# Note on Lindahl Equilibria and the Core of an Economy with a Public Good

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**Abstract**: The relationship between Lindahl equilibria and the core of an economy with a public good is examined under the assumption that every coalition plays a non-cooperative game against its complementary coalition in sharing the cost for the public good. It is shown that the Lindahl equilibrium is contained in the core if and only if it is unblocked by any one-person coalition, and this is interpreted in terms of the free rider.

1. In this note, we examine the wellknown relationship between Lindahl equilibria and the core of an economy with pure public goods under a specific assumption imposed on the activities of coalitions of agents. The result of Foley [2] that Lindahl equilibria are contained in the core was derived under a conservative assumption on permissible activities of the complementary coalition. That is, any coalition must produce the public goods for itself without assuming any contribution of the complementary coalition in blocking the allocation of the economy. As a result the core is quite large as mentioned by Foley himself. In fact, Champsauer [1] showed in a game theoretic framework that for every coalition, and hence for every one-person coalition, too, there exist core allocations which are attainable for it.

Alternative views on what a coalition can achieve for itself in situations involving externalities have been proposed and examined by Rosenthal [5] or Richter [4]. The latter derived necessary conditions for the core to disappear under the assumption that the complementary coalition contributes to the public good provision subject to certain rationality constraints. We shall also take an alternative view as to how the complementary coalition reacts, which might be a variation of the rationality constraint proposed by Richter [4]. Specifically, we consider the case where a coalition and its complement act as players of a two person non-cooperative game in sharing the cost of a public good to be produced. Then every coalition is associated with its utility level defined at the Nash equilibrium point in the set of the pairs of payments decided strategically. Thus we obtain a characteristic function game which describes the power of each coalition to block the outcome, i. e., utility levels assured under the Lindahl equilibrium, in our case.

In this framework it is shown that the Lindahl equilibrium belongs to the core if and only if it is unblocked by any one-person coalition. This result can be interpreted in terms of the free rider in a public goods economy defined precisely by e. g., Kaneko [3].

2. Our economy consists simply of n agents, one public good and freely transferable money. $N=\{1, \cdots, n\}$ is the set of all agents, and a coalition is a nonempty subset S of N. Each agent is assumed to have sufficiently large quantity of money. The public good is pure, i. e., consumed collectively and equally by all agents in N. The utility gained by agent i from consuming q amount of the public good is given by $u_i(q)$, which is assumed to be measured in terms of money. The cost needed to produce q amount

of the public good is given by $C(q)$ for all coalitions. We assume:

(a) $u_i(q)$ is differentiable on $(0,\infty)$ with $u_i'>0, u_i''<0$ and $\lim_{q\to+\infty} u_i'(q)=0$, and satisfies $u_i(0)=0$.

(b) $C(q)$ is linear and increasing in $q$ $(q\geqq 0)$, and $C(0)=0$.

The Lindahl equilibrium in this economy is the pair $(\bar{p};\bar{q})=(\bar{p}_1,\cdots\bar{p}_n;\bar{q})$ such that

$$u_i(\bar{q})-\bar{p}_i\bar{q}=\max_{q_i\geqq 0}(u_i(q_i)-\bar{p}_iq_i) \quad \text{for all } i\varepsilon N \tag{1}$$

$$\sum_{i\varepsilon N}\bar{p}_i\bar{q}-C(\bar{q})=\max_{q\geqq 0}(\sum_{i\varepsilon N}\bar{p}_iq-C(q)) \tag{2}$$

Lindahl imputation is the payoff vector $\bar{x}=(\bar{x}_1,\cdots,\bar{x}_n)$ defined by

$$\bar{x}_i=u_i(\bar{q})-\bar{p}_i\bar{q} \quad \text{for all } i\varepsilon N, \tag{3}$$

where $(\bar{p}:\bar{q})$ is the Lindahl equilibrium.

Let $Q(T)$ be an inverse function of $C(q)$, i. e., $C(Q(T))=T$. Let $f_s(t_s,t_{N-s})$ be a function defined for all $S\subset N$ by

$$f_s(t_s,t_{N-s})=\sum_{i\varepsilon S}u_i(Q(t_s+t_{N-s}))-t_s. \tag{4}$$

Let $t_s^\circ$ be defined for all $S\subset N$ by

$$f_s(t_s^\circ,0)=\max_{t_s\geqq 0} f_s(t_s,0). \tag{5}$$

We say $t_s^\circ$ is an individual cost for $S$, which is the cost needed for $S$ to produce the public good that attains maximal total utility without any contribution from $N-S$.

