Math. J. Toyama Univ.
Vol. 26(2003), 85-91

Some implications of indivisibility of special values of zeta functions of real quadratic fields

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Abstract. In this paper, we show some implications of Byeon's result. For example, we prove that, for an odd prime number p, there exist infinitely many real quadratic fields $\mathbf{Q}(\sqrt{D})$ which satisfy following properties: For each non-negative integer n, let $\mathbf{Q}(\sqrt{D})_n$ denote n-th layer of the cyclotomic $\mathbf{Z}_p$-extension over $\mathbf{Q}(\sqrt{D})$. Then, for each $n \geq 0$, there exist infinitely many CM-fields K whose maximal real subfield is $\mathbf{Q}(\sqrt{D})_n$ and whose relative Iwasawa λ- and μ-invariants for the cyclotomic $\mathbf{Z}_p$-extension over K are zero.

1. Introduction

We fix once and for all the algebraic closure $\overline{\mathbf{Q}}$ of the field of rational numbers $\mathbf{Q}$ in the field of complex numbers. All number fields of finite or infinite degree over $\mathbf{Q}$ are assumed to be subfields of $\overline{\mathbf{Q}}$. Let ζ_n denote the primitive n-th root of unity for a natural number n.

For any number field k of finite degree, let $\zeta_k(s)$ denote the Dedekind zeta function for k. For any rational prime p, let $\lambda_p(k), \mu_p(k)$ denote Iwasawa λ- and μ-invariants for the cyclotomic $\mathbf{Z}_p$-extension k_∞/k. For each nonnegative integer n, let k_n denote n-th layer of k_∞/k, that is, the unique

2000 *Mathematics Subject Classification.* Primary 11R23; Secondary 11R11, 11R29.
Key words and phrases. Iwasawa invariant, CM-field.
*Partially supported by Grant-in-Aid for Young Scientists (B), 14740009, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

intermediate field of k_∞/k of degree $[k_n : k] = p^n$. Let O_k denote the ring of integers of k.

Let F be a totally real number field. Let $w_F = 2^{n(2)+1}\prod_p p^{n(p)}$, where $n(p)$ is the maximal non-negative integer n such that the degree of extension $F(\zeta_{p^n})/F$ is at most 2. Serre [11] proved that $w_F\zeta_F(-1)$ is a rational integer.

Let K be a CM-field, that is, a totally imaginary quadratic extension over totally real subfield K^+. For any rational prime p, we define relative Iwasawa invariants for the cyclotomic $\mathbf{Z}_p$-extension of K as follows:

$$\begin{aligned} \lambda_p^-(K) &= \lambda_p(K) - \lambda_p(K^+), \\ \mu_p^-(K) &= \mu_p(K) - \mu_p(K^+). \end{aligned}$$

For example, if K is an imaginary quadratic field, $\lambda_p^-(K) = \lambda_p(K)$ and $\mu_p^-(K) = \mu_p(K)$ since $\lambda_p(\mathbf{Q}) = \mu_p(\mathbf{Q}) = 0$ for any prime p.

For a totally real field F and a rational prime p, let $\Omega_p(F)$ denote the set

$$\Omega_p(F) := \{K \,|\, K \text{ is a CM-field, } K^+ = F \text{ and } \lambda_p^-(K) = \mu_p^-(K) = 0\}.$$

Horie [3] proved that, for each odd prime p, the set $\Omega_p(\mathbf{Q})$ is an infinite set (see also Horie [4, §3]). Naito [9] extended Horie's argument and proved the following theorem:

Theorem 1.1 (Naito). *Let k be a totally real number field. Let p be an odd prime and suppose $p \nmid w_k\zeta_k(-1)$. Then $\Omega_p(k)$ is an infinite set.*

It is natural to consider about the hypothesis of theorem 1.1. Byeon [1] showed that infinitely many real quadratic fields satisfy the hypothesis:

Theorem 1.2 (Byeon). *Let p be an odd prime. Then there exist infinitely many real quadratic fields $\mathbf{Q}(\sqrt{D})$ for which $p \nmid w_{\mathbf{Q}(\sqrt{D})}\zeta_{\mathbf{Q}(\sqrt{D})}(-1)$ hold. For each of these real quadratic fields $\mathbf{Q}(\sqrt{D})$, thus, $\Omega_p(\mathbf{Q}(\sqrt{D}))$ is an infinite set.*

In this paper, we prove that, for each real quadratic field $F = \mathbf{Q}(\sqrt{D})$ which satisfies the hypothesis of theorem 1.1, and for an odd prime p (thus $\Omega_p(F)$ is an infinite set), all of the n-th layers F_n of the cyclotomic $\mathbf{Z}_p$-extension of F also satisfy the hypothesis:

