Some implications of indivisibility of special values of zeta functions of real quadratic fields

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Abstract. In this paper, we show some implications of Byeon’s result. For example, we prove that, for an odd prime number \( p \), there exist infinitely many real quadratic fields \( \mathbb{Q}(\sqrt{D}) \) which satisfy following properties: For each non-negative integer \( n \), let \( \mathbb{Q}(\sqrt{D})_n \) denote \( n \)-th layer of the cyclotomic \( \mathbb{Z}_p \)-extension over \( \mathbb{Q}(\sqrt{D}) \). Then, for each \( n \geq 0 \), there exist infinitely many CM-fields \( K \) whose maximal real subfield is \( \mathbb{Q}(\sqrt{D})_n \) and whose relative Iwasawa \( \lambda \)- and \( \mu \)-invariants for the cyclotomic \( \mathbb{Z}_p \)-extension over \( K \) are zero.

1. Introduction

We fix once and for all the algebraic closure \( \overline{\mathbb{Q}} \) of the field of rational numbers \( \mathbb{Q} \) in the field of complex numbers. All number fields of finite or infinite degree over \( \mathbb{Q} \) are assumed to be subfields of \( \overline{\mathbb{Q}} \). Let \( \zeta_n \) denote the primitive \( n \)-th root of unity for a natural number \( n \).

For any number field \( k \) of finite degree, let \( \zeta_k(s) \) denote the Dedekind zeta function for \( k \). For any rational prime \( p \), let \( \lambda_p(k), \mu_p(k) \) denote Iwasawa \( \lambda \)- and \( \mu \)-invariants for the cyclotomic \( \mathbb{Z}_p \)-extension \( k_\infty/k \). For each nonnegative integer \( n \), let \( k_n \) denote \( n \)-th layer of \( k_\infty/k \), that is, the unique

\[ k_n := \left( \mathbb{Q}(\zeta_p) \cap k_\infty \right)^{1/p^n}. \]

2000 Mathematics Subject Classification. Primary 11R23; Secondary 11R11, 11R29.

Key words and phrases. Iwasawa invariant, CM-field.

*Partially supported by Grant-in-Aid for Young Scientists (B), 14740009, The Ministry of Education, Culture, Sports, Science and Technology, Japan.
intermediate field of $k_{\infty}/k$ of degree $[k_\infty : k] = p^n$. Let $O_k$ denote the ring of integers of $k$.

Let $F$ be a totally real number field. Let $w_F = 2^{n(2)+1} \prod_pp^{n(p)}$, where $n(p)$ is the maximal non-negative integer $n$ such that the degree of extension $F(\wp^n)/F$ is at most 2. Serre [11] proved that $w_F\zeta_F(-1)$ is a rational integer.

Let $K$ be a CM-field, that is, a totally imaginary quadratic extension over totally real subfield $K^+$. For any rational prime $p$, we define relative Iwasawa invariants for the cyclotomic $\mathbb{Z}_p$-extension of $K$ as follows:

$$
\lambda_p^-(K) = \lambda_p(K) - \lambda_p(K^+), \\
\mu_p^-(K) = \mu_p(K) - \mu_p(K^+).
$$

For example, if $K$ is an imaginary quadratic field, $\lambda_p^-(K) = \lambda_p(K)$ and $\mu_p^-(K) = \mu_p(K)$ since $\lambda_p(\mathbb{Q}) = \mu_p(\mathbb{Q}) = 0$ for any prime $p$.

For a totally real field $F$ and a rational prime $p$, let $\Omega_p(F)$ denote the set

$$
\Omega_p(F) := \{K \mid K \text{ is a CM-field, } K^+ = F \text{ and } \lambda_p^-(K) = \mu_p^-(K) = 0\}.
$$

Horie [3] proved that, for each odd prime $p$, the set $\Omega_p(\mathbb{Q})$ is an infinite set (see also Horie [4, §3]). Naito [9] extended Horie’s argument and proved the following theorem:

**Theorem 1.1 (Naito).** Let $k$ be a totally real number field. Let $p$ be an odd prime and suppose $p \nmid w_k\zeta_k(-1)$. Then $\Omega_p(k)$ is an infinite set.

It is natural to consider about the hypothesis of theorem 1.1. Byeon [1] showed that infinitely many real quadratic fields satisfy the hypothesis:

**Theorem 1.2 (Byeon).** Let $p$ be an odd prime. Then there exist infinitely many real quadratic fields $\mathbb{Q}(\sqrt{D})$ for which $p \nmid w_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$ hold. For each of these real quadratic fields $\mathbb{Q}(\sqrt{D})$, thus, $\Omega_p(\mathbb{Q}(\sqrt{D}))$ is an infinite set.

