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Generalized lacunary strong Zweier convergent sequence spaces

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Abstract. In this paper we introduce generalized double Zweier lacunary convergent sequence spaces via sequence of Orlicz functions over n-normed spaces. We also make an effort to study some topological properties and inclusion relations between these spaces. Furthermore, we study the concept of double lacunary statistical Zweier convergence over n-normed spaces.

1. Introduction

In [10], Hardy introduced the concept of regular convergence for double sequences. Some important work on double sequences is also found by Bromwich [27]. By the convergence of a double sequence we mean the convergence of the Pringsheim sense i.e., a double sequence $x = (x_{ij})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{ij} - L| < \epsilon$ whenever $i, j > n$ [2]. In case $L = 0$, we say that double sequence $x = (x_{ij})$ is a Pringsheim null sequence. The double sequence $x = (x_{ij})$ is bounded if there exists a positive integer K such that $|x_{ij}| < K$ for all i and j. We denote by l^2_∞ the space of all bounded double sequences.

Definition 1.1. *[8] The double sequence $I_{r,s} = \{(k_r, l_s)\}$ is called double*

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lacunary if there exist two increasing integers sequences (k_r) *and* (l_s) *such that*

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \overline{h_s} = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Let $k_{r,s} = k_r l_s,\ h_{r,s} = h_r \overline{h_s}$, *and* $\theta_{r,s}$ *is determined by*

$$I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}},\ \overline{q_s} = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \overline{q_s}.$$

Definition 1.2. *An Orlicz function* $M : [0,\infty) \to [0,\infty)$ *is a continuous, non-decreasing and convex function such that* $M(0) = 0$, $M(x) > 0$ *for* $x > 0$ *and* $M(x) \longrightarrow \infty$ *as* $x \longrightarrow \infty$. *If convexity of Orlicz function is replaced by* $M(x+y) \leq M(x) + M(y)$, *then this function is called a modulus function.*
Lindenstrauss and Tzafriri [15] used the idea of Orlicz to define the sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

is known as an Orlicz sequence space. The space ℓ_M *is a Banach space with the norm*

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also it was shown in [15] that every Orlicz sequence space ℓ_M *contains a subspace isomorphic to* $\ell_p (p \geq 1)$. *An Orlicz function* M *can always be represented in the following integral form*

$$M(x) = \int_0^x \eta(t)dt,$$

where η *is known as the kernel of* M, *is a right differentiable for* $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η *is non-decreasing and* $\eta(t) \to \infty$ *as* $t \to \infty$.

The notion of difference sequence spaces was introduced by Kızmaz [14] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [23] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w denote the set of all real and complex sequences and n be a non-negative integer, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in Z\},$$

where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1}x_k - \Delta^{n-1}x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+v}.$$

Taking $n = 1$, we get the spaces studied by Et and Çolak [23]. Similarly, we can define difference operators on double sequences as:

$$\begin{aligned} \Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, \end{aligned}$$

and

$$\Delta^n x_{k,l} = \Delta^{n-1}x_{k,l} - \Delta^{n-1}x_{k,l+1} - \Delta^{n-1}x_{k+1,l} + \Delta^{n-1}x_{k+1,l+1}.$$

For more details about sequence spaces see ([17], [18], [26]) and references therein.

Definition 1.3. *A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be a Musielak-Orlicz function (see [21, 16]). A sequence $\mathcal{N} = (N_k)$ defined by*

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called a complementary function of the Musielak-Orlicz function (M_k). For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \textit{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \ for\ all\ \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ *is a convex modular defined by*

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ *equipped with the Luxemburg norm*

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \le 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

A Musielak-Orlicz function $\mathcal{M} = (M_k)$ *is said to satisfy* Δ_2*-condition if there exist constants* $a,\ K > 0$ *and a sequence* $c = (c_k)_{k=1}^{\infty} \in l_+^1$ *(the positive cone of* l^1*) such that the inequality*

$$M_k(2u) \le KM_k(u) + c_k$$

holds for all $k \in N$ *and* $u \in R^+$*, whenever* $M_k(u) \le a$.

Definition 1.4. *Let* X *be a linear metric space. A function* $p : X \to R$ *is called a paranorm, if*

1. $p(x) \ge 0$ *for all* $x \in X$*;*
2. $p(-x) = p(x)$ *for all* $x \in X$*;*
3. $p(x + y) \le p(x) + p(y)$ *for all* $x, y \in X$*;*
4. *if* (λ_n) *is a sequence of scalars with* $\lambda_n \to \lambda$ *as* $n \to \infty$ *and* (x_n) *is a sequence of vectors with* $p(x_n - x) \to 0$ *as* $n \to \infty$*, then* $p(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$*.*

A paranorm p *for which* $p(x) = 0$ *implies* $x = 0$ *is called a total paranorm and the pair* (X, p) *is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [5] Theorem 10.4.2, pp. 183).*

Definition 1.5. *A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| < 1$ for all $k \in N$.*

Definition 1.6. *A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of N.*

Definition 1.7. *A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.*

Definition 1.8. *A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.*

Definition 1.9. *Let $K = \{k_1 < k_2 < \cdots\} \subset N$ and let E be a sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (x_k) \in E\}$.*

Definition 1.10. *A canonical preimage of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in w$ defined by*

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0\ , & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all the elements in λ_K^E, that is, y is in the canonical preimage of λ_K^E if and only if y is a canonical preimage of some $x \in \lambda_K^E$.

