A Note on Lindahl Equilibria and the Core of a Game with a Crowded Public Good*

by

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1. This note considers the core of a game for an economy with a public good subject to crowding and its relationships to the Lindahl equilibria of that economy. Crowding means here that cost functions for providing the public good are nondecreasing with respect to a group of agents, namely its size, in which the public good is produced and consumed collectively excluding any of the nonmembers. This treatment of crowding is similar to that of Ellickson [1], with which he presented examples that the Lindahl equilibrium does not belong to the core and that the core itself is empty, contrary to the assertion of Foley [2] for the pure public good case. Crowding in this setting should involve a partition of agents into several "sharing groups" for the public good. Thus, in our game, players in each coalition are allowed to form a partition so as to maximize the net benefit of the coalition as a whole. We derive a necessary and sufficient condition for the game to have anonempty core, and a sufficient condition in a special

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case for the core to contain the Lindahl equilibria.

2. Let $N=\{1, \cdots, n\}$ be the set of players. Every nonempty subset T of N can produce a public good, for which we shall consider crowding, and consume it collectively excluding any of the members in $N-T$. T might be called a sharing group for the public good. We will find it convenient later to look upon T as a kind of clubs of players that is quite free from any geographical interpretation. The benefit of player $i \varepsilon N$ received from consuming q amount of the public good is given, in terms of money, by $u_i(q)$. We assume that $u_i(q)$ is monotone increasing, concave, continuous and bounded from above for $q(q \geqq 0)$. The cost of providing q amount of the public good in sharing group T is given by $C(q, T)$. Here $C(q, T)$ is a monotone nondecreasing function of $T \subset N$, *i. e.*, $T \subset R$ implies $C(q, T) \leqq C(q, R)$. This means that the addition of another player to a sharing group (possibly) increases the resources required to maintain the level of the public good consumed in the group. Pure public goods are those for which crowding does not occur, that is, $C(q, T)=C(q)$ for all $T \subset N$. We assume in addition that $C(q, T)$ is increasing and linear in $q(q \geqq 0)$.

In this setting, a coalition of players may form a partition into several sharing groups so as to maximize the net benefit from the public good. Denote by $\Pi(S)$ the collection of partitions of $S \subset N$. Then we define the game (N, v) as follows:

$$v(S)=\begin{cases} \max\limits_{\pi \varepsilon \Pi(S)} \sum\limits_{T \varepsilon \pi} B(T) & \text{for all nonempty } S \subset N, \\ 0 & \text{for } S \text{ an empty set,} \end{cases} \tag{1}$$

where

$$B(T)=\max_{q \geqq 0} \left(\sum_{i \varepsilon T} u_i(q)-C(q, T)\right). \tag{2}$$

The core of the game (N, v) is the set of payoff vectors $b=(b_1, \cdots b_n)$

satisfying

$$\sum_{i\varepsilon S} b_i \geqq v(S) \text{ for all } S \subset N, \text{ and } \sum_{i\varepsilon N} b_i \leqq v(N) \tag{3}$$

In Shapley and Shubik's term in [7], the game (N, v) is "the least super-additive majorant" of the game (N, B), where the function B is defined by (2) for all subsets of N. Thus the game (N, v) can be viewed as an example of the least superadditive majorant of games. For pure public goods it is clear that $v(S) = B(S)$. Then the core of the game (N, v) is nonempty and contains the Lindahl equilibria (defined later), since our model is a special version of that of Foley [2]. Kaneko [3] showed further that the game becomes the convex game in this pure public good case, which implies that the core is so large that it tells us almost nothing about the outcome the game would yield. However, as examples given by Ellickson [1] indicate, these properties are no longer true without any further condition if crowding is introduced.

3. Let (N, v') be the game defined by

$$v'(S) = \begin{cases} B(S) & \text{if } S \subsetneqq N, \\ v(N) & \text{if } S = N. \end{cases}$$

Note that the game (N, v') may not be superadditive.

Then we prove:

Proposition 1. The game (N, v) has a nonempty core if and only if the game (N, v') has a nonempty core.

Proof. The "only if" part is clear since $v'(S) \leqq v(S)$ for all $S \subset N$. Let $S \subset N$ be an arbitrary coalition and let $v(S) = \sum_{T \varepsilon \pi} B(T)$, where π is a partition of S. Then if $b = (b_1, \cdots, b_n)$ belongs to the core of the game (N, v'), we have

$$v(S)=\sum_{T\varepsilon\pi} B(T)\leqq\sum_{T\varepsilon\pi}\ \sum_{i\varepsilon T} b_i=\sum_{i\varepsilon S} b_i. \qquad \text{Q. E. D.}$$

It is wellknown that the core of a characteristic function game is nonempty if and only if the game is balanced, i. e.,

$$\sum_{S\subset N} x_s v(S)\leqq v(N) \text{ for all } \{x_s\geqq 0:\ S\subset N\}$$
$$\text{such that } \sum_{\substack{S\ni i\\ S\subset N}} x_s=1 \text{ for all } i\varepsilon N.$$

