Toyama Math. J.
Vol. 37(2015), 189-202

Skew centers of rank-one generalized quantum groups

Punita Batra and Hiroyuki Yamane

Dedicated to Professor Jun Morita on the occasion of his 60th birthday

Abstract. In this paper, we study skew centers of rank-one generalized quantum groups.

1. Introduction

Let $\mathbb{K}$ be a field. Let $\mathbb{K}^\times := \mathbb{K}\setminus\{0\}$. Let $\mathfrak{A}$ be a finite-rank free $\mathbb{Z}$-module. Let $\chi : \mathfrak{A} \times \mathfrak{A} \to \mathbb{K}^\times$ be a bi-homomorphism, that is,

$$\chi(\lambda, \mu + \mu') = \chi(\lambda, \mu)\chi(\lambda, \mu') \quad \text{and} \quad \chi(\lambda + \lambda', \mu) = \chi(\lambda, \mu)\chi(\lambda', \mu) \qquad (1)$$

for all λ, λ', μ, $\mu' \in \mathfrak{A}$. Let $q_\lambda := \chi(\lambda, \lambda)$ for $\lambda \in \mathfrak{A}$. Let $\varpi : \mathfrak{A} \to \mathbb{K}^\times$ be a $\mathbb{Z}$-module homomorphism.

Let $\delta_{a,b}$ means the Kronecker's delta, i.e., $\delta_{a,a} := 1$, and $\delta_{a,b} := 0$ if $a \neq b$. For a, $b \in \mathbb{R}$, let $J_{a,b} := \{\, n \in \mathbb{Z} \,|\, a \leq n \leq b \,\}$, and $J_{a,\infty} := \{\, n \in \mathbb{Z} \,|\, a \leq n \,\}$. Let $\mathbb{Z}_{\geq 0} := J_{0,\infty}$. Note $\mathbb{N} = J_{1,\infty}$.

For $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{K}$, let $(n)_x := \sum_{r=1}^{n} x^{r-1}$, and $(n)_x! := \prod_{r=1}^{n} (r)_x$. For $x \in \mathbb{K}^\times$, define $\hat{o}(x) \in \mathbb{Z}_{\geq 0} \setminus \{1\}$ by

$$\hat{o}(x) := \begin{cases} \min\{\, r' \in J_{2,\infty} \,|\, (r')_x! = 0 \,\} \text{ if } (r'')_x! = 0 \text{ for some } r'' \in J_{2,\infty}, \\ 0 \quad \text{otherwise.} \end{cases} \qquad (2)$$

2000 *Mathematics Subject Classification.* Primary 17B10; Secondary 17B37.
Key words and phrases. Quantum groups, Weyl groupoids, Lie superalgebras.

For $x \in \mathbb{K}^\times$, define $\check{o}(x) \in J_{2,\infty} \cup \{\infty\}$ by

$$\check{o}(x) := \begin{cases} \hat{o}(x) & \text{if } \hat{o}(x) \geq 2, \\ \infty & \text{otherwise.} \end{cases}$$

For an associative $\mathbb{K}$-algebra $\mathfrak{a}$ and $X, Y \in \mathfrak{a}$, let $[X,Y] := XY - YX$.

$$\begin{gathered}\text{Throughout this paper, fix } \alpha \in \mathfrak{A} \setminus \{0\}, \text{ and let } \bar{q} := q_\alpha, \\ \hat{o} := \hat{o}(\bar{q}),\ \check{o} := \check{o}(\bar{q}), \text{ and } \bar{\varpi} := \varpi(\alpha).\end{gathered} \tag{3}$$

Let $\mathcal{U} = \mathcal{U}(\chi;\alpha)$ be the associative $\mathbb{K}$-algebra (with 1) defined by generators K_λ, L_μ (λ, $\mu \in \mathfrak{A}$), E and F and relations:

$$\begin{aligned}
&K_0 = L_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu}, \quad L_\lambda L_\mu = L_{\lambda+\mu}, \quad L_\lambda K_\mu = K_\mu L_\lambda, \\
&K_\lambda E = \chi(\lambda,\alpha) E K_\lambda, \quad K_\lambda F = \chi(\lambda,-\alpha) F K_\lambda, \\
&L_\mu E = \chi(-\alpha,\mu) E L_\mu, \quad L_\mu F = \chi(\alpha,\mu) F L_\mu, \\
&[E,F] = -K_\alpha + L_\alpha, \\
&E^{\hat{o}} = F^{\hat{o}} = 0 \quad \text{if } \hat{o} \geq 2.
\end{aligned} \tag{4}$$

We call $\mathcal{U}$ *the rank-one generalized quantum group*, see Remark 3.2 below for a higher rank one. We can easily see:

Lemma 1.1. *As a* $\mathbb{K}$*-linear space, the elements* $F^n K_\lambda L_\mu E^m$ (λ, $\mu \in \mathfrak{A}$, m, $n \in J_{0,\check{o}-1}$) *form a* $\mathbb{K}$*-basis of* $\mathcal{U}$*.*

Let $\mathcal{U}^0 = \mathcal{U}^0(\chi;\alpha)$ be the $\mathbb{K}$-subalgebra of $\mathcal{U}$ generated by K_λ, L_μ (λ, $\mu \in \mathfrak{A}$). Define the $\mathbb{K}$-subalgebra $\mathcal{U}_0$ of $\mathcal{U}$ by $\mathcal{U}_0 := \oplus_{k=0}^{\check{o}-1} F^k \mathcal{U}^0 E^k$. Let

$$\bar{\mathfrak{Z}}_{\bar{\varpi}} = \bar{\mathfrak{Z}}_{\bar{\varpi}}(\chi;\alpha) := \{\, C \in \mathcal{U}_0 \mid \bar{\varpi}^{-1} CE - EC = \bar{\varpi} CF - FC = 0 \,\}. \tag{5}$$

We call an element of $\bar{\mathfrak{Z}}_{\bar{\varpi}}$ an ϖ*-skew central element* of $\mathcal{U}$. We give an explicit $\mathbb{K}$-basis of $\bar{\mathfrak{Z}}_{\bar{\varpi}}$ in Theorem 2.4.

We have the $\mathbb{K}$-algebra isomorphism $\bar{T} = \bar{T}^{\chi;-\alpha} : \mathcal{U}(\chi;-\alpha) \to \mathcal{U}(\chi;\alpha)$ defined by

$$\bar{T}(K_\lambda L_\mu) := K_\lambda L_\mu \,(\lambda,\mu \in \mathfrak{A}),\ \bar{T}(E) := F L_{-\alpha},\ \bar{T}(F) := K_{-\alpha} E. \tag{6}$$

We call $\bar{T}$ *a Lusztig isomorphism.* We also study an influence of $\bar{T}$ for $\bar{\mathfrak{Z}}_{\bar{\varpi}}$, which will be needed for future studies for higher-rank cases.

This paper is inspired by [4, Proof of Lemma 3.1.3].