Definition 1.11. *A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces.*

The concept of 2-normed spaces was initially developed by Gähler [25] in the mid of 1960's, while that for n-normed spaces one can see in Misiak [1]. Since then, many others have studied this concept and obtained various results, see Gunawan ([11], [12]) and Gunawan and Mashadi [13]. Let $n \in N$ and X be a real linear space of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \ldots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \ldots, x_n|| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X,

2. $||x_1, x_2, \ldots, x_n||$ is invariant under permutation,

3. $||\alpha x_1, x_2, \ldots, x_n|| = |\alpha| \ ||x_1, x_2, \ldots, x_n||$ for any $\alpha \in R$, and

4. $||x + x', x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||x', x_2, \ldots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \ldots, \cdot||)$ is called an n-normed space over the field R.
For example, we may take $X = R^n$ being equipped with the n-norm $||x_1, x_2, \ldots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \ldots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in R^n$ for each $i = 1, 2, \ldots, n$. Let $(X, ||\cdot, \ldots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \ldots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \ldots, x_{n-1}||_\infty = \max\{||x_1, x_2, \ldots, x_{n-1}, a_i|| : i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.
A sequence (x_k) in an n-normed space $(X, ||\cdot, \ldots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \ldots, z_{n-1}|| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in an n-normed space $(X, ||\cdot, \ldots, \cdot||)$ is said to be a Cauchy sequence if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \ldots, z_{n-1}|| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. A complete n-normed space is said to be an n-Banach space. For more details on n-normed spaces, see [19], [20] and references therein.

2. Lacunary strongly Zweier convergent sequence spaces

Zweier sequence spaces for single sequences were defined and studied by Şengönül [22], Esi and Sapsızoğlu [4], Khan et. al [28], [29]. Esi and Acikgoz [3] defined the double Zweier sequence spaces $[W^2, Z]$, $[N_{\theta_{r,s}}, Z]_0$, $[N_{\theta_{r,s}}, Z]$ and $[N_{\theta_{r,s}}, Z]_\infty$ as the set of all double sequences such that $Z-$transforms of them are in $[W^2]$, $[N_{\theta_{r,s}}]_0$, $[N_{\theta_{r,s}}]$ and $[N_{\theta_{r,s}}]_\infty$ which were introduced by Savaş in [7], Savaş and Patterson in [9].

We define the double sequences $v = (v_{ij})$ and $w = (w_{ij})$ which will be used throughout the paper, as Z-transform of a sequence $x = (x_{ij})$ and $y = (y_{ij})$ respectively i.e.,

$$v_{ij} = \frac{1}{2}(x_{ij} + x_{ij-1}) \quad \text{and} \quad w_{ij} = \frac{1}{2}(y_{ij} + y_{ij-1}); \quad (i, j \in N). \tag{2.1}$$

Let $(X, ||\cdot, \ldots, \cdot||)$ be an n-normed space and $W(n-X)$ denotes the space of X-valued sequences. Let $\mathcal{M} = (M_{ij})$ be a Musielak-Orlicz function, $p = (p_{ij})$ be a bounded double sequence of positive real numbers and $u = (u_{ij})$ be a double sequence of strictly positive real numbers. In the present paper we introduce the new double Zweier sequence spaces as follows:

$$[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0 = \Big\{x = (x_{ij}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} = 0 \text{ for some } \rho > 0\Big\},$$

$$[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|] = \Big\{x = (x_{ij}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} = 0 \text{ for some } L \text{ and } \rho > 0\Big\},$$

$$[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty = \Big\{x = (x_{ij}) : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty \text{ for some } \rho > 0\Big\}$$

and

$[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$

$$= \Big\{x=(x_{ij}) : P-\lim_{m,n}\frac{1}{mn}\sum_{i,j=1,1}^{m,n} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}=0 \text{ for some } L \text{ and } \rho>0\Big\}.$$

Remark 2.1. Let us consider a few special cases of the above sequence spaces:

(i) If $M_{ij}(x) = x$, for all $i,j \in N$, then above sequence space reduces to $[N_{\theta_{r,s}}, Z, \Delta^n, p, u, \|\cdot,\ldots,\cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \Delta^n, p, u, \|\cdot,\ldots,\cdot\|]$, $[N_{\theta_{r,s}}, Z, \Delta^n, p, u, \|\cdot,\ldots,\cdot\|]_\infty$ and $[W^2, Z, \Delta^n, p, u, \|\cdot,\ldots,\cdot\|]$.