In this paper, we prove that, for each real quadratic field $F = \mathbb{Q}(\sqrt{D})$ which satisfies the hypothesis of theorem 1.1, and for an odd prime $p$ (thus $\Omega_p(F)$ is an infinite set), all of the $n$-th layers $F_n$ of the cyclotomic $\mathbb{Z}_p$-extension of $F$ also satisfy the hypothesis:
Theorem 1.3. Let $p$ be an odd prime. Then there exist infinitely many real quadratic fields $F = \mathbb{Q}(\sqrt{D})$ such that, for each $n$-th layer $F_n$ ($n \geq 0$) of the cyclotomic $\mathbb{Z}_p$-extension over $F$, $\Omega_p(F_n)$ is an infinite set.

Remark 1.4. We note that, for any totally real field $F$, the condition $p \nmid w_F \zeta_F(-1)$ is a sufficient condition for $\Omega_p(F)$ to be an infinite set. For the case of $p = 3$, Horie and the author [5] proved that, for any totally real field $F$, $\Omega_3(F)$ is an infinite set for either case $w_F \zeta_F(-1)$ is divisible by 3 or not.

2. Proof of Theorem 1.3

Komatsu [7] proved the following theorem.

Theorem 2.1 (K. Komatsu). Let $F$ be a totally real number field, $p$ a prime and $F'/F$ $p$-extension. Let $\zeta_p$ be a primitive $p$-th root of 1. We assume that the Iwasawa $\mu$-invariant for $F(\zeta_p)$ is zero: $\mu_p(F(\zeta_p)) = 0$. Then the following assertion holds: If $[F(\zeta_p) : F] \neq 2$ and $F'/F$ is unramified outside $p$, then $p|K_2(O_F)$ if and only if $p|w_2(F)$.

His proof is based on an interpretation of Quillen’s $K$-group $K_2(O_F)$ of $O_F$ into certain ideal class group due to C. Soulé, and on the Riemann-Hurwitz type theorem in Iwasawa theory for $\mathbb{Z}_p$-extensions due to K. Iwasawa and Y. Kida. Of course for all but finitely many real quadratic field $F = \mathbb{Q}(\sqrt{D})$, $[F(\zeta_p) : F] \neq 2$ providing $p$ is given. Since $F(\zeta_p) = \mathbb{Q}(\sqrt{D}, \zeta_p)$ is an Abelian field (number field which are Abelian extension of $\mathbb{Q}$), $\mu_p(F(\zeta_p)) = 0$ is known by Ferrero-Washington [2].

On the other hand, one of the consequences of the main conjecture in Iwasawa theory (proved by Mazur-Wiles [8, Theorem 5] for Abelian fields and by Wiles [13, Theorem 1.5] for any Abelian extensions over totally real number fields) is the following equality:

$$w_2(F) \approx_p K_2(O_F) \sim p\zeta_F(-1),$$

(1)

where for any integers $a, b$, let $a \approx_p b$ denote $a/b$ is a $p$-adic unit.

Combining these results and theorem 1.2, we obtain the following proposition:
Proposition 2.2. Let $p$ be an odd prime number. Then there exist infinitely many real quartic fields $\mathbb{Q}(\sqrt{D})$ such that for any $p$-extension $F/\mathbb{Q}(\sqrt{D})$ unramified outside $p$, $p \nmid w_F \zeta_F(-1)$, that is, $\Omega_p(F)$ is an infinite set.

For any number field $k$, if a prime $p$ of $k$ is ramified in any $\mathbb{Z}_p$-extension over $k$, $p$ is lying above $p$, in other words, any $\mathbb{Z}_p$-extension over $k$ is unramified outside $p$ (see, e.g., Washington [12, Proposition 13.2]). Thus we are done.

For a totally real field $F$, let $B_p(F)$ denote the set of all CM-fields $K$ such that $K$ is bicyclic quartic extension of $F$ and satisfies $\lambda_p^-(K) = \mu_p^-(K) = 0$.

Corollary 2.3. For an odd prime $p$, there exist infinitely many real quadratic fields $F = \mathbb{Q}(\sqrt{D})$ such that, for each $n$-th layer $F_n$ ($n \geq 0$) of the cyclotomic $\mathbb{Z}_p$-extension over $F$, $B_p(F_n)$ is an infinite set.

Proof. As we showed in Horie and the author [5], the relative Iwasawa invariants behave additively for composition of two distinct CM-fields $K, K'$ whose maximal real subfields are coincide and for an odd prime $p$:

$$\lambda_p^-(K \cdot K') = \lambda_p^-(K) + \lambda_p^-(K'),$$
$$\mu_p^-(K \cdot K') = \mu_p^-(K) + \mu_p^-(K').$$

For $K, K' \in \Omega_p(F_n)$, $K \cdot K' \in B_p(F_n)$ by these formulae. Therefore if $\Omega_p(F_n)$ is an infinite set, so is $B_p(F_n)$.

