Remarks on Solutions of a Coupled Semilinear Parabolic System

Tokumori NANBU

1 Introduction

We are interested in the global solution $U(x,t) = (u(x,t), v(x,t))$ of the initial-
Dirichlet problem (1.1)-(1.3) for a coupled semilinear parabolic system

$$
(1.1) \quad \begin{cases} 
  u_t = \Delta u + f(u,v), & u \geq 0 \quad \text{in } Q = \Omega \times R^+,
  
  v_t = \Delta v + g(u,v), & v \geq 0 \quad \text{in } Q = \Omega \times R^+.
\end{cases}
$$

with the initial condition

$$
(1.2) \quad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in } \Omega
$$

and the boundary condition

$$
(1.3) \quad (u(x,t), v(x,t)) = (0,0) \quad \text{on } \partial \Omega \times R^+.
$$

Here $\Omega$ is a smoothly bounded domain in $R^N (3 \leq N)$, $R^+ = (0, +\infty)$, the functions $u_0(x)$ and $v_0(x)$ are of class $C_0^1(\Omega)$ and nonnegative in $\Omega$. $f(u,v)$ and $g(u,v)$ ($f, g \in C^1(R^+ \times R^+)$) satisfy some conditions which will be given later.

We state some known results on the asymptotic behaviour of solution of the initial-Dirichlet problem (1.1)-(1.3) for a semilinear parabolic system.

Lemma.([PW(p.190, Th.13)]) Let $u$ and $v$ be a pair of functions

$$
C^{2,1}(\Omega \times (0,T)) \cap C(\Omega \times (0,T))
$$

satisfying the inequalities

$$
- u_t + a \Delta u \leq \alpha u + \beta v
$$

$$
- v_t + b \Delta v \leq \gamma u + \delta v
$$

in $\Omega \times (0,T)$, where $a$ and $b$ are positive constants and $\alpha, \beta, \gamma$ and $\delta$ are bounded in $\Omega \times (0,T)$. Suppose further that

$$
\beta \leq 0, \quad \gamma \leq 0.
$$
Then the nonnegativity of $u$ and $v$ on $P$ implies the nonnegativity of $u$ and $v$ in $\Omega \times (0, T)$. Here $P=(\partial \Omega \times [0, T]) \cup (\Omega \times \{0\})$.

C. S. Kahane considered the existence, uniqueness and asymptotic behaviour for solutions of the initial-boundary value problem

\[(1.4)_1 \quad u_t = a \Delta u - j u v, \quad u \geq 0 \quad \text{in} \quad Q_T = \Omega \times (0, T),\]

\[(1.4)_2 \quad v_t = b \Delta v - k u v, \quad v \geq 0 \quad \text{in} \quad Q_T = \Omega \times (0, T)\]

with the nonnegative initial condition

\[(1.5) \quad (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega\]

and the boundary condition

\[(1.6) \quad u(x, t) = \psi_1(x, t), \quad v(x, t) = \psi_2(x, t) \quad \text{on} \quad \partial \Omega \times (0, T)\]

where $a, b, j, k$ are positive constants, $\psi_1, \psi_2, u_0$ and $v_0$ are nonnegative functions which satisfy $\psi_1(x, 0) = u_0(x), \psi_2(x, 0) = v_0(x)$ on $\partial \Omega$, and $0 < T \leq \infty$.

C. S. Kahane ([KA]) proved the existence of a local solution in time by using Green's function.

**Theorem 1.1.** ([KA]) Let $u$ and $v$ be a pair of functions

\[C^{2,1}(\Omega \times (0, T)) \cap C(\Omega \times (0, T))\]

satisfying (1.4)-(1.6).

Then the nonnegativity of $u$ and $v$ on $P$ implies

\[0 \leq u(x, t) \leq \sup_P u, \quad 0 \leq v(x, t) \leq \sup_P v\]

for $(x, t) \in \Omega \times (0, T)$. Here $P=(\partial \Omega \times [0, T]) \cup (\Omega \times \{0\})$.