(ii) By taking $(p_{ij}) = 1$, for all $i,j \in N$, then the above space becomes $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, u, \|\cdot,\ldots,\cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, u, \|\cdot,\ldots,\cdot\|]$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, u, \|\cdot,\ldots,\cdot\|]_\infty$ and $[W^2, Z, \mathcal{M}, \Delta^n, u, \|\cdot,\ldots,\cdot\|]$.

(iii) By taking $(u_{ij}) = 1$, for all $i,j \in N$, then we get the above space as $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, \|\cdot,\ldots,\cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, \|\cdot,\ldots,\cdot\|]$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, \|\cdot,\ldots,\cdot\|]_\infty$ and $[W^2, Z, \mathcal{M}, \Delta^n, p, \|\cdot,\ldots,\cdot\|]$.

(iv) If we take $M_{ij}(x) = x$, $(p_{ij}) = 1$, $(u_{ij}) = 1$, for all $i,j \in N$, and $n = 0$ then the above space reduces to $[N_{\theta_{r,s}}, Z, \|\cdot,\ldots,\cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \|\cdot,\ldots,\cdot\|]$, $[N_{\theta_{r,s}}, Z, \|\cdot,\ldots,\cdot\|]_\infty$ and $[W^2, Z, \|\cdot,\ldots,\cdot\|]$.

(v) Also, if we take $(p_{ij}) = 1$, $(u_{ij}) = 1$, for all $i,j \in N$, and $n = 0$ then the above space reduces to $[N_{\theta_{r,s}}, Z, \mathcal{M}, \|\cdot,\ldots,\cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \|\cdot,\ldots,\cdot\|]$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \|\cdot,\ldots,\cdot\|]_\infty$ and $[W^2, Z, \mathcal{M}, \|\cdot,\ldots,\cdot\|]$.

The following inequality will be used through out the paper. If $0 \le p_{ij} \le \sup p_{ij} = G$, $D = \max(1, 2^{G-1})$ then

$$|a_{ij} + b_{ij}|^{p_{ij}} \le D(|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}}) \qquad (2.2)$$

for all $i,j \in N$ and $a_{ij}, b_{ij} \in C$. Also $|a|^{p_{ij}} \le \max(1, |a|^G)$ for all $a \in C$.

The main purpose of this paper is to introduce double Zweier lacunary strongly convergent sequence spaces over n-normed spaces and study different properties of these spaces like linearity, paranorm, solidity and monotone etc. Some inclusion relations between these spaces are also established. Finally, we study the concept of the double Zweier lacunary statistical convergence over n-normed spaces.

3. Main Results

Theorem 3.1. *Let $\mathcal{M} = (M_{ij})$ be a sequence of Orlicz functions, $p = (p_{ij})$ be any bounded double sequence of positive real numbers and $u = (u_{ij})$ be a double sequence of strictly positive real numbers. Then the double Zweier sequence spaces $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$ and $[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$ are linear spaces over the field R of real numbers .*

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$. Let $\alpha, \beta \in R$. Then there exist positive real numbers ρ_1, ρ_2 such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho_1}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty,$$

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n w_{ij}}{\rho_2}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_{ij}'s are non-decreasing and convex so by using inequality (2.2), we have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n(\alpha v_{ij} + \beta w_{ij})}{\rho_3}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
\le\ & \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n \alpha v_{ij}}{\rho_3}, z_1, \ldots, z_{n-1}\Big\|\Big) \\
& + u_{ij}\Big(\Big\|\frac{\Delta^n \beta w_{ij}}{\rho_3}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
\le\ & D \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \frac{1}{2^{p_{ij}}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho_1}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
& + D \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \frac{1}{2^{p_{ij}}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n w_{ij}}{\rho_2}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
\le\ & D \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho_1}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
& + D \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \Big[u_{ij}\Big(\Big\|\frac{\Delta^n w_{ij}}{\rho_2}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
<\ & \infty.
\end{aligned}$$

Thus, $\alpha x+\beta y \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$. This proves that $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$ is a linear space. Similarly we can prove that $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$ and $[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$ are linear spaces. □

Theorem 3.2. *Let $\mathcal{M} = (M_{ij})$ be a sequence of Orlicz functions, $p = (p_{ij})$ be any bounded double sequence of positive real numbers and $u = (u_{ij})$ be a double sequence of strictly positive real numbers. Then the double Zweier sequence space $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$ is a paranormed space with paranormed defined by*

$$g(x) = \inf\left\{ (\rho)^{\frac{p_{ij}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}} \leq 1, \text{ for some } \rho > 0\right\},$$

where $0 < p_{ij} \leq \sup p_{ij} = G$ and $H = \max(1, G)$.

