ON THE ASYMPTOTIC PROPERTIES FOR
A SEMILINEAR HEAT EQUATION AT A CRITICAL EXPONENT

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1. Introduction

We consider the Cauchy problem for the following semilinear heat equation

\[
\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + u(t,x)^{1+\alpha}, \quad t > 0, \ x \in \mathbb{R}^N,
\]

with the initial condition

\[
u(0,x) = a(x), \ x \in \mathbb{R}^N.
\]

Assume that the initial value \( a(x) \) is non-negative bounded continuous. Then the following results are well known in case \( \alpha N > 2 \) (cf. H. Fujita [1] and K. Kobayashi–T. Sirao–H. Tanaka [3]).

(i) For sufficiently small initial value \( a(x) (\neq 0) \), the solution \( u(t,x) \) of (1.1) with (1.2) converges to 0
uniformly in $x$ as $t \to \infty$.

On the other hand,

(ii) for sufficiently large initial value $a(x)$, the solution $u(t,x)$ of (1.1) with (1.2) blows up in a finite time.

The purpose of this paper is to show the existence of separator-like initial value $a(x)$ into two cases stated above when $\alpha = 4/(N-2)$ and $N \geq 3$.

In this paper we assume that the initial value $a(x)$ is non-negative bounded continuous, the dimension $N$ is not less than 3 and $\alpha = 4/(N-2)$. Then, there exist positive radially symmetric stationary solutions $u_\lambda(x)$ of (1.1), which are described by

$$\tag{1.3} u_\lambda(x) = \left( \frac{1}{N(N-2)\lambda} |x|^{2+\lambda} \right)^{(2-N)/2}, \lambda > 0.$$

**THEOREM.** For fixed $\lambda > 0$, the radially symmetric stationary solution $u_\lambda(x)$ of (1.1) separates the initial values as follows.

(i) If $a(x) \leq \gamma u_\lambda(x)$ for some $\gamma (0 < \gamma < 1)$, then the solution $u(t,x)$ of (1.1) with (1.2) converges to 0 uniformly in $x$ as $t \to \infty$.

(ii) If $a(x) \geq \gamma u_\lambda(x)$ for some $\gamma (\gamma > 1)$, then the solution $u(t,x)$ of (1.1) with (1.2) blows up in a finite time.
2. Proof of Theorem

The heat equation (1.1) with (1.2) is transformed into the integral equation

\[ u(t,x) = H_t a(x) + \int_0^t H_{t-s} u(s, \cdot)^{1+\alpha}(x) \, ds, \quad x \in \mathbb{R}^N, \]

where

\[ \alpha = \frac{4}{(N-2)}, \quad N \geq 3, \]

\[ H_t a(x) = \int_{\mathbb{R}^N} H(t,x,y) a(y) \, dy, \]

\[ H(t,x,y) = (4\pi t)^{-N/2} \exp(-|x-y|^2/4t). \]

Let a positive constant \( \lambda \) be fixed. We assume that the solution \( u(t,x) \) of (2.1) does not blow up. For proving Theorem we prepare several lemmas.

**Lemma 1.** Let \( \beta > 0 \). If \( \gamma > 1 \), then

\[ \frac{\gamma^{1+2\beta} - \gamma^{1+\beta}}{\gamma^{1+2\beta} - 1} > \frac{\beta}{1 + 2\beta}. \]

**Proof.** Since the left hand side is monotone decreasing to the right hand side as \( \gamma \downarrow 1 \), Lemma 1 is trivial.

**Lemma 2.** For the stationary solution \( u_{\lambda}(x) \) of (1.1),
\[
\lim_{r \to \infty} \sup_{|x| \leq r} \frac{\int_0^\infty ds \int_{|y| > r} H(s, x, y) u_\lambda(y)^{\alpha} dy}{u_\lambda(x)} = 0 .
\]

**PROOF.** We first note the following (2.2), (2.3) and (2.4), which are derived from (1.3).

(2.2) There exist positive constants \( c_1, c_2 \) such that

\[
c_1 |x|^{2-N} \leq u_\lambda(x) \leq c_2 |x|^{2-N}
\]

for any \( |x| \geq 1 \).

