539.377.1

Thermal Stresses in a Plate Containing Two Circular Holes of Different Sizes under Uniform Heat Flow*

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This note deals with the thermal stresses in a plate containing two insulated circular holes of different sizes under uniform heat flow in an arbitrary direction. Bipolar coordinates are used, and the temperature distribution and the stress functions which are adjusted so as to satisfy the boundary conditions at the edge of one hole, are expanded into Fourier's series at the edge of another hole. The parametric coefficients involved in them are determined from the given boundary conditions at the edge of the hole and at infinity. Expression of the thermal stress along the edge of the hole is derived and the values of it are calculated. Some limiting cases are also discussed.

1. Introduction

Recently the thermal conditions for heat engine, aircraft and chemical machine have been severe in consequence of an increased performance of them and the thermal stresses due to large temperature differences are becoming very important. Concerning the thermal stress analysis, many theoretical investigations may be mentioned, and especially with respect to the problems on the state of two-dimensional steady heat flow, the solution for an infinite plate with a circular hole was given by Florence and Goodier(1) and for a hole of general shape it was given by Isida(2), Deresiewicz(3) or Takeuti and Noda(4). Goodier and Florence also analysed the case of a semi-infinite plate(5) with a circular hole using bipolar coordinates. Muramatsu and Atsumi treated the thermal stresses in a plate with three circular holes(6) or an infinite row of equal holes(7) using the method of successive approximation. These results show that the thermal stresses in a plate with many circular holes are greater than those in one with one circular hole.

In the present report, the thermal stresses in an infinite plate containing two insulated circular holes of different sizes under uniform heat flow in an arbitrary direction are solved. The method of solution consists in using bipolar coordinates and the expression of the thermal stress along the edge of hole is given in a form of Fourier's series.

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2. Temperature field

The temperature distribution in an infinite plate with one circular hole (hole II) of radius R_2 (see Fig. 1) under a steady heat flow with a constant temperature gradient τ in a direction at an angle ϕ to the x-axis at infinity is given in the form

$$T_0=\tau\cos\phi\left(1+\frac{R_2^2}{r_2^2}\right)r_2\cos\theta_2 + \tau\sin\phi\left(1+\frac{R_2^2}{r_2^2}\right)r_2\sin\theta_2 \quad \cdots\cdots(1)$$

For this problem we use the bipolar coordinates defined by

$$\left.\begin{aligned} &z=a\coth\frac{\bar{w}}{2},\quad z\equiv x+iy,\quad \bar{w}\equiv\alpha-i\beta,\\ &ah=\cosh\alpha-\cos\beta,\quad h(x+iy)=\sinh\alpha+i\sin\beta \end{aligned}\right\} \quad \cdots\cdots(2)$$

Let the boundaries of the circular holes I and II be defined respectively by the coordinate curves $\alpha=b>0$ and $\alpha=-c$ $(c>0)$. Since $R_2=a$ cosech c and $s_2=a$ coth c, we have

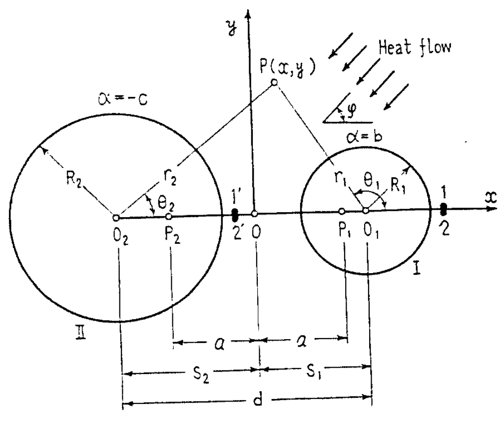

Fig. 1

$$\frac{r_2^2}{R_2^2}=\frac{\cosh(\alpha+2c)-\cos\beta}{\cosh\alpha-\cos\beta},\quad r_2\cos\theta_2=a\frac{\cosh(\alpha+c)-\cosh c\cos\beta}{\sinh c\,(\cosh\alpha-\cos\beta)},\quad r_2\sin\theta_2=a\frac{\sin\beta}{\cosh\alpha-\cos\beta} \quad\cdots(3)$$

The temperature distribution T_0 is expressed as follows in bipolar coordinates

$$T_0=a\tau\cos\phi\left\{\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}-\frac{\sinh(\alpha+2c)}{\cosh(\alpha+2c)-\cos\beta}\right\}+a\tau\sin\phi\left\{\frac{\sin\beta}{\cosh\alpha-\cos\beta}+\frac{\sin\beta}{\cosh(\alpha+2c)-\cos\beta}\right\} \quad\cdots(4)$$

Here the constant terms which produce no thermal dislocation are omitted. With the aid of the formulas

$$\left.\begin{aligned}\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}&=1+2\sum_{n=1}^{\infty}e^{-n\alpha}\cos n\beta \qquad (\alpha>0,\ -\pi\leqq\beta\leqq\pi)\\ \frac{\sin\beta}{\cosh\alpha-\cos\beta}&=2\sum_{n=1}^{\infty}e^{-n\alpha}\sin n\beta \qquad (\alpha>0,\ -\pi\leqq\beta\leqq\pi)\end{aligned}\right\} \quad\cdots(5)$$

Equation (4) is transformed into the following Fourier's series for $\alpha>0$

$$\frac{T_0}{4a}=\tau\cos\phi\sum_{n=1}^{\infty}e^{-n\xi}\sinh nc\cos n\beta+\tau\sin\phi\sum_{n=1}^{\infty}e^{-n\xi}\cosh nc\sin n\beta,\quad (\alpha>0),\quad \xi\equiv\alpha+c \quad\cdots(6)$$

As the auxiliary temperature distribution T_1 which satisfies the heat conduction equation $\nabla^2T_1=0$ and the boundary conditions i.e. $T_1=$constant at infinity and $\partial T_1/\partial r_2=0$ for $r_2=R_2$, we take the following equation

$$\frac{T_1}{4a}=\sum_{n=1}^{\infty}\{A_n\cosh n\xi\cos n\beta+B_n\cosh n\xi\sin n\beta\} \quad\cdots(7)$$