**Theorem 1.2.** ([KA]) The problem (1.4)-(1.6) has a unique nonnegative solution in $\Omega \times (0, T)$ assuming given nonnegative continuous data prescribed for $u$ and $v$ on $(\partial \Omega \times [0, T]) \cup (\Omega \times \{0\})$.

R. Martin posed a problem on the existence and uniform bounds of solution $U = (u, v)$ for the system

\[u_t = \Delta u - u v^\beta, \quad u \geq 0 \quad \text{in} \quad Q = \Omega \times R^+\]

\[v_t = \Delta v + u v^\beta, \quad v \geq 0 \quad \text{in} \quad Q = \Omega \times R^+\]

with the nonnegative initial condition

\[(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega\]
and the boundary condition
\[ u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on} \quad \partial \Omega \times R^+. \]

Here \( \beta \geq 1, u_0 \) and \( v_0 \) are nonnegative functions in \( \Omega \).

K. Masuda ([MAS]) extended the Martin's problem to the more general equations. He proved the existence and asymptotic behaviour of solutions of the following system

\[
\begin{align*}
(1.7)_1 & \quad u_t = \Delta u - f(u, v) \quad \text{in} \quad Q = \Omega \times R^+, \\
(1.7)_2 & \quad v_t = \Delta v + g(u, v) \quad \text{in} \quad Q = \Omega \times R^+
\end{align*}
\]

with the initial condition
\[(1.8) \quad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega \]
and the boundary condition
\[(1.9) \quad u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on} \quad \partial \Omega \times R^+. \]

Here Masuda assumed that the functions \( f(u, v), g(u, v) \) are non-negative, \( f(0, s) = g(s, 0) = 0 \) \( (s \in R^+) \) and \( f(u, v), g(u, v) \) satisfy some additional conditions: there is a monotonically increasing function \( \omega(s) (s \geq 0) \) and a positive constant \( r \) with \( g(u, v) \leq \omega_1(u)(v + v^r) \), and \( g(u, v) \leq \omega_1(u)f(u, v) \) \( (u, v) \in R^+ \times R^+ \).

M. Escobedo and M. A. Herrero ([EH]) considered the following equation
\[
\begin{align*}
(1.10)_1 & \quad u_t = \Delta u + v^p, \quad u \geq 0 \quad \text{in} \quad Q = \Omega \times (0, T), \\
(1.10)_2 & \quad v_t = \Delta v + u^q, \quad v \geq 0 \quad \text{in} \quad Q = \Omega \times (0, T).
\end{align*}
\]

Here \( p(> 0) \) and \( q(> 0) \) are positive constants and \( 0 < T \leq \infty \).

Recently, N. Bedjaoui and P. Souplet ([BS]) considered the existence of the solution \((u, v)\) of the following initial-Dirichlet problem
\[
\begin{align*}
u_t = \Delta u + v^p - au^r, \quad u \geq 0 \quad &\text{in} \quad Q_T = \Omega \times (0, \infty), \\
v_t = \Delta v + u^q - bv^s, \quad v \geq 0 \quad &\text{in} \quad Q_T = \Omega \times (0, \infty)
\end{align*}
\]

with the nonnegative initial conditions and the zero boundary condition.

Pao ([PAO]) considered the existence of solution of the following coupled parabolic system

\[
\begin{align*}
(1.10)_1 & \quad u_1_t = \Delta u_1 + f_1(x, t, u_1, u_2) \quad \text{in} \quad Q_T = \Omega \times (0, T), \\
(1.10)_2 & \quad u_2_t = \Delta u_2 + f_2(x, t, u_1, u_2) \quad \text{in} \quad Q_T = \Omega \times (0, T)
\end{align*}
\]
with the initial condition
\begin{equation}
(1.11) \quad (u_1(x, 0), u_2(x, 0)) = (u_{10}(x), u_{20}(x)) \quad \text{in } \Omega
\end{equation}
and the boundary condition
\begin{equation}
(1.12) \quad u_1(x, t) = u_2(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T).
\end{equation}
Here $0 < T \leq \infty$. The functions $f_1(x, t, u_1, u_2)$, $f_2(x, t, u_1, u_2)$ satisfy some monotone conditions which will be given later.