(2.3) There exists a positive constant \( c \) such that

\[
u_\lambda(rx) \leq cr^{2-N} u_\lambda(x)
\]

for any \( |x| \geq 1 \) and any \( r \geq 1 \).

(2.4) \[
u_\lambda(x) = \int_{\mathbb{R}^N}^{\infty} ds \int_{|y| > r} H(s, x, y) u_\lambda(y)^{\alpha} dy.
\]

Making the change of variable \( y=rz \), we have

\[
\int_0^\infty ds \int_{|y| > r} H(s, x, y) u_\lambda(y)^{\alpha} dy
\]

\[
= \int_0^\infty ds \int_{|z| > 1} (4\pi s)^{-N/2} e^{-|x/r-z|^2/(4s/r^2)} u_\lambda(rz)^{\alpha} r^N dz
\]

\[
\leq c_1^{\alpha} r^{(2-N)(1+\alpha)+N} \int_0^\infty ds \int_{|z| > 1} (4\pi s)^{-N/2} e^{-|x/r-z|^2/(4s/r^2)} u_\lambda(z)^{\alpha} dz,
\]

where we have used (2.3). Again, making the change of variable \( s=r^2 \tau \), the last line of the above inequality is
equal to

\[
c^{1+\alpha} r(2-N)(1+\alpha)+2 \int_0^\infty d\tau \int e^{-|x/r-z|^2/4\tau} u_\lambda(z)^{1+\alpha} dz \leq c^{1+\alpha} r(2-N)(1+\alpha)+2 \int_0^\infty d\tau \int_{R^N} H(\tau, x/r, z) u_\lambda(z)^{1+\alpha} dz = c^{1+\alpha} r(2-N)(1+\alpha)+2 u_\lambda(x/r) \leq c^{1+\alpha} r(2-N)(1+\alpha)+2 u_\lambda(0),
\]

where we have used (2.4). Since \( u_\lambda(x) \geq u_\lambda(x_r) \geq c_1 r^{2-N} \), \( |x_r| = r \), for \( |x| \leq r \), we obtain

\[
\sup_{|x| \leq r} \frac{\int_0^\infty ds \int e^{H(s, x, y) u_\lambda(y)^{1+\alpha} dy}}{u_\lambda(x)} \leq \frac{c^{1+\alpha} r(2-N)(1+\alpha)+2 u_\lambda(0)}{c_1 r^{2-N}} = c^{1+\alpha} c_1^{-1} r(2-N) \alpha+2 u_\lambda(0) = c^{1+\alpha} c_1^{-1} r^{-2} u_\lambda(0),
\]

which completes the proof of Lemma 2.

**Lemma 3.** Let \( \gamma > 1 \). If the initial value \( a(x) \geq \gamma u_\lambda(x) \), then also the following same inequality about the global solution \( u(t, x) \) of (2.1) holds:

\[
u(t, x) \geq \gamma u_\lambda(x) \quad \text{for } t > 0, \, x \in R^N.
\]
Proof. The solution $u(t,x)$ of (2.1) and the stationary solution $u_\lambda(x)$ can be constructed by iteration as follows. Putting

$$u_0(t,x) = H_t a(x),$$

$$u_n(t,x) = H_t a(x) + \int_0^t H_{t-s} u_{n-1}(s,\cdot)^{1+\alpha}(x) ds, \ n=0,1,2,\cdots,$$

$$u_{\lambda,0}(t,x) = H_t u_\lambda(x),$$

$$u_{\lambda,n}(t,x) = H_t u_\lambda(x) + \int_0^t H_{t-s} u_{\lambda,n-1}(s,\cdot)^{1+\alpha}(x) ds, \ n=0,1,2,\cdots,$$

we have $u_n(t,x) \uparrow u(t,x)$ and $u_{\lambda,n}(t,x) \uparrow u_\lambda(x)$ as $n \to \infty$.

