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Oscillation of nonlinear hyperbolic equations with distributed deviating arguments

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Abstract. Oscillations of solutions to nonlinear hyperbolic equations with continuous distributed deviating arguments are studied. By employing some integral means of solutions, the multi-dimensional oscillation problems are reduced to one-dimensional oscillation problems.

1. Introduction

Oscillation properties of hyperbolic equations without functional arguments were studied by Kreith, Kusano and Yoshida [5], Yoshida [12] by employing the averaging techniques. Parabolic equations with functional arguments were investigated in the paper Yoshida [13] by making use of the integral means of solutions.

The oscillation results for hyperbolic equations with delay were first obtained by Mishev and Bainov [7]. Recently there has been an increasing interest in studying the oscillation of hyperbolic equations with continuous distributed deviating arguments. We refer the reader to [3, 4, 9, 10] for linear hyperbolic equations with continuous distributed deviating arguments,

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and to [2, 6, 8, 11] for nonlinear hyperbolic equations with continuous distributed deviating arguments. Deng [2], Liu and Fu [6] and Wang and Yu [11] pertain to the hyperbolic equations of the form

$$\begin{aligned} &\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{\ell}h_i(t)u(x,\rho_i(t))\right)\right]-a(t)\Delta u(x,t)\\ &-\sum_{i=1}^{k}b_i(t)\Delta u(x,\tau_i(t))+\int_\gamma^\delta q(x,t,\zeta)\varphi\big(u(x,\sigma(t,\zeta))\big)d\omega(\zeta)\\ =\ & f(x,t), \end{aligned} \tag{1}$$

where $h_i(t) \geq 0$ and $q(x,t,\zeta) \geq 0$.

There appears to be no known oscillation results for the equation (1) with $h_i(t) \leq 0$ and $q(x,t,\zeta) \geq 0$. In this paper we are concerned with the oscillatory properties of solutions of hyperbolic equations with continuous distributed arguments

$$\begin{aligned} &\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u(x,t)-\int_\alpha^\beta h(t,\xi)u(x,\rho(t,\xi))d\eta(\xi)\right)\right]-a(t)\Delta u(x,t)\\ &-\sum_{i=1}^{k}b_i(t)\Delta u(x,\tau_i(t))+q_0(x,t)u(x,t)\\ &+\int_\gamma^\delta q(x,t,\zeta)\varphi\big(u(x,\sigma(t,\zeta))\big)d\omega(\zeta)\\ =\ & f(x,t),\ (x,t)\in\Omega\equiv G\times(0,\infty), \end{aligned} \tag{2}$$

where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G.

It is assumed that :

(A$_1$) $p(t) \in C([0,\infty);(0,\infty))$, $a(t) \in C([0,\infty);[0,\infty))$, $b_i(t) \in C([0,\infty);[0,\infty))$ $(i = 1,2,...,k)$, $h(t,\xi) \in C([0,\infty)\times[\alpha,\beta];[0,\infty))$, $q(x,t,\zeta) \in C(\overline{\Omega}\times[\gamma,\delta];[0,\infty))$, $q_0(x,t) \in C(\overline{\Omega};[0,\infty))$ and $f(x,t) \in C(\overline{\Omega};\mathbb{R})$;

(A$_2$) $\tau_i(t) \in C([0,\infty);\mathbb{R})$ $(i = 1,2,...,k)$, $\rho(t,\xi) \in C([0,\infty)\times[\alpha,\beta];\mathbb{R})$, $\sigma(t,\zeta) \in C([0,\infty)\times[\gamma,\delta];\mathbb{R})$ such that $\lim_{t\to\infty}\tau_i(t) = \infty$, $\lim_{t\to\infty}\min_{\xi\in[\alpha,\beta]}\rho(t,\xi) = \infty$ and $\lim_{t\to\infty}\min_{\zeta\in[\gamma,\delta]}\sigma(t,\zeta) = \infty$;

(A_3) $\eta(\xi) \in C([\alpha,\beta];\mathbb{R})$ and $\omega(\zeta) \in C([\gamma,\delta];\mathbb{R})$ are increasing functions on $[\alpha,\beta]$ and $[\gamma,\delta]$, respectively, and the integrals appearing in (2) are Stieltjes integrals ;

(A_4) $\varphi(s) \in C(\mathbb{R};\mathbb{R})$, $\varphi(-s) = -\varphi(s)$, $\varphi(s) > 0$ for $s > 0$, and $\varphi(s)$ is nondecreasing and convex in $(0,\infty)$.