Remark 1.2. In this remark, we assume that the rank of $\mathfrak{A}$ is one and $\mathfrak{A}$ is generated by α, so $\mathfrak{A} = \mathbb{Z}\alpha$. Let $\mathcal{U}^{\dagger,0}$ be the subalgebra of $\mathcal{U}^0$ generated by $K_\alpha L_{-\alpha}$ and $K_{-\alpha}L_\alpha$, so $\mathcal{U}^{\dagger,0} = \oplus_{r=-\infty}^{\infty}\mathbb{K}K_{r\alpha}L_{-r\alpha}$. Let $\mathcal{U}^\dagger$ be the $\mathbb{K}$-linear subspace of $\mathcal{U}$ by $\oplus_{k=0}^{\hat{o}-1}\mathrm{Span}_{\mathbb{K}}(F^k K_{-k\alpha}\mathcal{U}^\dagger E^k)$. We can easily see

$$\bar{\mathfrak{Z}}_{\varpi} = \oplus_{r=-\infty}^{\infty}(\bar{\mathfrak{Z}}_{\varpi} \cap K_{r\alpha}\mathcal{U}^\dagger).$$

Let $\mathcal{Z}$ be the two-sided ideal generated by $L_\alpha K_\alpha - 1$. Let $\check{\mathcal{U}}$ be the quotient algebra $\mathcal{U}/\mathcal{Z}$. Let $\pi : \mathcal{U} \to \check{\mathcal{U}}$ be the canonical map. Assume that $\bar{q} \neq 1$, and that there exists $q \in \mathbb{K}^\times$ with $q^2 = \bar{q}$. Let $\check{K} := \pi(K_\alpha)$, $\check{E} := \frac{(-1)}{q-q^{-1}}\pi(E)$ and $\check{F} := \pi(F)$. If $\hat{o} = 0$, $\check{\mathcal{U}}$ is isomorphic to $U_{\bar{q}}\mathrm{sl}_2$. If $\hat{o} \geq 2$, the quotient algebra $\check{\mathcal{U}}^{\langle r\rangle} := \check{\mathcal{U}}/(\check{K}^{r\hat{o}} - 1)\check{\mathcal{U}}$ for some $r \in \mathbb{N}$ is usually called *a small quantum group of type* A_1; $\check{\mathcal{U}}^{\langle 2\rangle}$ was introduced by [2]; in this paper, we treat not $\check{\mathcal{U}}^{\langle r\rangle}$ but $\check{\mathcal{U}}$. Let $\check{\bar{\mathfrak{Z}}}_{\varpi} := \{\, \check{C} \in \pi(\mathcal{U}_0) \,|\, \varpi^{-1}\check{C}\check{E} - \check{E}\check{C} = \varpi\check{C}\check{F} - \check{F}\check{C} = 0 \,\}$. Then the $\mathbb{K}$-linear map $f : (\bar{\mathfrak{Z}}_{\varpi} \cap \mathcal{U}^\dagger) \oplus (\bar{\mathfrak{Z}}_{\varpi} \cap K_\alpha\mathcal{U}^\dagger) \to \check{\bar{\mathfrak{Z}}}_{\varpi}$ defined by $f(X) := \pi(X)$ is bijective. So, from main results for $\mathcal{U}$ in this paper, we can easily corresponding ones for $\check{\mathcal{U}}$ via f.

2. Skew graded centers for the rank one generalized quantum group

For $n \in \mathbb{Z}_{\geq 0}$, $m \in J_{0,n}$ and $x \in \mathbb{K}$, define $\binom{n}{m}_x \in \mathbb{K}$ by $\binom{n}{0}_x := \binom{n}{n}_x := 1$, and $\binom{n}{m}_x := \binom{n-1}{m}_x + x^{n-m}\binom{n-1}{m-1}_x = x^m\binom{n-1}{m}_x + \binom{n-1}{m-1}_x$ (if $m \in J_{1,n-1}$). If $(m)_x!(n-m)_x! \neq 0$, then $\binom{n}{m}_x = \frac{(n)_x!}{(m)_x!(n-m)_x!}$. For x, y, $z \in \mathbb{K}$, and $n \in \mathbb{N}$, we have $\prod_{t=0}^{n-1}(y + x^t z) = \sum_{m=0}^{n} x^{\frac{m(m-1)}{2}}\binom{n}{m}_x y^{n-m}z^m$.

For $n \in \mathbb{Z}_{\geq 0}$, and x, $y \in \mathbb{K}$, let $(n; x, y) := 1 - x^{n-1}y$ and $(n; x, y)! := \prod_{m=1}^{n}(m; x, y)$.

Let $\varpi^{\chi}_{\lambda,\mu;\beta} := \varpi(\beta) \cdot \frac{\chi(\beta,\mu)}{\chi(\lambda,\beta)}$ for β, λ, $\mu \in \mathfrak{A}$.

For $r \in \mathbb{N}$, we have

$$\begin{aligned} [E, F^r] &= (r)_{\bar{q}}F^{r-1}(-\bar{q}^{-(r-1)}K_\alpha + L_\alpha), \\ [E^r, F] &= (r)_{\bar{q}}(-\bar{q}^{-(r-1)}K_\alpha + L_\alpha)E^{r-1}. \end{aligned} \tag{7}$$

Define a $\mathbb{K}$-algebra automorphism $\bar{\Upsilon} : \mathcal{U}^0 \to \mathcal{U}^0$ by

$$\bar{\Upsilon}(K_\lambda L_\mu) := \frac{\chi(\alpha,\mu)}{\chi(\lambda,\alpha)}K_\lambda L_\mu.$$

Let $k \in J_{0,\check{o}-1}$. Let $C \in \mathcal{U}_0$ be such that

$$C = \sum_{n=0}^{k} F^n Z_n E^n, \tag{8}$$

where $Z_n \in \mathcal{U}^0$. Then

$$\begin{aligned}
&\bar{\varpi}^{-1}CE - EC \\
&\quad = (\bar{\varpi}^{-1}Z_0 - \bar{\Upsilon}(Z_0))E \\
&\qquad + \sum_{m=1}^{k}\Big(\bar{\varpi}^{-1}F^m(Z_m - \bar{\Upsilon}(Z_m))E^{m+1} \\
&\qquad -(m)_{\bar{q}}F^{m-1}((-\bar{q}^{-(m-1)}K_\alpha + L_\alpha)Z_m)E^m\Big) \\
&\quad = \Big(\sum_{m=0}^{k-1} F^m(\bar{\varpi}^{-1}Z_m - \bar{\Upsilon}(Z_m) - (m+1)_{\bar{q}}(-\bar{q}^{-m}K_\alpha + L_\alpha)Z_{m+1})E^{m+1}\Big) \\
&\qquad + F^k(\bar{\varpi}^{-1}Z_k - \bar{\Upsilon}(Z_k))E^{k+1}
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&\bar{\varpi}CF - FC \\
&\quad = F(\bar{\varpi}\bar{\Upsilon}(Z_0) - Z_0) \\
&\qquad + \sum_{m=0}^{k-1}\Big(F^{m+1}(\bar{\varpi}\bar{\Upsilon}(Z_m) - Z_m)E^m \\
&\qquad + \bar{\varpi}(m)_{\bar{q}}F^{m-1}Z_m(-\bar{q}^{-(m-1)}K_\alpha + L_\alpha)E^m\Big) \\
&\quad = \Big(\sum_{m=0}^{k-1} F^{m+1}(\bar{\varpi}\bar{\Upsilon}(Z_m) - Z_m + \bar{\varpi}(m+1)_{\bar{q}}Z_{m+1}(-\bar{q}^{-m}K_\alpha + L_\alpha)E^m\Big) \\
&\qquad + F^{k+1}(\bar{\varpi}\bar{\Upsilon}(Z_k) - Z_k)E^k.
\end{aligned} \tag{10}$$

By (9) and (10), we have

Lemma 2.1. *The following conditions are equivalent.*

(1) $\bar{\varpi}^{-1}CE - EC = 0$.

(2) $\bar{\varpi}CF - FC = 0$.

