DIVISIBILITY OF ORDERS OF K_2 GROUPS ASSOCIATED TO QUADRATIC FIELDS

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Abstract. We discuss some divisibility results of orders of K-groups and cohomology groups associated to quadratic fields.

1. Introduction

In our previous paper [8], basing on Byeon's investigation [4], we discussed some indivisibility properties of the special values of Dedekind zeta functions $\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$ associated to real quadratic fields $\mathbb{Q}(\sqrt{D})$, $(D > 0)$ at -1. We were interested in the indivisibility of $\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$ because it is closely related to the existence of certain infinite family of CM-fields.

These values are, on the other hand, closely related to orders of the second algebraic K-groups $K_2(O_D)$ associated to the ring O_D of integers of $\mathbb{Q}(\sqrt{D})$ via Birch-Tate conjecture (now one of consequences of Iwasawa Main conjecture proved by Mazur-Wiles [12]).

We raised, in above mentioned paper, the following question (see Question 2.4 for more precise statement):

Question 1.1. *For any odd prime p, is there real quadratic field $\mathbb{Q}(\sqrt{D})$ such that p divide the numerator of $\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$? Further, are there infinitely many such real quadratic fields?*

In §2, we answer this question affirmatively for primes $p = 3, 5$. A similar problem for imaginary quadratic fields is considered in §3.

We use the following notations throughout this paper. For any set S, $\sharp S$ is the cardinality of S. If K/k is an extension of fields, $[K : k]$ is the degree of K/k. If further K/k is a Galois extension, $\mathrm{Gal}(K/k)$ is its Galois group. As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Z}_{\geq 0}$ is the set of non-negative rational integers.

Let us fix an algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers $\mathbb{Q}$ in the field of the complex numbers $\mathbb{C}$. We assume that any number field is a subfield of this $\overline{\mathbb{Q}}$. For any number field k, $D(k)$ is the discriminant, $\mathrm{Cl}(k)$ is the ideal class group, $h(k) = \sharp\mathrm{Cl}(k)$ is the class number, O_k is the ring of integers and $\zeta_k(s)$ is the Dedekind zeta function, of k. For any natural number n, $\zeta_n \in \overline{\mathbb{Q}}$ is a primitive n-th root of unity.

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Definition 1.2. *For any number field F of finite degree over $\mathbb{Q}$ and for $n \in \mathbb{Z}_{\geq 0}$, let $w_n(F)$ be as follows:*

$$w_n(F) := 2^{n(2)+1} \prod_p p^{n(p)}, \quad n(p) := \max\{m | [F(\zeta_{p^m}) : F] \leq n\}.$$

Note that $w_n(F) = \max\{m | \exp(\mathrm{Gal}(F(\zeta_m)/F)) | n\}$, where $\exp(G)$ is the exponent of a group G.

2. Divisibility of special values of zeta functions associated to real quadratic fields at -1

Definition 2.1. *For any totally real number field F of finite degree over $\mathbb{Q}$ and for any even integer $n \in \mathbb{Z}_{\geq 0}$, we define*

$$\xi_n(F) := w_n(F)\zeta_F(1-n).$$

Note that $\zeta_F(1-n) \in \mathbb{Q}$ is known by Siegel-Klingen, so $\xi_n(F) \in \mathbb{Q}$.

Theorem 2.2 (Serre [15])**.** *For any totally real number field F, $\xi_2(F) \in \mathbb{Z}$.*

Theorem 2.3 (Mazur-Wiles [12])**.** *The following equality (Birch-Tate conjecture at p) holds for any totally real Abelian field F and for any odd prime p:*

$$\sharp K_2(O_F) \sim_p \xi_2(F), \tag{1}$$

where for any $a, b \in \mathbb{Q}$ and for a prime p, $a \sim_p b$ means a/b is a p-adic unit.

