On $\mathbb{A}^1$-FORMS

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ABSTRACT. We survey a structure of algebraic curves whose base field extension to an algebraic closure is an affine line.

Let $k$ be a field and let $\overline{k}$ be an algebraic closure of $k$. A commutative $k$-algebra $A$ (or an algebraic scheme $\text{Spec} A$) is called an $\mathbb{A}^1$-form if $A \otimes_k \overline{k}$ is $\overline{k}$-isomorphic to a polynomial ring $\overline{k}[x]$ in one variable. The purpose of this article is to survey a $k$-algebraic structure of $\mathbb{A}^1$-forms. In particular we give a necessary and sufficient condition for a given plane curve (i.e. $\text{Spec} A$ of a $k$-algebra $A$ generated by two elements over $k$) to be an $\mathbb{A}^1$-form. For the detail see [1].

An $\mathbb{A}^1$-form $A$ is said to be separable (resp. purely inseparable), if there exists a finite separable (resp. purely inseparable) algebraic extension $k'/k$ such that

$$A \otimes_k k' \cong k'[x].$$

It is well-known that any separable $\mathbb{A}^1$-form is trivial, i.e., $A \cong k[x]$. Let $A$ be an $\mathbb{A}^1$-form. Then $A$ is an affine $k$-domain and hence there exists a finite Galois extension $k'$ over $k$ such that

$$A \otimes_k k' \cong k'[x].$$

So we can find an intermediate field $k_1$ between $k'$ and $k$ such that $k'/k_1$ is separable and $k_1/k$ is purely inseparable. Note that

$$(A \otimes_k k_1) \otimes_{k_1} k' \cong A \otimes_k k' \cong k'[x].$$

So $A \otimes_k k_1$ is a separable $k_1$-form and we have

$$A \otimes_k k_1 \cong k_1[x].$$

This shows that every $\mathbb{A}^1$-form is purely inseparable. From now on $k$ is always assumed to be a field of characteristic $p > 0$.

Given an $\mathbb{A}^1$-form $A$ over $k$, as mentioned above there exists an integer $e \geq 0$ such that

$$A \otimes_k k^{p^{-e}} \cong k^{p^{-e}}[x],$$

where $k^{p^{-e}}$ denotes the subfield of $\overline{k}$ consists of all elements $a \in \overline{k}$ with $a^{p^e} \in k$. The smallest $s$ among such integers $e$ is called the height of $A$ and is denoted height $A = s$ (c.f.[2]). It should be noticed that any $\mathbb{A}^1$-form $A$ is a normal affine $k$-domain such that the differential module $\Omega_k(A)$ is a projective $A$-module of rank one. So $\Omega_k(A)$ is isomorphic to some ideal of $A$.

1. $\mathbb{A}^1$-FORMS OF $B_{\nu}$-TYPE

Let $k$ be a field of characteristic $p > 0$ and let $(e, \nu, \lambda)$ be a set of positive integers such that $\nu \lambda \equiv 1 \pmod{p^e}$. Now let us set

$$B_\nu = k[x^{p^e}, u(x)^\nu v(x), u(x)v(x)^\lambda]$$

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as a $k$-subalgebra of $k^{p^{-e}}[x]$ where $u(x), v(x)$ are elements of $k^{p^{-e}}[x]$ satisfying the differential equation

$$\nu u(x)v(x) + u(x)v(x)' = 1.$$  

(1.1)

**Lemma 1.1.** (1) Given an integer $0 \leq n \leq e$ we have

$$k^{p^{-e}}[B_v] = k^{p^{-e}}[x^{p^n}, u(x)^i v(x), u(x)v(x)^j] \subset k^{p^{-e}}[x]$$

for any non-negative integers $m, i, j$ such that

$$m \geq e - n, \ i \equiv \nu \pmod{p^{e-n}}, \ j \equiv \lambda \pmod{p^{e-n}}.$$  

In particular we have

$$k^{p^{-e}}[B_v] = k^{p^{-e}}[x].$$

(2) $B_v$ is a free $k[x^{p^e}]$-module of rank $p^e$ with a basis

$$\{u(x)^{\langle i \rangle}v(x)^{\langle j \rangle} | i = 0, 1, \ldots, p^e - 1\}$$

where $\langle j \rangle$ means the integer such that $0 \leq \langle j \rangle < p^e$ and $\langle j \rangle \equiv j \pmod{p^e}$.  