(3) $\bar{\varpi}^{-1}Z_m - \bar{\Upsilon}(Z_m) - (m+1)_{\bar{q}}(-\bar{q}^{-m}K_\alpha + L_\alpha)Z_{m+1} = 0$ *for all* $m \in J_{0,k-1}$, *and, if* $k < \hat{o} - 1$, $\bar{\varpi}^{-1}Z_k - \bar{\Upsilon}(Z_k) = 0$.

Let

$$\mathcal{U}^\dagger := \bigoplus_{m=0}^{\infty} \bigoplus_{p=-\infty}^{\infty} \mathbb{K} F^m K_{(p-m)\alpha} L_{-p\alpha} E^m,$$

see also Remark 1.2.

By (9), we have

$$\bar{\mathfrak{Z}}_{\bar{\varpi}} = \sum_{(\lambda,\mu)\in\mathfrak{A}^2} (\bar{\mathfrak{Z}}_{\bar{\varpi}} \cap K_\lambda L_\mu \mathcal{U}^\dagger).$$

Lemma 2.2. *Let* $(\lambda, \mu) \in \mathfrak{A}^2$*. Let*

$$C = \sum_{m=0}^{\check{o}-1} F^m \Big(\sum_{p=-\infty}^{\infty} a_{m,p} K_{\lambda+(p-m)\alpha} L_{\mu-p\alpha} \Big) E^m \in \bar{\mathfrak{Z}}_{\bar{\varpi}} \cap K_\lambda L_\mu \mathcal{U}^\dagger,$$

where $a_{m,p} \in \mathbb{K}$*. Assume that* $C \in \bar{\mathfrak{Z}}_{\bar{\varpi}} \setminus \{0\}$ *and that*

$$a_{m,p} = 0 \quad \textit{for all } (m,p) \in J_{0,\check{o}-1} \times \mathbb{Z} \textit{ with } \varpi^{\chi}_{\lambda,\mu;\alpha} \bar{q}^{m-2p} = 1. \tag{11}$$

(Note that $\varpi^{\chi}_{\lambda,\mu;\alpha} \bar{q}^{m-2p} = \varpi^{\chi}_{\lambda+(p-m)\alpha,\mu-p\alpha;\alpha}$*.) Then*

$$\hat{o} \geq 2, \ \prod_{t=0}^{\hat{o}-1} (\varpi^{\chi}_{\lambda,\mu;\alpha} - \bar{q}^t) \neq 0, \ \textit{and } \{p \in \mathbb{Z} | a_{\hat{o}-1,p} \neq 0\} \neq \emptyset.$$

Proof. For $m \in J_{0,\hat{o}-1}$, let $X_m := \{ p \in \mathbb{Z} \,|\, a_{m,p} \neq 0\}$, and

$$p_m := \begin{cases} \min X_m & \text{if } X_m \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Let $n := \mathrm{Max}\{m \in J_{0,\check{o}-1} | X_m \neq \emptyset\}$. It follows from Lemma 2.1 and (5) that

$$-\bar{q}^{-(m-1)} (m)_{\bar{q}} a_{m,p} + (m)_{\bar{q}} a_{m,p+1} = \bar{\varpi}^{-1} (1 - \varpi^{\chi}_{\lambda,\mu;\alpha} \bar{q}^{m-1-2p}) a_{m-1,p}$$
$$\text{for all } m \in J_{1,\check{o}-1} \text{ and all } p \in \mathbb{Z}. \tag{12}$$

By (12), letting $m \in \mathbb{Z}_{\geq 0}$ be such that $X_m \neq \emptyset$, we have $X_r \neq \emptyset$ for all $r \in J_{m,\check{o}-1}$. Hence $\hat{o} \geq 2$ and $n = \hat{o}-1$, so (11) implies $\varpi^{\chi}_{\lambda,\mu;\alpha} \bar{q}^{-1-2p_{\hat{o}-1}} \neq 1$. Moreover by (12), we see that for $r \in J_{1,\hat{o}-1}$, with $X_r \neq \emptyset$,

$$p_{r-1} = p_r - 1 \text{ and } \varpi^{\chi}_{\lambda,\mu;\alpha} \bar{q}^{r-1-2p_{r-1}} \neq 1.$$

This completes the proof. □

Let $(\lambda,\mu)\in\mathfrak{A}^2$. Let $k\in J_{1,\check{o}-1}$. Let

$$\begin{aligned}&Z_{\bar{\varpi}}(\lambda,\mu;k,m)\\&:=\tfrac{\bar{\varpi}^{-m}}{(m)_{\bar{q}}!}\sum_{n=0}^{k-m}\bar{q}^{-(m-1)n}\binom{m+n}{n}_{\bar{q}}(m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n})!K_{\lambda+n\alpha}L_{\mu-(m+n)\alpha}\end{aligned}\tag{13}$$

for $m\in J_{0,k}$, and let

$$\begin{aligned}C_{\bar{\varpi}}(\lambda,\mu;k)&=C^{\chi;\alpha}_{\bar{\varpi}}(\lambda,\mu;k):=\sum_{m=0}^{k}F^mZ_{\bar{\varpi}}(\lambda,\mu;k,m)E^m\\&=(\sum_{t=0}^{k}\bar{q}^tK_{\lambda+t\alpha}L_{\mu-t\alpha})\\&\quad+(\sum_{m=1}^{k-1}F^mZ_{\bar{\varpi}}(\lambda,\mu;k,m)E^m)\\&\quad+\tfrac{\bar{\varpi}^{-k}}{(k)_{\bar{q}}!}(k;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha})!F^kK_\lambda L_{\mu-k\alpha}E^k.\end{aligned}\tag{14}$$

