Linear Problems and Green Function of the Derivative Nonlinear Schrödinger Equation

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Abstract

Linear problems associated with the derivative nonlinear Schrödinger (DNLS) equation are studied from the point of view of inverse scattering techniques. By means of the generalized inverse method we find the solution of a linear homogeneous equation corresponding to the first variational system of the DNLS equation. This solution is represented by the squared eigenfunctions of the Kaup-Newell eigenvalue problem. A Green function is defined for that linear equation and we obtain the solution of a nonhomogeneous linear equation which naturally arises in the perturbation calculations of the DNLS equation. Giving explicit formulae of Jost functions and potentials, we analyse a perturbation with such a background as pure one-soliton state. This perturbation has "stational" and "transitional" parts excited by the forced term and initial value, respectively. These both parts are classified into two components, "continuous" and "discrete" components. At the limit of large time, generally speaking, the continuous component results in a decaying oscillation, while the discrete one yields secular terms. The sufficient condition which suppressess the secular term is given by a simple formula.

§ 1. Introduction

By means of inverse scattering method several workers had studied the perturbation method for a class of nonlinear evolution equations,[1-4] Korteweg de-Vries, nonlinear Schrödinger and sine-Gordon equations, which are belonging to the AKNS class.[5] There is a discrete type of perturbations[6] relating to the Toda Lattice. The above treatments are roughly classified into two types, (1) a type by Kaup-Newll[1,2] and Karpman-Maslov[3] and (2) another type by Keener-McLaughlin.[4,7] For the first type (1) the effect of perturbations is obtained from a small variation of spectrums, while the second (2) directly estimates the lowest correction of solution by constructing a Green function.[7] The method of Keener-MacLaughlin has a remarkable similarity to the classical method for solving a linear partial differential equation. It is interesting to develop the same technique for other types of equations. Recently Kaup and Newell proposed a new type of eigenvalue problem and solved a new class of nonlinear equations; a derivative nonlinear Schrödinger (DNLS) equation[8] and a two-dimensional massive Thirring model.[9] As shown in ref. 7 the squared eigenfunction[5] plays an important role in the construction of a solution of the first variational system. The squared eigenfunction of the AKNS equation was worked out by Kaup[10] and he gave the completeness relation by which any function could be expanded to a series of squared eigenfunctions. Using the completeness by Kaup, Keener and McLaughlin constructed the Green function which gives the solution of a nonhomogeneous linear equation.

In our previous paper,[11] the Kaup-Newell equation was extensively analysed for the generalized inverse scattering theory and for the functional relations between variationals of potentials and scattering data. We also obtained the completeness of squared eigenfunctions. In this paper we study the linear problems associated with the DNLS equation. The solution of a linear homogeneous partial-differential equation regarded as the first variational system of the DNLS equation is given by using the squared eigenfunction. For that linear equation we define a Green function by which the solution of a nonhomogeneous equation is constructed. For actual applications of above discussion, it is necessary to give the squared eigenfunction with scattering data. In Appendix-A, we solve the pure one-soliton state according to Zakharov and Shabat[12] and give the list of Jost functions and potentials. As a simple but important case, we analyse the nonhomogeneous solution corresponding to the lowest correction of the pure one-soliton state. Generally speaking, the solution has two kinds of components, one is the continuous component decaying into linear oscillations while another is the discrete one regarded as a resonant effect from the one-soliton's background. We remark that the discrete component may yield secular terms growing as large as possible under the limit of sufficiently large time. The sufficient condition to suppress the secular term is given by a simple formula.