In this problem, the complete temperature distribution T is constructed in the form $T=T_0+T_1$ and the parametric coefficients involved in Eq. (7) are adjusted so as to satisfy the boundary conditions at the insulated circular hole I ($\partial T/\partial\alpha=0$ at $\alpha=b$). Consequently, the values of A_n and B_n are determined as

$$A_n=\tau\cos\phi e^{-n\xi_1}\sinh nc\,\mathrm{cosech}\,n\xi_1,\quad B_n=\tau\sin\phi e^{-n\xi_1}\cosh nc\,\mathrm{cosech}\,n\xi_1,\quad \xi_1\equiv b+c \quad\cdots(8)$$

From the temperature distribution T, the complex function W having T as its real part is

$$\begin{aligned}\frac{W}{a}=&\tau\cos\phi\left\{\frac{\zeta+1}{\zeta-1}-\frac{\zeta+e^{-2c}}{\zeta-e^{-2c}}+2\sum_{n=1}^{\infty}(e^{nc}\zeta^n+e^{-nc}\zeta^{-n})e^{-n\xi_1}\sinh nc\,\mathrm{cosech}\,n\xi_1\right\}\\ &-i\tau\sin\phi\left\{\frac{\zeta+1}{\zeta-1}+\frac{\zeta+e^{-2c}}{\zeta-e^{-2c}}-2\sum_{n=1}^{\infty}(e^{nc}\zeta^n-e^{-nc}\zeta^{-n})e^{-n\xi_1}\cosh nc\,\mathrm{cosech}\,n\xi_1\right\},\quad \zeta\equiv e^{\bar{w}}\end{aligned} \quad\cdots(9)$$

3. Thermal dislocations

The temperature distribution T mentioned above gives rise to some discontinuities of displacement around the holes I and II. For the hole I, if the plate is cut along the half-line ($\theta_1=0$) from the edge of the hole and is allowed to deform freely, the discontinuities are equal to the relative displacements of the points $1(r_1, 0)$ and $2(r_1, 2\pi)$ on the cut line. Similarly, for the hole II, they are the relative displacements of the points $1'(r_2, 0)$ and $2'(r_2, 2\pi)$. These relative displacements are given by[8]

$$\left.\begin{aligned}(u_2+iv_2)-(u_1+iv_1)&=k\int_1^2 W\,dz=-2ak\oint_{(\alpha=b)}\frac{W\,d\zeta}{(\zeta-1)^2}\\ (u_{2'}+iv_{2'})-(u_{1'}+iv_{1'})&=k\int_{1'}^{2'}W\,dz=-2ak\oint_{(\alpha=-c)}\frac{W\,d\zeta}{(\zeta-1)^2}\end{aligned}\right\} \quad\cdots(10)$$

In Eq. (10) u and v are the x and y-components of the displacement respectively and for plane stress k is equal to α_t, for plane strain k is equal to $(1+\nu)\alpha_t$, where α_t is the coefficient of thermal expansion and ν is Poisson's ratio of the material. We have from Eq. (9)

$$\left.\begin{aligned}u_2-u_1&=-8\pi a^2k\tau\sin\phi\sum_{n=1}^{\infty}ne^{-nb}\cosh nc\,\mathrm{cosech}\,n\xi_1\\ v_2-v_1&=8\pi a^2k\tau\cos\phi\sum_{n=1}^{\infty}ne^{-nb}\sinh nc\,\mathrm{cosech}\,n\xi_1\\ u_{2'}-u_{1'}&=-8\pi a^2k\tau\sin\phi\sum_{n=1}^{\infty}ne^{-nc}\cosh nb\,\mathrm{cosech}\,n\xi_1\\ v_{2'}-v_{1'}&=8\pi a^2k\tau\cos\phi\sum_{n=1}^{\infty}ne^{-nc}\sinh nb\,\mathrm{cosech}\,n\xi_1\end{aligned}\right\} \quad\cdots(11)$$

4. Stress function

Physically, the dislocations of the displacement given in Eq. (11) are impossible. These discontinuities are removed by superposing the components of the displacement obtained from Airy's stress function and a displacement one in an isothermal state, and this stress function is determined so as to satisfy the boundary conditions at both circular holes and at infinity. In the first place, for the hole II the first stress function χ_{21} which gives many-valued displacements and has no stress at infinity and at the edge of the hole II is

$$\frac{\chi_{21}}{a^2E_ek\tau}=C\cos\phi\left(\frac{R_2^2}{r_2^2}+\log r_2^2\right)r_2\cos\theta_2+D\sin\phi\left(\frac{R_2^2}{r_2^2}+\log r_2^2\right)r_2\sin\theta_2 \quad \cdots\cdots(12)$$

In Eq. (12), E_e is equal to E for plane stress and E_e is equal to $E/(1-\nu^2)$ for plane strain, where E is the modulus of longitudinal elasticity of the material. The displacement function ψ_{21} which is obtained from the relations $\partial/\partial r\{r(\partial\psi/\partial\theta)\}=\nabla^2\chi$ and $\nabla^2\psi=0$, can be represented as follows using Eq. (12)

$$\frac{\psi_{21}}{4a^2E_ek\tau}=C\cos\phi\left(\frac{\sin\theta_2}{r_2}\log r_2-\frac{\theta_2}{r_2}\cos\theta_2\right)-D\sin\phi\left(\frac{\cos\theta_2}{r_2}\log r_2+\frac{\theta_2}{r_2}\sin\theta_2\right) \quad \cdots\cdots(13)$$