For small primes $p$, one can easily find a real quadratic field $F = \mathbb{Q}(\sqrt{D})$ such that $p | w_F \zeta_F(-1)$ via a formula $w_F \zeta_F(-1) = B_{2,\chi}$ for $F = \mathbb{Q}(\sqrt{D})$, $D > 5$, where $\chi(\cdot) = (D/\cdot)$ is the Kronecker symbol and $B_{2,\chi}$ is the generalized Bernoulli number. For example, if $p = 5$ and $F = \mathbb{Q}(\sqrt{37})$, then $w_F \zeta_F(-1) = 2^2 \cdot 5$, if $p = 7$ and $F = \mathbb{Q}(\sqrt{40})$, then $w_F \zeta_F(-1) = 2^2 \cdot 7$ and if $p = 11$ and $F = \mathbb{Q}(\sqrt{61})$, then $w_F \zeta_F(-1) = 2^2 \cdot 11$. But we can say nothing about $\Omega_p(F)$ (see also Naito [10]).

In view of these numerical computations, it seems probable that for each rational prime $p$, there exists a real quadratic field $F = \mathbb{Q}(\sqrt{D})$ satisfying $p | w_F \zeta_F(-1) = B_{2,\chi}$. We end this paper with the following proposition which gives an equivalent condition of $p | w_F \zeta_F(-1)$.
Indivisibility of special values of zeta functions

Put $F(\zeta_p^\infty) = \cup_{n\geq 1} F(\zeta_p^n)$, then $F(\zeta_p^\infty)/F(\zeta_p)$ is the cyclotomic $\mathbb{Z}_p$-extension. Let $M_\infty$ be the maximal Abelian $p$-extension of $F(\zeta_p^\infty)$ unramified outside $p$, and $\mathcal{X}_\infty = \text{Gal}(M_\infty/F(\zeta_p^\infty))$. Put $\Delta = \text{Gal}(F(\zeta_p)/F)$. Since $\text{Gal}(F(\zeta_p^\infty)/F) = \text{Gal}(F(\zeta_p^\infty)/F(\zeta_p)) \times \Delta$ acts on $\mathcal{X}_\infty$ via conjugation, so does $\Delta$.

Let $\varepsilon_p := |\Delta|^{-1} \sum_{\delta \in \Delta} \omega^i(\delta)\delta^{-1}$ be an idempotent of $\mathbb{Z}_p[\Delta]$, where $\omega$ is the $p$-adic Teichmüller character.

**Proposition 2.4.** Let $p$ be an odd prime, $F = \mathbb{Q}(\sqrt{D})$ a real quadratic field. Then, $p|\omega_2(\zeta_F(-1)) = B_2, \chi$ if and only if $\varepsilon_2\mathcal{X}_\infty \neq 0$.

**Proof.** Let $A_\infty$ be the $p$-Sylow subgroup of the ideal class group of $F(\zeta_p^\infty)$, that is, $A_\infty = \lim_{\to} A_n$, where $A_n$ is the $p$-Sylow subgroup of the ideal class group of $F(\zeta_p^n)$. By the action induced from that of $\text{Gal}(F(\zeta_p^\infty)/F)$, $A_\infty$ becomes a $\mathbb{Z}_p[\Delta]$-module.

We can assume $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. Then

$$\Delta \cong \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times.$$

As we saw in (1), $p|\omega_2(F)\zeta_F(-1)$ if and only if $p|\sharp K_2(O_F)$. On the other hand, lemma 3 of Komatsu [7] states that, $p|\sharp K_2(O_F)$ if and only if $\varepsilon_2A_\infty \neq 0$.

Let $T$ be a projective limit of all $p$-power-th roots of unity (with respect to the $p$-power map) and put $\varepsilon_i\mathcal{X}_\infty(-1) := \varepsilon_i\mathcal{X}_\infty \otimes_{\mathbb{Z}_p} \text{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$. The standard argument in Iwasawa theory for $\mathbb{Z}_p$-extensions involving Kummer theory over $F(\zeta_p^\infty)$, one can show that, for $i + j \equiv 1 \pmod{|\Delta|}$, $i$ being an odd integer,

$$\varepsilon_j\mathcal{X}_\infty(-1) \cong \text{Hom}_{\mathbb{Z}_p}(\varepsilon_iA_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$$

as $\mathbb{Z}_p[[\text{Gal}(F(\zeta_p^\infty)/F(\zeta_p))]]$-modules (see, for example, [6], [12, Proposition 13.32]). Putting $i = p - 2$, we obtain our assertion since $\varepsilon_2\mathcal{X}_\infty(-1) = \varepsilon_2\mathcal{X}_\infty$ as Abelian groups.

**References**


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(Received September 2, 2003)