§ 2. Integrable Conditions and Variational System

The DNLS equation,[8)]

$$iq_t + q_{xx} - mi(|q|^2 q)_x = 0 \qquad (m = \pm 1), \quad \cdots\cdots(2.\ 1)$$

is solved exactly with rapidly vanishing conditions ($q \to 0$ as $x \to \pm\infty$) and the initial condition $q(x, 0) = q_0(x)$. The decoupling equations which give eq. (2. 1) are

$$\boldsymbol{u}_x = \lambda D\boldsymbol{u}, \quad D = -i\lambda\sigma_3 + Q, \quad Q = \begin{pmatrix} 0, & q \\ r, & 0 \end{pmatrix}, \quad r = mq^*, \quad \cdots\cdots(2.\ 2a)$$

$$\boldsymbol{u}_t = F\boldsymbol{u}, \quad F = A\sigma_3 + B, \quad B = \begin{pmatrix} 0, & b \\ c, & 0 \end{pmatrix}, \quad \cdots\cdots(2.\ 2b)$$

where λ is an eigenvalue, σ_3 is one of Pauli spin matrices and

$$A = a\lambda^4 - \frac{a}{4} < \boldsymbol{w}|\sigma_3|\boldsymbol{w} > \lambda^2, \quad a = -2i, \quad \boldsymbol{w} = \begin{pmatrix} r \\ q \end{pmatrix},$$

$$\boldsymbol{h} = \begin{pmatrix} c \\ b \end{pmatrix} = ia\boldsymbol{w}\lambda^3 + a\left(\frac{1}{2}\sigma_3 \boldsymbol{w}_x - \frac{i}{4} < \boldsymbol{w}|\sigma_3|\boldsymbol{w} > \boldsymbol{w}\right)\lambda. \quad \cdots\cdots(2.\ 3)$$

The notation $<\boldsymbol{u}|\boldsymbol{v}>$ means an inner product between a column vector $\boldsymbol{v}=|\boldsymbol{v}>$ and an adjoint raw vector $\boldsymbol{u}^A=<\boldsymbol{u}|$, and we also define an exterior product $|\boldsymbol{u}><\boldsymbol{v}|$,

$$<\boldsymbol{u}|\boldsymbol{v}> = (-u_2,\ u_1)\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -u_2 v_1 + u_1 v_2,$$

$$|\boldsymbol{u}><\boldsymbol{v}| = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\times(-v_2,\ v_1) = \begin{pmatrix} -u_1 v_2, & u_1 v_1 \\ -u_2 v_2, & u_2 v_1 \end{pmatrix}.$$

For an arbitrary matrix X, we note $X|\boldsymbol{u}> = |X\boldsymbol{u}>$ and $<\boldsymbol{u}|X = <X^A\boldsymbol{u}|$, where X^A is an adjoint matrix of X defined by

$$\begin{pmatrix} x_{11}, & x_{12} \\ x_{21}, & x_{22} \end{pmatrix}^A = \begin{pmatrix} x_{22}, & -x_{12} \\ -x_{21}, & x_{11} \end{pmatrix}.$$

The integrable condition of eqs. (2. 2) is

$$\lambda D_t - F_x + \lambda(DF - FD) = 0. \quad\cdots\cdots(2.\ 4)$$

Substituting eq. (2. 3) into eq. (2. 4), we obtain a general expression of eq. (2. 1),

$$\boldsymbol{w}_t = ia\frac{\partial}{\partial x}\left\{\left[\frac{1}{2i}\sigma_3\frac{\partial}{\partial x} - \frac{1}{2}|\,\boldsymbol{w}(x,\ t)>\int_{-\infty}^{x} dy<\boldsymbol{w}(y,\ t)|\,\sigma_3\frac{\partial}{\partial y}\right]\boldsymbol{w}\right\}. \quad\cdots\cdots(2.\ 5)$$

once we get a solution of eq. (2. 1), it becomes important to examine the stability of that solution. For this purpose we consider a function $\triangle q\,(=\tilde{q} - q)$, where $\tilde{q}$ is of course a solution of the DNLS equation but its initial value $\tilde{q}\,(x,\ 0)$ is different from $q\,(x,\ 0)$. For $\tilde{q}$ we provide another decoupling equations,