By Eqs. (12) and (13), the components of the displacement are

$$\left.\begin{aligned}
\frac{u_{r_2}}{a^2k\tau}=&C\cos\phi\left\{(1+\nu_e)\left(\frac{R_2^2}{r_2^2}-2\log r_2-2\right)\cos\theta_2+4(\cos\theta_2\log r_2-\cos\theta_2+\theta_2\sin\theta_2)\right\}\\
&+D\sin\phi\left\{(1+\nu_e)\left(\frac{R_2^2}{r_2^2}-2\log r_2-2\right)\sin\theta_2+4(\sin\theta_2\log r_2-\sin\theta_2-\theta_2\cos\theta_2)\right\}\\
\frac{v_{\theta_2}}{a^2k\tau}=&C\cos\phi\left\{(1+\nu_e)\left(\frac{R_2^2}{r_2^2}+2\log r_2\right)\sin\theta_2-4(\sin\theta_2\log r_2-\sin\theta_2-\theta_2\cos\theta_2)\right\}\\
&-D\sin\phi\left\{(1+\nu_e)\left(\frac{R_2^2}{r_2^2}+2\log r_2\right)\cos\theta_2-4(\cos\theta_2\log r_2-\cos\theta_2+\theta_2\sin\theta_2)\right\}
\end{aligned}\right\} \quad \cdots\cdots(14)$$

In Eq. (14), u_{r_2} and v_{θ_2} are the r_2 and θ_2-components of the displacement respectively and $\nu_e=\nu$ for plane stress and $\nu_e=\nu/(1-\nu)$ for plane strain. We have

$$(u_{r_2})_{\theta_2=2\pi}-(u_{r_2})_{\theta_2=0}=-8\pi a^2k\tau D\sin\phi,\qquad (v_{\theta_2})_{\theta_2=2\pi}-(v_{\theta_2})_{\theta_2=0}=8\pi a^2k\tau C\cos\phi \quad \cdots\cdots(15)$$

From Eqs. (11) and (15), the values of the coefficients C and D are determined to be

$$C=-\sum_{n=1}^{\infty}ne^{-nc}\sinh nb\,\mathrm{cosech}\,n\xi_1\quad(b>0,\ c>0),\qquad D=-\sum_{n=1}^{\infty}ne^{-nc}\cosh nb\,\mathrm{cosech}\,n\xi_1\quad(b>0,\ c>0) \cdots(16)$$

Let the second stress function χ_{22} which has a singular point at the center O_2 of the hole II and produces no stress along the edge of the hole II, be given by

$$\frac{\tanh c}{a^3E_ek\tau}\chi_{22}=K_2C\cos\phi\left(\frac{r_2^2}{R_2^2}-\log r_2^2\right) \quad \cdots\cdots(17)$$

Using Eqs. (2) and (3), we convert Eqs. (12) and (17) into the forms of bipolar coordinates. With the aid of Eq. (5) and

$$\log(\cosh\alpha-\cos\beta)=\alpha-\log 2-2\sum_{n=1}^{\infty}\frac{e^{-n\alpha}}{n}\cos n\beta \qquad (\alpha>0) \quad \cdots\cdots(18)$$

$h(\chi_{21}+\chi_{22})$ can be expanded as follows

$$\begin{aligned}
-\frac{\sinh c}{2a^2E_ek\tau}h(\chi_{21}+\chi_{22})=&C\cos\phi\Big[\sinh^3 c\cos\beta-2(K_2+1)e^{-\xi}\sinh\xi\sinh^2 c\cosh c\cos\beta\\
&-2\sinh c\sum_{n=2}^{\infty}\frac{e^{-n\xi}}{n}\{(n\cosh nc\sinh c+\sinh nc\cosh c)\sinh\xi\\
&-\sinh nc\sinh c(n\sinh\xi+\cosh\xi)\}\cos n\beta\\
&-2(K_2-1)\cosh c\sum_{n=2}^{\infty}\frac{e^{-n\xi}}{n(n^2-1)}\{(n^2-1)\sinh nc\sinh c\sinh\xi\\
&-(n\cosh nc\sinh c-\sinh nc\cosh c)(n\sinh\xi+\cosh\xi)\}\cos n\beta\Big]\\
&+D\sin\phi\Big[e^{-2\xi}\sinh^3 c\sin\beta-2\sinh c\sum_{n=2}^{\infty}\frac{e^{-n\xi}}{n^2-1}\{(n^2-1)\sinh nc\sinh c\sinh\xi\\
&-(n\cosh nc\sinh c-\sinh nc\cosh c)(n\sinh\xi+\cosh\xi)\}\sin n\beta\Big] \qquad (\alpha>0) \quad \cdots\cdots(19)
\end{aligned}$$

Here, the terms which produce no stress and strain are omitted. As the third stress function $h\chi_{23}$ which has no singular point in the domain of plate and produces no stress on the boundary $\alpha=-c$, we can take

$$\begin{aligned}
\frac{\sinh c}{2a^2E_ek\tau}h\chi_{23}=&B_{20}\{\xi(\cosh\alpha-\cos\beta)\cosh c+\sinh\xi(\cosh\alpha\cos\beta-\cosh^2 c)\}\\
&+A_{21}(\cosh 2\xi-1)\cos\beta+C_{21}(\cosh 2\xi-1)\sin\beta\\
&+\sum_{n=2}^{\infty}[A_{2n}\{\cosh(n+1)\xi-\cosh(n-1)\xi\}+B_{2n}\{(n-1)\sinh(n+1)\xi-(n+1)\sinh(n-1)\xi\}]\cos n\beta\\
&+\sum_{n=2}^{\infty}[C_{2n}\{\cosh(n+1)\xi-\cosh(n-1)\xi\}+D_{2n}\{(n-1)\sinh(n+1)\xi-(n+1)\sinh(n-1)\xi\}]\sin n\beta
\end{aligned} \quad \cdots\cdots(20)$$

The stress function $\chi_2=\chi_{21}+\chi_{22}+\chi_{23}$ for the hole II must satisfy the boundary conditions at the edge of the hole I

$$\left[\frac{\partial}{\partial\alpha}(h\chi_2)\right]_{\alpha=b}=\rho_2,\quad [h\chi_2]_{\alpha=b}=\rho_2\tanh b+\sigma_2(\cosh b\cos\beta-1)+\tau_2\sin\beta \qquad (21)$$