We can directly see that for $m\in J_{0,k-1}$,

$$\begin{aligned}&\bar{\varpi}^{-1}Z_{\bar{\varpi}}(\lambda,\mu;k,m)-\bar{\Upsilon}(Z_{\bar{\varpi}}(\lambda,\mu;k,m))\\&\ -(m+1)_{\bar{q}}(-\bar{q}^{-m}K_\alpha+L_\alpha)Z_{\bar{\varpi}}(\lambda,\mu;k,m+1)\\&\ =\tfrac{\bar{\varpi}^{-m}}{(m)_{\bar{q}}!}\Big(\textstyle\sum_{n=0}^{k-m}\bar{q}^{-(m-1)n}\binom{m+n}{n}_{\bar{q}}(m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n})!\\&\qquad\cdot\bar{\varpi}^{-1}(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(m+2n)})K_{\lambda+n\alpha}L_{\mu-(m+n)\alpha}\Big)\\&\quad-\tfrac{\bar{\varpi}^{-(m+1)}}{(m)_{\bar{q}}!}\Big(\textstyle\sum_{n=0}^{k-m-1}\bar{q}^{-mn}\binom{m+1+n}{n}_{\bar{q}}(m+1;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n})!\\&\qquad\cdot(-\bar{q}^{-m}K_\alpha+L_\alpha)K_{\lambda+n\alpha}L_{\mu-(m+n+1)\alpha}\Big)\\&\ =\tfrac{\bar{\varpi}^{-(m+1)}}{(m)_{\bar{q}}!}\Big(((m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha})!(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-m})\\&\qquad-(m+1;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha})!)K_\lambda L_{\mu-m\alpha}\\&\quad+\Big(\textstyle\sum_{n=1}^{k-m-1}\Big(\\&\qquad\bar{q}^{-(m-1)n}\binom{m+n}{n}_{\bar{q}}(m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n})!(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(m+2n)})\\&\qquad+\bar{q}^{-m(n-1)}\binom{m+n}{n-1}_{\bar{q}}(m+1;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n+1})!\bar{q}^{-m}\\&\qquad-\bar{q}^{-mn}\binom{m+n+1}{n}_{\bar{q}}(m+1;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n})!\Big)K_{\lambda+n\alpha}L_{\mu-(m+n)\alpha}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\Big(\bar{q}^{-(m-1)(k-m)}\tbinom{k}{k-m}_{\bar{q}}(m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(k-m)}) \\
&\qquad\cdot(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(2k-m)}) \\
&\quad+\bar{q}^{-m(k-m-1)}\tbinom{k}{k-m-1}_{\bar{q}}(m+1;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(k-m-1)})!\bar{q}^{-m}\Big) \\
&\qquad\cdot K_{\lambda+(k-m)\alpha}L_{\mu-k\alpha}\Big) \\
=&\ \tfrac{\bar{\varpi}^{-(m+1)}}{(m)_{\bar{q}}!}\Big(\Big(\textstyle\sum_{n=1}^{k-m-1}\bar{q}^{-mn}(m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n})!\Big(\\
&\quad\tbinom{m+n}{n}_{\bar{q}}(\bar{q}^{n}-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(m+n)}) \\
&\quad+\tbinom{m+n}{n-1}_{\bar{q}}(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-n+1})-\tbinom{m+n+1}{n}_{\bar{q}}(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(m+n)})\Big) \\
&\qquad\cdot K_{\lambda+n\alpha}L_{\mu-(m+n)\alpha}\Big) \\
&\quad+\bar{q}^{-m(k-m)}(m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(k-m)})!\Big(\tbinom{k}{k-m}_{\bar{q}}(\bar{q}^{k-m}-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-k}) \\
&\quad+\tbinom{k}{k-m-1}_{\bar{q}}(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(k-m-1)})\Big)K_{\lambda+(k-m)\alpha}L_{\mu-k\alpha}\Big) \\
=&\ \tfrac{\bar{\varpi}^{-(m+1)}}{(m)_{\bar{q}}!}\bar{q}^{-m(k-m)}(m;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-(k-m)})!\tbinom{k+1}{k-m}_{\bar{q}}(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-k}) \\
&\qquad\cdot K_{\lambda+(k-m)\alpha}L_{\mu-k\alpha} \\
=&\ \tfrac{\bar{\varpi}^{-(m+1)}\bar{q}^{-m(k-m)}}{(m)_{\bar{q}}!}\tbinom{k+1}{m+1}_{\bar{q}}(m+1;\bar{q},\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-k})!K_{\lambda+(k-m)\alpha}L_{\mu-k\alpha},
\end{aligned}
$$

and that

$$
\begin{aligned}
&\bar{\varpi}^{-1}Z_{\bar{\varpi}}(\lambda,\mu;k,k)-\bar{\Upsilon}(Z_{\bar{\varpi}}(\lambda,\mu;k,k)) \\
&\quad=\tfrac{\bar{\varpi}^{-k}}{(k)_{\bar{q}}!}(k;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha})!\bar{\varpi}^{-1}(1-\varpi^{\chi}_{\lambda,\mu;\alpha}\bar{q}^{-k})K_{\lambda}L_{\mu-k\alpha} \\
&\quad=\tfrac{\bar{\varpi}^{-(k+1)}}{(k)_{\bar{q}}!}(k+1;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;\alpha})!K_{\lambda}L_{\mu-k\alpha}.
\end{aligned}
$$

Hence by Lemma 2.1,

$$
C^{\chi;\alpha}_{\bar{\varpi}}(\lambda,\mu;k)\in\bar{\mathfrak{Z}}_{\bar{\varpi}}\quad\text{if and only if}\quad(k+1)_{\bar{q}}\cdot(\varpi^{\chi}_{\lambda,\mu;\alpha}-\bar{q}^{k})=0. \tag{15}
$$

Note

$$
C^{\chi;\alpha}_{1}(0,\alpha;1)=\bar{q}K_{\alpha}+L_{\alpha}+(1-\bar{q})FE\in\bar{\mathfrak{Z}}_{1}.
$$

For $t\in\mathbb{Z}$, let $\mathcal{H}_{\bar{\varpi},t}:=\{\,(\lambda,\mu)\in\mathfrak{A}^2\,|\,\varpi^{\chi}_{\lambda,\mu;\alpha}=\bar{q}^{t}\,\}$. Let $\mathcal{H}_{\bar{\varpi}}:=\cup_{t\in\mathbb{Z}}\mathcal{H}_{\bar{\varpi},t}$. Let

$$
\begin{aligned}
\bar{\mathfrak{Z}}'_{\bar{\varpi}}=\bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha)&:=\bar{\mathfrak{Z}}_{\bar{\varpi}}\cap(\oplus_{m=0}^{\check{o}-1}\oplus_{(\lambda,\mu)\in\mathcal{H}_{\bar{\varpi}}}\mathbb{K}F^{m}K_{\lambda}L_{\mu}E^{m}), \\
\bar{\mathfrak{Z}}''_{\bar{\varpi}}=\bar{\mathfrak{Z}}''_{\bar{\varpi}}(\chi;\alpha)&:=\bar{\mathfrak{Z}}_{\bar{\varpi}}\cap(\oplus_{m=0}^{\check{o}-1}\oplus_{(\lambda,\mu)\in\mathfrak{A}^2\setminus\mathcal{H}_{\bar{\varpi}}}\mathbb{K}F^{m}K_{\lambda}L_{\mu}E^{m}).
\end{aligned}
$$

Then, as a $\mathbb{K}$-linear space, we have

$$
\bar{\mathfrak{Z}}_{\bar{\varpi}}(\chi;\alpha)=\bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha)\oplus\bar{\mathfrak{Z}}''_{\bar{\varpi}}(\chi;\alpha). \tag{16}
$$

Note

$$
\mathcal{U}^0\cap\bar{\mathfrak{Z}}_{\bar{\varpi}}(\chi;\alpha)=\mathcal{U}^0\cap\bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha)=\oplus_{(\lambda,\mu)\in\mathcal{H}_{\bar{\varpi},0}}\mathbb{K}K_{\lambda}L_{\mu}.
$$

Lemma 2.3. *Assume* $\bar{q} = 1$. *Then* $\bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha) \subset \mathcal{U}^0$.

Proof. Let $C = \sum_{m=0}^{k} F^m Z_m E^m$ be as in (8). Assume $C \in \bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha)\setminus\{0\}$. Note that $\bar{\varpi}^{-1} Z_m = \bar{\Upsilon}(Z_m)$ for all $m \in J_{0,k}$. By Lemma 2.1 (3), $Z_m = 0$ for all $m \in J_{1,k}$. Hence $C \in \mathcal{U}^0$, as desired. $\square$

Theorem 2.4. *As* $\mathbb{K}$*-linear spaces, we have*

$$\bar{\mathfrak{Z}}''_{\bar{\varpi}}(\chi;\alpha) = \begin{cases} \{0\} & \text{if } \hat{o} = 0, \\ \oplus_{(\lambda,\mu)\in\mathfrak{A}^2\setminus\mathcal{H}_{\bar{\varpi}}} \mathbb{K} C^{\chi;\alpha}_{\bar{\varpi}}(\lambda,\mu;\hat{o}-1) & \text{if } \hat{o} \geq 2, \end{cases} \tag{17}$$

and

$$\bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha) = \begin{cases} \oplus_{m=0}^{\infty} \oplus_{(\lambda,\mu)\in\mathcal{H}_{\bar{\varpi},0}} \mathbb{K} K_\lambda L_\mu C^{\chi;\alpha}_1(0,m\alpha;m) & \text{if } \bar{q} \neq 1 \text{ and } \hat{o} = 0, \\ \oplus_{m=0}^{\hat{o}-1} \oplus_{(\lambda,\mu)\in\mathcal{H}_{\bar{\varpi},0}} \mathbb{K} K_\lambda L_\mu C^{\chi;\alpha}_1(0,m\alpha;m) & \text{if } \bar{q} \neq 1 \text{ and } \hat{o} \geq 2, \\ \oplus_{(\lambda,\mu)\in\mathcal{H}_{\bar{\varpi},0}} \mathbb{K} K_\lambda L_\mu & \text{if } \bar{q} = 1. \end{cases} \tag{18}$$