Assuming $u_{n-1}(t,x) \geq \gamma u_{\lambda,n-1}(t,x)$, we have

$$u_n(t,x) \geq \gamma H_t u_\lambda(x) + \gamma^{1+\alpha} \int_0^t H_{t-s} u_{\lambda,n-1}(s,\cdot)^{1+\alpha}(x) ds \geq \gamma u_{\lambda,n}(t,x).$$

Therefore we complete the proof of Lemma 3.

Lemma 4. Let $\gamma > 1$ and $\beta = \alpha/2$. If $a(x) \geq \gamma u_\lambda(x)$ for any $x \in \mathbb{R}^N$, then for any $r_0 > 0$ there exists positive number $T$ such that

$$u(t,x) \geq \gamma^{(1+\beta)n} u_\lambda(x)$$

(2.5)
for $|x| \leq r_0$, $t \geq nT$ and $n = 0, 1, 2, \cdots$.

PROOF. We prove the lemma by induction. When $n = 0$, we already have this fact for any $r_0 > 0$ in Lemma 3.

Let

$$f(u) = u^{1+\alpha},$$

$$J_0(x) = \frac{\beta}{2(1+2\beta)} \left( \int_0^\infty \int_{|y| > r} H(s, x, y) f(u_\lambda(y)) dy \right) / u_\lambda(x),$$

$$K_0(t, x) = \frac{\beta}{2(1+2\beta)} - \frac{H_t u_\lambda(x)}{u_\lambda(x)}.$$

First, choose $r \geq r_0$ such that $|x| \leq r$ implies $J_0(x) > 0$. Such an $r$ exists by Lemma 2. Next, pick $T > 0$ such that $|x| \leq r$ and $t \geq T$ imply $K_0(t, x) > 0$.

We shall prove (2.5) replacing $r_0$ by $r$. Assuming that (2.5) holds for $n$, we shall prove that (2.5) holds also for $n+1$.

We denote the solution $u(t, x)$ of the equation (1.1) with the initial condition (1.2) by $u(t, x; a, f)$. Since

$$u(t+nT, x; a, f) = u(t, x; u_{nT}, f), \quad u_{nT}(x) = u(nT, x),$$

we have, using the induction hypothesis,

$$u(t+nT, x) = H_t u(nT, x) + \int_0^t ds \int_{|y| > r} H(t-s, x, y) f(u(nT+s, y)) dy.$$
\[ u(t+nT, x) \geq \gamma^{(1+\beta)^n+1} u_\lambda(x) + (\gamma^{(1+\beta)^n(1+2\beta)} - \gamma^{(1+\beta)^{n+1}}) u_\lambda(x) + (1 - \gamma^{(1+\beta)^n(1+2\beta)}) H_t u_\lambda(x) \\
+ (1 - \gamma^{(1+\beta)^n(1+2\beta)}) \int_0^t ds \int_{|y|>r} H(s, x, y) f(u_\lambda(y)) dy \]

\[ = \gamma^{(1+\beta)^{n+1}} u_\lambda(x) + K(t, x, M) + J(t, x, r, M), \]

where

\[ K(t, x, M) = \frac{1}{2} (M^{1+2\beta} - M^{1+\beta}) u_\lambda(x) - (M^{1+2\beta} - 1) H_t u_\lambda(x), \]

\[ J(t, x, r, M) = \frac{1}{2} (M^{1+2\beta} - M^{1+\beta}) u_\lambda(x) - (M^{1+2\beta} - 1) \int_0^t ds \int_{|y|>r} H(s, x, y) f(u_\lambda(y)) dy, \]
\[ M = \gamma^{(1+\beta)^n} > 1. \]

