, and only if, $J(R)$ is nil. In particular, when R is commutative with $J(R) = N(R)$, note that the classes of JU rings and UU rings do coincide. For instance, this holds for *Hilbert rings*, that are commutative rings for which every prime ideal is an intersection of maximal ideals; e.g., the polynomial ring $R[X]$ is Hilbert, whenever R is a commutative ring. This also happens when R is both commutative and finitely generated as an algebra over either a field or $\mathbb{Z}$. In addition, as a special case, the ring $\mathbb{Z}(m)$ is JU (or respectively, UU) if, and only if, $m = 2^k$ for some positive integer k.

Likewise, we emphasize that the Jacobson radical of any artinian ring (in particular, of any finite ring) is nil. This can be subsumed by the following assertion.

Proposition 2.3. *Finite UU rings are JU and finite JU rings are UU.*

Proof. Let R be a finite ring. As aforementioned, $J(R) \subseteq Nil(R)$.

To show the first implication, consider an injective function $f : U(R) \to 1+U(R)$, defined by $f(u) = 1+u$, which is obviously a surjection and thus it is a bijection. Hence $|U(R)| = |1 + U(R)|$. Notice that $1 + Nil(R) = U(R)$ and so we deduce that $|U(R)| = |Nil(R)|$. We also have that $|U(R)| = |J(R)|$, which follows from the two facts that $|J(R)| \leq |U(R)|$ and that $R/J(R)$ being finite UU must be reduced giving that $Nil(R) \subseteq J(R)$ whence $|Nil(R)| \leq |J(R)|$, thus substantiating our claim. This finally assures that $1 + U(R) = J(R)$, as required.

Treating the second implication, $U(R) = 1 + J(R)$ obviously yields that $U(R) = 1 + Nil(R)$, as expected. ■

Resuming the proof, one may claim that if R is a finite UU ring or a finite JU ring, then $J(R) = Nil(R)$.

In the spirit of [12], recollect that a ring is said to be *clean* if each its element is a sum of an idempotent and a unit. Moreover, recall that a ring R is called *exchange* provided for any a in R there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Notice that clean rings are always exchange, whereas the converse is not true in general; however for abelian rings (that are rings with central idempotents) these two ring classes do coincide.

It is worthwhile noticing that in [4] was established that if R is an exchange UU ring, then $R/J(R)$ is boolean and thus reduced; in particular the same holds for finite UU rings (compare with the proof of Proposition 2.3 quoted above). Therefore, exchange UU rings are always JU. In addition, for finite commutative rings it is well known that $J(R) = N(R)$, so that a finite commutative ring is a UU ring if, and only if, it is a JU ring. Note that finite rings are always clean but not always UU or JU.

Recollect that rings R for which $J(R) = \{0\}$ are called *semiprimitive* rings; for example, any field, any von Neumann regular ring and any left or right primitive ring are semiprimitive. Also, so is the ring of integers $\mathbb{Z}$. Therefore, a semiprimitive ring is JU if, and only if, $U(R) = \{1\}$, and so $\mathbb{Z}$ and any field with at least 3 elements (e.g., $\mathbb{Q}$) is not JU.

On the other vein, if K is a field and R is the ring $\mathbb{T}_n(K)$ of all upper triangular $n \times n$ matrices with entries in K, then $J(R)$ consists of all upper triangular matrices with zeros on the main diagonal. Hence, excepting the case $K = \mathbb{F}_2$, such rings are not JU.

Moreover, it is clear that any ring R in which the identity is a sum of two units (in particular, if $2 \in U(R)$) is not JU. Also, it is well known that the identity in the full matrix $n \times n$ ring $\mathbb{M}_n(R)$ with $n > 1$ is a sum of two units. For example, we consult with Lemma 1 of [5], where it is shown that every diagonal matrix in $\mathbb{M}_n(R)$ with $n \geq 2$ is a sum of two units. Thus, we come to

Theorem 2.4. *No matrix ring over a ring with identity is JU.*

The following constructions illustrate that the JU and UU concepts are independent.