(3) If we set

$$I = (u(x)^\nu(x), u(x)v(x)^\lambda)$$

the ideal of $B_v$ generated by $u(x)^\nu u(x)$ and $u(x)v(x)^\lambda$, then we have

$$\Omega_k(B_v) \cong I$$

as $B_v$-modules.

**Theorem 1.2.** $B_v$ is an $A^1$-form of height $B_v \leq e$.

In Theorem 1.2 the isomorphism

$$B_v \otimes_k k^{p^{-e}} \cong k^{p^{-e}}[x]$$

is given by the canonical map

$$B_v \otimes_k k^{p^{-e}} \rightarrow k^{p^{-e}}[B_v] = k^{p^{-e}}[x].$$

**Definition 1.3.** An $A^1$-form $A$ which is $k$-isomorphic to some $B_v$ is called an $A^1$-form of $B_v$-type.

**Corollary 1.4.** The following two conditions are equivalent:

1. $A$ is an $A^1$-form such that the quotient field $K$ of $A$ is rational, i.e., $K \cong_k k(x)$.
2. $A$ is an $A^1$-form of $B_v$-type for $\nu = p^e - 1$ such that

$$A \cong_k k[x^{p^e}, x^\nu(ax + 1), x(ax + 1)^\nu]$$

for some $a \in k^{p^{-e}}$.

$k$-isomorphic to $S$ for some $p$-polynomial $f(x)$. In particular we have the following

**Corollary 2.1.** (1) For any integer $i \geq 0$ we have

$$k^{p^{-e}}[S] = k^{p^{-e}}[x].$$

(2) $S$ is a free $k[x^{p^e}]$-module of rank $p^e$ with a free basis

$$\{1, f(x), \ldots, f(x)^{p^e-1}\}$$

This is a very special case of Lemma 1.1 and the proof of this special case is easy.

**Corollary 2.2.** (1) $S$ is an $A^1$-form of $B_1$-type of height $S \leq e$.

(2) height $S = e$ if and only if
for any $b \in k^{p^e}$. 

In Corollary 2.2 (1) the isomorphism 

$$S \otimes_k k^{p^e} \cong k^{p^e}[x]$$

is given by the canonical map 

$$S \otimes_k k^{p^e} \to k^{p^e}[B] = k^{p^e}[x].$$

**Definition 2.3.** An $A^1$-form which is $k$-isomorphic to $S$ is called an $A^1$-form of $p$-polynomial type.

It should be noted that by Corollary 2.2 (2) the $A^1$-form $S$ is trivial if and only if there exists $b \in k^{p^e}$ such that $f(x + b) \in k[x]$.

**Lemma 2.4** (S.M.Bhatwadekar). Let $K[x,y]$ be a polynomial ring in two variables $x, y$ over a field $K$ of any characteristic and let $B$ be a $K$-algebra which is $K$-isomorphic to the residue $K[x,y]/(f(x,y))$ for some $f(x,y) \in K[x,y]$. Suppose $f(x,y) \notin K$ and $\Omega_K(B)$ is a projective $B$-module, then $\Omega_K(B)$ is a free $B$-module.

**Proof.** Consider the canonical exact sequence 

$$(f(x,y))/(f(x,y))^2 \to \Omega_K(K[x,y]) \otimes_K[y] B \to \Omega_K(B) \to 0$$

of projective $B$-modules where $(f(x,y))/(f(x,y))^2$ and $\Omega_K(K[x,y]) \otimes_K[y] B$ are free $B$-modules of rank one and rank two respectively. Note that the rank $\text{rk} \Omega_K(B)$ of the $B$-modules $\Omega_K(B)$ is less than or equal to two. If $\text{rk} \Omega_K(B) = 2$, then $\phi$ is the zero map and hence $\psi$ is an isomorphism, which implies that $\text{rk} \Omega_K(B)$ is also free. If $\text{rk} \Omega_K(B) = 1$, then $\phi$ is injective and hence the isomorphism 

$$\Omega_K(K[x,y]) \otimes_K[y] B \cong (f(x,y))/(f(x,y))^2 \oplus \Omega_K(B)$$

of $B$-modules follows from the exact sequence just above. Taking the exterior product on both sides of this isomorphism we easily verify that $\Omega_K(B) \cong B$, as desired. \qed

**Lemma 2.5.** If $A$ is an $A^1$-form with $\Omega_K(A) = A$, then there exists a $k$-derivation $d$ of $A$ such that $d(f) = 1$ for some $f \in A$.