Consequently, the values of the coefficients in Eq. (20) are

$$\left.\begin{aligned}
B_{20}&=-C\cos\phi\left\{\frac{\sinh^3 c\coth\xi_1}{\cosh c(\sinh^2 b+\sinh^2 c)}-(K_2+1)\frac{\sinh^2 c}{\sinh^2 b+\sinh^2 c}\right\}\\
A_{21}&=C\cos\phi\left\{\frac{\sinh b\sinh^3 c}{2\sinh\xi_1\cosh c(\sinh^2 b+\sinh^2 c)}\right.\\
&\quad\left.+(K_2+1)\sinh^2 c\frac{e^{-2\xi_1}\sinh c\sinh(b-c)-2e^{-\xi_1}\sinh\xi_1\sinh b}{2\sinh\xi_1(\sinh^2 b+\sinh^2 c)}\right\}\\
C_{21}&=-D\sin\phi\sinh^3 c(\coth 2\xi_1-1)\\
A_{2n}&=C\cos\phi\left[\frac{\sinh c}{n}\left\{n(n^2-1)\sinh nc\sinh c\frac{\sinh^2\xi_1}{\Delta_n}\right.\right.\\
&\quad\left.+(n\cosh nc\sinh c+\sinh nc\cosh c)\left(1-\frac{\sinh 2n\xi_1-n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\\
&\quad+(K_2-1)\frac{\cosh c}{n}\left\{n(n\cosh nc\sinh c-\sinh nc\cosh c)\frac{\sinh^2\xi_1}{\Delta_n}\right.\\
&\quad\left.\left.+\sinh nc\sinh c\left(1-\frac{\sinh 2n\xi_1-n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\right]\\
B_{2n}&=C\cos\phi\left[\frac{\sinh c}{n}\left\{n(n\cosh nc\sinh c+\sinh nc\cosh c)\frac{\sinh^2\xi_1}{\Delta_n}\right.\right.\\
&\quad\left.+\sinh nc\sinh c\left(1-\frac{\sinh 2n\xi_1+n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\\
&\quad+(K_2-1)\frac{\cosh c}{n(n^2-1)}\left\{n(n^2-1)\sinh nc\sinh c\frac{\sinh^2\xi_1}{\Delta_n}\right.\\
&\quad\left.\left.+(n\cosh nc\sinh c-\sinh nc\cosh c)\left(1-\frac{\sinh 2n\xi_1+n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\right]\\
C_{2n}&=D\sin\phi\sinh c\left\{n(n\cosh nc\sinh c-\sinh nc\cosh c)\frac{\sinh^2\xi_1}{\Delta_n}\right.\\
&\quad\left.+\sinh nc\sinh c\left(1-\frac{\sinh 2n\xi_1-n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\\
D_{2n}&=D\sin\phi\frac{\sinh c}{n^2-1}\left\{n(n^2-1)\sinh nc\sinh c\frac{\sinh^2\xi_1}{\Delta_n}\right.\\
&\quad\left.+(n\cosh nc\sinh c-\sinh nc\cosh c)\left(1-\frac{\sinh 2n\xi_1+n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\\
\Delta_n&\equiv\sinh^2 n\xi_1-n^2\sinh^2\xi_1
\end{aligned}\right\} \qquad (22)$$

At infinity $\alpha=\beta=0$, the stresses derived from the stress function $\chi_{22}+\chi_{23}$ must be zero and so the coefficient K_2 is determined from this condition. It requires $h(\chi_{22}+\chi_{23})=0$, then we have

$$\begin{aligned}&K_2C\cos\phi\sinh^2 c\cosh c-B_{20}\sinh^3 c+2A_{21}\sinh^2 c\\
&+2\sum_{n=2}^{\infty}\{A_{2n}\sinh nc\sinh c+B_{2n}(n\cosh nc\sinh c-\sinh nc\cosh c)\}=0\end{aligned} \qquad (23)$$

Substituting Eq. (22) in (23) and using the formula

$$\sum_{n=1}^{\infty}p^n=\frac{p}{1-p},\quad (|p|<1) \qquad (24)$$

we arrive at the equation

$$\begin{aligned}&(K_2-1)\left[\frac{2\sinh^3 b\sinh^3 c}{\sinh\xi_1(\sinh^2 b+\sinh^2 c)}+\sum_{n=2}^{\infty}\{(\sinh^2 nb-n^2\sinh^2 b)(\sinh 2nc-n\sinh 2c)\right.\\
&\left.+(\sinh^2 nc-n^2\sinh^2 c)(\sinh 2nb-n\sinh 2b)\}\frac{1}{n(n^2-1)\Delta_n}\right]\\
&+\tanh c\left\{\frac{\sinh^2 b\sinh^2 c}{\sinh\xi_1(\sinh^2 b+\sinh^2 c)}(3\sinh\xi_1-4\sinh c\cosh b)\right.\\
&\left.+2\sum_{n=2}^{\infty}(\sinh^2 nb\sinh^2 c-\sinh^2 nc\sinh^2 b)\frac{1}{\Delta_n}\right\}=0\qquad(c>0)\end{aligned} \qquad (25)$$

Subsequently, we consider the hole I in the same way as mentioned above. The equations corresponding to

Eqs. (12), (15), (16), (17) and (20) become

$$\frac{\chi_{11}}{a^2E_ek\tau}=F\cos\phi\left(\frac{R_1^2}{r_1^2}+\log r_1^2\right)r_1\cos\theta_1+H\sin\phi\left(\frac{R_1^2}{r_1^2}+\log r_1^2\right)r_1\sin\theta_1 \quad\cdots\cdots(26)$$

$$(u_{r_1})_{\theta_1=2\pi}-(u_{r_1})_{\theta_1=0}=-8\pi a^2k\tau H\sin\phi,\qquad (v_{\theta_1})_{\theta_1=2\pi}-(v_{\theta_1})_{\theta_1=0}=8\pi a^2k\tau F\cos\phi \quad\cdots\cdots(27)$$

$$F=-\sum_{n=1}^{\infty} ne^{-nb}\sinh nc\,\mathrm{cosech}\,n\xi_1 \quad (b>0,\ c>0),\qquad H=-\sum_{n=1}^{\infty} ne^{-nb}\cosh nc\,\mathrm{cosech}\,n\xi_1 \quad (b>0,\ c>0) \quad\cdots\cdots(28)$$