We plainly see that, for any fundamental discriminant $D > 5$,

$$\xi_2(\mathbb{Q}(\sqrt{D})) = w_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1) = B_{2,\chi},$$

where χ is Kronecker symbol associated to $\mathbb{Q}(\sqrt{D})$, $B_{2,\chi}$ is the generalized Bernoulli number associated to χ. Numerical computation of $\xi_2(\mathbb{Q}(\sqrt{D}))$ is equivalent to the computation of generalized Bernoulli number $B_{2,\chi}$ and is easy (one can use Cohen's formula [7, Proposition 4.1]

$$\zeta_{\mathbb{Q}(\sqrt{D})}(-1) = \frac{1}{60}\sum_s \sigma_1\left(\frac{D-s^2}{4}\right),$$

where the sum is taken over all s so that $D \geq s^2$ and $\sigma_1(\cdot)$ is usual sum-of-divisors function).

The table below shows, for small primes $p = 5, 7, 11, 13, 17, 19, 23$, there exists real quadratic field $\mathbb{Q}(\sqrt{D})$ such that $\xi_2(\mathbb{Q}(\sqrt{D}))$ is divisible by p.

p	5	7	11	13	17	19	23
D	37	40	61	89	97	76	88
$\xi_2(\mathbb{Q}(\sqrt{D}))$	$2^2 \cdot 5$	$2^2 \cdot 7$	$2^2 \cdot 11$	$2^3 \cdot 13$	$2^3 \cdot 17$	$2^2 \cdot 19$	$2^2 \cdot 23$

It may be natural to ask the following question:

Question 2.4. *For any odd prime p, is there always real quadratic field $\mathbb{Q}(\sqrt{D})$ such that $p | \xi_2(\mathbb{Q}(\sqrt{D}))$ holds? Further, are there always infinitely many such real quadratic fields?*

We interpreted the divisibility $p|\xi_2(\mathbb{Q}(\sqrt{D}))$ by means of ideal class groups or Galois groups in [8]. We need some notations.

Let $F(\zeta_{p^\infty}) = \cup_{n\geq 1} F(\zeta_{p^n})$ be the field generated by all p-power-th root of unity over a totally real number field F, then $F(\zeta_{p^\infty})/F(\zeta_p)$ is the cyclotomic $\mathbb{Z}_p$-extension. Let $A_\infty = \varinjlim A_n$, where A_n is the p-Sylow subgroup of $\mathrm{Cl}(F(\zeta_{p^n}))$. Let M_∞ denote the maximal Abelian p-extension unramified outside p over $F(\zeta_{p^\infty})$, and $\mathcal{X}_\infty$ its Galois group over $F(\zeta_{p^\infty})$:

$$\mathcal{X}_\infty = \mathrm{Gal}(M_\infty/F(\zeta_{p^\infty})).$$

We also put $\Delta = \mathrm{Gal}(F(\zeta_p)/F)$, then $\mathrm{Gal}(F(\zeta_{p^\infty})/F) = \mathrm{Gal}(F(\zeta_{p^\infty})/F(\zeta_p)) \times \Delta$ acts on $\mathcal{X}_\infty$ via conjugation, so Δ does.

Let ε_i be an idempotent

$$\varepsilon_i := \frac{1}{\sharp\Delta} \sum_{\delta\in\Delta} \omega^i(\delta)\delta^{-1} \in \mathbb{Z}_p[\Delta],$$

where ω is a p-adic Teichmüller character. Then we have

Proposition 2.5 (*cf.* Kimura[8]). *Let p be an odd prime, $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field. Then, the following 3 assertions are equivalent:*

- $p|\xi_2(F) = B_{2,\chi}$,
- $\varepsilon_2\mathcal{X}_\infty \neq 0$,
- $\varepsilon_{p-2}A_\infty \neq 0$.

We can answer affirmatively the question 2.4 for $p = 3, 5$.

Theorem 2.6.

$$\sharp\{0 < D < X \mid 3 \mid \sharp K_2(O_D))\} \gg X^{\frac{7}{8}-\varepsilon}.$$

Before giving a proof, we state two theorems which is essential to prove theorem 2.6.

Theorem 2.7 (Browkin[1], Lu[11], Queen[13], Urbanowicz[18]). *For any real quadratic field $\mathbb{Q}(\sqrt{D})$, the following two assertions are equivalent:*

- $3 \mid \sharp K_2(O_D)$
- $3 \mid h(\mathbb{Q}(\sqrt{-3D}))$ *or* $D \equiv 6 \pmod 9$.