$$\tilde{\boldsymbol{u}}_x = \lambda\tilde{D}\tilde{\boldsymbol{u}}, \qquad \tilde{\boldsymbol{u}}_t = \tilde{F}\tilde{\boldsymbol{u}}, \quad\cdots\cdots(2.\ 6)$$

where the superscript "~" is used for denoting all quantities relating to $\tilde{q}$. The generalized inverse scattering theory is powerful to calculate the solution $\triangle q$ with the knowledge of q. If the norm $\|\triangle q\,(x,\ t)\|$ is sufficiently small, the meaning of the generalized theory becomes more clear. The mapping from $\delta q\,(x,\ 0)\ (\equiv\triangle q\,(x,\ 0),\ \|\triangle q\|\ll 1)$ to $\delta q\,(x,\ t)$ may be linear and the behaviour of $\delta q\,(x,\ t)$ can be described by the following inverse decoupling manner,

$$\delta\boldsymbol{u}_x = \lambda D\cdot\delta\boldsymbol{u} + \lambda\delta D\cdot\boldsymbol{u}, \qquad \delta\boldsymbol{u}_t = F\cdot\delta\boldsymbol{u} + \delta F\cdot\boldsymbol{u}. \quad\cdots\cdots(2.\ 7)$$

From eq. (2. 4) and the cross-differentiation of eq. (2. 7) we get

$$(\lambda\delta D_t - \delta F_x) + \lambda\,(\delta D\cdot F - \delta F\cdot D) + \lambda\,(D\cdot\delta F - F\cdot\delta D) = 0. \quad\cdots\cdots(2.\ 8)$$

From eqs. (2. 2) and (2. 3) we can reduce eq. (2. 8) to

$$\boldsymbol{L}(x,\ t)\,\delta\boldsymbol{w}(x,\ t) = 0. \quad\cdots\cdots(2.\ 9)$$

where $\boldsymbol{L}(x,\ t)$ is a linear partial differential operator,

$$\boldsymbol{L}(x,\ t) = \partial_t - \frac{a}{2}\sigma_3\partial_x^2 + iaW(x,\ t)\,\partial_x + iaW_x(x,\ t), \quad\cdots\cdots(2.\ 10)$$

$$W(x,\ t) = \frac{1}{2} | \boldsymbol{w} >< \boldsymbol{w} | \sigma_3 + \frac{1}{4} < \boldsymbol{w} | \sigma_3 | \boldsymbol{w} > .$$

We remark that eq. (2. 9) is equivalent to the first variational system of the DNLS equation (2. 1),

$$i\delta q_t + \delta q_{xx} - mi(2|q|^2\delta q + q^2\delta q^*)_x = 0. \quad \cdots\cdots (2.\ 11)$$

§ 3. Jost Functions, S-Matrix and Squared Eigenfunctions

For the Kaup-Newell equation (2. 2a) we define Jost (matrix) functions $\Phi^{\pm}$ and a scattering matrix S by

$$\Phi_x^{\pm}(\lambda,\ x) = \lambda D(\lambda,\ x)\ \Phi^{\pm}(\lambda,\ x), \quad \cdots\cdots (3.\ 1a)$$

$$\Phi^{\pm}(\lambda,\ x) \to J(\lambda^2 x) \equiv \exp(-i\lambda^2 x\sigma_3) \text{ as } x \to \pm\infty \quad \cdots\cdots (3.\ 1b)$$

$$\Phi^{-}(\lambda,\ x) = \Phi^{+}(\lambda,\ x)\ S(\lambda), \quad \cdots\cdots (3.\ 2)$$

where t is omitted for simplicity and

$$\Phi = (\phi_1,\ \phi_2) = \begin{pmatrix} \phi_{11}, & \phi_{12} \\ \phi_{21}, & \phi_{22} \end{pmatrix}, \quad S = \begin{pmatrix} s_{11}, & s_{12} \\ s_{21}, & s_{22} \end{pmatrix}.$$