The following two kinds of boundary conditions are considered :

(B_1) $u = \psi \quad \text{on} \quad \partial G \times (0,\infty)$,

(B_2) $\dfrac{\partial u}{\partial \nu} + \mu u = \tilde{\psi} \quad \text{on} \quad \partial G \times (0,\infty)$,

where $\psi,\ \tilde{\psi} \in C(\partial G \times (0,\infty);\mathbb{R})$, $\mu \in C(\partial G \times (0,\infty);[0,\infty))$ and ν denotes the unit exterior normal vector to ∂G.

Definition 1. By a *solution* of equation (2) we mean a function $u(x,t) \in C^2(\overline{G} \times [t_{-1},\infty);\mathbb{R}) \cap C(\overline{G} \times [\tilde{t}_{-1},\infty);\mathbb{R})$ which satisfies (2), where

$$\begin{aligned} t_{-1} &= \min\left\{0,\ \min_{1\le i\le k}\left\{\inf_{t\ge 0}\tau_i(t)\right\},\ \min_{\xi\in[\alpha,\beta]}\left\{\inf_{t\ge 0}\rho(t,\xi)\right\}\right\}, \\ \tilde{t}_{-1} &= \min\left\{0,\ \min_{\zeta\in[\gamma,\delta]}\left\{\inf_{t\ge 0}\sigma(t,\zeta)\right\}\right\}. \end{aligned}$$

Definition 2. A solution $u(x,t)$ of equation (2) is said to be *oscillatory* in Ω if $u(x,t)$ has a zero in $G \times (t,\infty)$ for any $t > 0$.

In Section 2 we reduce the multi-dimensional oscillation problems to one-dimensional oscillation problems for functional differential inequalities. In Section 3 we derive sufficient conditions for functional differential inequalities to have no eventually positive unbounded solutions. Oscillation results for boundary value problems (2), (B_i) ($i = 1,2$) are presented in Section 4.

2. Reduction to one-dimensional oscillation problems

In this section we reduce the multi-dimensional oscillation problems for (2) to the nonexistence of eventually positive unbounded solutions of functional differential inequalities.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta v &= \lambda v \quad \text{in} \quad G, \\ v &= 0 \quad \text{on} \quad \partial G \end{aligned}$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G (see Courant and Hilbert [1]).

The following notation will be used :

$$\begin{aligned} F(t) &= \left(\int_G \Phi(x)dx \right)^{-1} \int_G f(x,t)\Phi(x)dx, \\ \Psi(t) &= \left(\int_G \Phi(x)dx \right)^{-1} \int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu}(x)dS, \\ \tilde{F}(t) &= \frac{1}{|G|} \int_G f(x,t)dx, \\ \tilde{\Psi}(t) &= \frac{1}{|G|} \int_{\partial G} \tilde{\psi}\, dS, \end{aligned}$$

where $|G| = \displaystyle\int_G dx$.