For arbitrary $S$ fixed, the pair $(t_s^*,t_{N-s}^*)$ is a Nash equilibrium if

$$f_s(t_s^*,t_{N-s}^*)=\max_{t_s\geqq 0} f_s(t_s,t_{N-s}^*) \quad \text{for } S \text{ and } N-S. \tag{6}$$

We need the following Lemma which characterizes the Nash equilibrium.

*Lemma.* Let $t^*=(t_s^*,t_{N-s}^*)$ be the Nash equilibrium. Then,

(i) $t_s^*+t_{N-s}^*=\max\{t_s^\circ,t_{N-s}^\circ\}$,

(ii) $t_s^*=0$ if $t_s^\circ<t_{N-s}^\circ$.

Though the proof follows straightforwardly from Kaneko [3, Corollary], we shall state it here for completeness.

*Proof.* Let $T^*=t_s^*+t_{N-s}^*$, and let $T^\circ=\max\ \{t_s^\circ, t_{N-s}^\circ\}$.

(i) Suppose $T^*>T^\circ$.

Them, $f_s(T^*, 0)<f_s(T^\circ, 0)$ for $S$ and $N-S$.

Since every $u_i$ is strictly concave, we have

$$\left.\frac{df_s(T,0)}{dT}\right|_{T=T^*}=\left.\frac{\partial f_s(t_s, t_{N-s})}{\partial t_s}\right|_{t=t^*}<0.$$

Since $T^*>0$, we may well assume $t_s^*>0$.

Then for sufficiently small $\delta>0$, we have

$$\begin{aligned}&f_s(t_s^*-\delta, t_{N-s}^*)-f_s(t_s^*, t_{N-s}^*)\\&\qquad=\left.\frac{\partial f_s}{\partial t_s}\right|_{t=t^*}(-\delta)+o(\delta)>0,\end{aligned}$$

where $\lim_{\delta\to 0} o(\delta)/\delta=0$.

This contradicts that $t^*$ is a Nash equilibrium.

Suppose $T^*<T^\circ$. Then for $S$ with $t_s^\circ=T^\circ$, we have

$$f_s(T^\circ, 0)>f_s(T^*, 0).$$

Then we are led to a contradiction in a similar way.

(ii) It suffices to show that

$$\left.\frac{\partial f_s(t_s, t_{N-s})}{\partial t_s}\right|_{t=t^*}<0.$$

This follows immediately from the proof of (i), since $T^*=t_{N-s}^\circ>t_s^\circ$. *Q. E. D.*

Under the Nash equilibrium the coalition that bears the cost for the public good is the one for which the individual cost is greater, and the complementary coalition bears no cost.

However if it happens that $t_s^\circ=t_{N-s}^\circ$, then $t^*$ is not uniquely determined. To avoid this difficulty for our purpose, we shall simply assume that

(c) $t_S^\circ \neq t_{N-S}^\circ$ for all $S \subset N$.

We can then define utility level $v(S)$ each coalition $S$ can assure under the Nash equilibrium by $v(S) = f_S(t_S^*, t_{N-S}^*)$, *i. e.*,

$$v(S) = \begin{cases} \sum_{i \in S} u_i(Q(t_S^\circ)) - t_S^\circ & \text{if } t_S^\circ > t_{N-S}^\circ \\ \sum_{i \in S} u_i(Q(t_{N-S}^\circ)) & \text{if } t_S^\circ < t_{N-S}^\circ \end{cases} \tag{7}$$

Thus we obtain a game in characteristic function form represented by $(N, v)$. For $S$ *with* $t_S^\circ > t_{N-S}^\circ$, $v(s)$ is the same to the one defined under the usual asssumption on the activities of $N-S$. When $t_S^\circ < t_{N-S}^\circ$, the coalition $S$ is just free riding on the benefit spilled over from $N-S$.

The core of the game $(N, v)$ is the set of payoff vectors $x = (x_1, \cdots, x_n)$ satisfying

$$\sum_{i \in S} x_i \geqq v(S) \quad \text{for all } S \subset N \text{ and } \sum_{i \in N} x_i = v(N).$$

We want to know if the core of this game contains the Lindahl equilibrium of this economy.

3. We prove:

*Proposition.* Assume that

(d) $t^\circ_{\{i\}} < t_{N-\{i\}}^\circ$ for all $i \varepsilon N$.

