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SOME REMARKS ON THE MAXIMUM EIGENVALUE OF 3 × 3 PAIRWISE COMPARISON MATRIX

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SOME REMARKS ON THE MAXIMUM EIGENVALUE OF $3 \times 3$
PAIRWISE COMPARISON MATRIX

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Abstract $3 \times 3$ random matrices make the completely same order of two consistency index. In this short paper, we show this phenomenon theoretically holds.

Keywords: AHP, characteristic polynomial, consistency index, Cardano’s method

1. Introduction: Experiments and observation
We made 10 random positive reciprocal matrices of $3 \times 3$ and computed Saaty’s C.I.(Consistency Index) and $c_{mod}$. Easily seen that the rank of C.I. and the rank of $c_{mod}$ are completely the same. See table 1.

Table 1: dim $n=3$, $N=5$

<table>
<thead>
<tr>
<th>C.I.</th>
<th>$c_{mod}$</th>
<th>C.I.’s rank</th>
<th>$c_{mod}$’s rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2603215</td>
<td>6.4533333</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0.2178446</td>
<td>5.1428571</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.115187</td>
<td>61.015873</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1.115187</td>
<td>61.015873</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0.2804168</td>
<td>7.1111111</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2: dim $n=4$, $N=5$

<table>
<thead>
<tr>
<th>C.I.</th>
<th>$c_{mod}$</th>
<th>C.I.’s rank</th>
<th>$c_{mod}$’s rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2489949</td>
<td>16.8324515</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0.9642403</td>
<td>146.2095238</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2.076002</td>
<td>292.075586</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>0.2661867</td>
<td>16.7619048</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2.468707</td>
<td>882.923457</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

The aim of the present paper is to show the observations from this toy experiment is theoretically true. The result holds only when the matrix is $3 \times 3$. Table 2 shows this is not the case the dimension of matrices is greater than 4.

Next we made rather big experiment. In this time, number of random matrices are 5,000. Figure 1 in section 2 below shows cubic polynomial completely fits plotted data($R^2 = 1$). I show this fact through relationships between C.I. and $c_{mod}$. As a consequence, we know C.I. is a monotone function of $c_{mod}$ and vise versa.

As a byproduct of the proof, we can show the representing formula of C.I. by the element of the matrix $(a_{ij})$ (Morris [7]).
2. The largest of eigenvalue of $3 \times 3$ pairwise comparison matrix and $c_{mod}$

In general, the characteristic polynomial $P_A(\lambda)$ of $n \times n$ matrix $A$ has a following form (Saito[10]):

$$P_A(\lambda) = \lambda^n - \text{trace}(A)\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + (-1)^n\det(A)$$  \hspace{1cm} (2.1)

In pairwise comparison matrix, $\text{trace}A = n$. We showed $c_2 = 0$ in (Shiraishi et al. [13]). we also reviewed

$$c_3 = \sum_{i<j<k} \left( 2 - \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}a_{ij}}{a_{jk}} \right) \right). \hspace{1cm} (2.2)$$

Well-known relationships between arithmetic mean and geometric mean implies $c_3 \leq 0$. In the sequel, we will set $c_{mod} = -c_3$. Computational experiments by the several authors suggest that $c_{mod}$ can be used a new consistency index. See Obata et al.[8], Brunelli et al. [2] and Pel`aez et al. [9].

If $n = 3$,

$$P_A(\lambda) = \lambda^3 - 3\lambda^2 + c_3 = \lambda^3 - 3\lambda^2 - c_{mod}.$$  
and $c_{mod} = \det A$. The maximum eigenvalue satisfies following equation.

$$\lambda_{max}^3 - 3\lambda_{max}^2 = c_{mod} \hspace{1cm} (2.3)$$

I consider the function $f(x) = x^3 - 3x^2 = x^2(x - 3)$. Since $f'(x) = 3x(x - 2)$, we see that this function is monotone increasing when $x > 2$. From the well-known results of pairwise comparison matrix, $\lambda_{max}$ varies greater than 3. Hence I have the followings.

**Proposition 2.1.** $c_{mod}$ is a monotone increasing w.r.t. $\lambda_{max}$ i.e. C.I..

(2.3) can be rewritten as follows. Because $C.I. = \frac{\lambda_{max} - 3}{2}$, I have $\lambda_{max} = 2C.I. + 3$. By substituting to (2.3), it is easily seen that

$$c_{mod} = 2C.I.\left(2C.I. + 3\right)^2 = 8C.I.3 + 24C.I.2 + 18C.I.$$$$ \hspace{1cm} (2.4)$$

With the experiment of producing 5,000 random matrices, the cubic regression suggested (2.4). See Fig.1.