**Proof.** We may assume that $A$ is a $k$-subalgebra of $\overline{k}[x]$ and the canonical map 

$$A \otimes_k \overline{k} \to \overline{k}[A]$$

is an isomorphism. If we write $\text{Der}_k(A)$ the $k$-derivation $A$-module of $A$, then by assumption $\text{Der}_k(A)$ is also a free $A$-module of rank one. On the other hand the isomorphism 

$$A \otimes_k \overline{k} \cong \overline{k}[x]$$

yields canonically an isomorphism 

$$\text{Der}_k(A) \otimes_k \overline{k} \cong \text{Der}_k(\overline{k}[x]).$$

(2.1) Since $\text{Der}_k(A)$ may be considered as a submodule of $\text{Der}_k(A) \otimes_k \overline{k}$ through the canonical injection

$$\text{Der}_k(A) \to \text{Der}_k(A) \otimes_k \overline{k},$$

so identifying $\text{Der}_k(A) \otimes_k \overline{k}$ with $\text{Der}_k(\overline{k}[x])$ under the isomorphism 2.1 we may regard $d$ as a $\overline{k}$-derivation of $\overline{k}[x]$. Furthermore it is easy to see that $d$ is also a free basis of $\text{Der}_k(\overline{k}[x])$, that is $d(x) = c$ for some $0 \neq c \in \overline{k}$. Replacing $x$ by $c^{-1}x$ we may assume $c = 1$ from the first. Note that 

$$d^{p-1} : A \to A$$

is well-defined as a $k[A^p]$-module homomorphism. Now by the canonical ring homomor-
phisms
\[ k[A^p] \cong k^{p^{-1}}[A] \cong A \otimes_k k^{p^{-1}} \]
we see that \( k[A^p] \) is an \( \mathbb{A}^1 \)-form over the field \( k^p \), so that \( k[A^p] \) is normal. Recall that the canonical map
\[ k[A^p] \otimes_k \overline{k} \longrightarrow A \otimes_k \overline{k} \]
is injective and
\[ \overline{k}[x^p] \cong \overline{k}[A^p], \ A \otimes_k \overline{k} \cong \overline{k}[x]. \]
This shows that the degree of the quotient field \( \text{q}t(A) \) of \( A \) over the quotient field \( \text{q}t(k[A^p]) \) is equal to the characteristic \( p \). So we have
\[ d^{p^{-1}}(A) \subset k[A^p] = A \cap \overline{k}[x^p]. \]
In other words \( d^{p^{-1}}(A) \) may be considered as a principal ideal of \( k[A^p] \) such that its extension \( \overline{k}d^{p^{-1}}(A) \) to \( \overline{k}[A](= \overline{k}[x]) \) is a unit ideal of \( \overline{k}[A] \). This implies that \( d^{p^{-1}}(A) \) is a unit ideal because the ring extension \( \overline{k}[A]/k[A^p] \) is integral. In particular we can choose an element
\[ f \in d^{p^{-2}}(A) \subset A \]
so that \( d(f) = 1 \), as required. \( \square \)

**Theorem 2.6.** Let \( k \) be a field of characteristic \( p > 0 \) and let \( A \) be a \( k \)-algebra. Then the following three conditions are equivalent:
(1) \( A \) is an \( \mathbb{A}^1 \)-form generated by two elements over \( k \), i.e., \( \text{Spec} \ A \) is a plane curve.
(2) \( A \) is an \( \mathbb{A}^1 \)-form such that \( \Omega_k(A) \) is a free \( A \) module.
(3) \( A \) is of \( p \)-polynomial type.