$$\frac{\tanh b}{a^3E_ek\tau}\chi_{12}=K_1F\cos\phi\left(\frac{r_1^2}{R_1^2}-\log r_1^2\right) \quad\cdots\cdots(29)$$

$$\begin{aligned}\frac{\sinh b}{2a^2E_ek\tau}h\chi_{13}=&B_{10}\{\eta(\cosh\alpha-\cos\beta)\cosh b+\sinh\eta(\cosh\alpha\cos\beta-\cosh^2 b)\}\\&+A_{11}(\cosh 2\eta-1)\cos\beta+C_{11}(\cosh 2\eta-1)\sin\beta\\&+\sum_{n=2}^{\infty}[A_{1n}\{\cosh(n+1)\eta-\cosh(n-1)\eta\}+B_{1n}\{(n-1)\sinh(n+1)\eta-(n+1)\sinh(n-1)\eta\}]\cos n\beta\\&+\sum_{n=2}^{\infty}[C_{1n}\{\cosh(n+1)\eta-\cosh(n-1)\eta\}+D_{1n}\{(n-1)\sinh(n+1)\eta-(n+1)\sinh(n-1)\eta\}]\sin n\beta\\&\eta\equiv\alpha-b\end{aligned} \quad\cdots\cdots(30)$$

Using Eqs. (5) and (18), we write $h(\chi_{11}+\chi_{12})$ in the form of Fourier's series for $\alpha<0$ and the stress function for the hole I is constructed in the form $\chi_1=\chi_{11}+\chi_{12}+\chi_{13}$. The parametric coefficients included in Eq. (30) are determined so as to satisfy the boundary conditions at the edge of the hole II, and we find

$$\left.\begin{aligned}B_{10}=&-F\cos\phi\left\{\frac{\sinh^3 b\coth\xi_1}{\cosh b(\sinh^2 b+\sinh^2 c)}+(K_1-1)\frac{\sinh^2 b}{\sinh^2 b+\sinh^2 c}\right\}\\A_{11}=&-F\cos\phi\left\{\frac{\sinh^3 b\sinh c}{2\sinh\xi_1\cosh b(\sinh^2 b+\sinh^2 c)}\right.\\&\left.+(K_1-1)\sinh^2 b\frac{e^{-2\xi_1}\sinh b\sinh(b-c)+2e^{-\xi_1}\sinh\xi_1\sinh c}{2\sinh\xi_1(\sinh^2 b+\sinh^2 c)}\right\}\\C_{11}=&-H\sin\phi\sinh^3 b(\coth 2\xi_1-1)\\A_{1n}=&-F\cos\phi\left[\frac{\sinh b}{n}\left\{n(n^2-1)\sinh nb\sinh b\frac{\sinh^2\xi_1}{\Delta_n}+(n\cosh nb\sinh b\right.\right.\\&\left.+\sinh nb\cosh b)\left(1-\frac{\sinh 2n\xi_1-n\sinh 2\xi_1}{2\Delta_n}\right)\right\}-(K_1+1)\frac{\cosh b}{n}\left\{n(n\cosh nb\sinh b\right.\\&\left.\left.-\sinh nb\cosh b)\frac{\sinh^2\xi_1}{\Delta_n}+\sinh nb\sinh b\left(1-\frac{\sinh 2n\xi_1-n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\right]\\B_{1n}=&F\cos\phi\left[\frac{\sinh b}{n}\left\{n(n\cosh nb\sinh b+\sinh nb\cosh b)\frac{\sinh^2\xi_1}{\Delta_n}+\sinh nb\sinh b\right.\right.\\&\left.\times\left(1-\frac{\sinh 2n\xi_1+n\sinh 2\xi_1}{2\Delta_n}\right)\right\}-(K_1+1)\frac{\cosh b}{n(n^2-1)}\left\{n(n^2-1)\sinh nb\sinh b\frac{\sinh^2\xi_1}{\Delta_n}\right.\\&\left.\left.+(n\cosh nb\sinh b-\sinh nb\cosh b)\left(1-\frac{\sinh 2n\xi_1+n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\right]\\C_{1n}=&H\sin\phi\sinh b\left\{n(n\cosh nb\sinh b-\sinh nb\cosh b)\frac{\sinh^2\xi_1}{\Delta_n}\right.\\&\left.+\sinh nb\sinh b\left(1-\frac{\sinh 2n\xi_1-n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\\D_{1n}=&-H\sin\phi\frac{\sinh b}{n^2-1}\left\{n(n^2-1)\sinh nb\sinh b\frac{\sinh^2\xi_1}{\Delta_n}\right.\\&\left.+(n\cosh nb\sinh b-\sinh nb\cosh b)\left(1-\frac{\sinh 2n\xi_1+n\sinh 2\xi_1}{2\Delta_n}\right)\right\}\end{aligned}\right\} \quad\cdots\cdots(31)$$

The value of the coefficient K_1 is obtained from the condition $h(\chi_{12}+\chi_{13})=0$ for $\alpha=\beta=0$.

$$\begin{aligned}(K_1+1)&\left[\frac{2\sinh^3 b\sinh^3 c}{\sinh\xi_1(\sinh^2 b+\sinh^2 c)}+\sum_{n=2}^{\infty}\{(\sinh^2 nb-n^2\sinh^2 b)(\sinh 2nc-n\sinh 2c)+(\sinh^2 nc\right.\\&\left.-n^2\sinh^2 c)(\sinh 2nb-n\sinh 2b)\}\frac{1}{n(n^2-1)\Delta_n}\right]+\tanh b\left\{\frac{\sinh^2 b\sinh^2 c}{\sinh\xi_1(\sinh^2 b+\sinh^2 c)}\right.\\&\left.\times(4\sinh b\cosh c-3\sinh\xi_1)+2\sum_{n=2}^{\infty}(\sinh^2 nb\sinh^2 c-\sinh^2 nc\sinh^2 b)\frac{1}{\Delta_n}\right\}=0\quad(b>0)\end{aligned} \quad\cdots\cdots(32)$$