Theorem 1. *Assume that the hypotheses* (A$_1$)–(A$_4$) *hold. If the functional differential inequalities*

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(y(t) - \int_\alpha^\beta h(t,\xi)y(\rho(t,\xi))d\eta(\xi) \right) \right] + \int_\gamma^\delta Q(t,\zeta)\varphi\big(y(\sigma(t,\zeta))\big)d\omega(\zeta) \le \pm G(t) \tag{3}$$

have no eventually positive unbounded solutions, then every solution u of the boundary value problem (2), (B$_1$) *with unbounded $U(t)$ is oscillatory in Ω, where*

$$\begin{aligned} Q(t,\zeta) &= \min_{x \in \overline{G}} q(x,t,\zeta), \\ G(t) &= F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(t)\Psi(\tau_i(t)), \\ U(t) &= \left(\int_G \Phi(x)dx \right)^{-1} \int_G u(x,t)\Phi(x)dx. \end{aligned}$$

Proof. Suppose to the contrary that there exists a nonoscillatory solution u of the problem (2), (B_1) with the property that $U(t)$ is unbounded. First we assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Then there is a number $t_1 \geq t_0$ such that $u(x, \tau_i(t)) > 0$ in $G \times [t_1, \infty)$ $(i = 1, 2, ..., k)$, $u(x, \sigma(t, \zeta)) > 0$ in $G \times [t_1, \infty) \times [\gamma, \delta]$. Multiplying (2) by $\left(\int_G \Phi(x)dx\right)^{-1} \Phi(x)$ and then integrating over G yields

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(U(t) - \int_\alpha^\beta h(t,\xi)U(\rho(t,\xi))d\eta(\xi)\right)\right]$$
$$-a(t)K_\Phi \int_G \Delta u(x,t)\Phi(x)dx - \sum_{i=1}^k b_i(t)K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx$$
$$+K_\Phi \int_G q_0(x,t)u(x,t)\Phi(x)dx$$
$$+\int_\gamma^\delta Q(t,\zeta)K_\Phi \int_G \varphi\big(u(x,\sigma(t,\zeta))\big)\Phi(x)dxd\omega(\zeta) \leq F(t),\ t \geq t_1, \quad (4)$$

where $K_\Phi = \left(\int_G \Phi(x)dx\right)^{-1}$. It follows from Green's formula that

$$K_\Phi \int_G \Delta u(x,t)\Phi(x)dx = -\Psi(t) - \lambda_1 U(t), \quad t \geq t_1, \quad (5)$$

$$K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx = -\Psi(\tau_i(t)) - \lambda_1 U(\tau_i(t)),\ t \geq t_1 \quad (6)$$

(see, e.g., [14, p.79]). An application of Jensen's inequality shows that

$$K_\Phi \int_G \varphi\big(u(x,\sigma(t,\zeta))\big)\Phi(x)dx \geq \varphi(U(\sigma(t,\zeta))), \quad t \geq t_1. \quad (7)$$

Combining (4)–(7) yields

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(U(t) - \int_\alpha^\beta h(t,\xi)U(\rho(t,\xi))d\eta(\xi)\right)\right]$$
$$+\lambda_1 a(t)U(t) + \lambda_1 \sum_{i=1}^k b_i(t)U(\tau_i(t)) + K_\Phi \int_G q_0(x,t)u(x,t)\Phi(x)dx$$
$$+\int_\gamma^\delta Q(t,\zeta)\varphi\big(U(\sigma(t,\zeta))\big)d\omega(\zeta) \leq G(t), \quad t \geq t_1,$$

and therefore

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(U(t) - \int_\alpha^\beta h(t,\xi)U(\rho(t,\xi))d\eta(\xi)\right)\right]$$
$$+\int_\gamma^\delta Q(t,\zeta)\varphi\big(U(\sigma(t,\zeta))\big)d\omega(\zeta) \leq G(t), \quad t \geq t_1.$$

It is clear that $U(t) > 0$ on $[t_1, \infty)$. Hence, $U(t)$ is an eventually positive unbounded solution of (3) with $+G(t)$. This contradicts the hypothesis. If $u < 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$, we observe that $V(t) = -U(t)$ is an eventually positive unbounded solution of (3) with $-G(t)$. This also contradicts the hypothesis. The proof is complete.