Since the potential rapidly vanishies at infinity ($x = \pm\infty$), we get the following analytical property ; "*a set of functions* $\{\phi_1^-(\lambda,\ x),\ \phi_2^+(\lambda,\ x),\ s_{11}(\lambda)\}$ *is analytic as to* λ *on the region* $\mathrm{Im}(\lambda^2) \geqq 0$, *while* $\{\phi_2^-(\lambda, x),\ \phi_1^+(\lambda,\ x),\ s_{22}(\lambda)\}$ *is analytic on* $\mathrm{Im}(\lambda^2) \leqq 0$." Since $\det\Phi^{\pm} = 1$ and $([\Phi^{\pm}]^{-1})_x = -\lambda[\Phi^{\pm}]^{-1}D$, anjoint Jost functions $\Phi^{A\pm}$ are naturally defined by $\Phi^{A\pm} = [\Phi^{\pm}]^{-1}$. Using the bra-ket notation,

$$\Phi = (|\phi_1>, |\phi_2>), \qquad \Phi^A = \begin{pmatrix} -<\phi_2| \\ <\phi_1| \end{pmatrix},$$

for eq. (2. 2a), we get

$$\frac{\partial}{\partial x}|\phi_j^{\pm}> = \lambda D|\phi_j^{\pm}>, \qquad \frac{\partial}{\partial x}<\phi_j^{\pm}| = -\lambda<\phi_j^{\pm}|\ D \qquad (j = 1,\ 2). \quad \cdots\cdots (3.\ 3)$$

We introduce the squared eigenfunction as the element of an exterior product of Jost vectors, $|\phi_j^{\pm}><\phi_j^{\pm}|$. We specially define a scalar type $\Omega_j^{\pm}$ and a vector type $\Phi_j^{\pm}$,

$$\Omega_j = -\phi_{1j}\phi_{2j}, \qquad \boldsymbol{\Phi}_j = \begin{pmatrix} \phi_{1j}\phi_{1j} \\ \phi_{2j}\phi_{2j} \end{pmatrix}. \quad \cdots\cdots (3.\ 4)$$

The equation (3. 1a) is rewritten by these squared eigenfunctions,

$$\frac{\partial}{\partial x}\Omega_j^{\pm} = i\lambda < \boldsymbol{\Phi}_j^{\pm} | \sigma_2 \boldsymbol{w} >,$$

$$\frac{\partial}{\partial x}\boldsymbol{\Phi}_j^{\pm} + 2i\lambda^2\sigma_3\boldsymbol{\Phi}_j^{\pm} = -2\lambda\Omega_j^{\pm}\sigma_1\boldsymbol{w} \qquad (j = 1,\ 2). \quad \cdots\cdots (3.\ 5)$$

The boundary condition is given by $\Omega_j^{\pm}(\lambda,\ x) \to 0$ as $x \to \pm\infty$ and

$$\Phi_1^{\pm}(\lambda, x) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2i\lambda^2 x}, \quad \Phi_2^{\pm}(\lambda, x) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2i\lambda^2 x} \quad \text{as} \quad x \to \pm\infty. \quad \cdots\cdots(3.\ 6)$$

A set of functions $\{\Omega_1^-(\lambda, x), \Omega_2^+(\lambda, x), \Phi_1^-(\lambda, x), \Phi_2^+(\lambda, x)\}$ is analytic as to λ on the region $\mathrm{Im}(\lambda^2) \geqq 0$, while $\{\Omega_1^+(\lambda, x), \Omega_2^-(\lambda, x), \Phi_1^+(\lambda, x), \Phi_2^-(\lambda, x)\}$ is analytic on $\mathrm{Im}(\lambda^2) \leqq 0$. Considering eqs. (2. 2b) and (3. 1b), we obtain the temporal behaviour of squared eigenfunctions,

$$\frac{\partial}{\partial t}\Omega_j^{\pm} = 2(-1)^j A_\infty \Omega_j^{\pm} + i < \Phi_j^{\pm} \mid \sigma_2 \boldsymbol{w} >,$$

$$\frac{\partial}{\partial t}\Phi_j^{\pm} - 2\{A\sigma_3 + (-1)^j A_\infty\}\Phi_j^{\pm} = -2\Omega_j^{\pm}\sigma_1 \boldsymbol{h}, \quad \cdots\cdots(3.\ 7)$$

where $A_\infty = \lim_{x\to\pm\infty} A = -2i\lambda^4$.