- There is a UU ring which is not a JU ring.

If one takes Bergman's ring R as showed in [4], then R is UU as pointed out there. But since $J(R) = \{0\}$, the only J-unit is 1, while $U(R) \neq \{1\}$. This shows that R is not JU.

- There is a JU ring which is not a UU ring.

If R is a local ring with residue field $\mathbb{F}_2$, then R is JU, but it is not UU unless its Jacobson radical is nil.

The next properties are also helpful:

(1) The local ring $(R, \mathbf{m})$ is JU if and only if $R/\mathbf{m} \cong \mathbb{Z}_2$.

In fact, in both directions the factor-ring $R/\mathbf{m}$ has trivial unit group and is a division ring, whence it must be isomorphic to the field of two elements.

Following [2], a ring R is said to be *J-reduced* if $Nil(R) \subseteq J(R)$.

(2) If R is a JU ring, then $R/J(R)$ is reduced. In particular, R is J-reduced.

In addition, J-reduced UU rings are JU rings.

Indeed, as we have seen above, $1 + Nil(R/J(R)) \subseteq U(R/J(R)) = 1$, so that $Nil(R/J(R)) = 0$. Thus $R/J(R)$ does not contain non-trivial nilpotents and it follows at once that R is J-reduced. However, this can be derived directly by observing that $1 + Nil(R) \subseteq U(R) = 1 + J(R)$ and hence $Nil(R) \subseteq J(R)$.

The last observation follows immediately.

(3) If R is a JU ring, then 2 lies in $J(R)$.

Indeed, -1 being a unit can be written as $-1 \in 1 + J(R)$ which gives the wanted claim.

(4) If R is a JU ring such that $p \in J(R)$ (in particular, if R is of prime characteristic p), then $p = 2$.

In fact, by assumption, $R/J(R)$ has characteristic p. On the other hand, in view of the preceding point, $R/J(R)$ must have characteristic 2, whence $p = 2$ as stated.

(5) if I is an ideal of a ring R such that R/I has no nontrivial units, then $I \supseteq J(R)$.

In addition, if I is nil and R is JU, then $I = J(R)$.

To prove this, given $u \in U(R)$, it must be that $u+I$ is a unit in R/I and hence $1-u \in I$. Thus $U(R) \subseteq 1+I$. But $1+J(R) \subseteq U(R)$ whence $J(R) \subseteq I$, as asserted. To show the additional part, since $1 + I \subseteq U(R) = 1 + J(R)$, it follows that $I \subseteq J(R)$ which is tantamount to the equality $I = J(R)$, as claimed.

(6) For an ideal I of R the implication R/I is JU $\Rightarrow$ R is JU generally fails.

Indeed, $\mathbb{Z}/2\mathbb{Z}$ is JU but as demonstrated above $\mathbb{Z}$ is not.

Reciprocally, if R is JU, then R/I may not be JU even if I is a nil ideal.

Indeed, appealing to Theorem 4.1 below, the abelian 2-group $G = \mathbb{Z}(2^{k_1}) \oplus \mathbb{Z}(2^{k_2}) \oplus \cdots \oplus \mathbb{Z}(2^{k_n})$ with $k_1 < k_2 < \cdots < k_n$, $n \in \mathbb{N}$, has JU endomorphism ring $E(G)$, but $E(G)/2E(G) \cong \mathbb{M}_n(\mathbb{F}_2)$ is not JU by consulting with Theorem 2.4. Notice that here $2E(G)$ is a nil ideal.

3. Main Results

The leitmotif of one of our chief results is to describe exchange JU rings. The intersection between these two classes, however, gives nothing new. Specifically, the following is valid:

Theorem 3.1. *A ring R is an exchange JU ring if, and only if, it is J-clean.*

Proof. The sufficiency follows from Example 2.2 and from the obvious fact that J-clean rings being always clean are thus exchange (see [13] and [12]).

