NOTE ON PARETO OPTIMA

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It is well-known that a Pareto optimum is attained where a positive linear combination of utility functions of all individuals achieves its maximum (see, e. g., Takayama [4, Theorem 1. E. 6.]). But, to state the converse of this theorem we usually need the assumption of concavity of utility functions (see, Takayama [4, Theorem 1. E. 4.]). The purpose of this note is to give a simple alternative characterization for a Pareto optimum in terms of some real-valued function. The necessary and sufficient condition we give here requires no convexity assumptions at all.

Now, let $N=\{1, 2, \ldots, n\}$ be the set of all individuals. For each $i \in N$, $u_i(x)$ is a continuous real-valued function defined on R^m_+, the nonnegative orthant of m-dimensional Euclidean space. Let X be a nonempty subset of R^m_+. X is interpreted as a feasible set of "social states", and $u_i(x)$, the utility of individual i under a social state x. Let us normalize the utility functions, so that $u_i(0)=0$ for each $i \in N$. We assume that:

Assumption 1. If $x \in X$ and $x \neq 0$, then $u_i(x) > 0$ for all $i \in N$.

This is a much weaker assumption than strict monotonicity. We say $x^* \in X$ is Pareto optimal if $u_i(x) > u_i(x^*)$ for all $i \in N$ implies $x \notin X$. Note that the optimality is used in a weak sense. To characterize a Pareto optimum, we define the following real-valued function $v(x, a)$:

$$v(x, a) = \min_i \left\{ \frac{u_i(x)}{a_i} \right\}, \text{ for all } x \in X \text{ and for all } a \in A = \{a \in R^n_+ : \sum_{i \in N} a_i = 1\}.$$

Here the notation $\min_i$ is used to mean the minimum taken over all $i \in N$ such that $a_i > 0$. Note that $v(x, a)$ is continuous on $X \times A$. Then we can state:

Theorem. $x^* \in X$ is Pareto optimal if and only if there is an $a \in A$ such that

$$v(x^*, a) = \max_{x \in X} v(x, a). \quad (1)$$

Proof. Suppose x^* is not Pareto optimal. Then, there is an $x \in X$ such that $u_i(x) > u_i(x^*)$ for all $i \in N$. Hence $u_i(x)/a_i > u_i(x^*)/a_i$ for all $i \in N$ with $a_i > 0$. Hence we have $v(x, a) > v(x^*, a)$, a contradiction.

Conversely, let x^* be Pareto optimal. By Assumption 1, if $x^* \neq 0$, we have $u_i(x^*) > 0$ for all $i \in N$. Hence for any $x \in X$ we have $\min_{i \in N}\{u_i(x)/u_i(x^*)\} \leqq 1$, for we would otherwise have a contradiction that $u_i(x) > u_i(x^*)$ for all $i \in N$. Now, define $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n) \in A$ by

$$\bar{a}_i = u_i(x^*)/\sum_{j \in N} u_j(x^*), \text{ for each } i \in N$$

Then, for any $x \in X$ we have

$$\begin{aligned} v(x^*, \bar{a}) &= \min_{i \in N}\{u_i(x^*)/\bar{a}_i\} = \sum_{j \in N} u_j(x^*) \geqq \sum_{j \in N} u_j(x^*)[\min_{i \in N}\{u_i(x)/u_i(x^*)\}] \\ &= \min_{i \in N}\{[u_i(x)/u_i(x^*)]\sum_{j \in N} u_j(x^*)\} = v(x, \bar{a}), \end{aligned}$$

or

$$v(x^*, \bar{a}) = \max_{x \in X} v(x, \bar{a}).$$

If $x^* = 0$, then for any $x \in X$ there is an $i \in N$ such that $u_i(x) = 0$, since x^* is Pareto optimal. But, then, it must not be true that $x \neq 0$, for we would otherwise have $u_i(x) > 0$ for all $i \in N$ by Assumption 1. Hence $X = \{x^*\}$, and this implies that $v(x^*, a) = \max_{x \in X} v(x, a)$ for any $a \in A$.