In particular, if $\bar{q} \neq 1$,

$$\bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha) = \bigoplus_{(\lambda,\mu)\in\mathcal{H}_{\bar{\varpi},0}} \bigoplus_{m=0}^{\check{o}-1} \mathbb{K} K_\lambda L_\mu C^{\chi;\alpha}_1(0,\alpha;1)^m. \tag{19}$$

Proof. This can be easily proved by using Lemmas 2.2 and 2.3 and (15) and paying attention to the coefficients of the highest terms of $C(\lambda,\mu;m)$'s. $\square$

Define the $\mathbb{K}$-linear map $\bar{\mathfrak{Sh}} = \bar{\mathfrak{Sh}}^{\chi;\alpha} : \mathcal{U} \to \mathcal{U}^0$ by

$$\bar{\mathfrak{Sh}}(F^n K_\lambda L_\mu E^m) := \delta_{n,0}\delta_{m,0} K_\lambda L_\mu \quad (\lambda,\, \mu \in \mathfrak{A},\, m,\, n \in \mathbb{Z}_{\geq 0}).$$

Define the $\mathbb{K}$-algebra homomorphism $\bar{\mathfrak{HC}}_{\bar{\varpi}} = \bar{\mathfrak{HC}}^{\chi;\alpha}_{\bar{\varpi}} : \bar{\mathfrak{Z}}_{\bar{\varpi}} \to \mathcal{U}^0$ by $\bar{\mathfrak{HC}}_{\bar{\varpi}} := \bar{\mathfrak{Sh}}_{|\bar{\mathfrak{Z}}_{\bar{\varpi}}}$.

Lemma 2.5. *For* $(\lambda,\mu) \in \mathfrak{A}^2$, *let*

$$X_{\lambda,\mu} := \bar{\mathfrak{HC}}^{\chi;\alpha}_{\bar{\varpi}}(\bar{\mathfrak{Z}}_{\bar{\varpi}} \cap K_\lambda L_\mu \mathcal{U}^\dagger).$$

(1) *Assume that* $\bar{q} \neq 1$ *and* $\hat{o} = 0$. *If* $\varpi^{\chi}_{\lambda,\mu;\alpha} = 1$, *then the elements*

$$K_\lambda L_\mu, \quad K_\lambda L_\mu((K_\alpha L_{-\alpha})^{-m} + \bar{q}^{2m}(K_\alpha L_{-\alpha})^m) \quad (m \in \mathbb{N}) \tag{20}$$

form a $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*. If* $\varpi^{\chi}_{\lambda,\mu;\alpha} = \bar{q}$*, then the elements*

$$K_\lambda L_\mu((K_\alpha L_{-\alpha})^{1-m} + \bar{q}^{2m-1}(K_\alpha L_{-\alpha})^m) \quad (m \in \mathbb{N}) \tag{21}$$

form a $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*. If* $\varpi^{\chi}_{\lambda,\mu;\alpha} \neq \bar{q}^t$ *for all* $t \in \mathbb{Z}$*, then* $X_{\lambda,\mu} = \{0\}$*.*

(2) *Assume that* $\bar{q} \neq 1$ *and* $\hat{o} \neq 0$*. If* $\hat{o} \in 2\mathbb{N}+1$ *and* $\varpi^{\chi}_{\lambda,\mu;\alpha} = 1$*, then the elements*

$$\begin{aligned}&K_\lambda L_\mu(K_\alpha L_{-\alpha})^{r\hat{o}},\\ &K_\lambda L_\mu(K_\alpha L_{-\alpha})^{r\hat{o}}((K_\alpha L_{-\alpha})^{-m} + \bar{q}^{2m}(K_\alpha L_{-\alpha})^m),\\ &K_\lambda L_\mu(K_\alpha L_{-\alpha})^{r\hat{o}}((K_\alpha L_{-\alpha})^{\frac{\hat{o}+1}{2}-m} + \bar{q}^{2m-1}(K_\alpha L_{-\alpha})^{\frac{\hat{o}-1}{2}+m})\\ &(r \in \mathbb{Z},\ m \in J_{1,\frac{\hat{o}-1}{2}})\end{aligned} \tag{22}$$

form a $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*. If* $\hat{o} \in 2\mathbb{N}$ *and* $\varpi^{\chi}_{\lambda,\mu;\alpha} = 1$*, then the elements*

$$\begin{aligned}&K_\lambda L_\mu(K_\alpha L_{-\alpha})^{\frac{r\hat{o}}{2}},\\ &K_\lambda L_\mu(K_\alpha L_{-\alpha})^{\frac{r\hat{o}}{2}}((K_\alpha L_{-\alpha})^{-m} + \bar{q}^{2m}(K_\alpha L_{-\alpha})^m)\\ &(r \in \mathbb{Z},\ m \in J_{1,\frac{\hat{o}}{2}-1})\end{aligned} \tag{23}$$

form a $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*. If* $\hat{o} \in 2\mathbb{N}$ *and* $\varpi^{\chi}_{\lambda,\mu;\alpha} = \bar{q}$*, then the elements*

$$\begin{aligned}&K_\lambda L_\mu(K_\alpha L_{-\alpha})^{\frac{r\hat{o}}{2}}((K_\alpha L_{-\alpha})^{-m} + \bar{q}^{2m+1}(K_\alpha L_{-\alpha})^{m+1})\\ &(r \in \mathbb{Z},\ m \in J_{0,\frac{\hat{o}}{2}-1})\end{aligned} \tag{24}$$

form a $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*. If* $\varpi^{\chi}_{\lambda,\mu;\alpha} \neq \bar{q}^t$ *for all* $t \in J_{0,\hat{o}-1}$*, then the elements*

$$(K_\alpha L_{-\alpha})^r \cdot \sum_{t=0}^{\hat{o}-1} \bar{q}^t K_{\lambda+t\alpha} L_{\mu-t\alpha} \quad (r \in \mathbb{Z}) \tag{25}$$

form a $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*.*

(3) *Assume that* $\bar{q} = 1$*. If* $\varpi^{\chi}_{\lambda,\mu;\alpha} = 1$*, then the elements* $K_\lambda L_\mu(K_\alpha L_{-\alpha})^r$ $(r \in \mathbb{Z})$ *form a* $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*. If* $\varpi^{\chi}_{\lambda,\mu;\alpha} \neq 1$ *and* $\hat{o} = 0$*, then* $X_{\lambda,\mu} = \{0\}$*. If* $\varpi^{\chi}_{\lambda,\mu;\alpha} \neq 1$ *and* $\hat{o} \neq 0$*, then the elements* $(K_\alpha L_{-\alpha})^r \cdot \sum_{t=0}^{\hat{o}-1} K_{\lambda+t\alpha} L_{\mu-t\alpha}$ $(r \in \mathbb{Z})$ *form a* $\mathbb{K}$*-basis of* $X_{\lambda,\mu}$*.*