Proof. (i) Clearly $g(x) \geq 0$ for $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$. Since $M_{ij}(0) = 0$, we get $g(0) = 0$.

(ii) $g(-x) = g(x)$.

(iii) Let $x = (x_{ij})$ and $y = (y_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$, then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho_1}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \leq 1$$

and

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n w_{ij}}{\rho_2}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n (v_{ij}+w_{ij})}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n (v_{ij}+w_{ij})}{\rho_1+\rho_2}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\leq \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho_1+\rho_2}, z_1, \ldots, z_{n-1}\Big\|\Big)
\end{aligned}$$

$$+u_{ij}\Big(\Big\|\frac{\Delta^n w_{ij}}{\rho_1+\rho_2},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho_1},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n w_{ij}}{\rho_2},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$\leq 1$$

and thus

$$g(x+y)$$

$$= \inf\left\{(\rho)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n (v_{ij}+w_{ij})}{\rho_1+\rho_2},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}}\leq 1\right\}$$

$$\leq \inf\left\{(\rho_1)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho_1},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}}\leq 1\right\}$$

$$+\inf\left\{(\rho_2)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n w_{ij}}{\rho_2},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}}\leq 1\right\}.$$

Therefore, $g(x+y)\leq g(x)+g(y)$. Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\left\{(\rho)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n \lambda v_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}}\leq 1\right\}$$

$$= \inf\left\{(|\lambda| t)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{t},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}}\leq 1\right\},$$

where $t=\frac{\rho}{|\lambda|}>0$. Since $|\lambda|^{p_{ij}}\leq \max(1,|\lambda|^{\sup p_{ij}})$, we have
$g(\lambda x)\leq \max(1,|\lambda|^{\sup p_{ij}})$

$$\inf\left\{t^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{t},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}}\leq 1\right\}.$$

So, the fact that the scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 3.3. *If* $0<p_{ij}<q_{ij}<\infty$ *for each* i *and* j*, then we have* $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot,\ldots,\cdot\|]_\infty \subset [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, q, u, \|\cdot,\ldots,\cdot\|]_\infty$.

Proof. Let $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]_\infty$. Then there exists $\rho > 0$ such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty$$

This implies that $M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < 1$ for sufficiently large values of i and j. Since M_{ij}'s are non-decreasing, we get

$$\begin{aligned} &\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{q_{ij}} \\ \le\ & \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ <\ & \infty. \end{aligned}$$

Thus, $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, q, u, \|\cdot, \dots, \cdot\|]_\infty$. This completes the proof. □

Theorem 3.4. *Suppose $\mathcal{M} = (M_{ij})$ be a sequence of Orlicz functions, $p = (p_{ij})$ be a bounded double sequence of positive real numbers and $u = (u_{ij})$ be a double sequence of strictly positive real numbers. Then*
(i) *If $0 < \inf p_{ij} < p_{ij} \le 1$, then*
$[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]_\infty \subset [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, u, \|\cdot, \dots, \cdot\|]_\infty$.
(ii) *If $1 \le p_{ij} \le \sup p_{ij} < \infty$, then*
$[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, u, \|\cdot, \dots, \cdot\|]_\infty \subset [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]_\infty$.

Proof. (**i**) Let $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]_\infty$. Since $0 < \inf p_{ij} \le 1$, we obtain the following

$$\begin{aligned} &\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big] \\ \le\ & \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ <\ & \infty, \end{aligned}$$

and hence $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, u, \|\cdot, \dots, \cdot\|]_\infty$.
(**ii**) Let $p_{ij} \ge 1$ for each i and j and $\sup p_{ij} < \infty$. Let $x = (x_{ij}) \in$

$[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, u, \|\cdot, \ldots, \cdot\|]_\infty$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big] \leq \epsilon < 1 \text{ for all } r, s \geq N.$$

This implies that

$$\begin{aligned}\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ \leq \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big] \\ < \infty.\end{aligned}$$

Therefore, $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$. This completes the proof. □

Theorem 3.5. *Let $\mathcal{M}' = (M'_{ij})$ and $\mathcal{M}'' = (M''_{ij})$ be two sequences of Orlicz functions, then we have $[N_{\theta_{r,s}}, Z, \mathcal{M}', \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty \cap [N_{\theta_{r,s}}, Z, \mathcal{M}'', \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty \subset [N_{\theta_{r,s}}, Z, \mathcal{M}' + \mathcal{M}'', \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$.*

Proof. Let $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}', \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty \cap [N_{\theta_{r,s}}, Z, \mathcal{M}'', \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$. Then

