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A remark on rational operator monotone functions

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Abstract. Recently, M. Nagisa presented the general form of rational operator monotone functions for any open interval in $\mathbb{R} = (-\infty, \infty)$, directly deduced from the property of Pick function. Using his results, we, in particular, determine the form of positive rational operator monotone functions on $(0, \infty)$.

1. Introduction

A (bounded linear) operator A acting on a Hilbert space H is said to be positive, denoted by $A \geq 0$, if $(Av, v) \geq 0$ for all $v \in H$. The definition of positivity induces the order $A \geq B$ for self-adjoint operators A and B on H. Let I be an open interval in $\mathbb{R}$. A real-valued continuous function f on I is *operator monotone,* if $A \leq B$ implies $0 \leq f(A) \leq f(B)$ for self-adjoint operators A and B with spectra in I. The set of such functions, that is, real continuous operator monotone functions defined on I are denoted by $OM(I)$. In particular, for $I = (0, \infty)$, we denote by $OM_+(0, \infty)$ the set of (strictly) positive functions f on I satisfying $f \in OM(I)$.

Recently, M. Nagisa presented the general form of rational operator monotone functions for any open interval I in $\mathbb{R}$ as follows ([5]):

Theorem N. *The following are equivalent.*
(1) $f \in OM(I)$ *is rational.*

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(2) There exist $b_0 \in \mathbb{R}$*, nonnegative numbers* $a_0, a_1, \ldots, a_n$ *and real numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ *such that*

$$f(x) = a_0 x + b_0 - \sum_{i=1}^{n} \frac{a_i}{x - \alpha_i}.$$

(3) There exist $a_0, c \geq 0, b_0 \in \mathbb{R},\ \alpha_1, \alpha_2, \ldots, \alpha_n \notin I$ *and* $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \mathbb{R}$ *satisfying that*

$$f(x) = a_0 x + b_0 - \frac{c(x-\beta_1)(x-\beta_2)\cdots(x-\beta_{n-1})}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}$$

and

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n.$$

For a general positive operator monotone function on $(0, \infty)$, it is well-known as Löwner's integral representation theorem ([2]) that

$$f(x) = \alpha x + \beta + \int_0^\infty \frac{x}{x+\lambda} d\mu(\lambda) \tag{1.1}$$

with nonnegative α, β and a positive measure μ on $(0, \infty)$. From the integral representation above, we see that $f(x)$ is approximated by $\alpha x + \beta + \Sigma_{\epsilon,E}(x)$, where

$$\Sigma_{\epsilon,E}(x) := \sum_{i=1}^{n} \frac{x}{x+\lambda_i} m_i \quad (\epsilon = \lambda_0 < \lambda_1 < \ldots < \lambda_n = E) \tag{1.2}$$

with $m_i = \mu((\lambda_{i-1}, \lambda_i])$, which is an approximate sum of $J_{\epsilon,E}(x) := \int_\epsilon^E \frac{x}{x+\lambda} \times d\mu(\lambda)$ for $0 < \epsilon < E < \infty$.

We assume that all operator monotone functions f are defined on $(0, \infty)$ and positive, and $f(0) = \lim_{x \to 0} f(x)$ if necessary.

In this paper, we will determine the form of a rational function in $OM_+(0, \infty)$, that is, positive rational operator monotone function on $(0, \infty)$, using Theorem N and Löwner's integral representation. We show that *the sum of a linear function and an approximate sum of the integral* is nothing but the form of such a function.

2. Main results

If $f \in OM_+(0,\infty)$ is rational, then, from (1.1) and (1.2), we can expect that f is represented as

$$f(x) = \alpha x + \beta + \sum_{i=1}^{n} \frac{m_i x}{x+\lambda_i}, \tag{2.1}$$

with some $\alpha, \beta \geq 0$, and $m_i \geq 0, \lambda_i > 0$ for $i = 1, 2, ..., n$. (At least one of coefficients $\alpha, \beta, m_1, ..., m_n$ is nonzero.)