**Proof.** The fact (1) \( \Rightarrow \) (2) follows from Lemma 2.4. Now suppose \( A \) satisfies the condition (2). We may assume that \( A \) is a \( k \)-subalgebra of \( k^{p^{-1}}[x] \) such that \( k^{p^{-1}}[A] = k^{p^{-1}}[x] \) and the canonical surjection
\[ A \otimes_k k^{p^{-1}} \rightarrow k^{p^{-1}}[A] \]
is an isomorphism. Then by Lemma 2.5 there exists a \( p \)-polynomial \( f \in A \). We set \( k[x^p, f] = D \) for short. Notice that \( D \) is of \( p \)-polynomial type and therefore \( B_1 \)-type, which implies that \( D \) is an \( \mathbb{A}^1 \)-form by Theorem 1.2. So we have the canonical isomorphism
\[ D \otimes_k k^{p^{-1}} \cong k^{p^{-1}}[x] \cong A \otimes_k k^{p^{-1}}. \]
This shows that \( D = A \) because \( D \subset A \), which completes the proof of both implications (2) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (1). \( \square \)

**Corollary 2.7.** If \( A \) is an \( \mathbb{A}^1 \)-form such that the group \( \text{Aut}_k A \) of all \( k \)-automorphisms of \( A \) is infinite, then \( A \) is of \( p \)-polynomial type.

There exists a non-trivial \( k \)-forms of the additive group \( \mathbb{G}_a \) over \( k \). Such \( k \)-forms are completely described by P. Russell. Using Theorem 2.6 we obtain another proof of his result.

**Corollary 2.8** (P. Russell). The following two conditions are equivalent:
(1) \( A \) is a \( k \)-form of \( \mathbb{G}_a \).
(2) $A$ is an $A^1$-form of $p$-polynomial type such that

$$A \cong k[x^p, x + a_1x^p + a_2x^{p^2} + \cdots + a_nx^{p^n}] \quad (a_i \in k^{p^{-e}}).$$

**Corollary 2.9.** Let $k$ be a field of char $k = p > 0$ and let $A$ be an $A^1$-form. Then the following two conditions are equivalent:

1. $A \cong k[x]$.
2. $A \otimes_k A$ is UFD.

### 3. $A^1$-FORMS OF CHARACTERISTIC $p = 2$

The structure of $A^1$-forms of char $k = p = 2$ is quite different from those of char $k = p > 2$ and in this section we treat only the case of $p = 2$. First we show the case of height one.

**Theorem 3.1.** Let $k$ be a field of characteristic $p = 2$ and let $A$ be an $A^1$-form. If height $A = 1$, then $A$ is of $p$-polynomial type.

In general there exists a close connection between $A^1$-forms $A$ of char $k = p = 2$ and the $A^1$-forms of $B_3$-type.

Recall that $B_3$ is defined as a special case of $B_{\nu}$. That is

$$B_3 = k[x^p, u(x)^3 v(x), u(x)v(x)^{\lambda}] \quad (\subset k^{p^{-e}}[x]),$$

where

$$\lambda = \begin{cases} 
(p^e + 1)/3 & (e \equiv 1 \pmod{2}) \\
(p^{e+1} + 1)/3 & (e \equiv 0 \pmod{2})
\end{cases}$$

and $u(x), v(x)$ are elements of $k^{p^{-e}}[x]$ with

$$u(x)'v(x) + u(x)v(x)' = 1. \quad (3.1)$$

**Remark 3.2.** In case of $p = 2$, the following three conditions are equivalent:

1. $(u(x), v(x))$ is a solution of the differential equation (3.1).
2. The product $u(x)v(x)$ is a $p$-polynomial.
3. $u(x)$ and $v(x)$ are written as

$$u(x) = u_0(x^p) + u_1(x^p)x, \quad v(x) = v_0(x^p) + v_1(x^p)x,$$

where

$$u_0(x^p), u_1(x^p), v_0(x^p), v_1(x^p) \in k^{p^{-e}}[x^p]$$

with

$$u_0(x^p)v_1(x^p) + u_1(x^p)v_0(x^p) = 1.$$

The following theorem gives the structure of $A \otimes_k k^{p^{-1}}$ for an $A^1$-form $A$ in the case of char $k = p = 2$.

**Theorem 3.3.** Let $k$ be a field of characteristic $p = 2$. Then the following three conditions are equivalent:

1. $A$ is an $A^1$-form of height $A \leq e$.
2. $A$ is a $k$-algebra such that

$$A \otimes_k k^{p^{-1}} \cong B_3 \otimes_k k^{p^{-1}}.$$

3. $A \otimes_k k^{p^{-1}}$ is of $B_3$-type as an $A^1$-form over $k^{p^{-1}}$.