Theorem 2. *Assume that the hypotheses* (A_1)–(A_4) *hold. If the functional differential inequalities*

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(y(t) - \int_\alpha^\beta h(t,\xi)y(\rho(t,\xi))d\eta(\xi)\right)\right] + \int_\gamma^\delta Q(t,\zeta)\varphi\big(y(\sigma(t,\zeta))\big)d\omega(\zeta) \le \pm\tilde{G}(t) \tag{8}$$

have no eventually positive unbounded solutions, then every solution u *of the boundary value problem* (2), (B_2) *with unbounded* $\tilde{U}(t)$ *is oscillatory in* Ω, *where*

$$\tilde{G}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(t)\tilde{\Psi}(\tau_i(t)),$$

$$\tilde{U}(t) = \frac{1}{|G|}\int_G u(x,t)dx.$$

Proof. Assume on the contrary that there is a nonoscillatory solution u of the problem (2), (B_2) with the property that $\tilde{U}(t)$ is unbounded. First we assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Then there is a number $t_1 \ge t_0$ such that $u(x, \tau_i(t)) > 0$ in $G \times [t_1, \infty)$ $(i = 1, 2, ..., k)$, $u(x, \sigma(t,\zeta)) > 0$ in $G \times [t_1, \infty) \times [\gamma, \delta]$. Dividing (2) by $|G|$ and then integrating over G yields

$$\begin{aligned}
&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\tilde{U}(t) - \int_\alpha^\beta h(t,\xi)\tilde{U}(\rho(t,\xi))d\eta(\xi)\right)\right] \\
&-a(t)\frac{1}{|G|}\int_G \Delta u(x,t)dx - \sum_{i=1}^{k} b_i(t)\frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx \\
&+\frac{1}{|G|}\int_G q_0(x,t)u(x,t)dx \\
&+\int_\gamma^\delta Q(t,\zeta)\frac{1}{|G|}\int_G \varphi\big(u(x,\sigma(t,\zeta))\big)dxd\omega(\zeta) \le \tilde{F}(t), \ t \ge t_1.
\end{aligned} \tag{9}$$

The divergence theorem implies that

$$\begin{aligned}\frac{1}{|G|}\int_G \Delta u(x,t)dx &= \frac{1}{|G|}\int_{\partial G}\frac{\partial u}{\partial \nu}(x,t)dS \\ &= \frac{1}{|G|}\int_{\partial G}\left(-\mu\cdot u(x,t)+\tilde{\psi}\right)dS \\ &\le \tilde{\Psi}(t), \quad t \ge t_1. \end{aligned} \tag{10}$$

Analogously we obtain

$$\frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx \le \tilde{\Psi}(\tau_i(t)), \quad t \ge t_1. \tag{11}$$

An application of Jensen's inequality yields

$$\frac{1}{|G|}\int_G \varphi\big(u(x,\sigma(t,\zeta))\big)dx \ge \varphi(\tilde{U}(\sigma(t,\zeta))), \quad t \ge t_1. \tag{12}$$

Combining (9)–(12) and taking account of the hypothesis (A_1), we have

$$\begin{aligned}&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\tilde{U}(t)-\int_\alpha^\beta h(t,\xi)\tilde{U}(\rho(t,\xi))d\eta(\xi)\right)\right] \\ &\qquad + \int_\gamma^\delta Q(t,\zeta)\varphi\big(\tilde{U}(\sigma(t,\zeta))\big)d\omega(\zeta) \le \tilde{G}(t), \quad t \ge t_1. \end{aligned} \tag{13}$$

Consequently we observe that $\tilde{U}(t)$ is an eventually positive unbounded solution of (8) with $+\tilde{G}(t)$. This contradicts the hypothesis. The case where $u < 0$ can be treated similarly, and we are led to a contradiction. The proof is complete.

3. Functional differential inequalities

In this section we derive sufficient conditions for the functional differential inequality

$$\begin{aligned}&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(y(t)-\int_\alpha^\beta h(t,\xi)y(\rho(t,\xi))d\eta(\xi)\right)\right] \\ &\qquad + \int_\gamma^\delta Q(t,\zeta)\varphi\big(y(\sigma(t,\zeta))\big)d\omega(\zeta) \le H(t) \end{aligned} \tag{14}$$

to have no eventually positive unbounded solution, where $H(t)$ is a continuous function.