Using Lemma 1, we have

\[ J(t, x, r, M) \]

\[ = (M^{1+2\beta} - 1)u_\lambda(x) \left\{ \frac{1}{2} \frac{M^{1+2\beta} - M^{1+\beta}}{M^{1+2\beta} - 1} - \frac{\int_t^\infty \int_{|y|>r} H(s, x, y) f(u_\lambda(y)) \, dy \, u_\lambda(x)}{u_\lambda(x)} \right\} \]

\[ \geq (M^{1+2\beta} - 1)u_\lambda(x) \left\{ \frac{\beta}{2(1+2\beta)} - \frac{\int_t^\infty \int_{|y|>r} H(s, x, y) f(u_\lambda(y)) \, dy \, u_\lambda(x)}{u_\lambda(x)} \right\} \]

\[ \geq (M^{1+2\beta} - 1)u_\lambda(x) J_0(x) > 0 \]

for \( |x| \leq r \), and

\[ K(t, x, M) = (M^{1+2\beta} - 1)u_\lambda(x) \left\{ \frac{1}{2} \frac{M^{1+2\beta} - M^{1+\beta}}{M^{1+2\beta} - 1} - \frac{H_t u_\lambda(x)}{u_\lambda(x)} \right\} \]

\[ \geq (M^{1+2\beta} - 1)u_\lambda(x) \left\{ \frac{\beta}{2(1+2\beta)} - \frac{H_t u_\lambda(x)}{u_\lambda(x)} \right\} \]

\[ = (M^{1+2\beta} - 1)u_\lambda(x) K_0(t, x) > 0 \]

for \( |x| \leq r \) and \( t \geq T \).

Therefore we obtain

\[ u(t+nT, x) \geq \gamma^{(1+\beta)^{n+1}} u_\lambda(x) \]
for $|x| \leq r$ and $t \geq T$, namely

$$(2.8) \quad u(t,x) \geq \gamma^{(1+\beta)\frac{n+1}{n}} u_\lambda(x)$$

for $|x| \leq r$ and $t \geq (n+1)T$, which completes the proof of Lemma 4.

**PROOF OF THEOREM.** Since the function $f(u)$ is non-decreasing and satisfies the following condition

$$(2.9) \quad \int_\epsilon^\infty \frac{du}{f(u)} < \infty \quad \text{for some } \epsilon > 0,$$

(2.8) contradicts that $u(t,x)$ does not blow up. (See Theorem 4.1 in K. Kobayashi-T. Sirao-H. Tanaka [3].) This completes the proof of Part (ii) of Theorem. We can prove Part (i) of Theorem in a similar way using the following lemma.

**LEMMA 4'.** Let $\gamma < 1$ and $\beta = \alpha/2$. If $a(x) \leq \gamma u_\lambda(x)$ for any $x \in \mathbb{R}^N$, then for any $r_0 > 0$ there exists positive number $T$ such that

$$(2.5) \quad u(t,x) \leq \gamma^{(1+\beta)\frac{n}{n}} u_\lambda(x)$$

for $|x| \leq r_0$, $t \geq nT$ and $n = 0,1,2,\ldots$.

We consider a class of monotone radially symmetric functions as follows:
\( \mathcal{A} = \{ \alpha \in C(\mathbb{R}^N): \alpha(x) \geq 0, \xi(0), \alpha(x) \geq \alpha(y) \text{ for } |x| \leq |y| \} \).

Since \( \gamma u_\alpha(x) \in \mathcal{A} \), it is sufficient to prove (i) for initial values \( \alpha(x) \in \mathcal{A} \) by the comparison theorem. Then, since \( u(t,x) = u(t,x;\alpha,\xi) \in \mathcal{A} \) (see Lemma 3.2 in [3]), Lemma 4' implies

\[
    u(t,x) \leq \gamma^{(1+\beta)n} u_\alpha(0) \text{ for } t \geq nT \text{ and } n = 0,1,2,\ldots.
\]

Thus the proof of Theorem is completed.

References


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(Received May 30, 1989)