**Example 3.4.** Let $k_0$ be a perfect field of char $k_0 = p = 2$ and let $e$ be an integer with
If we set $k = k_0(t^p)$ as a subfield of the rational function field $k_0(t)$ in one variable $t$, then we have $k^{p^{-e}} = k_0(t)$. We consider the $A^1$-form

$$A = k[x^p, x^3(1 + tx), x(1 + tx)^3]$$

of $B_3$-type. In this case $u(x)$ and $v(x)$ are set as $u(x) = x$ and $v(x) = 1 + tx$. Note that $u(x)v(x) = x + tx^p$ is a $p$-polynomial. Then height $A = e$ and the ideal

$$I = (u(x)^3u(x), u(x)v(x)^3)$$

of $A$ is not principal and hence by Lemma 1.1 (3) and Theorem 2.6 we see that $A$ is not of $p$-polynomial type. In particular Spec $A$ is not a plane curve.

Now in order to describe the $k$-algebraic structure of an $A^1$-form of ch $k = p = 2$ itself we need to extend the notion of $B_3$-type as follows:

Let $k$ be a field of characteristic $p = 2$ and let $e$ be a positive integer. Let us define a $k$-subalgebra $C_3$ of a polynomial ring $k^{p^{-e}}[x]$ by

$$C_3 = k[x^p, u(x)^3(u(x)v(x) + g(x)^2), v(x)^{\lambda-1}(u(x)v(x) + g(x)^2)] \subset k^{p^{-e}}[x],$$

where $\lambda$ is a positive integer with

$$\lambda = \begin{cases} (p^e + 1)/3 & (e \equiv 1 \pmod{2}) \\ (p^{e+1} + 1)/3 & (e \equiv 0 \pmod{2}) \end{cases}$$

and $u(x), v(x), g(x)$ are the elements of $k^{p^{-e}}[x]$ such that the product $u(x)v(x)$ is a $p$-polynomial and $u(x)g(x)$ is contained in

$$k^{p^{-2}}[x^{p^{e-2}}, u(x)^3v(x), u(x)v(x)^{\lambda}].$$

**Remark 3.5.** In case of $e > 1$ if we set

$$R = k^{p^{-2}}[x^{p^{e-2}}, u(x)^3v(x), u(x)v(x)^{\lambda}],$$

then by Lemma 1.1(1) we have

$$R = k^{p^{-2}}[x^{p^{e-2}}, u(x)^3v(x), u(x)v(x)^r]$$

for any integer $r \geq 0$ with $3r \equiv 1 \pmod{p^{e-2}}$, and $g(x)$ just above is obtained as an element of the (rational) ideal

$$(u(x)^{p^{e-2}} - 1, u(x)^2v(x), v(x)^r)R$$

of $R$.

**Theorem 3.6.** (1) $C_3$ is an $A^1$-form.

(2) height $C_3 = e$ if and only if

$$f(x + b) \not\in k^{p^{-e}}[x]$$

for any $b \in k^{p^{-e}}$.

The isomorphism

$$C_3 \otimes k k^{p^{-e}} \cong k^{p^{-e}}[x]$$

is given explicitly as follows: We have

$$k^{p^{-1}}[C_3] = k^{p^{-1}}[B_3]$$

and the canonical map

$$C_3 \otimes k k^{p^{-1}} \to k^{p^{-1}}[C_3] = k^{p^{-1}}[B_3]$$
is an isomorphism. This shows that
\[ C_3 \otimes_k k^{p^{-1}} \cong B_3 \otimes_k k^{p^{-1}}. \]

**Definition 3.7.** If \( A \) is an \( A^1 \)-form of height \( A = e \) which is \( k \)-isomorphic to \( C_3 \), then \( A \) is called an \( A^1 \)-form of \( C_3 \)-type.

**Remark 3.8.** With respect to the \( k \)-algebra \( C_3 \) if we set \( g(x) = 0 \), then we have \( C_3 = B_3 \). Thus an \( A^1 \)-form of \( B_3 \)-type is always of \( C_3 \)-type.