Pao([PAO]) states the following definitions.

**Definition 1** ([PAO]) Let $J_1 \times J_2$ be a bounded subset in $R^2$. A vector function $(f_1(u_1, u_2), f_2(u_1, u_2))$ is defined in a bounded subset in $R^2$.

(i) A function $f_k(u_1, u_2)$ is said to be quasimonotone nonincreasing if for fixed $u_k$, the function $f_k(u_1, u_2)$ is nonincreasing in $u_j$ for $j \neq k$.

(ii) A function $f_k(u_1, u_2)$ is said to be quasimonotone nondecreasing if for fixed $u_k$, the function $f_k(u_1, u_2)$ is nondecreasing in $u_j$ for $j \neq k$.

**Definition 2** ([PAO]) Let $J_1 \times J_2$ be a bounded subset in $R^2$.

(i) A function $F = (f_1, f_2)$ is called quasimonotone nonincreasing in $J_1 \times J_2$ if both $f_1$ and $f_2$ are quasimonotone nonincreasing for $(u_1, u_2) \in J_1 \times J_2$.

(ii) A function $F = (f_1, f_2)$ is called quasimonotone nondecreasing in $J_1 \times J_2$ if both $f_1$ and $f_2$ are quasimonotone nondecreasing for $(u_1, u_2) \in J_1 \times J_2$.

(iii) A function $F = (f_1, f_2)$ is called mixed quasimonotone in $J_1 \times J_2$ if $f_1$ is quasimonotone nonincreasing and $f_2$ is quasimonotone nondecreasing for $(u_1, u_2) \in J_1 \times J_2$ (or vice versa).

Pao([PAO]) gives the definition on the ordered upper solution and lower solution for the problem (1.10)-(1.12) as follows:

**Definition 3**. ([PAO] p.383) A pair of function $\bar{u} = (\bar{u}_1, \bar{u}_2)$, $\underline{u} = (u_1, u_2)$ in $C(\overline{Q_T}) \cap C^{1,2}(Q_T)$ are called ordered upper and lower solutions of (1.10)-(1.12) if they satisfy the relation $\bar{u} \geq \underline{u}$ and

\begin{equation}
\bar{u}_k \geq \psi_k \geq u_k \quad (k = 1, 2) \quad \text{on } \partial \Omega \times (0, T),
\end{equation}
and if

(i) \begin{align*}
\bar{u}_1 & - \Delta \bar{u}_1 - f_1(\bar{u}_1, \bar{u}_2) \geq 0 \geq u_1 - \Delta u_1 - f_1(u_1, u_2) \\
\bar{u}_2 & - \Delta \bar{u}_2 - f_2(\bar{u}_1, \bar{u}_2) \geq 0 \geq u_2 - \Delta u_2 - f_2(u_1, u_2)
\end{align*}
when $(f_1, f_2)$ is quasimonotone nondcreasing

(ii) \begin{align*}
\bar{u}_1 & - \Delta \bar{u}_1 - f_1(\bar{u}_1, \bar{u}_2) \geq 0 \geq u_1 - \Delta u_1 - f_1(u_1, \bar{u}_2) \\
\bar{u}_2 & - \Delta \bar{u}_2 - f_2(u_1, \bar{u}_2) \geq 0 \geq u_2 - \Delta u_2 - f_2(u_1, u_2)
\end{align*}
Remarks on Solutions of a Coupled Semilinear Parabolic System

when \((f_1, f_2)\) is quasimonotone nonincreasing, and

\[
\begin{align*}
\bar{u}_1, \Delta \bar{u}_1 - f_1(\bar{u}_1, \bar{u}_2) & \geq 0 \geq u_1, \Delta u_1 - f_1(u_1, \bar{u}_2) \\
\bar{u}_2, \Delta \bar{u}_2 - f_2(\bar{u}_1, \bar{u}_2) & \geq 0 \geq u_2, \Delta u_2 - f_2(u_1, u_2)
\end{align*}
\]

when \((f_1, f_2)\) is mixed quasimonotone.