Then the Lindahl imputation $\bar{x}$ belongs to the core if and only if

$$x_i \geqq v(\{i\}) \quad \text{for all } i \varepsilon N.$$

*Proof.* The "only if" part is trivial. To show the converse, we first prove that

$$Q(t_R^\circ) \leqq Q(t_S^\circ) \text{ if } R \subset S. \tag{8}$$

Suppose $Q(t_R^\circ) > Q(t_S^\circ)$. Then by the monotonicity of $u_i$, we have

$$u_i(Q(t_R^\circ)) > u_i(Q(t_S^\circ)).$$

Hence,

$$
\begin{aligned}
f_s(t_s^\circ, 0) &= \sum_{i\varepsilon S} u_i(Q(t_s^\circ)) - t_s^\circ \\
&= \sum_{i\varepsilon R} u_i(Q(t_s^\circ)) - t_s^\circ + \sum_{i\varepsilon S-R} u_i(Q(t^\circ)) \\
&< \sum_{i\varepsilon R} u_i(Q(t_R^\circ)) - t_R^\circ + \sum_{i\varepsilon S-R} u_i(Q(t_R^\circ)) \\
&= \sum_{i\varepsilon S} u_i(Q(t_R^\circ)) - t_R^\circ \\
&\leqq f_s(t_s^\circ, 0).
\end{aligned}
$$

This is a contradiction.

Let $t_s^\circ < t_{N-s}^\circ$. Then in view of $(d)$ and (8) we have

$$
\begin{aligned}
\sum_{i\varepsilon S} x_i \geqq \sum_{i\varepsilon S} \mathrm{v}(\{i\}) &= \sum_{i\varepsilon S} u_i(Q(t_{N-\{i\}}^\circ)) \\
&\geqq \sum_{i\varepsilon S} u_i(t_{N-s}^\circ)) = v(s).
\end{aligned}
$$

Let $t_s^\circ > t_{N-s}^\circ$. Then by the definition of the Lindahl equilibrium and the linearity of $C(\mathrm{q})$ we have

$$
\begin{aligned}
\sum_{i\varepsilon S} x_i &= \sum_{i\varepsilon S} (u_i(\bar{q}) - \bar{p}_i q) \\
&\geqq \sum_{i\varepsilon S} u_i(Q(t_s^\circ)) - \sum_{i\varepsilon S} p_i Q(t_s^\circ) \\
&= \sum_{i\varepsilon S} u_i(Q(t_s^\circ)) - \sum_{i\varepsilon S} \bar{p}_i C(Q(t_s^\circ)) / \sum_{i\varepsilon N} \bar{p}_i \\
&\geqq \sum_{i\varepsilon S} u_i(Q(t_s^\circ)) - t_s^\circ \\
&= v(S).
\end{aligned}
$$

The equality $\sum_{i\varepsilon N} \bar{x}_i = v(N)$ is clear. *Q. E. D.*

Thus if the Lindahl imputation is unblocked by any one-person coalition, it is contained in the core. Under the assumption (d), $v(\{i\})$ describes the benefit of agent i gained by consuming the public good produced out of cooperation of all agents except i. In other words, $v(\{i\})$ is a gain to i obtainable by acting unilaterally as a free rider. Then the condition for the Lindahl equilibrium to be contained in the core of the game $(N, v)$ amounts to saying that no agent be incited to act as a free rider under the Lindahl equilibrium.

Nonemptiness of the core of this game depends on the shape of each utility function. The following example indicates that it is rather an exce-ption under the assumption (d). This in turn implies that the Lindahl epuilibrium is not likely to be achieved, as the free rider problem in a public goods economy suggests.

Consider the example:

$$N=\{1,2,3\}$$

$$u_i(\mathrm{q})=a_i\sqrt{q} \quad (a_i>0)$$

$$C(q)=cq \quad (c>0)$$

Then the followings are easy to obtain:

$$v(N)=(a_1+a_2+a_3)^2/4c$$

$$v(\{i\})=a_i(a_j+a_k)/2c$$

$$a_i<a_j+a_k$$

The core is empty, since

$$\begin{aligned} v(N)-\sum_{i\varepsilon N} v(\{i\}) &= [(a_1+a_2+a_3)^2-4(a_1a_2+a_1a_3+a_2a_3)]/4c \\ &= \{[a_1-(a_2+a_3)]a_1+[a_2-(a_3+a_1)]a_2+[a_3-(a_1+a_2)]a_3\}/4c \\ &<0 \end{aligned}$$

## REFERENCES

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