Proof. We first note that

$$\sum_{t=0}^{m} \bar{q}^t K_{t\alpha} L_{(m-t)\alpha} = (\bar{q}^m K_{m\alpha} + L_{m\alpha}) + \bar{q}^m K_\alpha L_\alpha \cdot \sum_{t=0}^{m-2} \bar{q}^t K_{t\alpha} L_{(m-2-t)\alpha}$$

for $m \in J_{2,\infty}$. For $m := 2m'$ for some $m' \in \mathbb{N}$,

$$\bar{q}^m K_{m\alpha} + L_{m\alpha} = (K_\alpha L_\alpha)^{m'}((K_\alpha L_{-\alpha})^{-m'} + \bar{q}^m (K_\alpha L_{-\alpha})^{m'}).$$

For $m := 2m' - 1$ for some $m' \in \mathbb{N}$,

$$\bar{q}^m K_{m\alpha} + L_{m\alpha} = (K_\alpha L_\alpha)^{m'-1} L_\alpha((K_\alpha L_{-\alpha})^{1-m'} + \bar{q}^m (K_\alpha L_{-\alpha})^{m'}).$$

If $\hat{o} = 2k' - 1$ for some $k' \in \mathbb{N}$, then for $m := 2m' - 1$ with some $m' \in \mathbb{N}$,

$$\bar{q}^m K_{m\alpha} + L_{m\alpha} = (K_\alpha L_\alpha)^{m'-k'} L_\alpha^{\hat{o}}((K_\alpha L_{-\alpha})^{k'-m'} + \bar{q}^m (K_\alpha L_{-\alpha})^{k'+m'-1}).$$

Define $f \in \mathrm{Ch}(\mathcal{U}^0)$ by $f(K_{\lambda'} L_{\mu'}) := \frac{\chi(\alpha,\mu')}{\chi(\lambda',\alpha)}$ $((\lambda', \mu') \in \mathfrak{A}^2)$. Note that

$$f((K_\alpha L_\alpha)^x L_\alpha^y (K_\alpha L_{-\alpha})^z) = \bar{q}^{y-2z} \quad (x, y, z \in \mathbb{Z}).$$

Then, using f, we can easily have (20)-(24) by (14) and (18).

We can easily see (25) by (14) and (17).

The other statements can also be proved similarly. □

By Lemma 2.5, we can easily see

Lemma 2.6. *Let*

$$X := \sum_{(\lambda',\mu') \in \mathfrak{A}^2} a_{(\lambda',\mu')} K_{\lambda'} L_{\mu'} \in \mathcal{U}^0 \quad (a_{(\lambda',\mu')} \in \mathbb{K}).$$

Then $X \in \mathrm{Im}\bar{\mathfrak{H}}\mathfrak{C}_{\varpi}$ *if and only if for all* $(\lambda, \mu) \in \mathfrak{A}^2$ *with* $\varpi^\chi_{\lambda,\mu;\alpha} \neq 1$, *all the equations in* (i)-(iv) *below are satisfied.*

(i) *In case that* $\bar{q} \neq 1$, $\hat{o} = 0$ *and* $\varpi^\chi_{\lambda,\mu;\alpha} = \bar{q}^t$ *for some* $t \in \mathbb{Z} \setminus \{0\}$,

$$a_{(\lambda+t\alpha,\mu-t\alpha)} = \bar{q}^t \cdot a_{(\lambda,\mu)}.$$

(ii) *In case that* $\bar{q} \neq 1$, $\hat{o} \geq 2$ *and* $\varpi^\chi_{\lambda,\mu;\alpha} = \bar{q}^t$ *for some* $t \in J_{1,\hat{o}-1}$,

$$\sum_{x=-\infty}^{\infty} a_{(\lambda+(\hat{o}x+t)\alpha,\mu-(\hat{o}x+t)\alpha)} = \bar{q}^t \cdot \sum_{y=-\infty}^{\infty} a_{(\lambda+\hat{o}y\alpha,\mu-\hat{o}y\alpha)}.$$

(iii) *In case that* $\hat{o} = 0$ *and* $\varpi^\chi_{\lambda,\mu;\alpha} \neq \bar{q}^t$ *for all* $t \in \mathbb{Z}$, $a_{(\lambda,\mu)} = 0$.

(iv) *In case that* $\hat{o} \geq 2$ *and* $\varpi^\chi_{\lambda,\mu;\alpha} \neq \bar{q}^t$ *for all* $t \in J_{0,\hat{o}-1}$,

$$\sum_{x=-\infty}^{\infty} a_{(\lambda+(\hat{o}x+m)\alpha,\mu-(\hat{o}x+m)\alpha)} = \bar{q}^m \cdot \sum_{y=-\infty}^{\infty} a_{(\lambda+\hat{o}y\alpha,\mu-\hat{o}y\alpha)} \quad (m \in J_{1,\hat{o}-1}).$$

3. Influence of a Lusztig isomorphism

Recall $\bar{T}$ from (6). We can easily see

$$\bar{T}(\bar{\mathfrak{Z}}'_{\bar{\varpi}^{-1}}(\chi;-\alpha)) = \bar{\mathfrak{Z}}'_{\bar{\varpi}}(\chi;\alpha) \quad \text{and} \quad \bar{T}(\bar{\mathfrak{Z}}''_{\bar{\varpi}^{-1}}(\chi;-\alpha)) = \bar{\mathfrak{Z}}''_{\bar{\varpi}}(\chi;\alpha). \tag{26}$$

Define the $\mathbb{K}$-linear isomorphism $\bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}^{-1}} : \mathcal{U}^0(\chi;-\alpha) \to \mathcal{U}^0(\chi;\alpha)$ by

$$\bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(K_\lambda L_\mu) := (\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1} K_\lambda L_\mu \qquad (\lambda,\, \mu \in \mathfrak{A}).$$

Lemma 3.1. *We have*

$$(\bar{\mathfrak{H}\mathfrak{C}}^{\chi;\alpha}_{\bar{\varpi}} \circ \bar{T}^{\chi;-\alpha})(X) = (\bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}} \circ \bar{\mathfrak{H}\mathfrak{C}}^{\chi;-\alpha}_{\bar{\varpi}^{-1}})(X) \qquad (X \in \bar{\mathfrak{Z}}_{\bar{\varpi}^{-1}}(\chi;-\alpha)). \tag{27}$$

Proof. Note

$$\bar{T}^{\chi;-\alpha}(C^{\chi;-\alpha}_1(0,-\alpha;1)) = K_{-\alpha}L_{-\alpha}C^{\chi;\alpha}_1(0,\alpha;1). \tag{28}$$