**Theorem 3.9.** Let \( k \) be a field of characteristic \( p = 2 \). Then every \( A^1 \)-form is of \( C_3 \)-type.

### 4. \( A^1 \)-FORMS OF CHARACTERISTIC \( p > 2 \)

Let \( k \) be a field of characteristic \( p > 2 \). Recall that an \( A^1 \)-form of \( B_2 \)-type is \( k \)-isomorphic to
\[ B_1 = k[x^p, u(x)^2v(x), u(x)v(x)^\lambda] \quad (\lambda = (p^e + 1)/2) \]
where \( u(x), v(x) \) are elements of \( k^{p^{-1}}[x] \) satisfying the differential equation
\[ 2u(x)'v(x) + u(x)v(x)' = 1. \tag{4.1} \]

**Remark 4.1.** Every solution \( (u(x), v(x)) \) of (4.1) is given as follows:
Choose \( h(x) \in k^{p^{-e}}[x] \) such that \( \gcd(h(x)', h(x)''') = 1 \). Then \( h(x)' \) is separable and hence we have
\[ k^{p^{-e}}[x] \cong k^{p^{-e}}[x^p] \pmod{h(x)'k^{p^{-e}}[x]} . \]
So \( h(x) \) is of the form
\[ h(x) = g(x^p) + h(x)'w(x) \]
for some \( g(x^p) \in k^{p^{-e}}[x^p] \) and \( w(x) \in k^{p^{-e}}[x] \). Thus
\[ h(x)' = h(x)'''w(x) + h(x)'w(x)'. \]

From the assumption we have
\[ w(x) \in h(x)'k^{p^{-e}}[x]. \]
If we set \( u(x) = h(x)' \) and \( w(x) = u(x)v(x) \), then we can easily see that \( (u(x), v(x)) \) is a required solution. Conversely every solution is given through the process.

**Theorem 4.2.** Let \( k \) be a field of characteristic \( p > 2 \). Then every \( A^1 \)-form is of \( B_2 \)-type.

Let \( k[X, Y, Z] \) be a polynomial ring in three variables \( X, Y, Z \) and let
\[ \phi: k[X, Y, Z] \to B_1 \]
be a surjective \( k \)-homomorphism defined by
\[ \phi: (X, Y, Z) \to (x^p, u^2v, uv^\lambda). \]

Notice that \( u(x)^p, v(x)^p \in k[x^p] \). Therefore we can take \( U(X), V(X) \in k[X] \) so that \( \phi(U(X)) = u(x)^p \) and \( \phi(V(X)) = v(x)^p \). Note that \( U(X), V(X) \) also satisfy the condition
\[ 2U(X)'V(X) + U(X)V(X)' = 1. \]

If we set
\[ P = (Y^\lambda - U(X)Z, Z^2 - V(X)Y, Y^\lambda - U(X)V(X)) \quad (\lambda = (p^e + 1)/2) \tag{4.2} \]
as an ideal of \( k[X, Y, Z] \), then we have \( \ker \phi = P \) and hence

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P = (Y^\lambda - U(X)Z, Z^2 - V(X)Y, Y^\lambda - U(X)V(X)) (\lambda = (p^e + 1)/2)
\[ B_2 \cong k[X, Y, Z]/P. \]

Therefore as a corollary of Theorem 4.2 we have the following

**Corollary 4.3.** Let \( k \) be a field of characteristic \( p > 2 \) and let \( A \) be a \( k \)-algebra. Then the following two conditions are equivalent:

1. \( A \) is an \( A^1 \)-form of height \( A \leq e \).
2. There exists \( U(X), V(X) \in k[X] \) such that

\[
2U(X)V(X) + U(X)V(X)' = 1
\]

and

\[ A \cong k[X, Y, Z]/P \]

where \( P \) is the prime ideal defined as in 4.2.

It should be noticed that

\[ P = \text{rad}(Y^p - U(X)^2V(X), Z^p - U(X)V(X)^{\lambda}). \]

In particular \( P \) is a set theoretic complete intersection.

**Theorem 4.4.** Let \( P \) be as in Corollary 4.3. Then the following two conditions are equivalent:

1. \( P \) is an ideal theoretic complete intersection.
2. \( A \) is generated by two elements over \( k \).

Recall that the condition (2) is equivalent to saying that \( A \) is an \( A^1 \)-form of \( p \)-polynomial type by Theorem 2.6.