![Figure 1: polynomial fitting N=5,000](image-url)
We can confirm this result considering the other function \( g(x) = 8x^3 + 24x^2 + 18x \) in the same manner. By differentiating \( g \), we have \( g'(x) = 24x^2 + 48x + 18 = 6(4(x + 2)^2 - 1) \) which shows \( g(x) \) takes its local minimum at \( x = -\frac{1}{2} \), and monotone increasing \( x \geq -\frac{1}{2} \).

Next, we show the converse result through Cardano’s method. See Ueno[14]. If we set \( \lambda = x + 1 \) in (2.1), we have

\[
(x + 1)^3 - 3(x + 1)^2 + c_3 = (x^3 + 3x^2 + 3x + 1) - 3(x^2 + 2x + 1) - c_{\text{mod}}
\]

\[
= x^3 - 3x - 2 - c_{\text{mod}}. \tag{2.5}
\]

Next we set \( x = u + v \), then we have

\[
(u + v)^3 - 3(u + v) - (c_{\text{mod}} + 2) = (u^3 + v^3 - (c_{\text{mod}} + 2)) + 3u^2v + 3uv^2 - 3(u + v)
\]

\[
= (u^3 + v^3 - (c_{\text{mod}} + 2)) + 3(u + v)(uv - 1). \]

Thus we reach the following system of equation.

\[
0 = u^3 + v^3 - (c_{\text{mod}} + 2) \tag{2.6}
\]

\[
0 = (u + v)(uv - 1) \tag{2.7}
\]

If \( u + v = 0 \), (2.6) implies \( c_{\text{mod}} = -2 \). It contradicts \( c_{\text{mod}} \geq 0 \). From (2.7), we have \( v = \frac{1}{u} \). By substituting to (2.6), we obtain

\[
0 = u^3 + \frac{1}{u^3} - (c_{\text{mod}} + 2)
\]

\[
0 = (u^3)^2 - (c_{\text{mod}} + 2)u^3 + 1 \tag{2.8}
\]

From (2.8), we get two solution

\[
u = \sqrt[3]{\frac{c_{\text{mod}} + 2 + \sqrt{(c_{\text{mod}} + 2)^2 - 4}}{2}}
\]

\[
v = \sqrt[3]{\frac{c_{\text{mod}} + 2 - \sqrt{(c_{\text{mod}} + 2)^2 - 4}}{2}}.
\]

Thus, considering symmetricity of \( u \) and \( v \), we get

\[
x = \sqrt[3]{\frac{c_{\text{mod}} + 2}{2}} + \sqrt[3]{\frac{(c_{\text{mod}} + 2)^2 - 4}{2}} \quad + \sqrt[3]{\frac{(c_{\text{mod}} + 2)^2 - 4}{2}} + 1 \tag{2.9}
\]

since \( \lambda = x + 1 \), we finally get the followings.

\[
\lambda = \sqrt[3]{\frac{c_{\text{mod}} + 2}{2}} + \sqrt[3]{\frac{(c_{\text{mod}} + 2)^2 - 4}{2}} + 1
\]
Proposition 2.2. $\lambda_{\text{max}}(i.e. \ C.I.)$ is monotone increasing w.r.t. $c_{\text{mod}}$.

Proof. case 1: $x > 0$

We consider the function $f(x) = \sqrt[3]{x + 2 + \sqrt{(x + 2)^2 - 4}} + \sqrt{x + 2 - \sqrt{(x + 2)^2 - 4}}$, and show its monotonicity. Calculation of the derivative of $f(x)$ shows

$$f'(x) = \frac{1}{3} \left[ \frac{1 + \frac{(x + 2)}{\sqrt{(x + 2)^2 - 4}}}{\left(\sqrt{x + 2 + \sqrt{(x + 2)^2 - 4}}\right)^2} + \frac{1 - \frac{(x + 2)}{\sqrt{(x + 2)^2 - 4}}}{\left(\sqrt{x + 2 + \sqrt{(x + 2)^2 - 4}}\right)^2} \right]$$

$$= \frac{1}{3\sqrt{(x + 2)^2 - 4}} \left[ \frac{x + 2 + \sqrt{(x + 2)^2 - 4}}{\left(\sqrt{x + 2 + \sqrt{(x + 2)^2 - 4}}\right)^3} - \frac{(x + 2)\sqrt{(x + 2)^2 - 4}}{\left(\sqrt{x + 2 + \sqrt{(x + 2)^2 - 4}}\right)^3} \right]$$

$$= \frac{1}{3\sqrt{(x + 2)^2 - 4}} \left[ \frac{(x + 2) + \sqrt{(x + 2)^2 - 4}}{\sqrt{x + 2 + \sqrt{(x + 2)^2 - 4}}} \right] > 0 \text{ for } x > 0.$$

This means $f(x)$ is monotone increasing for $x > 0$.

case 2 : $x \geq 0$

The relationship between arithmetic and geometric means shows that

$$f(x) = \sqrt[3]{x + 2 + \sqrt{(x + 2)^2 - 4}} + \sqrt{x + 2 - \sqrt{(x + 2)^2 - 4}}$$

$$\geq 2\sqrt{\sqrt{x + 2 + \sqrt{(x + 2)^2 - 4}}\sqrt{x + 2 - \sqrt{(x + 2)^2 - 4}}}$$

$$= 2\sqrt[3]{4} = 2\sqrt[3]{2} = (\sqrt[3]{2} + \sqrt[3]{2}) = f(0).$$

The following theorem is obvious from propositions 2.1 and 2.2.