The time evolution of S-matrix is given by

$$S(\lambda, t) = e^{A_\infty t\sigma_3} S(\lambda, 0) e^{-A_\infty t\sigma_3}. \quad \cdots\cdots(3.\ 8)$$

Reflectional coefficients $\rho_{1,2}$ are introduced by $\rho_1 = s_{21}/s_{11}$ and $\rho_2 = s_{12}/s_{22}$, respectively, and $\rho_1(\lambda, t) = \rho_1(\lambda, 0)\exp(2A_\infty t)$, $\rho_2(\lambda, t) = \rho_2(\lambda, 0)\exp(-2A_\infty t)$. For the generalized case the function $\triangle\rho_j (=\tilde{\rho}_j - \rho_j)$ is used for defining the spectral function (see the next section),

$$\triangle\rho_1(\lambda, t) = \triangle\rho_1(\lambda, 0) e^{2A_\infty t}, \quad \triangle\rho_2(\lambda, t) = \triangle\rho_2(\lambda, 0) e^{-2A_\infty t}. \quad \cdots\cdots(3.\ 9)$$

The Jost matrices and S-matrix have symmetries about λ, $\Phi^{\pm}(\lambda, x) = \sigma_3\Phi^{\pm}(-\lambda, x)\sigma_3$ and $S(\lambda) = \sigma_3 S(-\lambda)\sigma_3$. Furthermore, from $r(x) = mq^*(x)$ we get another type of symmetries, $\Phi^{\pm}(\lambda, x) = \sigma_m[\Phi^{\pm}(\lambda^*, x)]^*\sigma_m$ and $S(\lambda) = \sigma_m S^*(\lambda^*)\sigma_m$, where σ_m means σ_1 or σ_2 according to $m = +1$ or -1, respectively. We specially list

$$\phi_2^{\pm}(\lambda, x) = -\sigma_3\phi_2^{\pm}(-\lambda, x) = \begin{pmatrix} 0, & m \\ 1, & 0 \end{pmatrix}[\phi_1^{\pm}(\lambda^*, x)]^*, \quad \cdots\cdots(3.\ 10a)$$

$$\triangle\rho_2(\lambda) = -\triangle\rho_2(-\lambda) = m\triangle\rho_1^*(\lambda^*), \quad \cdots\cdots(3.\ 10b)$$

$$\Phi_j^{\pm}(\lambda, x) = \Phi_j^{\pm}(-\lambda, x), \quad \Phi_1^{\pm}(\lambda, x) = \sigma_1[\Phi_2^{\pm}(\lambda^*, x)]^*. \quad \cdots\cdots(3.\ 10c)$$

§ 4. Generalized Gel'fand-Levitan Integral Equation

The solution of generalized inverse theory is obtained by solving a general type of Gel'fand-Levitan equations,

$$\begin{aligned} &H_1(x, y) - M_d(x, y)\sigma_3 - iM_o(x, y)Q(x) \\ &\qquad + \int_\infty^x \{M_d(x, z)H_1(z, y) + M_o(x, z)H_2(z, y)\}\,dz = 0, \\ &H_0(x, y) - M_0(x, y)\sigma_3 \\ &\qquad + \int_\infty^x \{M_d(x, z)H_0(z, y) + M_o(x, z)H_1(z, y)\}\,dz = 0, \quad (y > x) \quad \cdots\cdots(4.\ 1) \end{aligned}$$