Dealing with the necessity, since R is exchange, by [12] we deduce that all idempotents in R can be lifted modulo $J(R)$. Moreover, combining again [12] with one of the equivalencies above for a ring to be JU, we obtain that the factor-ring $R/J(R)$ is exchange with trivial unit group. Therefore, Corollary 4.2 of [4] applies to show that $R/J(R)$ is boolean. We finally apply [13] to get the desired claim. ■

Remark. This result can also be deduced from [9, Theorem 13 (3)] by showing that in JU rings each non-zero idempotent cannot be written as the sum of two units. To show this, in a way of contradiction, given a non-zero idempotent e in R which is a sum of two units, say $e = u_1 + u_2$. Thus one writes that $e = (j_1 + 1) + (j_2 + 1)$, where both j_1, j_2 are from $J(R)$. Hence $1 - e = j - 1 = -(1 + (-j))$, where $j = -j_1 - j_2$ is in $J(R)$, whence $1 - e$ must be a unit. Consequently, $1 - e = 1$ and hence $e = 0$, a contradiction.

Actually, it is worthwhile noticing that Theorem 13 (3) from [9] is equivalent to our Theorem 3.1 by using the methods for proof developed in [4].

The following assertion gives an element-wise description in parallel to Theorem 3.1.

Proposition 3.2. *Any semi-boolean element is clean. For JU rings, the converse holds.*

Proof. Write $r = j + e = (1 - e) + (j + 2e - 1)$, where $j \in J(R)$ and $e \in Idemp(R)$. Since $1 - e$ is an idempotent and $(2e - 1)^2 = 1$, with Lemma 2.1 at hand we derive that $j + 2e - 1$ belongs to $U(R)$ and hence this is a clean decomposition.

Conversely, write $r = u + e = 1 + j + e = (1 - e) + (j + 2e)$, where $u \in U(R)$ and $e \in Idemp(R)$. Since in view of property (3) the element 2 lies in $J(R)$, and so $j + 2e$ is in $J(R)$, the claim follows. ∎

The following two technical statements can be seen in [8], and thereby we omit the proofs leaving them to the interested reader.

Lemma 3.3. *Let A, B be subsets of a ring R with A a subgroup of R^+. If $a \in A$, then $a + (A \cap B) = A \cap (a + B)$.*

Lemma 3.4. *Let $0 \neq e = e^2$ in a ring R. Then*
(a) $J(eRe) = (eRe) \cap J(R) = eJ(R)e$.
(b) $U(eRe) = (eRe) \cap (1 - e + U(R))$.

And so, we are in a position to proceed by proving of the following.

Proposition 3.5. *JU rings passes to corners, that is, each corner ring of a JU ring is again JU.*

Proof. According to the Definition, and with Lemmas 3.3 and 3.4 at hand, we deduce that $e + U(eRe) = e + [(eRe) \cap (1 - e + U(R))] = (eRe) \cap (1 - e + e + U(R)) = (eRe) \cap (1 + U(R)) = (eRe) \cap J(R) = J(eRe)$, as required. ∎

4. Applications to Abelian Groups

Theorem 4.1. *Suppose G is a finite abelian group. Then $E(G)$ is a JU ring if, and only if, $G \cong \mathbb{Z}(2^{k_1}) \oplus \mathbb{Z}(2^{k_2}) \oplus \cdots \oplus \mathbb{Z}(2^{k_n})$ with $k_1 < k_2 < \cdots < k_n$, $n \in \mathbb{N}$.*

Proof. Concerning the necessity, since in view of point (3) the element 2 is always a nilpotent in a finite JU ring, the multiplication by 2 must always be a nilpotent endomorphism, so that G has to be a bounded 2-group. Since by [7] we have $E(\mathbb{Z}(2) \oplus \mathbb{Z}(2)) \cong \mathbb{M}_2(\mathbb{F}_2)$, according to Theorem 2.4 it is not a JU ring. Hence an appeal to Proposition 3.5 guarantees that G must have the above given form, as stated.

As to the sufficiency, it follows by a combination of Theorem 4.1 from [1] and Proposition 2.3. ∎

It is worthwhile noticing the surprising fact that, in view of Theorem 4.1 from [1], a finite abelian group G is a JU group exactly when it is a UU group.