Let \((\bar{u}, u)\) be the lower solution and upper solution.

Define the sector

\[
\langle \bar{u}, u \rangle \equiv \{(u_1, u_2) \in C(\bar{D}_T) \times C(\bar{D}_T); (u_1, u_2) \leq (u_1, u_2) \leq (\bar{u}, \bar{u})\}.
\]

Pao ([PAO]) assumed that there exist bounded continuous function \(K_i \equiv K_i(t, x)\) such that \((f_1, f_2)\) satisfies the Lipschitz condition

\[
(HF) \quad |f_i(t, x, u_1, u_2) - f_i(t, x, v_1, v_2)| \leq K_i(|u_1 - v_1| + |u_2 - v_2|)
\]

for \((u_1, u_2), (v_1, v_2) \in \langle \bar{u}, u \rangle\) and \((t, x) \in \bar{D}_T\) \((i = 1, 2)\).

Pao ([PAO]) proved the existence of solution for the problem (1.10)-(1.12).

**Theorem 1.3.** (Theorem 3.1 of Chapter 8 in [PAO]) Let \((\bar{u}_1, \bar{u}_2), (u_1, u_2)\) be ordered upper and lower solutions of (1.10)-(1.12), and let \((f_1, f_2)\) be quasimonotone nondecreasing in \(\langle \bar{u}, u \rangle\) and satisfy the condition (HF). Then the problem has a unique solution \(u = (u_1, u_2)\) in \(\langle \bar{u}, u \rangle\) such that

\[
(\bar{u}_1, \bar{u}_2) \leq (u_1, u_2) \leq (\bar{u}_1, \bar{u}_2) \quad \text{in} \quad \bar{D}_T.
\]

**Theorem 1.4.** (Theorem 3.2 of Chapter 8 in [PAO]) Let \((\bar{u}_1, \bar{u}_2), (u_1, u_2)\) be ordered upper and lower solutions of (1.10)-(1.12), and let \((f_1, f_2)\) be quasimonotone nonincreasing in \(\langle \bar{u}, u \rangle\) and satisfy the condition (HF). Then the problem has a unique solution \(u = (u_1, u_2)\) in \(\langle \bar{u}, u \rangle\) such that

\[
(\bar{u}_1, \bar{u}_2) \leq (u_1, u_2) \leq (\bar{u}_1, \bar{u}_2) \quad \text{in} \quad \bar{D}_T.
\]

**Theorem 1.5.** (Theorem 3.3 of Chapter 8 in [PAO]) Let \((\bar{u}_1, \bar{u}_2), (u_1, u_2)\) be ordered upper and lower solutions of (1.10)-(1.12), and let \((f_1, f_2)\) be mixed quasimonotone in \(\langle \bar{u}, u \rangle\) and satisfy the condition (HF). Then the problem has a unique solution \(u = (u_1, u_2)\) in \(\langle \bar{u}, u \rangle\) such that

\[
(\bar{u}_1, \bar{u}_2) \leq (u_1, u_2) \leq (\bar{u}_1, \bar{u}_2) \quad \text{in} \quad \bar{D}_T.
\]

2 Semilinear parabolic system of the quasimonotone nonincreasing type

We consider the existence and the asymptotic behaviour of solution \(U(x, t) = (u(x, t), v(x, t))\) of the initial-Dirichlet problem for a semilinear parabolic system

\[
(2.11) \quad u_t = \Delta u - j u^p v^q, \quad u \geq 0 \quad \text{in} \quad Q = \Omega \times R^+,
\]

\[
- 75 -
\]
(2.1)

\[ v_t = \Delta v - k u v^p, \quad v \geq 0 \quad \text{in} \quad Q = \Omega \times R^+ \]

with the initial condition

(2.2)

\[ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega \]

and the boundary condition

(2.3)

\[ u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on} \quad \partial \Omega \times R^+. \]

Here \( u_0 \) and \( v_0 \) are nonnegative functions which belong to \( C_0^1(\bar{\Omega}) \), and

\[ j > 0, k > 0, p \geq 1, q \geq 1, r \geq 1, s \geq 1 \]

are constants.