Theorem 1.3. *Let p be an odd prime. Then there exist infinitely many real quadratic fields $F = \mathbf{Q}(\sqrt{D})$ such that, for each n-th layer F_n $(n \geq 0)$ of the cyclotomic $\mathbf{Z}_p$-extension over F, $\Omega_p(F_n)$ is an infinite set.*

Remark 1.4. *We note that, for any totally real field F, the condition $p \nmid w_F\zeta_F(-1)$ is a sufficient condition for $\Omega_p(F)$ to be an infinite set. For the case of $p = 3$, Horie and the author* [5] *proved that, for any totally real field F, $\Omega_3(F)$ is an infinite set for either case $w_F\zeta_F(-1)$ is divisible by* 3 *or not.*

2. Proof of Theorem 1.3

Komatsu [7] proved the following theorem.

Theorem 2.1 (K. Komatsu). *Let F be a totally real number field, p a prime and F'/F p-extension. Let ζ_p be a primitive p-th root of* 1*. We assume that the Iwasawa μ-invariant for $F(\zeta_p)$ is zero: $\mu_p(F(\zeta_p)) = 0$. Then the following assertion holds: If $[F(\zeta_p) : F] \neq 2$ and F'/F is unramified outside p, then $p | \sharp K_2(O_F)$ if and only if $p | \sharp K_2(O_{F'})$.*

His proof is based on an interpretation of Quillen's K-group $K_2(O_F)$ of O_F into certain ideal class group due to C. Soulé, and on the Riemann-Hurwitz type theorem in Iwasawa theory for $\mathbf{Z}_p$-extensions due to K. Iwasawa and Y. Kida. Of course for all but finitely many real quadratic field $F = \mathbf{Q}(\sqrt{D})$, $[F(\zeta_p) : F] \neq 2$ providing p is given. Since $F(\zeta_p) = \mathbf{Q}(\sqrt{D}, \zeta_p)$ is an Abelian field (number field which are Abelian extension of $\mathbf{Q}$), $\mu_p(F(\zeta_p)) = 0$ is known by Ferrero-Washington [2].

On the other hand, one of the consequences of the main conjecture in Iwasawa theory (proved by Mazur-Wiles [8, Theorem 5] for Abelian fields and by Wiles [13, Theorem 1.5] for any Abelian extensions over totally real number fields) is the following equality:

$$\sharp K_2(O_F) \sim_p w_2(F)\zeta_F(-1), \tag{1}$$

where for any integers a, b, let $a \sim_p b$ denote a/b is a p-adic unit.

Combining these results and theorem 1.2, we obtain the following proposition:

Proposition 2.2. *Let p be an odd prime number. Then there exist infinitely many real quartic fields $\mathbf{Q}(\sqrt{D})$ such that for any p-extension $F/\mathbf{Q}(\sqrt{D})$ unramified outside p, $p \nmid w_F\zeta_F(-1)$, that is, $\Omega_p(F)$ is an infinite set.*

For any number field k, if a prime $\mathfrak{p}$ of k is ramified in any $\mathbf{Z}_p$-extension over k, $\mathfrak{p}$ is lying above p, in other words, any $\mathbf{Z}_p$-extension over k is unramified outside p (see, e.g., Washington [12, Proposition 13.2]). Thus we are done. □

For a totally real field F, let $\mathcal{B}_p(F)$ denote the set of all CM-fields K such that K is bicyclic quartic extension of F and satisfies $\lambda_p^-(K) = \mu_p^-(K) = 0$.

Cororally 2.3. *For an odd prime p, there exist infinitely many real quadratic fields $F = \mathbf{Q}(\sqrt{D})$ such that, for each n-th layer F_n $(n \geq 0)$ of the cyclotomic $\mathbf{Z}_p$-extension over F, $\mathcal{B}_p(F_n)$ is an infinite set.*

Proof. As we showed in Horie and the author [5], the relative Iwasawa invariants behave additively for composition of two distinct CM-fields K, K' whose maximal real subfields are coincide and for an odd prime p:

$$\begin{aligned} \lambda_p^-(K \cdot K') &= \lambda_p^-(K) + \lambda_p^-(K'), \\ \mu_p^-(K \cdot K') &= \mu_p^-(K) + \mu_p^-(K'). \end{aligned}$$

For $K, K' \in \Omega_p(F_n)$, $K \cdot K' \in \mathcal{B}_p(F_n)$ by these formulae. Therefore if $\Omega_p(F_n)$ is an infinite set, so is $\mathcal{B}_p(F_n)$. □