Now we show the following, as expected above:

Proposition 2.1. *If $f \in OM_+(0,\infty)$ is rational, then f has the form of (2.1).*

Proof. We use (2) of Theorem N. Let $\alpha_i = -\lambda_i$ ($\lambda_i > 0$). Then we have

$$f(x) = a_0 x + b_0 - \sum_{i=1}^{n} \frac{a_i}{x+\lambda_i}.$$

Now, we assume that

$$\alpha x + \beta + \sum_{i=1}^{n} \frac{m_i x}{x+\lambda_i} = a_0 x + b_0 - \sum_{i=1}^{n} \frac{a_i}{x+\lambda_i}.$$

Then we want to determine α, β and m_i. By putting $x = 0$, we obtain $\beta = b_0 - \sum_{i=1}^{n} \frac{a_i}{\lambda_i} (= f(0) \geq 0)$, so that we have

$$\alpha x + b_0 - \sum_{i=1}^{n} \frac{a_i}{\lambda_i} + \sum_{i=1}^{n} \frac{m_i x}{x+\lambda_i} = a_0 x + b_0 - \sum_{i=1}^{n} \frac{a_i}{x+\lambda_i}.$$

Hence, from *the principal of identity,* we obtain $\alpha = a_0$ (≥ 0) and furthermore, since

$$\sum_{i=1}^{n} \frac{(m_i\lambda_i - a_i)x - a_i\lambda_i}{\lambda_i(x+\lambda_i)} = -\sum_{i=1}^{n} \frac{a_i}{x+\lambda_i},$$

we obtain $m_i = \frac{a_i}{\lambda_i}$ (≥ 0). Therefore we have

$$f(x) = a_0 x + \left(b_0 - \sum_{i=1}^{n} \frac{a_i}{\lambda_i}\right) + \sum_{i=1}^{n} \frac{(a_i/\lambda_i)x}{x+\lambda_i}.$$

This is the desired.

The next result is obtained directly from the general case given in Corollary 2.3 in [5]. Here, by an elementary argument, we show the fact for $f \in OM_+(0,\infty)$ as the particular case.

Proposition 2.2. *If $f \in OM_+(0,\infty)$ is rational and has an expression as (2.1), then $f(x)$ has one of the following forms:*

$$\begin{cases} \dfrac{\alpha(x+\mu_1)(x+\mu_2)\cdots(x+\mu_n)(x+\mu_{n+1})}{(x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n)} & \text{if } \alpha > 0, \\ \dfrac{(\beta+\Sigma_{i=1}^n m_i)(x+\mu_1)(x+\mu_2)\cdots(x+\mu_n)}{(x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n)} & \text{if } \alpha = 0, \end{cases}$$

where $\mu_1, \mu_2, ..., \mu_{n+1}$ satisfy

$$0 \le \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots < \mu_n < \lambda_n < \mu_{n+1}. \ (\beta = 0 \Longleftrightarrow \mu_1 = 0.)$$

Proof. We can put

$$f(x) = \frac{g(x)}{(x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n)},$$

where

$$\begin{aligned} g(x) = &(\alpha x+\beta)(x+\lambda_1)\cdots(x+\lambda_n) \\ &+\sum_{i=1}^n m_i x(x+\lambda_1)\cdots(x+\lambda_{i-1})(x+\lambda_{i+1})\cdots(x+\lambda_n). \end{aligned}$$

We may assume that $m_i > 0$ for all i.

If $\alpha > 0, \beta > 0$ and n is odd, then we have

$$\begin{aligned} &g(0) = \beta\lambda_1\cdots\lambda_n > 0, \\ &g(-\lambda_1) = m_1(-\lambda_1)(-\lambda_1+\lambda_2)\cdots(-\lambda_1+\lambda_n) < 0, \\ &g(-\lambda_2) = m_2(-\lambda_2)(-\lambda_2+\lambda_1)(-\lambda_2+\lambda_3)\cdots(-\lambda_2+\lambda_n) > 0, \\ &\qquad\vdots \\ &g(-\lambda_n) = m_n(-\lambda_n)(-\lambda_n+\lambda_1)(-\lambda_n+\lambda_2)\cdots(-\lambda_n+\lambda_{n-1}) < 0. \end{aligned}$$