Now, we can derive the thermal stresses in a plate from the complete stress function $\chi=\chi_1+\chi_2$. In particular, the thermal stress $[\sigma_\beta]_b$ along the edge of the hole I is given by

$$a[\sigma_\beta]_b=(\cosh b-\cos\beta)\left[\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\beta^2}-1\right)(h\chi)\right]_b \quad\cdots(33)$$

and so we have

$$\frac{[\sigma_\beta]_b}{E_ek\tau R_1}=4(\cosh b-\cos\beta)\Big[C\cos\phi\sinh b\sinh c\frac{\sinh^2 b+\sinh(b-c)\sinh c\cos\beta}{\sinh\xi_1(\sinh^2 b+\sinh^2 c)}$$
$$+F\cos\phi\sinh^2 b\left\{\sinh b\frac{\sinh c-\sinh(b-c)\cos\beta}{\sinh\xi_1(\sinh^2 b+\sinh^2 c)}+2\cosh c\,\mathrm{cosech}\,\xi_1\cos\beta\right\}$$
$$+(K_2C\sinh b\cosh c+K_1F\cosh b\sinh c)\cos\phi\sinh b\sinh c\frac{\sinh\xi_1+2\sinh c\cos\beta}{\sinh\xi_1(\sinh^2 b+\sinh^2 c)}$$
$$-2D\sin\phi\sinh b\sinh^2 c\,\mathrm{cosech}\,2\xi_1\sin\beta+2H\sin\phi\sinh^2 b\sinh(\xi_1+c)\,\mathrm{cosech}\,2\xi_1\sin\beta$$
$$+2(C+F)\cos\phi\sinh b\sum_{n=2}^{\infty}\{n(n\cosh nb\sinh c-\sinh nb\cosh c)\sinh\xi_1+(n\cosh nc\sinh b$$
$$-\sinh nc\cosh b)\sinh n\xi_1\}\frac{\cos n\beta}{\Delta_n}-2\{(K_2-1)C\sinh b\coth c+(K_1+1)F\cosh b\}\cos\phi$$
$$\times\sum_{n=2}^{\infty}(n\sinh nb\sinh\xi_1\sinh c-\sinh n\xi_1\sinh nc\sinh b)\frac{\cos n\beta}{\Delta_n}-2(D+H)\sin\phi\sinh b$$
$$\times\sum_{n=2}^{\infty}n(n\sinh nb\sinh\xi_1\sinh c-\sinh n\xi_1\sinh nc\sinh b)\frac{\sin n\beta}{\Delta_n}\Big]\qquad(b>0,\ c>0)\quad\cdots(34)$$

5. Limiting cases

We can consider some limiting cases.

(1) Taking $c\to\infty$, we have the case that the hole II is infinitesimal at the pole P_2 (Fig. 1). From Eqs. (16), (28) and (34), we find

$$\frac{[\sigma_\beta]_b}{E_ek\tau R_1}=-\cos(\phi-\theta_1)\quad\cdots(35)$$

Equation (35) is equal to the thermal stress along the edge of the hole for a plate with one circular hole(1).

(2) Taking $b\to\infty$, we have the case that the hole I is infinitesimal at the pole P_1 (Fig. 1). The thermal stress along the edge of the infinitesimal hole I becomes

$$-\frac{[\sigma_\beta]_b}{E_ek\tau R_2}=e^{-c}\cos\phi+2e^{-3c}\cos\phi\cos 2\beta+2(e^{-c}-e^{-3c})\sin\phi\sin 2\beta\quad\cdots(36)$$

When a plate with one hole of radius R_2 is under a heat flow of gradient τ, the thermal stress components near the pole P_1 are given by the forms(1),

$$\left.\begin{aligned}&-\frac{\sigma_x}{E_ek\tau R_2}=\frac{1}{2}(e^{-c}-e^{-3c})\cos\phi,\qquad -\frac{\sigma_y}{E_ek\tau R_2}=\frac{1}{2}(e^{-c}+e^{-3c})\cos\phi\\&-\frac{\tau_{xy}}{E_ek\tau R_2}=-\frac{1}{2}(e^{-c}-e^{-3c})\sin\phi\end{aligned}\right\}\quad\cdots(37)$$

Equation (36) is equal to the stress equation along the edge of the hole for a plate which has one circular hole and is under a uniform stress state shown in Eq. (37).

(3) Taking $\phi=90°$ and $c\to 0$ or $\phi=90°$ and $b\to 0$, we have the case that a small hole is extremely near to an infinitely large hole. When $c\to 0$, we have from Eqs. (16), (28) and (34)

$$C=D=F=0,\qquad H=-\sum_{n=1}^{\infty}ne^{-nb}\,\mathrm{cosech}\,nb\quad\cdots(38)$$

$$\frac{[\sigma_\beta]_b}{E_ek\tau R_1}=4H\sinh b\tanh b(\cosh b-\cos\beta)\sin\beta\quad\cdots(39)$$

When $b\to 0$, we can write similarly

$$C=F=H=0,\qquad D=-\sum_{n=1}^{\infty}ne^{-nc}\,\mathrm{cosech}\,nc\quad\cdots(40)$$

$$\frac{[\sigma_\beta]_b}{E_ek\tau R_2}=-4D\sinh c\tanh c(1-\cos\beta)\sin\beta\quad\cdots(41)$$

Equations (39) and (41) give the thermal stress equations along the edge of a small hole and along the edge of an infinitely large hole respectively. The gradient of the heat flow near the small hole is 2τ from Eq. (1). Then Eqs. (39) and (41) are equal to the thermal stress equations along the edge of a hole and along the straight edge when a semi-infinite plate with a circular hole is under a heat flow of gradient 2τ and is parallel to the straight edge[5].