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M'_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}, \text{ for some } \rho_1 > 0$$

and

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M''_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the inequality

$$\begin{aligned}\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (M'_{ij} + M''_{ij})\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ = \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M'_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ + \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M''_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\end{aligned}$$

$$\begin{aligned} &\leq D\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M'_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ &\quad + D\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M''_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ &< \infty. \end{aligned}$$

Thus, $\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}(M'_{ij}+M''_{ij})\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}<\infty$. Therefore, $x=(x_{ij})\in[N_{\theta_{r,s}},Z,\mathcal{M}'+\mathcal{M}'',\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]_\infty$. This completes the proof. □

Theorem 3.6. *For a sequence of Orlicz functions $\mathcal{M}=(M_{ij})$, $p=(p_{ij})$ be any bounded double sequence of positive real numbers and $u=(u_{ij})$ be a double sequence of strictly positive real numbers. Then*
(i) $[N_{\theta_{r,s}},Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]_0\subset[N_{\theta_{r,s}},Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]_\infty$
(ii) $[N_{\theta_{r,s}},Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]\subset[N_{\theta_{r,s}},Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]_\infty$

Proof. The proof is easy so we omit it. □

Theorem 3.7. *The double Zweier sequence space $[N_{\theta_{r,s}},Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]_\infty$ is solid.*

Proof. Suppose $x=(x_{ij})\in[N_{\theta_{r,s}},Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]_\infty$

$$\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}<\infty,\ \text{for some } \rho>0.$$

Let (α_{ij}) be a double sequence of scalars such that $|\alpha_{ij}|\leq 1$ for all $i,j\in N$. Then we get

$$\begin{aligned} &\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n \alpha_{ij}v_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ &\leq \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\ &< \infty. \end{aligned}$$

This completes the proof. □

Theorem 3.8. *The double Zweier sequence space* $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$ *is monotone.*

Proof. The proof is trivial so we omit it. □

Theorem 3.9. *The double Zweier sequence spaces* $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$ *and* $[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$ *are linearly isomorphic to the double sequence spaces* $[N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$, $[N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$, $[N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$ *and* $[W^2, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$, *respectively, i.e.,*

(i) $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0 \approx [N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$,
(ii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|] \approx [N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$,
(iii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty \approx [N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$
(iv) $[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|] \approx [W^2, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$.

Proof. We consider only $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$. We should show the existence of a linear bijection between the double sequence spaces $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$ and $[N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$. Consider the transformation Z from $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$ to $[N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$ by

$$x \mapsto Zx = v, \quad v = (v_{ij})$$

and

$$v_{ij} = \frac{1}{2}(x_{ij} + x_{ij-1}); \quad (i, j \in N).$$

The linearity of Z is clear. Further, it is trivial that $x = 0$ whenever $Zx = 0$ and hence Z is injective. Let $v = (v_{ij}) \in [N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0$ and define the sequence $x = (x_{ij})$ by

$$x_{ij} = 2\sum_{k=0}^{j}(-1)^{j-k} v_{ik}, \quad (\forall i \in N).$$

Then

$\|x\|_{[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]_0}$

$$= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n \frac{1}{2}(x_{ij}+x_{ij-1})}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n \frac{1}{2}(2\sum_{k=0}^{j}(-1)^{j-k}v_{ik} + 2\sum_{k=0}^{j}(-1)^{(j-1)-k}v_{ik})}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

which says that $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]_0$. Additionally, we observe that $\|x\|_{[N_{\theta_{r,s}}, Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\dots,\cdot\|]_\infty} = \|v\|_{[N_{\theta_{r,s}},\mathcal{M},\Delta^n,p,u,\|\cdot,\dots,\cdot\|]_\infty}$. Thus, we have that the transform Z is surjective. Hence, Z is a linear bijection which therefore says us the double sequence spaces $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]_0$ and $[N_{\theta_{r,s}}, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]_0$ are linearly isomorphic. The others can be proved similarly. This completes the proof. □

Theorem 3.10. *Let $\theta_{r,s}$ be a double lacunary sequence. Then*
(i) $[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|] \subset [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]$ *if* $\liminf q_r > 1$ *and* $\liminf \overline{q_s} > 1$*;*
(ii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|] \subset [W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]$ *if* $\limsup q_r < \infty$ *and* $\limsup \overline{q_s} < \infty$*;*
(iii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|] = [W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]$ *if* $1 < \liminf q_r < \infty$ *and* $1 < \limsup \overline{q_s} < \infty$*.*