5. Applications to Commutative Group Rings

Throughout this section, for a commutative ring R, let $zd(R) = \{p \mid pr = 0, \exists r \in R\}$ and, for an abelian group G, let $supp(G) = \{p \mid G_p \neq 1\}$, where G_p is the p-torsion component of G.

Finding a criterion when the group ring $R(G)$ of G over R is a UU ring and a JU ring will be the theme here. The best we can offer at this stage is the following:

Proposition 5.1. *Suppose R is a commutative ring and G is an abelian group such that $zd(R) \cap supp(G) = \emptyset$. The following are true:*

(a) $R(G)$ is a UU ring if, and only if, R is a UU ring and G is trivial.

(b) $R(G)$ is a JU ring if, and only if, R is a JU ring and G is trivial.

Proof. (a) The "if" half-part is elementary, because $R(G) \cong R$. To treat the "and only if" one, utilizing [10], we know that $N(R(G)) = N(R)(G)$ and thus $R(G)/N(R(G)) \cong (R/N(R))(G)$. So $U((R/N(R))(G) = \{1\}$ and therefore $U(R/N(R)) = \{1\}$ and $G = \{1\}$, as asserted.

(b) The "if" half-part is straightforward, because $R(G) \cong R$. To deal with "and only if" one, using [6], we know that $J(R(G)) = J(R)(G)$ and so $R(G)/J(R(G)) \cong (R/J(R))(G)$. But then $U((R/J(R))(G) = \{1\}$ and consequently $U(R/J(R)) = \{1\}$ and $G = \{1\}$, as claimed. ∎

A part of the preceding result is valid even in the general case.

Proposition 5.2. *Suppose that R is a commutative ring and G is an abelian group. The following two items hold:*

(1) If $R(G)$ is a UU ring, then R is a UU ring.

(2) If $R(G)$ is a JU ring, then R is a JU ring.

Proof. (1) Since $U(R(G)) = 1 + Nil(R(G))$ and $U(R) \subseteq U(R(G))$, then $U(R) = (1+Nil(R(G)))\cap R = 1+Nil(R(G))\cap R = 1+Nil(R)$, as required.

(2) Since $U(R(G)) = 1 + J(R(G))$ and $U(R) \subseteq U(R(G))$, then $U(R) = (1 + J(R(G))) \cap R = 1 + J(R(G)) \cap R = 1 + J(R)$ because the inclusion $J(R(G)) \cap R \subseteq J(R)$ is always fulfilled, as required. ∎

Another confirmation to this fact is that R is an epimorphic image of $R(G)$ and thus using the property that a homomorphic image of a UU ring (resp., a JU ring) is again a UU ring (resp., a JU ring), we are done.

In contrast to Proposition 5.1, we will now consider the case when $zd(R)\cap supp(G) \neq \emptyset$, say some prime p belongs to this intersection. Specifically, the following is valid:

Theorem 5.3. *Suppose R is a commutative local ring with $p \in J(R)$ and G is an abelian p-group. Then $R(G)$ is a JU ring if, and only if, R is a JU ring and $p = 2$.*

Proof. It is well known that $R/J(R)$ is a field of characteristic p. According to [11], $R(G)$ is a local group ring too, i.e., $R(G)$ has a unique maximal ideal, say $J(R(G))$. Also, appealing to [6], $J(R(G)) = J(R)(G) + I$, where I is the augmentation ideal of $R(G)$. Therefore,

$$R(G)/J(R(G)) \cong R/J(R)$$

because of the sequence $R(G) \to R \to R/J(R)$ of epimorphisms which composite $\Phi : R(G) \to R/J(R)$, defined by $\Phi(\sum_{g\in G} r_g g) = (\sum_{g\in G} r_g) + J(R)$, is again an epimorphism, and whose kernel is $J(R)(G)+I = J(R(G))$ which fact can be verified by a routine technical check.