**Example 4.5.** For any \( \alpha_i(x^p) \in k^{p^{-e}}[x^p] \) \( (i = 1, ..., n) \), we set

\[
u(I) = x,
\]

\[
2v(I) = 1 + \alpha_1(x^p)x^{p-2} + \cdots + \alpha_n(x^p)x^{np-2}.
\]

Then \( (u(I), v(I)) \) is a solution of the differential equation (4.1) and hence the \( A^1 \)-form \( B_2 \) is well-defined. Furthermore if \( \alpha_i(x^p) \not\in k[x^p] \) for some \( i = 1, ..., n \), then \( A \) is not generated by two elements over \( k \). The prime ideal \( P \) corresponding to this \( B_2 \) is not a complete intersection.

5. \( A^1 \)-FORMS OF CHARACTERISTIC \( p = 3 \)

In this section we consider the isomorphism class of all \( A^1 \)-forms of height one when \( \text{ch} k = p = 3 \).

If \( k \) is a field of characteristic \( p = 3 \), then \( (u(x), v(x)) (u(x), v(x) \in k^{p^{-e}}[x]) \) is a solution of the differential equation (4.1) if and only if \( u(x) \) and \( v(x) \) are written as

\[
u(x) = \alpha(x^p) + \beta(x^p)x, \quad v(x) = \gamma(x^p) + \delta(x^p)x
\]

for

\[
\begin{pmatrix}
\alpha(x^p) & \beta(x^p) \\
\gamma(x^p) & \delta(x^p)
\end{pmatrix} \in SL_2(k^{p^{-e}}[x^p]).
\]

In this way a matrix of \( SL_2(k^{p^{-e}}[x^p]) \) determines a solution \( (u(x), v(x)) \), in other words there exists a map from \( SL_2(k^{p^{-e}}[x^p]) \) onto the isomorphism classes \( B_2 \) of all \( k \)-algebra \( B_2 \), which will be denoted

\[ \Phi : SL_2(k^{p^{-e}}[x^p]) \to B_2. \]

Note that by Theorem 4.2 \( B_2 \) may be considered as isomorphism classes of all \( A^1 \)-forms.
A of height $A \leq e$.

The equivalence relation in $SL_2(k^{3\varepsilon}[x^p])$ induced from the map $\Phi$ will be denoted by $\sim$.

When $e = 1$, this equivalence relation $\sim$ is given explicitly as follows: Let $\sigma$ be a $k^{p^{-1}}$-automorphism of $k^{p^{-1}}[x]$. Then we have

$$\sigma(x) = ax + b \quad (a, b \in k^{p^{-1}}, \ a \neq 0).$$

Let us set the $2 \times 2$ matrix $[\sigma]$ by

$$[\sigma] = \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \in GL_2(k^{p^{-1}}) \subset GL_2(k^{p^{-1}}[x^p]).$$

Given a matrix

$$M = \begin{pmatrix} \alpha(x^p) & \beta(x^p) \\ \gamma(x^p) & \delta(x^p) \end{pmatrix} \in SL_2(k^{p^{-\varepsilon}}[x^p]),$$

the action $\sigma M$ by $\sigma$ on $M$ is defined as

$$\sigma M = \begin{pmatrix} \sigma(\alpha(x^p)) & \sigma(\beta(x^p)) \\ \sigma(\gamma(x^p)) & \sigma(\delta(x^p)) \end{pmatrix} \in GL_2(k^{p^{-1}}[x^p]).$$

**Theorem 5.1.** Let $k$ be a field of characteristic $p = 3$ and let $M, N \in GL_2(k^{p^{-1}}[x^p])$. Then the following two conditions are equivalent:

1. $M \sim N$.
2. $M = \alpha L^\sigma N[\sigma]$ for some $L \in GL_2(k[x^p])$ and some $a, b \in k^{p^{-1}}, a \neq 0$, where $\sigma$ is a $k^{p^{-1}}$-automorphism of $k^{p^{-1}}[x]$ defined by $\sigma(x) = ax + b$.

As an immediate consequence of this theorem, for example, we can see the cardinal number of the set of $B_2$ is equal to or larger than that of $k$ in case $k$ is not perfect. In particular the cardinal number of the set of $k$-isomorphism classes of all $A^1$-forms of height one is equal to that of $k$.

**References**

[2] T. Kambayashi and M. Miyanishi, On forms of the affine line over a field, Lectures in Mathematics Kyoto University 1977