The case \( f(u,v) = -j u^{p}v^{q}, g(u,v) = -k u^{r}v^{s} \) is quasimonotone nonincreasing.

**Theorem 2.1.** The problem (2.1)-(2.3) has a unique nonnegative solution in \( \Omega \times R^+ \), and the solution \( (u,v) \) satisfies

\[ |u(\cdot,t)|_{\infty} + |v(\cdot,t)|_{\infty} \leq C_1 \exp(-\lambda_0 t) \quad (0 < t) \]

where \( C_1 \) depends only on \( u_0, v_0 \).

Here \( \lambda_0 \) is the smallest eigenvalue of the problem

\[ -\Delta \phi_0 = \lambda_0 \phi_0, \quad \phi_0(x) > 0 \quad \text{in} \quad \Omega \]

with

\[ \phi_0(x) = 0 \quad (x \in \partial \Omega). \]

**Proof.** For suitable constants \( C_1, C_2 \) the functions \( \bar{u} = (C_1 \exp(-\lambda_0 t \phi_0(x), C_2 \exp(-\lambda_0 t) \phi_0(x)), \underline{u} = (0,0) \) are ordered upper and lower solutions for the problem (2.1)-(2.3). The existence of solution of the problem (2.1)-(2.3) is proved by the theorem 1.4. The decay estimate is obtained by the upper solution method.

3 Semilinear parabolic system of the mixed quasimonotone type

We consider the existence and the asymptotic behaviour of solution \( U(x,t) = (u(x,t), v(x,t)) \) of the initial-Dirichlet problem for a quasilinear parabolic system

(3.1)

\[ u_t = \Delta u - j u^p v^q - v^\alpha, \quad u \geq 0 \quad \text{in} \quad Q = \Omega \times R^+, \]

(3.1)

\[ v_t = \Delta v + u^q + k u^r v^s, \quad v \geq 0 \quad \text{in} \quad Q = \Omega \times R^+ \]

- 76 -
Remarks on Solutions of a Coupled Semilinear Parabolic System

with the initial condition

\[(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega, \]

and the boundary condition

\[u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on} \quad \partial \Omega \times R^+.\]

Here \(u_0\) and \(v_0\) are nonnegative functions which belong to \(C_0^1(\bar{\Omega})\), and

\[(H.3.1) \quad j > 0, k > 0, p \geq 1, q \geq 1, \alpha \geq 1, \beta \geq 1, r \geq 1, 1 \leq s < \frac{N+2}{N-2}.

We remark that the case \((f,g) = (-j u^p v^q - v^\alpha, k u^r v^\beta + u^\beta)\) is mixed quasimonotone.

For \(u \in H_0^1(\Omega)\), we define

\[J_0(u) = \frac{1}{2} \|\nabla u\|^2 - k \frac{1}{1+s} \|u\|^{1+s}_{1+s} \]

and

\[J_1(u) = \|u\|^2 - \|u\|^{1+s}_{1+s} \cdot \]

By Sobolev’s Lemma, we can define

\[d = \inf_{u \in H_0^1(\Omega), u \neq 0} \sup_{\lambda \geq 0} J_0(\lambda u)(>0). \]

Since \(s \in [1, \frac{N+2}{N-2})\), we set the potential well set as

\[W = \{u|u \in H_0^1(\Omega), 0 < J_1(u), 0 \leq J_0(\lambda u) < d, \text{for} \lambda \in [0,1]\}. \]