(4) Taking R_1/R_2 as a constant value and $b\to 0$, we have the case of two contacted circular holes of different sizes. In this case, $R_1/R_2=\rho=c/b$ and taking $nb\equiv u$, Eqs. (16), (25), (28) and (32) become

$$\left.\begin{aligned}
&C\sinh^2 b=-\int_0^\infty ue^{-u\rho}\sinh u\,\mathrm{cosech}\,u(1+\rho)\,du\equiv C'\\
&D\sinh^2 b=-\int_0^\infty ue^{-u\rho}\cosh u\,\mathrm{cosech}\,u(1+\rho)\,du\equiv D'\\
&F\sinh^2 b=-\int_0^\infty ue^{-u}\sinh u\rho\,\mathrm{cosech}\,u(1+\rho)du\equiv F'\\
&H\sinh^2 b=-\int_0^\infty ue^{-u}\cosh u\rho\,\mathrm{cosech}\,u(1+\rho)\,du\equiv H'\\
&(K_1'+1)\int_0^\infty\{(\sinh^2 u-u^2)(\sinh 2u\rho-2u\rho)+(\sinh^2 u\rho-u^2\rho^2)(\sinh 2u-2u)\}\frac{du}{u^3\Delta_u'}\\
&\quad+2\int_0^\infty(\rho^2\sinh^2 u-\sinh^2 u\rho)\frac{du}{\Delta_u'}=0\\
&(K_2'-1)\int_0^\infty\{(\sinh^2 u-u^2)(\sinh 2u\rho-2u\rho)+(\sinh^2 u\rho-u^2\rho^2)(\sinh 2u-2u)\}\frac{du}{u^3\Delta_u'}\\
&\quad+2\rho\int_0^\infty(\rho^2\sinh^2 u-\sinh^2 u\rho)\frac{du}{\Delta_u'}=0\\
&\Delta_u'\equiv\sinh^2 u(1+\rho)-u^2(1+\rho)^2
\end{aligned}\right\}\quad\cdots\cdots(42)$$

Moreover we have

$$\cos\beta=\frac{\cosh b\cos\theta_1+1}{\cosh b+\cos\theta_1},\quad \sin\beta=\frac{\sinh b\sin\theta_1}{\cosh b+\cos\theta_1}\quad\cdots\cdots(43)$$

and taking $b\to 0$ in Eq. (43), we obtain the following limit values.

$$\frac{\cosh b-\cos\beta}{\sinh^2 b}=\frac{1}{2}\sec^2\left(\frac{\theta_1}{2}\right),\quad n\beta=u\tan\left(\frac{\theta_1}{2}\right)\quad\cdots\cdots(44)$$

Then the stress equation (34) along the edge of the hole I is

$$\begin{aligned}
\frac{[\sigma_\beta]_b}{E_e k\tau R_1}=&4\sec^2\left(\frac{\theta_1}{2}\right)\Big[(C'+F')\cos\phi\int_0^\infty\{u(1+\rho)(u\rho\cosh u-\sinh u)\\
&+(u\cosh u\rho-\sinh u\rho)\sinh u(1+\rho)\}\cos\left(u\tan\frac{\theta_1}{2}\right)\frac{du}{\Delta_u'}-\left\{(K_2'-1)\frac{C'}{\rho}+(K_1'+1)F'\right\}\cos\phi\\
&\times\int_0^\infty\{u\rho(1+\rho)\sinh u-\sinh u\rho\sinh u(1+\rho)\}\cos\left(u\tan\frac{\theta_1}{2}\right)\frac{du}{\Delta_u'}-(D'+H')\sin\phi\\
&\times\int_0^\infty u\{u\rho(1+\rho)\sinh u-\sinh u\rho\sinh u(1+\rho)\}\sin\left(u\tan\frac{\theta_1}{2}\right)\frac{du}{\Delta_u'}\Big]
\end{aligned}\quad\cdots\cdots(45)$$

6. Numerical results

Let us represent the ratio of the radius of two holes by $\rho=R_1/R_2$ and the distance between them by $\mu=(R_1+R_2)/d$. Hence, $\rho=1$ shows a case of two equal circular holes, $\rho=\infty$ shows also a case of one hole as mentioned in the limiting case (1) and $\rho=0$ (then $\mu=e^{-c}$) shows the limiting case (2). Further, $\mu=1$ shows a case of two contacted circular holes as mentioned in the limiting case (5) and $\mu=0$ shows a case of two holes which are infinitely distant from each other. The terms of series in Eqs. (25), (32) and (34) were reformed to converge rapidly and the integrals (42) and (45) were evaluated by means of Simpson's rule.

When the heat flow is parallel to the center line of two holes, the extreme values of the thermal stress occur at the points A, B, A′ and B′. Figure 2 shows the thermal stresses at the points A and B of the hole I ($R_1<R_2$, $R_1=R_2$ or $R_1>R_2$).

(1) When the hole I is small and the hole II is large ($R_1<R_2$, $\rho<1$), the values of the thermal stresses at the points A and B are obtained from the curves for the cases of $\rho=0.1$, 0.2 and 0.5, while the negative values of them at the points A′ and B′ are obtained for the cases of $\rho=10$, 5 and 2. When the two holes are distant from each other, $|\sigma|_{max}$ occurs at the point A′ of a large hole and when μ exceeds 0.65, 0.70 and 0.85 for $\rho=0.1$, 0.2 and 0.5 respectively, $|\sigma|_{max}$ occurs at the point B′ of it. Moreover, as the two holes approach more and more ($\mu\to 1$), $|\sigma|_{max}$ occurs at the point B of a small hole and the value of it becomes great remarkably. These values are larger than that of a plate with one hole ($\mu=0$).

(2) When the hole I is large in comparison with the hole II, the thermal stresses are obtained by changing the direction of heat flow.

Figure 3 shows the maximum compressive thermal stresses of the holes ($R_1<R_2$, $R_1=R_2$ or $R_1>R_2$) for the case of heat flow perpendicular to the center line of two holes and the extreme values of them occur near the points shown in Table 1. From this graph, $|\sigma|_{max}$ occurs always at the edge of a large hole and the larger the ratio of radius ρ is, the greater the values of $|\sigma|_{max}$ are. When $\rho>10$, they tend to be constant independently of the distance between two holes. When $10>\rho>1$, they are greater than those of a plate with one hole.