Proof. (**i**) Suppose that $\liminf q_r > 1$ and $\liminf \overline{q_s} > 1$. Then there exists $\delta > 0$ such that both $q_r > 1+\delta$ and $\overline{q_s} > 1+\delta$. This implies $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h_s}}{l_s} \geq \frac{\delta}{1+\delta}$. If $x = (x_{ij}) \in [W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \dots, \cdot\|]$, then we obtain the following:

$$A_{rs} = \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij} - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$= \frac{1}{h_{r,s}} \sum_{i=1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij} - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$
\begin{aligned}
& -\frac{1}{h_{r,s}}\sum_{i=1}^{k_{r-1}}\sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
& -\frac{1}{h_{r,s}}\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
& -\frac{1}{h_{r,s}}\sum_{j=l_{s-1}+1}^{l_s}\sum_{i=1}^{k_{r-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
= \; & \frac{k_r l_s}{h_{r,s}}\Big(\frac{1}{k_r l_s}\sum_{i=1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) \\
& -\frac{k_{r-1}l_{s-1}}{h_{r,s}}\Big(\frac{1}{k_{r-1}l_{s-1}}\sum_{i=1}^{k_{r-1}}\sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) \\
& -\frac{1}{h_r}\sum_{i=k_{r-1}+1}^{k_r}\frac{l_{s-1}}{\overline{h_s}}\frac{1}{l_{s-1}}\sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
& -\frac{1}{\overline{h_s}}\sum_{j=l_{s-1}+1}^{l_s}\frac{k_{r-1}}{h_r}\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}
\end{aligned}
$$

Since $x=(x_{ij})\in[W^2,Z,\mathcal{M},\Delta^n,p,u,\|\cdot,\ldots,\cdot\|]$ the last two terms tend to zero in the pringsheim sense, thus

$$
\begin{aligned}
A_{rs} \; = \; & \frac{k_r l_s}{h_{r,s}}\Big(\frac{1}{k_r l_s}\sum_{i=1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) \\
& -\frac{k_{r-1}l_{s-1}}{h_{r,s}}\Big(\frac{1}{k_{r-1}l_{s-1}}\sum_{i=1}^{k_{r-1}}\sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) \\
& +O(1)
\end{aligned}
$$

Since $h_{r,s}=k_r l_s-k_r l_{s-1}-k_{r-1}l_s+k_{r-1}l_{s-1}$ we are granted for the following:

$$
\frac{k_r l_s}{h_{r,s}}\le\Big(\frac{1+\delta}{\delta}\Big)^2 \quad\text{and}\quad \frac{k_{r-1}l_{s-1}}{h_{r,s}}\le\frac{1}{\delta}.
$$

The terms

$$
\frac{k_r l_s}{h_{r,s}}\Big(\frac{1}{k_r l_s}\sum_{i=1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big)
$$

and

$$\frac{k_{r-1}l_{s-1}}{h_{r,s}}\Big(\frac{1}{k_{r-1}l_{s-1}}\sum_{i=1}^{k_{r-1}}\sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big)$$

are both pringsheim null sequences. Thus $A_{r,s}$ is a pringsheim null sequence.
Therefore, $x=(x_{ij})\in[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot,\ldots,\cdot\|]$.
(**ii**) Suppose that $\limsup q_r<\infty$ and $\limsup \overline{q_s}<\infty$, then there exists $K>0$ such that $q_r\le K$, $\overline{q_s}\le K$ for all r and s. Let $x=(x_{ij})\in[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot,\ldots,\cdot\|]$ and $\epsilon>0$. Also there exist $r_0>0$ and $s_0>0$ such that for every $k\ge r_0$ and $l\ge s_0$

$$A_{k,l}=\frac{1}{h_{k,l}}\sum_{(i,j)\in I_{k,l}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}<\epsilon.$$

Let $N=\max\{A_{k,l}:1\le k\le r_0 \text{ and } 1\le l\le s_0\}$ and p and q be such that $k_{r-1}<p\le k_r$ and $l_{s-1}<q\le l_s$. Then we obtain the following

$$\begin{aligned}
&\tfrac{1}{pq}\sum_{i,j=1,1}^{p,q} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\le \frac{1}{k_{r-1}l_{s-1}}\sum_{i=1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\le \frac{1}{k_{r-1}l_{s-1}}\sum_{p,q=1,1}^{r,s}\Bigg(\sum_{(i,j)\in I_{p,q}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{\Delta^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Bigg)\\
&= \frac{1}{k_{r-1}l_{s-1}}\sum_{p,q=1,1}^{r_0,s_0} h_{p,q}A_{p,q}+\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\le r)\cup(s_0<q\le s)} h_{p,q}A_{p,q}\\
&\le \frac{N}{k_{r-1}l_{s-1}}\sum_{p,q=1,1}^{r_0,s_0} h_{p,q}+\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\le r)\cup(s_0<q\le s)} h_{p,q}A_{p,q}\\
&\le \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\Big(\sup_{(p\ge r_0)\cup(q\ge s_0)} A_{p,q}\Big)\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\le r)\cup(s_0<q\le s)} h_{p,q}\\
&\le \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\epsilon\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\le r)\cup(s_0<q\le s)} h_{p,q}\\
&\le \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\epsilon K^2.
\end{aligned}$$