It is assumed that :

(A_5) there exists a positive constant h_0 satisfying

$$\int_\alpha^\beta h(t,\xi)d\eta(\xi) \le h_0 < 1 \ ;$$

(A_6) $\rho(t,\xi) \le t \quad$ for $(t,\xi) \in (0,\infty) \times [\alpha,\beta]$;

(A_7) $\tilde{\sigma}(t) \equiv \min_{\zeta\in[\gamma,\delta]} \sigma(t,\zeta)$ is a nondecreasing continuous function.

Theorem 3. *Assume that the hypotheses* (A_1)–(A_7) *hold, and that the following hypothesis is satisfied* :

(A_8) *there is a* C^2*-function* $\theta(t)$ *such that* $\theta(t)$ *is bounded and*

$$\big(p(t)\theta'(t)\big)' = H(t).$$

If the following conditions is satisfied :

$$\int_c^\infty \left[\int_\gamma^\delta Q(t,\zeta)d\omega(\zeta)\right] dt = +\infty \tag{15}$$

for some $c > 0$, *then* (14) *has no eventually positive unbounded solution.*

Proof. Suppose that (14) has an eventually positive unbounded solution $y(t)$. Letting

$$z(t) = y(t) - \int_\alpha^\beta h(t,\xi)y(\rho(t,\xi))d\eta(\xi) - \theta(t)$$

and taking into account (A_8), we find that

$$\begin{aligned}\big(p(t)z'(t)\big)' &\le -\int_\gamma^\delta Q(t,\zeta)\varphi\big(y(\sigma(t,\zeta))\big)d\omega(\zeta) \\ &\le 0.\end{aligned} \tag{16}$$

Therefore, $p(t)z'(t) \ge 0$ or $p(t)z'(t) < 0$ eventually. Since $p(t) > 0$, we see that $z'(t) \ge 0$ or $z'(t) < 0$. Hence, $z(t)$ is a monotone function, and $z(t) > 0$ or $z(t) \le 0$ eventually. We claim that $\lim_{t\to\infty} z(t) = \infty$. Hence, $z(t) > 0$ eventually. Since $y(t)$ is unbounded from above, there exists a sequence

$\{t_n\}_{n=1}^{\infty}$ satisfying $\lim\limits_{n\to\infty} t_n = \infty$, $\lim\limits_{n\to\infty} y(t_n) = \infty$ and $\max\limits_{t_0\le t\le t_n} y(t) = y(t_n)$. The hypotheses ($A_5$) and ($A_6$) imply that

$$\begin{aligned} z(t_n) &= y(t_n) - \int_\alpha^\beta h(t_n,\xi) y(\rho(t_n,\xi)) d\eta(\xi) - \theta(t_n) \\ &\ge y(t_n) - y(t_n)\int_\alpha^\beta h(t_n,\xi) d\eta(\xi) - \theta(t_n) \\ &= \left(1 - \int_\alpha^\beta h(t_n,\xi) d\eta(\xi)\right) y(t_n) - \theta(t_n) \\ &\ge (1-h_0) y(t_n) - \theta(t_n) \end{aligned}$$

for sufficiently large n. Since $\theta(t)$ is bounded and $\lim\limits_{n\to\infty}(1-h_0)y(t_n) = \infty$, we find that $\lim\limits_{t\to\infty} z(t_n) = \infty$. This combined with the monotonicity property of $z(t)$ implies that $\lim\limits_{t\to\infty} z(t) = \infty$. In this case it is easily seen that $z'(t) \ge 0$. Since $\theta(t)$ is bounded and $\lim\limits_{t\to\infty} z(t) = \infty$, for any $\varepsilon > 0$ there is a sufficiently large number T such that $\theta(t) \ge -\varepsilon z(t)$ $(t \ge T)$. Hence we see that