Theorem 2.8 (Soundararajan[17]). *Let g be any natural number. We define $\mathcal{N}_g(X)$ as*

$$\mathcal{N}_g(X) := \sharp\{0 < d < X \mid d \text{ is square free and satisfies the condition } (**)\},$$
$$(**) : \quad \mathrm{Cl}(\mathbb{Q}(\sqrt{-d})) \text{ has an element of order } g.$$

Then, for all sufficiently large real number X, the following inequality holds:

$$\mathcal{N}_g(X) \gg \begin{cases} X^{\frac{1}{2}+\frac{2}{g}-\varepsilon} & \text{if } g \equiv 0 \pmod 4, \\ X^{\frac{1}{2}+\frac{3}{g+2}-\varepsilon} & \text{if } g \equiv 2 \pmod 4. \end{cases}$$

Proof. By theorem 2.7, it is sufficient to count imaginary quadratic fields whose class numbers are divisible by 3 and in which 3 is ramified.

Theorem 2.8 for $g = 6$ then shows that $\mathcal{N}_6(X) \gg X^{7/8-\varepsilon}$. Note that 3 is ramified in the imaginary quadratic fields counted in theorem 2.8 (see his paper p.683, formula (1.3)). □

We next treat the case of $p = 5$.

Theorem 2.9. $\{\mathbb{Q}(\sqrt{D}) \mid D > 0,\, 5 \mid \sharp K_2(O_D)\}$ *is an infinite set.*

Proof. By (1), we have

$$\sharp K_2(O_D) \sim_5 \xi_2(\mathbb{Q}(\sqrt{D})).$$

We see that, by the theorem of Yamamoto [21], the set $\mathcal{Q}_5^+$ of real quadratic fields F that satisfy the condition "$25 \mid h(F)$ and 5 is inert in F" is an infinite set. On the other hand, Lu (*loc. cit.*) proved

Theorem 2.10 (Lu). *Suppose* $5 \nmid D_0$, $D_0 > 0$, *then we have*

$$-12\zeta_{\mathbb{Q}(\sqrt{5D_0})}(-1) \equiv \left(1 - \frac{\left(\frac{D_0}{5}\right)_K}{5}\right) \frac{2h(\mathbb{Q}(\sqrt{D_0}))\log_5 \varepsilon_0}{\sqrt{D_0}} \pmod 5,$$

where ε_0 *is a fundamental unit of* $\mathbb{Q}(\sqrt{D_0})$, $\log_5$ *is* 5*-adic log,* $(\cdot/\cdot)_K$ *is a Kronecker symbol.*

We have, for real quadratic fields $F \in \mathcal{Q}_5^+$, letting ε_0 be the fundamental unit of F,

$$\left|\frac{2\log_5(\varepsilon_0)}{\sqrt{D(F)}}\right|_5 \le 1.$$

(*cf.* Washington [19, Proposition 5.33]). Since 5 is inert in $F \in \mathcal{Q}_5^+$,

$$\left(1 - \frac{\left(\frac{D_0}{5}\right)_K}{5}\right) \frac{2h(F)\log_5 \varepsilon}{\sqrt{D(F)}} = \left(1 - \frac{-1}{5}\right) h(F) \frac{2\log_5(\varepsilon_0)}{\sqrt{D(F)}} \equiv 0 \pmod 5.$$

For $D = 5D(F), (F \in \mathcal{Q}_5^+)$, we thus have $5 \mid \sharp K_2(O_D)$. □

Remark 2.11. *(1) By the theorem 2.7, if positive fundamental discriminant* D *satisfies* $D \equiv 6 \pmod 9$, $3 \mid \sharp K_2(O_D)$. *Such fundamental discriminants have positive proportion. Browkin* [3] *computed a density of discriminants which satisfy* $3 \mid \sharp K_2(O_D)$ *assuming Cohen-Lenstra heuristics* [6].

(2) One can prove theorem 2.9 by using theorem 5.5 (iii) of Browkin [2]. *His theorem states that if* $5 \mid h(\mathbb{Q}(\sqrt{5D}))$ *then* $5 \mid \sharp K_2(O_D)$. *There are, by Yamamoto's theorem (loc. cit.), infinitely many real quadratic fields whose class numbers and discriminants are divisible by* 5*, the theorem follows.*

These remarks are due to J. Browkin. The author express his gratitude to him for pointing out these clarifications.