Thus $U(R(G)/J(R(G))) \cong U(R/J(R))$. However, the quotient $R/J(R)$ must be a finite field, that is, $R/J(R) \cong \mathbb{Z}_p$ such that $U(R/J(R)) = 1$. But this is obviously tantamount to $p = 2$, as stated. ■

6. Some Related Concepts

We will consider here some closely related concepts, starting with the following:

• $Nil_*(R) = P(R)$ is the *lower nil-radical* (the *prime radical*) that is the intersection of all prime ideals of R.

• $Nil^*(R)$ is the *upper nil-radical* that is the sum of all nil-ideals of R. In particular, $Nil^*(R)$ is itself also a nil-ideal.

It is well known that the following containments hold:

$Nil_*(R) \subseteq Nil^*(R) \subseteq Nil(R) \cap J(R)$.

So, we may further define

• A ring R is called *a LUU ring* if $U(R) = 1 + Nil_*(R)$.

• A ring R is called *a UUU ring* if $U(R) = 1 + Nil^*(R)$.

Clearly any LUU ring is a UUU ring, and the latter is both a UU ring and a JU ring.

Recall that a ring R is said to be 2-*primal* provided that $Nil(R) = P(R)$, that is, $R/Nil_*(R)$ is reduced.

The following two characterization statements are valid:

Theorem 6.1. *The following are equivalent:*

(i) R is a LUU ring;

(ii) R is a 2-primal UU ring;

(iii) $U(R)$ is a 2-group, $2 \in Nil(R)$ and R is a 2-primal ring.

Proof. About the implication (i) ⇒ (ii), we observe that $1 + Nil(R) \subseteq U(R) = 1 + Nil_*(R)$, i.e., $Nil(R) = Nil_*(R)$, as required.

To show that the implication (ii) ⇒ (i) is true, one sees that $U(R) = 1 + Nil(R)$ with $Nil(R) = Nil_*(R)$ gives that $U(R) = 1 + Nil_*(R)$, as needed.

Finally, the equivalence (ii) $\iff$ (iii) follows directly from [4]. ∎

Mimicking [14], a ring R is said to be *a NI ring* provided $Nil(R)$ is an ideal of R. Observe that 2-primal rings are always NI.

We are now in a position to proceed with proving of the next assertion.

Theorem 6.2. *The following are equivalent:*

(i) R is a UUU ring;

(ii) R is a UU ring and a NI ring;

(iii) $U(R)$ is a 2-group, $2 \in Nil(R)$ and R is a NI ring.

Proof. It follows using the same idea as that in Theorem 6.1. ∎

7. Concluding Discussion and Open Questions

In conclusion, we shall separately comment some basic facts that were presented above, and thus some light will be shed for a possible further exploration. In fact, it was shown in [4] that in exchange UU rings each non-zero idempotent is not the sum of two units. However, this was not shown directly, that is really rather difficult and unknown to the authors at this moment; it was just proved for rings which are boolean modulo their Jacobson radical which is, in fact, one of the main results stated as Theorem B in [4] provided the Jacobson radical is nil. That is why [4, Theorem 4.3] is independent to [9, Theorem 13] in the sense that it is not deducible from it.

A crucial query which now immediately arises is whether the same property holds only for UU rings (by dropping off the exchange condition) that amounts to the requirement that in UU rings does there exist a non-zero idempotent which is a sum of two nilpotents (compare with the Remark above where things going smoothly).

We now end with several problems of interest:

Problem 1. Find a necessary and sufficient condition when the commutative group ring $R(G)$ is a UU ring.

Problem 2. Find a necessary and sufficient condition when the commutative group ring $R(G)$ is a JU ring.

Problem 3. Characterize those rings R whose $U(R) = J(R) \pm 1$.

We shall call such rings as rings with *weak Jacobson units* or just *weakly JU rings* abbreviated as *WJU rings*.

Problem 4. Describe those rings R whose units are the sum of a unipotent and a nilpotent, i.e., the sum of 1 and two nilpotents.

Being a natural generalization of UU rings, we shall call them *UNU rings*.

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