Thus, Proposition 1 implies that the core of the game (N, v) is nonempty if and only if

$$\sum_{T\subset N} x_T B(T)\leqq v(N) \text{ for all } \{x_T\geqq 0:\ T\subset N\}$$
$$\text{such that } \sum_{\substack{T\ni i\\ T\subset N}} x_T=1 \text{ for all } i\varepsilon N. \qquad (4)$$

To interpret the condition (4), let us suppose that the society $N=\{1,\cdots,n\}$ seeks for an optimal pattern of sharing under which the net benefit from consuming the public good is maximized. Following Littlechild [4], a fractional membership is also allowed as a possible pattern of sharing. That is, every player can join a sharing group T at a proportion of time indicated by the balancing weight x_T. In Littlechild's terminology, T is "a part-time club" if $x_T\neq 1$. When $x_T=1$, T is "a full-time club" in the sense that every player in T devotes all of his time to T. Then $\sum_{T\subset N} x_T B(T)$ is the net benefit obtained when each player participates each sharing group T fractionally x_T amount of time. Thus condition (4) for the nonempty core states that the maximal net benefit over all possible pattern of sharing including such fractional memberships must be attained under a partition of N so that each player participates fully in a particular sharing group. In other words, any sharing pattern involving part-time clubs must not be optimal.

4. Next, we consider the relationship between the core and Lindahl equilibria in this economy. Let π be a partition of N. Then the Lindahl

equilibrium is the triplet $(p^*; q^*_\pi; \pi)=(p^*_1,\cdots,p^*_n; q^*_{T_1}\cdots,q^*_{T_m}; T_1,\cdots,T_m)$ such that for each $T\varepsilon\pi$,

$$u_i(q^*_T)-p^*_i q^*_T=\max_{qi\geqq 0}\ (u_i(q_i)-p_i^* q_i)\ \text{ for all } i\varepsilon T,$$

and

$$\sum_{i\varepsilon T} p^*_i q^*_T - C(q^*_T,T)=\max_{q\geqq 0}\ (\sum_{i\varepsilon T} p^*_i q - C(q,\ T)).$$

In view of Ellickson's nonpathological examples indicating that the Lindahl equilibrium is not contained in the core, it seems unsnccessful even in this simple framework to investigate generally the conbition for the core to contain the Lindahl equilibrium. Rather, we shall limit to the case where formation of sharing groups across the optimal partition of N is not advantageous. Let $v(N)=\sum_{T^*\varepsilon\pi} B(T^*)$. We assume that

$$\sum_{T^*\varepsilon\pi} B(T^*\cap T)\geqq B(T)\qquad \text{for all } T\subset N. \tag{5}$$

Under this assumption sharing groups across π would not form since it does no better than stay in each T^*. Let the cost function $C(q,T)$ be given by

$$C(q,\ T)=a_T q,\ a_T>0\qquad \text{for each } T\subset N.$$

Then we prove:

Proposition 2. Under assumption (5), the Lindahl equilibrium $(p^*; q^*_\pi; \pi)$ belongs to the core of the game $(N,\ v)$ if

$$\sum_{i\varepsilon T} p^*_i\leqq a_T \qquad \text{foɪ all } T\subset N. \tag{6}$$

Proof. Let $\alpha_i=u_i(q^*_{T^*})-p^*_i q^*_{T^*}$ if $i\varepsilon T^*$, and let $B(R)=\sum_{i\varepsilon R} u_i(q_R)-C(q_R,\ R)$ for $R\subset N$ arbitrary fixed. By Proposition 1 it suffices to show that $\sum_{i\varepsilon T}\alpha_i\geqq B(T)$ for all $T\subsetneqq N$, $\sum_{i\varepsilon N}\alpha_i\leqq v(N)$. The latter is clear. By the definition of the Lindahl equilibrium and condition (6), we have

$$\begin{aligned}\sum_{i\varepsilon T}\alpha_i &= \sum_{T^*\varepsilon\pi}\ \sum_{i\varepsilon T\cap T^*}(u_i(q^*_{T^*})-p^*_i q^*_{T^*})\\ &\geqq \sum_{T^*\varepsilon\pi}\ \sum_{i\varepsilon T\cap T^*}(u_i(q_{T\cap T^*})-p^*_i q_{T\cap T^*})\\ &= \sum_{T^*\varepsilon\pi}\ (\sum_{i\varepsilon T\cap T^*} u_i(q_{T\cap T^*})-\sum_{i\varepsilon T\cap T^*} p^*_i C(q_{T\cap T^*},\ T\cap T^*)/a_{T\cap T^*})\end{aligned}$$

$$\geqq \sum_{T^* \epsilon \pi} \left(\sum_{i \epsilon T \cap T^*} u_i(q_{T \cap T^*}) - C(q_{T \cap T^*}, T \cap T^*) \right)$$
$$= \sum_{T^* \epsilon \pi} B(T \cap T^*)$$
$$\geqq B(T). \qquad Q. E. D.$$

Thus assumption (6) together with the condition that the sum of the Lindahl prices p^*_i never exceeds the marginal cost of production for the public good in any sharing group assures the Lindahl equilibrium to be in the core of the game (N, v). The result for a pure public good follows immediately from $\pi = \{N\}$ and $a_T = a_N$ for all $T \subset N$.

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