3. Divisibility of special values of L-functions associated to imaginary quadratic fields

In this section, we treat similar problem for imaginary quadratic fields.

Theorem 3.1. *Let $p \geq 7$ be a prime $\equiv 3 \pmod 4$. Then, the set*

$$\{\mathbb{Q}(\sqrt{D}) \mid D < 0,\, p \mid L(1 - \frac{p-1}{2}, \chi_D)\}$$

is an infinite set.

We need the following theorem due to Lu (*loc. cit*).

Theorem 3.2 (Lu)**.** *Let $p \geq 7$ be a prime. Suppose $p \nmid D_0$, then we have*

$$\left(1 - \frac{\left(\frac{D_0}{p}\right)_K}{p}\right) \frac{h(\mathbb{Q}(\sqrt{D_0}))\log_p \varepsilon}{\sqrt{D_0}} \equiv B_{\frac{p-1}{2},\chi} \pmod p,$$

where $\chi(\cdot) = ((-1)^{(p-1)/2})pD_0/\cdot)_K$.

Proof. The proof is parallel to the proof of theorem 2.9. By Yamamoto's theorem [21], the set $\mathcal{Q}_p^+$ of real quadratic fields $F = \mathbb{Q}(\sqrt{D_0})$ that satisfy the condition "$p^2 \mid h(D_0)$ and p is inert in F" is an infinite set. Then we see by the same argument,

$$\left(1 - \frac{\left(\frac{D_0}{p}\right)_K}{p}\right) \frac{h(D_0)\log_p \varepsilon}{\sqrt{D_0}} \equiv 0 \equiv B_{\frac{p-1}{2},\chi} \pmod p,$$

where $\chi(\cdot) = ((-1)^{(p-1)/2}pD_0/\cdot)_K$. If we take $D = (-1)^{(p-1)/2}pD_0$, $\mathbb{Q}(\sqrt{D})$ is a member of the set in question. □

Definition 3.3. *For any odd integer $n \in \mathbb{Z}_{\geq 0}$, we define imaginary counterpart of $\xi_n(F)$ by*

$$\xi_n(\mathbb{Q}(\sqrt{D})) := w_n(\mathbb{Q}(\sqrt{D}))L(1-n,\chi), \qquad D < 0,$$

where χ is a Kronecker symbol associated to $\mathbb{Q}(\sqrt{D})$.

Corollary 3.4. *Let $p \geq 7$ be a prime $\equiv 3 \pmod 4$. Then, the set*

$$\{\mathbb{Q}(\sqrt{D}) \mid D < 0,\, p \mid \xi_{\frac{p-1}{2}}(\mathbb{Q}(\sqrt{D}))\}$$

is an infinite set.

Kolster [9] proved the *higher class number formula* for totally imaginary Abelian fields. We need some notations to state his formula for imaginary quadratic fields (see, Examples given in his paper).

For a number field k and a rational prime p, $O_k[1/p]$ is a ring of p-integers in k. Let $\mathrm{H}^2_{\text{ét}}(O_k[1/p], \mathbb{Z}_p(n))$ be the second étale cohomology group. We then define two groups $\mathrm{H}^2_{\text{ét}}(O_D, \mathbb{Z}(n))$ and $\mathrm{H}^2_{\text{ét}}(\mathbb{Z}, \mathbb{Z}(n))$ as:

$$\begin{aligned} \mathrm{H}^2_{\text{ét}}(O_D, \mathbb{Z}(n)) &= \prod_p \mathrm{H}^2_{\text{ét}}(O_D[1/p], \mathbb{Z}_p(n)), \\ \mathrm{H}^2_{\text{ét}}(\mathbb{Z}, \mathbb{Z}(n)) &= \prod_p \mathrm{H}^2_{\text{ét}}(\mathbb{Z}[1/p], \mathbb{Z}_p(n)). \end{aligned}$$

These are known to be finite Abelian groups. We then difine the higher relative class number as the ratio of these quantities:

$$h_n^-(\mathbb{Q}(\sqrt{D})) := \frac{\sharp \mathrm{H}^2_{\text{ét}}(O_D, \mathbb{Z}(n))}{\sharp \mathrm{H}^2_{\text{ét}}(\mathbb{Z}, \mathbb{Z}(n))}.$$