Then we have real numbers $\mu_1, \mu_2, .., \mu_n$ such that

$$-\mu_{n+1} < -\lambda_n < -\mu_n < -\lambda_{n-1} < \cdots < -\lambda_2 < -\mu_2 < -\lambda_1 < -\mu_1 < 0.$$

Hence we see that:

$$g(x) = \alpha(x+\mu_1)(x+\mu_2)\cdots(x+\mu_{n+1}),$$

where

$$0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots < \mu_n < \lambda_n < \mu_{n+1}.$$

Further, if $\alpha > 0, \beta = 0$, then similarly we have the following:

$$0 = \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots < \mu_n < \lambda_n < \mu_{n+1}.$$

Again, further with a similar method as above, if $\alpha > 0, \beta > 0$ and n is even, then we have

$g(0) = \beta\lambda_1 \cdots \lambda_n > 0,$
$g(-\lambda_1) = m_1(-\lambda_1)(-\lambda_1 + \lambda_2) \cdots (-\lambda_1 + \lambda_n) < 0,$
$g(-\lambda_2) = m_2(-\lambda_2)(-\lambda_2 + \lambda_1)(-\lambda_2 + \lambda_3) \cdots (-\lambda_2 + \lambda_n) > 0,$
$\vdots$
$g(-\lambda_n) = m_n(-\lambda_n)(-\lambda_n + \lambda_1)(-\lambda_n + \lambda_2) \cdots (-\lambda_n + \lambda_{n-1}) > 0.$

Therefore, we have real numbers $\mu_1, \mu_2, ..., \mu_{n+1}$ such that

$$-\mu_{n+1} < -\lambda_n < -\mu_n < -\lambda_{n-1} < \cdots < -\lambda_2 < -\mu_2 < -\lambda_1 < -\mu_1 < 0.$$

Hence, we obtain the same result as before for $\alpha > 0, \beta > 0$ and odd n.

Furthermore, if $\alpha = 0$ and $\beta \geq 0$, then we see that:

$$g(x) = \left(\beta + \sum_{i=1}^{n} m_i\right)(x + \mu_1) \cdots (x + \mu_n),$$

where

$$0 \leq \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots < \mu_n < \lambda_n < \mu_{n+1}, \; (\beta = 0 \Longleftrightarrow \mu_1 = 0,)$$

which is desired.

Related to Proposition 2.2, we can obtain the following:

Corollary 2.3. *The operator monotone function $f(x)$ on $(0, \infty)$ given by (2.1) has one of the following forms:*

$$\begin{cases} \alpha\left(x + \dfrac{\mu_1 \cdots \mu_{n+1}}{\lambda_1 \cdots \lambda_n} + \displaystyle\sum_{i=1}^{n}\left(-\frac{1}{\lambda_i}\frac{A_{n+1}(-\lambda_i)}{\frac{d}{dx}B_n(x)|_{x=-\lambda_i}}\right)\frac{x}{x + \lambda_i}\right) \\ \hfill \text{if } \alpha > 0, \\ \left(\beta + \displaystyle\sum_{i=1}^{n} m_i\right)\left(\dfrac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} + \displaystyle\sum_{i=1}^{n}\left(-\frac{1}{\lambda_i}\frac{A_n(-\lambda_i)}{\frac{d}{dx}B_n(x)|_{x=-\lambda_i}}\right)\frac{x}{x + \lambda_i}\right) \\ \hfill \text{if } \alpha = 0, \end{cases}$$

where

$$0 \le \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots < \mu_n < \lambda_n < \mu_{n+1}, \ (\beta = 0 \Longleftrightarrow \mu_1 = 0,)$$

$$A_{n+1}(x) = (x+\mu_1)(x+\mu_2)\cdots(x+\mu_{n+1}),$$

$$A_n(x) = (x+\mu_1)(x+\mu_2)\cdots(x+\mu_n), \ B_n(x) = (x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n).$$