Theorem 2.1. The order of C.I. and the order of $c_{\text{mod}}$ are completely the same in all $3 \times 3$ pairwise comparison matrix.
3. Byproduct results

Shiraishi et al.[13] gave $c_3 = 2 - \left( \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} \right)$. So $c_{\text{mod}} = \left( \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} \right) - 2$. From (2.9), one can transform it as follows.

$$(c_{\text{mod}} + 2)^2 - 4 = \left( \frac{a_{12}a_{23}}{a_{13}} \right)^2 + 4 + \left( \frac{a_{13}}{a_{12}a_{23}} \right)^2 - 4$$

$$= \left( \frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}} \right)^2$$

$$c_{\text{mod}} + 2 = \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}}$$

**Case 1**

When $\frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}} > 0$, one has

$$\lambda = \sqrt[3]{\frac{1}{2} \left( \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} \right)}$$

$$\lambda = \sqrt[3]{\frac{1}{2} \left( \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} \right) + 1}$$

$$\lambda = \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} + \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} + 1.$$

**Case 2**

When $\frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}} < 0$, one has

$$\lambda = \sqrt[3]{\frac{1}{2} \left( \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} + \frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}} \right)}$$

$$\lambda = \sqrt[3]{\frac{1}{2} \left( \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} + \frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}} \right) + 1}$$

$$\lambda = \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} + \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} + 1.$$

In any cases the following result holds.

**Proposition 3.1** (Morris[7], Crowford et al. [5], Fujihara et al.[6]).

$$\lambda_{\text{max}} = \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} + \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} + 1. \quad (3.1)$$

**Corollary 3.1.** one has

$$C.I. \leq \frac{c_{\text{mod}}}{2} - 1.$$  

The equality holds if and only if $a_{13} = a_{12}a_{23}$.

**Proof.** It is obvious from the inequality $\sqrt[3]{a} \leq a$.  

□
From (3.1), one has
\[
\lambda_{\text{max}} = \sqrt{\frac{a_{12}a_{23}}{a_{13}}} \left( 1 + \left( \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} \right)^2 + 3 \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} \right)
\]
\[
= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} \left( 1 + \left( \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} \right)^2 + \frac{2}{3} \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} \right) - 1
\]
\[
= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} \left( \frac{\sqrt[3]{a_{13}}}{\sqrt[3]{a_{12}a_{23}}} + 1 \right)^2 - 1.
\]

**Proposition 3.2.** The following formula holds.

\[
\lambda_{\text{max}} + 1 = \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} \left( \frac{\sqrt[3]{a_{13}}}{\sqrt[3]{a_{12}a_{23}}} + 1 \right)^2
\]
\[
= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} \left( \frac{\sqrt[3]{a_{13}}}{a_{12}a_{23}} + 1 \right)^2.
\]

4. **Conclusion**

In several articles, the relationships between C.I. and \( c_{\text{mod}} \) has been investigated[2, 8]. Brunelli et al. has investigated linear correlation computed on 10,000 randomly generated pairwise comparison matrices of order 6. The linear correlation is 0.952 which reveals best performance rather than another proposed consistency indeces. The results is my starting point of the present paper. Obata et al. [8] proposed use of \( c_{\text{mod}} \) as a new consistent index. With the aid of Excel\textsuperscript{TM}, one can compute the threshold according to C.I. value. If C.I. = 0.1, one has \( c_{\text{mod}} \approx 0.8929 \). If C.I. = 0.15, one has \( c_{\text{mod}} \approx 1.0291 \). So, we can say that if \( c_3 \) is less than 0.9, the pairwise comparison matrix is consistent in Saaty's sence.

By the way, \( c_{\text{mod}} \) has some shortcomings. First, value of \( c_3 \)'s become rather large\(^1\). Computation on 500 randomly generated pairwise comparison matrices gives maximum value of \( c_3 \) to be 727.0014. So one may hesitate use of \( c_{\text{mod}} \). Some normalization should be needed.

Second, in general, one can observe \( c_{\text{mod}} \) gets greater as the order of matrices. This also occurs on C.I.\(^2\) So one needs new consistent index whose value is independent to the order of matrices.

I believe that more modification of \( c_{\text{mod}} \) may solve these shortcomings. The issue remains for further research.

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\(^1\)C.I. has an upper limit. See Sekitani et al. [11]

\(^2\)To overcome this shortcoming, in AHP, CR is also used. See[1]
References


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