For $(\lambda,\mu) \in \mathcal{H}_{\bar{\varpi},0}$ and $m \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}
&(\bar{\mathfrak{H}\mathfrak{C}}^{\chi;\alpha}_{\bar{\varpi}} \circ \bar{T}^{\chi;-\alpha})(K_\lambda L_\mu C^{\chi;-\alpha}_1(0,-\alpha;1)^m) \\
&\quad = \bar{\mathfrak{H}\mathfrak{C}}^{\chi;\alpha}_{\bar{\varpi}}(K_\lambda L_\mu (K_{-\alpha}L_{-\alpha}C^{\chi;\alpha}_1(0,\alpha;1))^m) \quad \text{(by (28))} \\
&\quad = K_\lambda L_\mu (K_{-\alpha}L_{-\alpha}(\bar{q}K_\alpha + L_\alpha))^m \\
&\quad = \textstyle\sum_{t=0}^m \binom{m}{t}_1 \bar{q}^{m-t} K_{\lambda-t\alpha}L_{\mu-(m-t)\alpha} \\
&\quad = \bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(\textstyle\sum_{t=0}^m \binom{m}{t}_1 \bar{q}^{m-t}(\varpi_{\lambda-t\alpha,\mu-(m-t)\alpha;\alpha})^{-(\hat{o}-1)} K_{\lambda-t\alpha}L_{\mu-(m-t)\alpha}) \\
&\quad = \bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(\textstyle\sum_{t=0}^m \binom{m}{t}_1 \bar{q}^{m-t} \cdot (\bar{q}^{-m+2t})^{-(\hat{o}-1)} \cdot K_{\lambda-t\alpha}L_{\mu-(m-t)\alpha}) \\
&\qquad\quad (\text{since } \varpi^{\chi}_{\lambda,\mu;\alpha} = 1) \\
&\quad = \bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(\textstyle\sum_{t=0}^m \binom{m}{t}_1 \bar{q}^{t} K_{\lambda-t\alpha}L_{\mu-(m-t)\alpha}) \\
&\quad = \bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(K_\lambda L_\mu (\bar{q}K_{-\alpha} + L_{-\alpha})^m) \\
&\quad = (\bar{\gamma}^{\chi;-\alpha}_{\bar{\varpi}^{-1}} \circ \bar{\mathfrak{H}\mathfrak{C}}^{\chi;-\alpha}_{\bar{\varpi}^{-1}})(K_\lambda L_\mu C^{\chi;-\alpha}_1(0,-\alpha;1)^m).
\end{aligned} \tag{29}$$

Using (19) and (29), we see that (27) for $X \in \bar{\mathfrak{Z}}'_{\bar{\varpi}^{-1}}(\chi;-\alpha)$ is true.

Assume $\hat{o} \geq 2$. Let $(\lambda, \mu) \in \mathfrak{A}^2 \setminus \mathcal{H}$. We need the following calculation.

$$
\begin{aligned}
&\bar{T}^{\chi;-\alpha}\Big(\tfrac{(\bar{\varpi}^{-1})^{-(\hat{o}-1)}}{(\hat{o}-1)_{\bar{q}}!}(\hat{o}-1;\bar{q}^{-1},\varpi^{\chi}_{\lambda,\mu;-\alpha})!F^{\hat{o}-1}K_{\lambda}L_{\mu-(\hat{o}-1)(-\alpha)}E^{\hat{o}-1}\Big) \\
&= \bar{T}^{\chi;-\alpha}\Big(\tfrac{\bar{\varpi}^{\hat{o}-1}}{(\hat{o}-1)_{\bar{q}}!}(\hat{o}-1;\bar{q}^{-1},(\varpi^{\chi}_{\lambda,\mu;\alpha})^{-1})!F^{\hat{o}-1}K_{\lambda}L_{\mu-(\hat{o}-1)(-\alpha)}E^{\hat{o}-1}\Big) \\
&= \tfrac{\bar{\varpi}^{\hat{o}-1}}{(\hat{o}-1)_{\bar{q}}!}\Big(\prod_{p=0}^{\hat{o}-2}(1-\bar{q}^{-p}(\varpi^{\chi}_{\lambda,\mu;\alpha})^{-1})\Big) \\
&\quad \cdot(K_{-\alpha}E)^{\hat{o}-1}K_{\lambda}L_{\mu-(\hat{o}-1)(-\alpha)}(FL_{-\alpha})^{\hat{o}-1} \\
&= \tfrac{\bar{\varpi}^{\hat{o}-1}}{(\hat{o}-1)_{\bar{q}}!}\Big(\prod_{p=0}^{\hat{o}-2}(1-\bar{q}^{-p}(\varpi^{\chi}_{\lambda,\mu;\alpha})^{-1})\Big)E^{\hat{o}-1}K_{\lambda-(\hat{o}-1)\alpha}L_{\mu}F^{\hat{o}-1} \\
&\equiv \tfrac{\bar{\varpi}^{\hat{o}-1}}{(\hat{o}-1)_{\bar{q}}!}(-1)^{\hat{o}-1}\bar{q}^{-\frac{(\hat{o}-1)(\hat{o}-2)}{2}}(\varpi^{\chi}_{\lambda,\mu;\alpha})^{-(\hat{o}-1)}(\hat{o}-1;\bar{q},\varpi^{\chi}_{\lambda,\mu;\alpha})! \\
&\quad \cdot\tfrac{\chi(\alpha,\mu)^{2(\hat{o}-1)}}{\chi(\lambda,\alpha)^{2(\hat{o}-1)}}\bar{q}^{2(\hat{o}-1)^2}F^{\hat{o}-1}K_{\lambda-(\hat{o}-1)\alpha}L_{\mu}E^{\hat{o}-1} \\
&\qquad (\mathrm{mod}\ \oplus_{m=0}^{\hat{o}-2}F^m\mathcal{U}^0E^m) \\
&= \tfrac{\bar{\varpi}^{-(\hat{o}-1)}}{(\hat{o}-1)_{\bar{q}}!}(-1)^{\hat{o}-1}\bar{q}^{1-\frac{\hat{o}(\hat{o}-1)}{2}}(\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}(\hat{o}-1;\bar{q},\varpi^{\chi}_{\lambda,\mu;\alpha})! \\
&\quad \cdot F^{\hat{o}-1}K_{\lambda-(\hat{o}-1)\alpha}L_{\mu}E^{\hat{o}-1} \\
&\qquad (\text{since } \bar{q}^{2(\hat{o}-1)^2-\frac{(\hat{o}-1)(\hat{o}-2)}{2}} = \bar{q}^{2-\frac{(\hat{o}-1)(\hat{o}-2)}{2}} = \bar{q}^{2+(\hat{o}-1)-\frac{\hat{o}(\hat{o}-1)}{2}} \\
&\qquad\qquad = \bar{q}^{1-\frac{\hat{o}(\hat{o}-1)}{2}}) \\
&= (\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}\bar{q}\tfrac{\bar{\varpi}^{-(\hat{o}-1)}}{(\hat{o}-1)_{\bar{q}}!}(\hat{o}-1;\bar{q},\varpi^{\chi}_{\lambda,\mu;\alpha})!F^{\hat{o}-1}K_{\lambda-(\hat{o}-1)\alpha}L_{\mu}E^{\hat{o}-1} \\
&\qquad (\text{since } \bar{q}^{\frac{\hat{o}(\hat{o}-1)}{2}} = (-1)^{\hat{o}-1}) \\
&= (\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}\bar{q}\tfrac{\bar{\varpi}^{-(\hat{o}-1)}}{(\hat{o}-1)_{\bar{q}}!}(\hat{o}-1;\bar{q},\bar{q}^2\varpi^{\chi}_{\lambda-(\hat{o}-1)\alpha,\mu+(\hat{o}-1)\alpha;\alpha})! \\
&\quad \cdot F^{\hat{o}-1}K_{\lambda-(\hat{o}-1)\alpha}L_{\mu}E^{\hat{o}-1} \\
&\qquad (\text{since } \bar{q}^2\varpi^{\chi}_{\lambda-(\hat{o}-1)\alpha,\mu+(\hat{o}-1)\alpha;\alpha} = \varpi^{\chi}_{\lambda,\mu;\alpha}) \\
&= (\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}\bar{q}\tfrac{\bar{\varpi}^{-(\hat{o}-1)}}{(\hat{o}-1)_{\bar{q}}!}(\hat{o}-1;\bar{q}^{-1},\varpi^{\chi}_{\lambda-(\hat{o}-1)\alpha,\mu+(\hat{o}-1)\alpha;\alpha})! \\
&\quad \cdot F^{\hat{o}-1}K_{\lambda-(\hat{o}-1)\alpha}L_{(\mu+(\hat{o}-1)\alpha)-(\hat{o}-1)\alpha}E^{\hat{o}-1}
\end{aligned} \tag{30}
$$