$$y(t) \ge z(t) + \theta(t) \ge (1-\varepsilon) z(t)$$

and therefore

$$y(\sigma(t,\zeta)) \ge (1-\varepsilon) z(\sigma(t,\zeta)).$$

The inequality (16) implies that

$$\begin{aligned} \big(p(t)z'(t)\big)' &\le -\int_\gamma^\delta Q(t,\zeta)\varphi\big((1-\varepsilon)z(\sigma(t,\zeta))\big) d\omega(\zeta) \\ &\le -\varphi\big((1-\varepsilon)z(\tilde{\sigma}(t))\big)\int_\gamma^\delta Q(t,\zeta) d\omega(\zeta) \\ &\le -\varphi\big((1-\varepsilon)z(\tilde{\sigma}(T))\big)\int_\gamma^\delta Q(t,\zeta) d\omega(\zeta) \\ &\equiv -C_0 \int_\gamma^\delta Q(t,\zeta) d\omega(\zeta), \quad t \ge T, \end{aligned} \tag{17}$$

where $T > 0$ sufficiently large and $C_0 > 0$ by (A_4). Integrating (17) over $[T,t]$, we obtain

$$p(t)z'(t) - p(T)z'(T) \le -C_0 \int_T^t \left[\int_\gamma^\delta Q(s,\zeta) d\omega(\zeta)\right] ds$$

which yields

$$p(T)z'(T) \geq C_0 \int_T^t \left[\int_\gamma^\delta Q(s,\zeta) d\omega(\zeta) \right] ds.$$

Letting $t \to \infty$ in the above inequality, we obtain

$$\int_T^\infty \left[\int_\gamma^\delta Q(s,\zeta) d\omega(\zeta) \right] ds \leq \frac{1}{C_0} p(T)z'(T) < \infty,$$

which contradicts the hypothesis (15). The proof is complete.

4. Oscillation results

In this section we present oscillation results for the boundary value problems for (2), (B_i) ($i = 1, 2$) by combining the results in Sections 2 and 3.

Theorem 4. *Assume that the hypotheses* (A_1)–(A_7) *hold, and that there exists a* C^2*-function* $\theta(t)$ *such that* $\theta(t)$ *is bounded and*

$$\big(p(t)\theta'(t)\big)' = G(t).$$

If the condition (15) *is satisfied, then every solution* u *of the boundary value problem* (2), (B_1) *with unbounded* $U(t)$ *is oscillatory in* Ω.

Proof. The conclusion follows by combining Theorem 1 with Theorem 3.

Theorem 5. *Assume that the hypotheses* (A_1)–(A_7) *hold, and that there exists a* C^2*-function* $\theta(t)$ *such that* $\theta(t)$ *is bounded and*

$$\big(p(t)\theta'(t)\big)' = \tilde{G}(t).$$

If the condition (15) *is satisfied, then every solution* u *of the boundary value problem* (2), (B_2) *with unbounded* $\tilde{U}(t)$ *is oscillatory in* Ω.

Proof. A combination of Theorem 2 and Theorem 3 yields the conclusion.

Example. We consider the problem

$$\begin{aligned} &\frac{\partial}{\partial t}\left[p_0 \frac{\partial}{\partial t}\left(u(x,t) - \int_0^\pi \frac{1}{4} \cdot u(x, t-2\pi+\xi) d\xi \right) \right] \\ &\quad - e^{-t} \frac{\partial^2 u}{\partial x^2}(x,t) + q_0 u(x,t) + \int_0^{\pi/2} u(x, t-\pi+\zeta) d\zeta \\ &\quad = (\sin x) \sin t, \quad (x,t) \in (0,\pi) \times (0,\infty), \end{aligned} \tag{18}$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0, \tag{19}$$