**Theorem 3.1.** Assume that \(v_0(x)\) is small enough in the sense of potential well. Under \((H.3.1)\) the problem \((3.1)-(3.3)\) has a unique nonnegative solution in \(\Omega \times R^+\), and the solution \((u,v)\) satisfies

\[(3.4) \quad |u(t)|_{\infty} \leq C_1 \exp(-\lambda_0 t) \quad (0 < t) \]

and

\[(3.5) \quad \|\nabla v(t)\|_2 \leq C \exp(-\lambda^* t) \quad (0 < t). \]

Here \(\lambda_0\) is the smallest eigenvalue of the problem

\[-\Delta \phi_0 = \lambda_0 \phi_0, \phi_0(x) > 0 \quad \text{in} \quad \Omega \]

with

\[\phi_0(x) = 0 \quad (x \in \partial \Omega). \]

\(\lambda^*\) is a constant which is determined by the given data.

**proof.** By the standard way (Cf.[LSU]) we can show the existence of a local solution in time for the problem \((3.1)-(3.3). \) Then from \((3.1)_1\) we can obtain

(i) \[|u(t)|_{\infty} \leq C_1 \exp(-\lambda_0 t) \quad (0 < t). \]

From \((3.1)_2\), using the estimate (i) and the \(L^p\) method (Cf.[NAN1], [NAN2] and [NAN3]), we can prove that

\[\|\nabla v(t)\|_2 \leq C \exp(-\lambda^* t) \quad (0 < t). \]
4 Semilinear parabolic system of the quasimono-
tone nondecreasing type

We consider the existence and the asymptotic behavior of solution $U(x,t) = (u(x,t), v(x,t))$ of the initial-Dirichlet problem for a quasilinear parabolic system

\begin{align}
(4.1)_1 & \quad u_t = \Delta u + av^p - bu^r, \quad u \geq 0 \quad \text{in} \quad Q_T = \Omega \times R^+ , \\
(4.1)_2 & \quad v_t = \Delta v + cu^q - dv^s, \quad v \geq 0 \quad \text{in} \quad Q_T = \Omega \times R^+
\end{align}

with the initial condition

\begin{equation}
(4.2) \quad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega
\end{equation}

and the boundary condition

\begin{equation}
(4.3) \quad u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on} \quad \partial \Omega \times R^+.
\end{equation}

Here $u_0$ and $v_0$ are nonnegative functions which belong to $C^1_0(\overline{\Omega})$, and

\begin{equation}
(H.4.1.) \quad p \geq 1, \quad q \geq 1, \quad r \geq 1, \quad s \geq 1, \quad a > 0, \quad b > 0, \quad c > 0, \quad d > 0
\end{equation}

are constants. We assume that

\begin{equation}
(H.4.2) \quad 1 \leq pq < rs
\end{equation}

and

\begin{equation}
(H.4.3) \quad 1 \leq pq = rs, \quad b^q d^r \geq b^q d^r.
\end{equation}

We remark that the case $(f(u,v), g(u,v)) = (av^p - bu^r, cu^q - dv^s)$ is quasimonotone nondecreasing.

**Theorem 4.1.** Under $(H.4.1), (H.4.2)$, the problem $(4.1)-(4.3)$ has a unique nonnegative solution in $\Omega \times (0,\infty)$.

**Proof.** We can easily construct ordered upper and lower solutions for the problem $(4.1)-(4.3)$. Then by Theorem 1.3 we can prove Theorem 4.1.

**Theorem 4.2.** Under $(H.4.1), (H.4.3)$, the problem $(4.1)-(4.3)$ has a unique nonnegative solution and we have

\begin{equation}
|u(\cdot,t)|_\infty + |v(\cdot,t)|_\infty \leq C \exp(-\lambda_0 t) (0 < t).
\end{equation}

**Proof.** We can easily construct ordered upper and lower solutions for the problem $(4.1)-(4.3)$. The decay estimate can be proved by the uppersolution method.
Remarks on Solutions of a Coupled Semilinear Parabolic System

References