By (17) and (26), we have

$$
\begin{aligned}
&\bar{T}^{\chi;-\alpha}(C^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(\lambda,\mu;\hat{o}-1)) \\
&\quad = \bar{q}\cdot(\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}\cdot C^{\chi;\alpha}_{\bar{\varpi}}(\lambda-(\hat{o}-1)\alpha,\mu+(\hat{o}-1)\alpha;\hat{o}-1),
\end{aligned} \tag{31}
$$

since the top terms of LHS and RHS coincide by (30). By (31), we have

$$\begin{aligned}
&(\bar{\mathfrak{H}\mathfrak{C}}^{\chi;\alpha}_{\bar{\varpi}} \circ \bar{T}^{\chi;-\alpha})(C^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(\lambda,\mu;\hat{o}-1)) \\
&= \bar{q}\cdot(\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}\cdot\textstyle\sum_{t=0}^{\hat{o}-1}\bar{q}^{t}K_{(\lambda-(\hat{o}-1)\alpha)+t\alpha}L_{(\mu+(\hat{o}-1)\alpha)-t\alpha} \\
&\qquad \text{(by (31) and (14))} \\
&= (\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}\cdot\textstyle\sum_{t=0}^{\hat{o}-1}\bar{q}^{-(\hat{o}-1-t)}K_{\lambda+(\hat{o}-1-t)(-\alpha)}L_{\mu-(\hat{o}-1-t)(-\alpha)} \\
&= (\varpi^{\chi}_{\lambda,\mu;\alpha})^{\hat{o}-1}\cdot\textstyle\sum_{t=0}^{\hat{o}-1}\bar{q}^{-t}K_{\lambda+t(-\alpha)}L_{\mu-t(-\alpha)} \\
&= \textstyle\sum_{t=0}^{\hat{o}-1}\bar{q}^{t}\cdot(\varpi^{\chi}_{\lambda+t(-\alpha),\mu-t(-\alpha);\alpha})^{\hat{o}-1}K_{\lambda+t(-\alpha)}L_{\mu-t(-\alpha)} \\
&= (\bar{\gamma}^{\chi;-\alpha}\circ\bar{\mathfrak{H}\mathfrak{C}}^{\chi;-\alpha}_{\bar{\varpi}^{-1}})(C^{\chi;-\alpha}_{\bar{\varpi}^{-1}}(\lambda,\mu;\hat{o}-1)) \quad \text{(by (14))}.
\end{aligned}$$

Hence (27) for $X \in \bar{\mathfrak{Z}}''(\chi;-\alpha)$ is true. This completes the proof by (16). □

Remark 3.2. (*A generalized quantum group.*) Let I be a finite set. Assume that $\mathfrak{A}$ is a finite-rank free $\mathbb{Z}$-module with $|I| = \mathrm{rank}_{\mathbb{Z}}\mathfrak{A}$. Let $\{\,\alpha_i \mid i \in I\,\}$ be a (set) $\mathbb{Z}$-basis of $\mathfrak{A}$. In the same way as in the Lusztig's definition [3, 3.1.1] of the quantum groups, to the pair (χ,Π), we can associate a unique $\mathbb{K}$-algebra $U = U(\chi,\Pi)$ characterized by the following properties.

(i) As a $\mathbb{K}$-algebra, U has generators K_λ, L_λ ($\lambda \in \mathfrak{A}$), E_i, F_i ($i \in I$) satisfying the equations $K_0 = 1$, $K_\lambda L_\mu = L_\mu K_\lambda$, $K_\lambda K_\mu = K_{\lambda+\mu}$, $L_\lambda L_\mu = L_{\lambda+\mu}$, $K_\lambda E_i = \chi(\lambda,\alpha_i)E_iK_\lambda$, $K_\lambda F_i = \chi(\lambda,-\alpha_i)F_iK_\lambda$, $L_\lambda E_i = \chi(-\alpha_i,\lambda)E_iL_\lambda$, $L_\lambda F_i = \chi(\alpha_i,\lambda)F_iL_\lambda$, $E_iF_j - F_jE_i = \delta_{ij}(-K_{\alpha_i} + L_{\alpha_i})$. As a $\mathbb{K}$-linear space, the elements $K_\lambda L_\mu$ (λ, $\mu \in \mathfrak{A}$) are linearly independent.

(ii) Let U^0 be the subalgebra of U generated by K_λ, L_λ ($\lambda \in \mathfrak{A}$). Let U^+ (resp. U^-) be the subalgebra of U generated by E_i (resp. F_i) ($i \in I$) and 1. One has a $\mathbb{K}$-linear isomorphism $U^- \otimes_{\mathbb{K}} U^0 \otimes_{\mathbb{K}} U^+ \to U$ with $Y \otimes Z \otimes X \mapsto YZX$. One also has $\mathbb{K}$-linear subspaces U_λ ($\lambda \in \mathfrak{A}$) such that $U = \oplus_{\lambda\in\mathfrak{A}}U_\lambda$, $U_\lambda U_\mu \subset U_{\lambda+\mu}$, $U^0 \subset U_0$, and $E_i \in U_{\alpha_i}$, $F_i \in U_{-\alpha_i}$ ($i \in I$).

(iii) Let $U^{\geq}$ (resp. $U^{\leq}$) be the subalgebra of U generated by U^+ and K_λ's (resp. U^- and L_λ's). One has a Drinfeld bilinear map $\vartheta = \vartheta^{\chi,\Pi}$: $U^{\geq} \times U^{\leq} \to \mathbb{K}$ such that $\vartheta_{|U^+\times U^-} : U^+ \times U^- \to \mathbb{K}$ is non-degenerate, and $\vartheta(K_\lambda, L_\mu) = \chi(\lambda,\mu)$, $\vartheta(E_i, F_j) = \delta_{ij}$, $\vartheta(E_i, L_\mu) = \vartheta(K_\lambda, F_j) = 0$.

We call U *a generalized quantum group*, see also [1, (4.8)].

Acknowledgment. This research was in part supported by Japan's Grand-in-Aid for Scientific Research (C), 22540040. The authors thank for the referee for valuable comments.

References

[1] S. Azam, H. Yamane and M. Yousofzadeh, Classification of Finite-Dimensional Irreducible Representations of Generalized Quantum Groups via Weyl Groupoids, Publ. RIMS Kyoto Univ. 51 (2015), 59-130

[2] G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata, 35 (1990), 89-144.

[3] ———, *Introduction to quantum groups*, Birkhäuser Boston, Inc., Boston, MA, 1993.

[4] T. Tanisaki, Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras, Infinite Analysis (Proceedings of RIMS symposium, Kyoto, Japan, June-August 1991) ed, by A. Tsuchiya, T. Eguchi and M. Jimbo, World scientific Singapore, 1992, 941-961.

Punita Batra,
Harish-Chandra Research Institute,
Chhatnag Road, Jhunsi, Allahabad 211 019, India
e-mail: batra@hri.res.in

Hiroyuki Yamane,
Department of Mathematics,
Faculty of Science,
University of Toyama,
3190 Gofuku, Toyama-shi, Toyama 930-8555, Japan,
e-mail: hiroyuki@sci.u-toyama.ac.jp

(Received April 30, 2015)