Vandiver's conjecture for p which asserts that, as is well known, $p \nmid h(\mathbb{Q}(\zeta_p + \zeta_p^{-1}))$, is equivalent to $p \nmid \sharp \mathrm{H}^2_{\text{ét}}(\mathbb{Z}, \mathbb{Z}(n))$ (*cf.* Kurihara [10]). If we assume Vandiver's conjecture for all p, we plainly see that $h_n^-(\mathbb{Q}(\sqrt{D})) = \sharp \mathrm{H}^2_{\text{ét}}(O_D, \mathbb{Z}(n))$. It is known that $\mathrm{H}^2_{\text{ét}}(\mathbb{Z}, \mathbb{Z}(3))$ is trivial unconditionally (*cf.* Rognes [14], Soulé [16]).

For imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, $D \neq -1, -8$ and for any *odd* integer $n \in \mathbb{Z}_{\geq 0}$, Kolster's higher class number formula can be read as follows:

$$\xi_n(\mathbb{Q}(\sqrt{D})) = \pm 4 h_n^-(\mathbb{Q}(\sqrt{D})). \tag{2}$$

The following explicit formula holds for $n = 3$,

$$\xi_3(\mathbb{Q}(\sqrt{D})) = \pm 4 \sharp \mathrm{H}^2_{\text{ét}}(O_D, \mathbb{Z}(3)) = \pm 4 h_3^-(\mathbb{Q}(\sqrt{D})).$$

By theorem 3.1, we immediately see the following corollary.

Corollary 3.5. *Let $p \geq 7$ be a prime $\equiv 3 \pmod 4$. Then, the set*

$$\{\mathbb{Q}(\sqrt{D}) \mid D < 0,\, p \mid h^-_{\frac{p-1}{2}}(\mathbb{Q}(\sqrt{D}))\}$$

is an infinite set. If $p = 7$, the set

$$\{\mathbb{Q}(\sqrt{D}) \mid D < 0,\, 7 \mid \sharp \mathrm{H}^2_{\acute{e}t}(O_D, \mathbb{Z}(3))\}$$

is an infinite set.

Remark 3.6. *For any positive number X, let $\mathcal{Q}^+(X)$ (resp. $\mathcal{Q}^-(X)$) be the set of all real (resp. imaginary) quadratic fields F which satisfy $|D(F)| < X$. There is longstanding conjecture called Cohen-Lenstra heuristics* [6] *which asserts that*

$$\begin{aligned} \lim_{X \to \infty} \frac{\sharp\{F \in \mathcal{Q}^+(X) | \, p \mid h(F)\}}{\sharp \mathcal{Q}^+(X)} &= 1 - \prod_{n \geq 2} (1 - p^{-n}), \\ \lim_{X \to \infty} \frac{\sharp\{E \in \mathcal{Q}^-(X) | \, p \mid h(E)\}}{\sharp \mathcal{Q}^-(X)} &= 1 - \prod_{n \geq 1} (1 - p^{-n}). \end{aligned}$$

Some progress has been made with the conjecture (see Yu [22]*, Soundararajan (loc. cit.), Byeon-Koh* [5]*, and others). It may be interesting to consider a generalization of Cohen-Lenstra heuristics in the following way. A version of*

Lichtenbaum conjecture (proved by Wiles [20]) asserts that $w_{2n}(K)\zeta_K(1-2n) \sim_p \sharp \mathrm{H}^2_{\acute{e}t}(O_K[1/p], \mathbb{Z}_p(2n))$ *for any totally real number field* K *and for any odd prime* p*. What do we expect*

$$\sharp\{F \in \mathcal{Q}^+(X) | \, p \mid \sharp \mathrm{H}^2_{\acute{e}t}(O_F[\frac{1}{p}], \mathbb{Z}_p(2n))\} \quad (X \to \infty)?$$

We can ask similar question for imaginary quadratic case:

$$\sharp\{E \in \mathcal{Q}^-(X) | \, p \mid h_n^-(E)\} \quad (X \to \infty)?$$

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