Q. E. D.

Note that we do not need the concavity of u_i, nor the convexity of X. In practice, however, X is often a convex set, and utility functions are assumed to be quasiconcave. In this case, we can show that the function

$v(x, a)$ is also quasiconcave on X, i. e., $x, y \in X$ and $0 \leqq h \leqq 1$ imply $v(hx + (1-h)y, a) \geqq min\{v(x, a), v(y, a)\}$.

Proposition. Let X be convex. If $u_i(x)$ is quasiconcave on X for each $i \in N$, then $v(x, a)$ is also quasiconcave on X.

Proof. Let $x, y \in X$, and let $0 \leqq h \leqq 1$. Then,

$$\begin{aligned} v(hx+(1-h)y, a) &= \min_i \{u_i(hx+(1-h)y)/a_i\} \\ &\geqq \min_i \{(1/a_i) min\{u_i(x), u_i(y)\}\} \\ &\geqq min\{\min_i \{u_i(x)/a_i\}, \min_i \{u_i(y)/a_i\}\} \\ &= min\{v(x, a), v(y, a)\}. \end{aligned}$$

Q. E. D.

The theorem and the proposition together define an upper hemi-continuous correspondence that associates to every $a \in A$ a convex set of Pareto optimal states; namely,

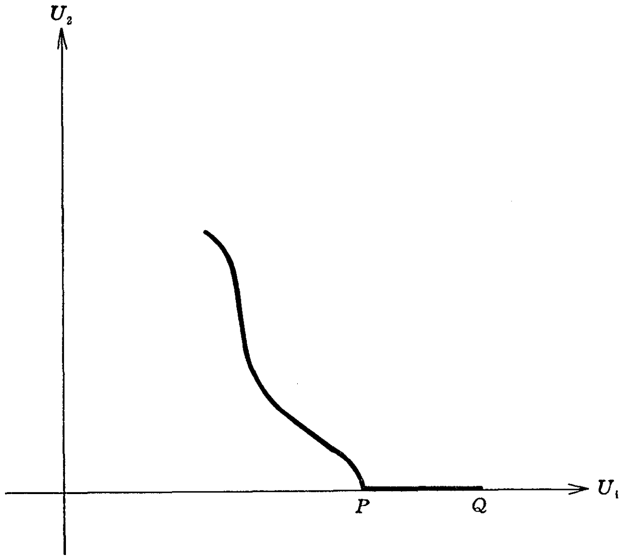
$$X(a) = \{x \in X : v(x, a) \geqq v(\bar{x}, a) \text{ for all } \bar{x} \in X\}.$$

Since $v(x, a)$ is continuous, $X(a)$ is a closed correspondence with a compact range X. Thus it is upper hemi-continuous (see Hildenbrand and Kirman [1, p. 194]). Such a correspondence is useful in some existence proofs (see, e. g., Nakayama [2]).

Finally, note that Assumption 1 is crucial in stating our theorem. Without this assumption, the "only if" part of the theorem is not always true as illustrated in Fig. 1. The weak Pareto frontier may contain the segment PQ. But any point on PQ other than Q can not be expressed as a maximum of $v(x, a)$ for any $a \in A$. The occurrence of such a case may also be avoided by the following assumption.

Assumption 1′ The interior of X is nonempty, and $u_i(x) > 0$ for all $i \in N$ if x is an interior point of X.

Under this assumption, whenever x^* is Pareto optimal there will be an



$x \in X$ in any neighborhood of x^* such that $u_i(x) > 0$ for all $i \in N$. Thus, the above possibility is ruled out.

foot note 1). This characterization is reminiscent of the well-known difference principle of Rawls [3]. Thus, $v(x, a)$ can be viewed as a Rawlsian social welfare function while the linear combination $\sum_{i \in N} a_i u_i(x)$ is interpreted as a utilitarian social welfare function.

REFERENCES

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