where

$$
\begin{aligned}
p_0 &= e^{-\pi}(e^{\pi/2}+1)\left[4+\frac{1}{2}e^{-2\pi}(e^{\pi}+1)\right]^{-1} > 0,\\
q_0 &= \frac{e^{-\pi}(e^{\pi/2}-1)}{2} - p_0\frac{e^{-2\pi}(e^{\pi}+1)}{2}\\
&= \frac{e^{-\pi}\left[(e^{\pi/2}-1)4e^{2\pi}-\frac{1}{2}(e^{\pi/2}+3)(e^{\pi}+1)\right]}{8e^{2\pi}+e^{\pi}+1}\\
&> \frac{e^{-\pi}\left(4e^{2\pi}-2e^{\pi/2}e^{\pi}\right)}{8e^{2\pi}+e^{\pi}+1}\\
&= \frac{2e^{\pi/2}(2e^{\pi/2}-1)}{8e^{2\pi}+e^{\pi}+1} > 0.
\end{aligned}
$$

Here $n=1$, $G=(0,\pi)$, $\Omega=(0,\pi)\times(0,\infty)$, $p(t)=p_0$, $[\alpha,\beta]=[0,\pi]$, $h(t,\xi)=1/4$, $\rho(t,\xi)=t-2\pi+\xi$, $\eta(\xi)=\xi$, $b_i(t)\equiv 0$, $a(t)=e^{-t}$, $q_0(x,t)=q_0$, $q_i(x,t)\equiv 0$, $[\gamma,\delta]=[0,\pi/2]$, $q(x,t,\zeta)=Q(t,\zeta)=1$, $\varphi(s)=s$, $\sigma(t,\zeta)=t-\pi+\zeta$, $\omega(\zeta)=\zeta$ and $f(x,t)=(\sin x)\sin t$. It is easily seen that $\lambda_1=1$ and $\Phi(x)=\sin x$. Since

$$\int_0^{\pi} h(t,\xi)d\eta(\xi) = \int_0^{\pi}\frac{1}{4}\,d\xi = \frac{\pi}{4} < 1,$$

we can choose $h_0=\pi/4$, and hence (A_5) is satisfied. It is easy to check that

$$\rho(t,\xi)=t-2\pi+\xi \le t-2\pi+\pi = t-\pi \le t,$$

and hence (A_6) is satisfied. Since

$$\tilde{\sigma}(t)=\min_{\zeta\in[0,\pi]}\big(t-\pi+\zeta\big)=t-\pi,$$

we find that (A_7) holds. An easy computation shows that

$$G(t)=F(t)=\frac{\pi}{4}\sin t.$$

Choosing $\theta(t)=-(\pi/4)\sin t$, we observe that $\theta''(t)=G(t)$ and $\theta(t)$ is bounded. It is obvious that

$$\int_c^{\infty}\Big[\int_{\gamma}^{\delta}Q(t,\zeta)d\omega(\zeta)\Big]dt=\int_c^{\infty}\frac{\pi}{2}dt=+\infty$$

and hence the condition (15) holds. It follows from Theorem 4 that every solution of (18), (19) with unbounded $U(t)$ is oscillatory in $(0,\pi)\times(0,\infty)$. In fact,

$$u = (\sin x)e^t \sin t$$

is such a solution.

Remark. The following restrictions have been made in [2], [6], [11] :

(R_1) $\tilde{\sigma}(t) \equiv \min_{\zeta\in[\gamma,\delta]} \sigma(t,\zeta)$ is a nondecreasing C^1-function such that

$$\begin{aligned} &\tilde{\sigma}(t) \geq t, \\ &\tilde{\sigma}'(t) \geq \frac{1}{\sigma_0} \quad \text{for some } \sigma_0 > 0 \ ; \end{aligned}$$

(R_2)

$$\int_c^\infty \frac{1}{\varphi(v)} dv < \infty \quad \text{for some } c > 0;$$

or there is a constant K_0 such that $\dfrac{\varphi(v)}{v} \geq K_0 > 0$ for $v \neq 0$.

However, in present paper we remove the above two restrictions.

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