Figures 4 and 5 show the thermal stress distributions along the edge of the hole I in a plate having two equal circular holes ($\rho=1$, $\mu=0.4$) and being under a heat flow parallel or perpendicular to the center line of two holes respectively. The figures also show comparisons with the cases of one hole(1), three equal holes(6) and an infinite row of equal holes(7). The stress distribution along the periphery of two holes is similar to that of outer hole in the case of three holes.

The thermal stresses in an infinite plate containing two insulated circular holes under uniform heat flow were analysed and the influences of the size and distance of two holes upon the thermal stresses were discusses concretely. Consequently, it is noted that a high thermal stress occurs at the edge of a large hole when the two holes are distant. It occurs at the edge of a small hole when the holes approach each other in the case of heat flow parallel to the center line of two holes, but it occurs at the edge of a large hole for a perpendicular heat flow. In any case, it is larger than that of a plate with one circular hole.

Table 1 Locations of extreme values ($\phi=90°$)

ρ	$\theta_1°$	ρ	$\theta_1°$
0	45	2	82
0.1	48	5	89
0.2	52	10	90
0.5	61	∞	90
1	72		

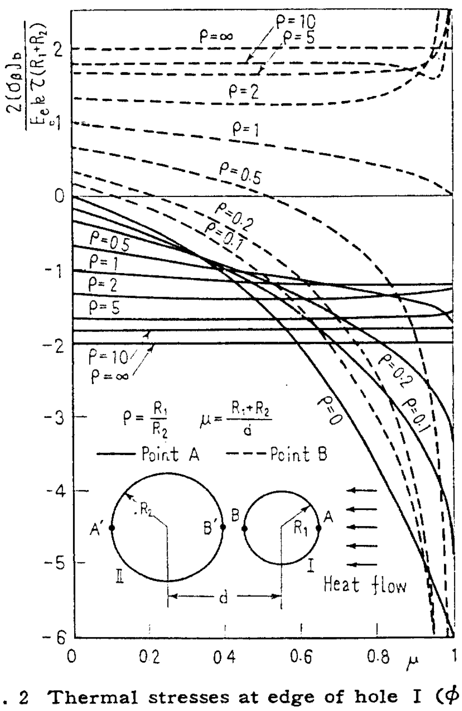

Fig. 2 Thermal stresses at edge of hole I ($\phi=0°$)

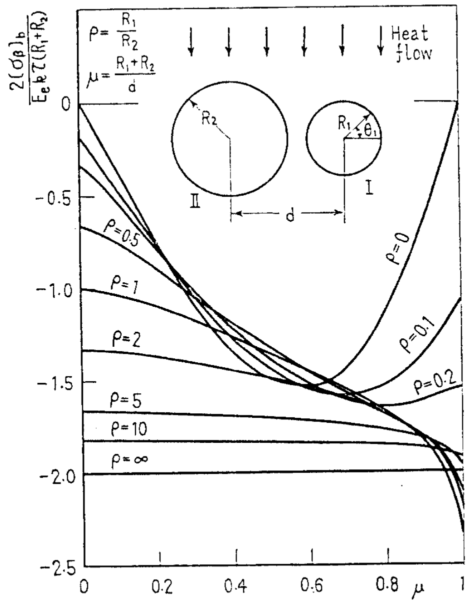

Fig. 3 Thermal stresses at edge of hole I ($\phi=90°$)

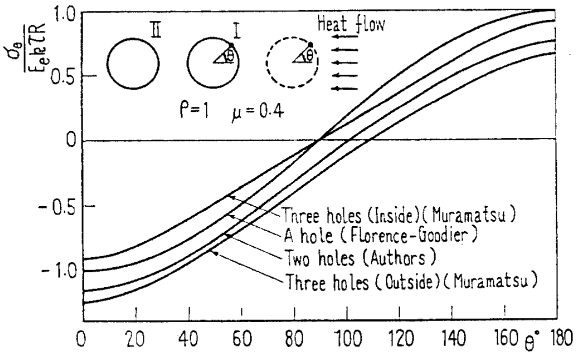

Fig. 4 Thermal stress distributions around equal holes ($\phi=0°$)

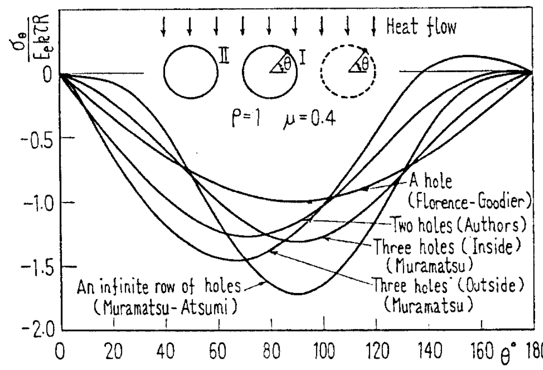

Fig. 5 Thermal stress distributions around equal holes ($\phi=90°$)

The authors wish to express their hearty thanks to Mr. N. Taniguti for his great cooperation.

References

(1) Florence, A.L. and Goodier, J.N., *Trans. ASME*, Ser. E, Vol. 26, No. 2 (1959-6), p. 293.
(2) Isida, M., *Preprint of Japan Soc. Aeron. Sci.* (in Japanese), (1960-2-20), p. 45.
(3) Deresiewicz, H., *Trans. ASME*, Ser. E, Vol. 28, No. 1 (1961-3), p. 147.
(4) Takeuti, Y. and Noda, N., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 34, No. 259 (1968-3), p. 411.
(5) Goodier, J.N. and Florence, A.L., *Quart. Jour. Mech. Appl. Math.*, Vol. 16, No. 3 (1963), p. 273.
(6) Muramatsu, M., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 32, No. 234 (1966-2), p. 163.
(7) Muramatsu, M. and Atsumi, A., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 28, No. 191 (1962-7), p. 807.
(8) Timoshenko, S. and Goodier, J.N., *Theory of Elasticity*, (1953), p. 430, McGraw-Hill.