Since k_r and l_s both approach infinity as both r and s approach infinity, it follows that

$$P - \lim_{p,q} \frac{1}{pq} \sum_{i,j=1,1}^{p,q} M_{ij} \Big[u_{ij} \Big(\Big\| \frac{\Delta^n v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{p_{ij}}$$

Therefore, $x = (x_{ij}) \in [W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$.
(**iii**) Combining (i) and (ii), we can easily prove (iii).
Example: Suppose $\liminf_r q_r = 1$ or $\liminf_s \overline{q_s} = 1$, and assume without loss of generality that $\liminf_r q_r = 1$ [6]; then there exists an ordinary subsequence $\{k_{\alpha_j}\}$ of the lacunary sequences θ_r such that $\frac{k_{\alpha_j}}{k_{\alpha_j}-1} < 1 + \frac{1}{j}$ and $\frac{k_{\alpha_j}-1}{k_{\alpha_j}-1} > j$ where $\alpha_j \geq \alpha_{j-1} + 2$. Let us define x as follows:

$$x_{ij} = \begin{cases} 1, & \text{if } i \in I_{\alpha_j} \text{ and } j \in N \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly the rows are not in $[N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$ but each row is such that x is in $|\sigma_{1,1}|$; where $|\sigma_{1,1}| = \Big\{ x : \text{ for some } L, P - \lim_{m,n} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} |x_{i,j} - L| = 0 \Big\}$. Therefore, each row is in $[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$. Since the double lacunary sequence $\theta_{r,s}$ is factorable, so we have $[W^2, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|] \not\subset [N_{\theta_{r,s}}, Z, \mathcal{M}, \Delta^n, p, u, \|\cdot, \ldots, \cdot\|]$. □

4. Lacunary Zweier statistical convergence sequence space

The following definition was presented by Mursaleen and Edely in [24]:

Definition 4.1. *[24] A real double sequence $x = (x_{ij})$ is said to be statistically convergent to L, provided that for each $\epsilon > 0$*

$$P - \lim_{m,n} \frac{1}{mn} |\{(i,j) : i \leq m \text{ and } j \leq n, |x_{i,j} - L| \geq \epsilon\}| = 0$$

where the vertical bars indicate the number of elements in the closed set. In this case we write $st_2 - \lim_{ij} x_{ij} = L$ and we denote the set of all $P-$ statistical convergent double sequences by st_2.

Remark 4.2. *(a) If x is a convergent double sequences then it is also statistically convergent to the same number. Since there are only a finite* *number of bounded(unbounded) rows and/or columns,*

$$K(m,n) \leq s_1 m + s_2 n,$$

where s_1 and s_2 are finite numbers, which can conclude that x is statistically convergent.
(b) If x is statistically convergent to the number L, then L is determined uniquely.
(c) If x is statistically convergent, then x need not be convergent. Also it is not necessarily bounded. For example, let $x = (x_{ij})$ be defined as

$$x_{ij} = \begin{cases} ij, & \text{if } i \text{ and } j \text{ are squares} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st_2 - \lim x_{ij} = 1$, since the cardinality of the set $\{(i,j) : |x_{i,j} - 1| \geq \epsilon\} \leq \sqrt{i}\sqrt{j}$ for every $\epsilon > 0$ but x is neither convergent or bounded.

Recently, in [7], Savaş defined double lacunary statistical convergence as follows:

Definition 4.3. *[7] A real double sequence $x = (x_{ij})$ is said to be $S_{\theta_{r,s}}$-convergent to L, provided that for each $\epsilon > 0$*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(i,j) \in I_{r,s} : |x_{i,j} - L| \geq \epsilon\}| = 0.$$

Definition 4.4. *A real double sequence $x = (x_{ij})$ is said to be double lacunary statistical Zweier convergent to L, provided that for each $\epsilon > 0$*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(i,j) \in I_{r,s} : |v_{i,j} - L| \geq \epsilon\}| = 0$$

where $v_{i,j}$ is the form in (2.1).

Theorem 4.5. *Let $\theta_{r,s}$ be a double lacunary sequence. If $x_{ij} \to L([N_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|])$, then $x_{ij} \to L([S_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|])$.*

Proof. If $\epsilon > 0$ and $x_{ij} \to L([N_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|])$ then we can write

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \|v_{ij} - L, z_1, \ldots, z_{n-1}\|$$

$$\geq \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s} \& \|\frac{1}{2}(x_{i,j}+x_{i,j})-L,z_1,\ldots,z_{n-1}\|\geq\epsilon} \|v_{ij} - L, z_1, \ldots, z_{n-1}\|$$

$$\geq \frac{1}{h_{r,s}} |\Big\{(i,j) \in I_{r,s} : \|v_{ij} - L, z_1, \ldots, z_{n-1}\| \geq \epsilon\Big\}|$$

It follows that $x_{ij} \to L([S_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|])$, that is $[N_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|] \subset [S_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|]$ and the inclusion is strict. To show this, we can establish an example as follows.