For small primes p, one can easily find a real quadratic field $F = \mathbf{Q}(\sqrt{D})$ such that $p | w_F\zeta_F(-1)$ via a formula $w_F\zeta_F(-1) = B_{2,\chi}$ for $F = \mathbf{Q}(\sqrt{D}), D > 5$, where $\chi(\cdot) = (D/\cdot)$ is the Kronecker symbol and $B_{2,\chi}$ is the generalized Bernoulli number. For example, if $p = 5$ and $F = \mathbf{Q}(\sqrt{37})$, then $w_F\zeta_F(-1) = 2^2 \cdot 5$, if $p = 7$ and $F = \mathbf{Q}(\sqrt{40})$, then $w_F\zeta_F(-1) = 2^2 \cdot 7$ and if $p = 11$ and $F = \mathbf{Q}(\sqrt{61})$, then $w_F\zeta_F(-1) = 2^2 \cdot 11$. But we can say nothing about $\Omega_p(F)$ (see also Naito [10]).

In view of these numerical computations, it seems probable that for each rational prime p, there exists a real quadratic field $F = \mathbf{Q}(\sqrt{D})$ satisfying $p | w_F\zeta_F(-1) = B_{2,\chi}$. We end this paper with the following proposition which gives an equivalent condition of $p | w_F\zeta_F(-1)$.

Put $F(\zeta_{p^\infty}) = \cup_{n\geq 1} F(\zeta_{p^n})$, then $F(\zeta_{p^\infty})/F(\zeta_p)$ is the cyclotomic $\mathbf{Z}_p$-extension. Let M_∞ be the maximal Abelian p-extension of $F(\zeta_{p^\infty})$ unramified outside p, and $\mathcal{X}_\infty = \mathrm{Gal}(M_\infty/F(\zeta_{p^\infty}))$. Put $\Delta = \mathrm{Gal}(F(\zeta_p)/F)$. Since $\mathrm{Gal}(F(\zeta_{p^\infty})/F) = \mathrm{Gal}(F(\zeta_{p^\infty})/F(\zeta_p)) \times \Delta$ acts on $\mathcal{X}_\infty$ via conjugation, so does Δ.

Let $\varepsilon_p := |\Delta|^{-1} \sum_{\delta\in\Delta} \omega^i(\delta)\delta^{-1}$ be an idempotent of $\mathbf{Z}_p[\Delta]$, where ω is the p-adic Teichmüller character.

Proposition 2.4. *Let p be an odd prime, $F = \mathbf{Q}(\sqrt{D})$ a real quadratic field. Then, $p|w_F\zeta_F(-1) = B_{2,\chi}$ if and only if $\varepsilon_2\mathcal{X}_\infty \neq 0$.*

Proof. Let A_∞ be the p-Sylow subgroup of the ideal class group of $F(\zeta_{p^\infty})$, that is, $A_\infty = \varinjlim_{n\geq 1} A_n$, where A_n is the p-Sylow subgroup of the ideal class group of $F(\zeta_{p^n})$. By the action induced from that of $\mathrm{Gal}(F(\zeta_{p^\infty})/F)$, A_∞ becomes a $\mathbf{Z}_p[\Delta]$-module.

We can assume $F \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$. Then

$$\Delta \cong \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q}) \cong (\mathbf{Z}/p\mathbf{Z})^\times.$$

As we saw in (1), $p|w_2(F)\zeta_F(-1)$ if and only if $p|\sharp K_2(O_F)$. On the other hand, lemma 3 of Komatsu [7] states that, $p|\sharp K_2(O_F)$ if and only if $\varepsilon_{p-2}A_\infty \neq 0$.

Let T be a projective limit of all p-power-th roots of unity (with respect to the p-power map) and put $\varepsilon_i\mathcal{X}_\infty(-1) := \varepsilon_i\mathcal{X}_\infty \otimes_{\mathbf{Z}_p} \mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p)$. The standard argument in Iwasawa theory for $\mathbf{Z}_p$-extensions involving Kummer theory over $F(\zeta_{p^\infty})$, one can show that, for $i + j \equiv 1 \pmod{|\Delta|}$, i being an odd integer,

$$\varepsilon_j\mathcal{X}_\infty(-1) \cong \mathrm{Hom}_{\mathbf{Z}_p}(\varepsilon_i A_\infty, \mathbf{Q}_p/\mathbf{Z}_p)$$

as $\mathbf{Z}_p[[\mathrm{Gal}(F(\zeta_{p^\infty})/F(\zeta_p))]]$-modules (see, for example, [6], [12, Proposition 13.32]). Putting $i = p-2$, we obtain our assertion since $\varepsilon_2\mathcal{X}_\infty(-1) = \varepsilon_2\mathcal{X}_\infty$ as Abelian groups. □

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(Received September 2, 2003)