More simply, for (the first case) $\alpha > 0$, putting $\alpha = 1$, we have

$$\frac{(x+\mu_1)(x+\mu_2)\cdots(x+\mu_n)(x+\mu_{n+1})}{(x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n)} = x + \frac{\mu_1\cdots\mu_{n+1}}{\lambda_1\cdots\lambda_n} + \sum_{i=1}^{n}\left(-\frac{1}{\lambda_i}\frac{A_{n+1}(-\lambda_i)}{\frac{d}{dx}B_n(x)|_{x=-\lambda_i}}\right)\frac{x}{x+\lambda_i}$$

and for (the second case) $\alpha = 0$, putting $\beta + \sum_{i=1}^{n} m_i = 1$, we have

$$\frac{(x+\mu_1)(x+\mu_2)\cdots(x+\mu_n)}{(x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n)} = \frac{\mu_1\cdots\mu_n}{\lambda_1\cdots\lambda_n} + \sum_{i=1}^{n}\left(-\frac{1}{\lambda_i}\frac{A_n(-\lambda_i)}{\frac{d}{dx}B_n(x)|_{x=-\lambda_i}}\right)\frac{x}{x+\lambda_i}.$$

Examples of rational functions in $OM_+(0,\infty)$.

Let $n = 3$. For the first case, let $\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 6, \mu_1 = 1$, $\mu_2 = 3, \mu_3 = 5, \mu_4 = 7$. Then we have

$$\frac{(x+1)(x+3)(x+5)(x+7)}{(x+2)(x+4)(x+6)}\left(= x + \frac{35}{16} + \frac{\frac{15}{16}x}{x+2} + \frac{\frac{9}{16}x}{x+4} + \frac{\frac{5}{16}x}{x+6}\right).$$

For the second case, let $\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 6, \mu_1 = 1, \mu_2 = 3, \mu_3 = 5$. Then we have

$$\frac{(x+1)(x+3)(x+5)}{(x+2)(x+4)(x+6)}\left(= \frac{5}{16} + \frac{\frac{3}{16}x}{x+2} + \frac{\frac{3}{16}x}{x+4} + \frac{\frac{5}{16}x}{x+6}\right).$$

Remark. For a strictly positive function f on $(0,\infty)$ we can define $f^\circ(x) := xf(1/x)$ (transpose), $f^*(x) := 1/f(1/x)$ (adjoint) and $f^\perp(x) := x/f(x)$ (dual). Then the four functions f, f°, f^* and $f^\perp$ are equivalent to one another with respect to operator monotonicity ([4], [2]).

For the function in Proposition 2.2, if $\alpha, \beta > 0$, then the following three functions f°, f^* and $f^\perp$ are also operator monotone:

$$f^\circ(x) = \frac{\alpha\mu_1\mu_2\cdots\mu_{n+1}}{\lambda_1\lambda_2\cdots\lambda_n} \frac{\left(x+\frac{1}{\mu_1}\right)\left(x+\frac{1}{\mu_2}\right)\cdots\left(x+\frac{1}{\mu_{n+1}}\right)}{\left(x+\frac{1}{\lambda_1}\right)\left(x+\frac{1}{\lambda_2}\right)\cdots\left(x+\frac{1}{\lambda_n}\right)},$$

$$f^*(x) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{\alpha\mu_1\mu_2\cdots\mu_{n+1}} \frac{x\left(x+\frac{1}{\lambda_1}\right)\left(x+\frac{1}{\lambda_2}\right)\cdots\left(x+\frac{1}{\lambda_n}\right)}{\left(x+\frac{1}{\mu_1}\right)\left(x+\frac{1}{\mu_2}\right)\cdots\left(x+\frac{1}{\mu_{n+1}}\right)},$$

and

$$f^\perp(x) = \frac{x(x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n)}{\alpha(x+\mu_1)(x+\mu_2)\cdots(x+\mu_{n+1})}.$$

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