Example 4.6. Let v_{ij} be the form of (2.1) and $v = (v_{ij})$ is defined as follows:

$$v_{ij} = \begin{pmatrix} 1 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \cdots \\ 2 & 2 & 3 & \cdots & [h_{r,s}] & 0 & 0 & \cdots \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 2 & [\sqrt[3]{h_{r,s}}] & [\sqrt[3]{h_{r,s}}] & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

It is clear that $x = (x_{ij})$ is an unbounded double sequence and for $\epsilon > 0$ and for every $z_1, \ldots, z_{n-1} \in X$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\Big\{(i,j) \in I_{r,s} : \|v_{ij} - L, z_1, \ldots, z_{n-1}\| \geq \epsilon\Big\}|$$

$$= P - \lim_{r,s} \frac{1}{h_{r,s}} \frac{[\sqrt[3]{h_{r,s}}]}{h_{r,s}} = 0.$$

Therefore, $x_{ij} \to 0([S_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|]$. But

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \|v_{ij}, z_1, \ldots, z_{n-1}\|$$

$$= P - \lim_{r,s} \frac{[\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}] + 1))}{2h_{r,s}} = \frac{1}{2}.$$

Therefore $x_{ij} \nrightarrow 0[N_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|]$. This completes the proof.

□

Theorem 4.7. *Let $\theta_{r,s}$ be a double lacunary sequence. If $x = (x_{ij}) \in l^2_\infty$ and $x_{ij} \to L([S_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|])$, then $x_{ij} \to L([N_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|])$.*

Proof. Suppose that $x = (x_{ij}) \in l^2_\infty$, then there exists a positive integer K such that $\|v_{ij} - L, z_1, \dots, z_{n-1}\| < K$ for all $i, j \in N$. Therefore we have, for every $\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} \|v_{ij} - L, z_1, \dots, z_{n-1}\|$$

$$\begin{aligned} &= \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s} \& \|\frac{1}{2}(x_{i,j} + x_{i,j}) - L, z_1, \dots, z_{n-1}\| \geq \epsilon} \|v_{ij} - L, z_1, \dots, z_{n-1}\| \\ &\quad + \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s} \& \|\frac{1}{2}(x_{i,j} + x_{i,j}) - L, z_1, \dots, z_{n-1}\| < \epsilon} \|v_{ij} - L, z_1, \dots, z_{n-1}\| \\ &\leq \frac{K}{h_{r,s}} \Big| \Big\{ (i,j) \in I_{r,s} : \|v_{ij} - L, z_1, \dots, z_{n-1}\| \geq \epsilon \Big\} \Big| + \epsilon. \end{aligned}$$

Therefore, $x = (x_{ij}) \in l^2_\infty$ and $x_{ij} \to L([S_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|])$ implies $x_{ij} \to L([N_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|])$. □

Corollary 4.8. *Let $\theta_{r,s}$ be a double lacunary sequence, then*

$$[N_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|] \cap l^2_\infty = [S_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|] \cap l^2_\infty.$$

Proof. It follows directly from Theorem 4.4. and Theorem 4.6. □

Theorem 4.9. *For any sequence of Orlicz functions $\mathcal{M}$, $[N_{\theta_{r,s}}, Z, \mathcal{M}, \|\cdot, \dots, \cdot\|] \subset [S_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|])$.*

Proof. Let $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, \|\cdot, \dots, \cdot\|]$. Then for $\epsilon > 0$ and for every $z_1, \dots, z_{n-1} \in X$

$$\frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} M_{ij} \Big[\Big\| \frac{v_{ij} - L}{\rho}, z_1, \dots, z_{n-1} \Big\| \Big]$$

$$\begin{aligned} &\geq \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s} \& \|\frac{1}{2}(x_{i,j} + x_{i,j}) - L, z_1, \dots, z_{n-1}\| \geq \epsilon} M_{ij} \Big[\Big\| \frac{v_{ij} - L}{\rho}, z_1, \dots, z_{n-1} \Big\| \Big] \\ &> \frac{1}{h_{r,s}} M_{ij} \Big(\frac{\epsilon}{\rho} \Big) | \Big\{ (i,j) \in I_{r,s} : \|v_{ij} - L, z_1, \dots, z_{n-1}\| \geq \epsilon \Big\} |. \end{aligned}$$

This shows that $x = (x_{ij}) \in [S_{\theta_{r,s}}, Z, \|\cdot, \ldots, \cdot\|])$. □

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