where diagonal matrix M_d and off-diagonal one M_o are unknown kernels and H_n ($n = 0, 1, 2$) is a spectral function defined by $H_n = H_n^+ - H_n^-$ and

$$H_n^+(x, y) = \frac{1}{2\pi}\int_{\Gamma^+} |\phi_2(\lambda, x)> \Delta\rho_1(\lambda) < \phi_2(\lambda, y)| \lambda^n d\lambda,$$

$$H_n^-(x, y) = \frac{1}{2\pi}\int_{\Gamma^-} |\phi_1(\lambda, x)> \Delta\rho_2(\lambda) < \phi_1(\lambda, y)| \lambda^n d\lambda. \quad \cdots\cdots(4.2)$$

The function ϕ_j of eq. (4. 2) is the right-defined Jost vectors ϕ_j^+ and Γ^+ (Γ^-) is a large counterclock circular path defined on the first and third (the second and fourth) quadrants of the λ-plane. The kernel M_o is related to the potential,

$$2M_o(x, x) = M^{-1}(x)\tilde{Q}(x)M(x) - Q(x), \quad \cdots\cdots(4.3)$$

$$M(x) = \exp\left\{\frac{i}{2}\int_{\infty}^{x}[\tilde{q}(y)\tilde{r}(y) - q(y)r(y)]dy \cdot \sigma_3\right\}.$$

The time evolution is obtained from the replacement of $\Delta\rho_j$ with that of eq. (3. 9). The kernels M_d and M_o are originally defined to give a mapping between two kinds of Jost matrices,

$$\tilde{\Phi}(\lambda, x) = M(x)\{\Phi(\lambda, x) + \int_{\infty}^{x}[M_d(x, y) + \lambda M_o(x, y)]\Phi(\lambda, y)dy\}. \quad \cdots\cdots(4.4)$$

Considering the symmetry $r = mq^*$, we can set the kernels as

$$M_d + M_o = \begin{pmatrix} M_1, & M_2 \\ mM_2^*, & M_1^* \end{pmatrix}. \quad \cdots\cdots(4.5)$$

On the other hand, from eqs. (3. 10) we get a symmetry about spectral functions,

$$H_n(x, y) = -\sigma_m H_n^*(x, y)\sigma_m, \qquad H_n^-(x, y) = \sigma_m[H_n^+(x, y)]^*\sigma_m. \quad \cdots\cdots(4.6)$$

Both symmtries satisfy eq. (4. 1) self-consistently. There is another type of symmetry for $H_n^+(x, y)$,

$$H_n^+(x, y) = -(-1)^n\sigma_3 H_n^+(x, y)\sigma_3, \qquad H_n^{(3)} = -(-1)^n\sigma_3 H_n^{(1)}\sigma_3, \quad \cdots\cdots(4.8)$$

where $H_n^+ = H_n^{(1)} + H_n^{(3)}$ and $H_n^{(k)}$ ($k = 1, 3$) is given by integrating along the path Γ_k ($\Gamma^+ = \Gamma_1 + \Gamma_3$) on the k-th quadrant. From eqs. (4. 6) and (4. 8) we remark that H_0, H_2 are off-diagonal and H_1 is diagonal. Then we can reduce eq. (4. 1) to

$$f_1(x, y) - M_1(x, y) - imM_2(x, y)q^*(x)$$
$$+\int_{\infty}^{x}\{M_1(x, z)f_1(z, y) - mM_2(x, z)f_2^*(z, y)\}dz = 0,$$

$$f_0(x, y) + M_2(x, y)$$
$$+\int_{\infty}^{x}\{M_1(x, z)f_0(z, y) - M_2(x, z)f_1^*(z, y)\}dz = 0, \quad \cdots\cdots(4.9)$$

where f_n is a reduced spectral function,

$$f_n(x,\ y) = \frac{1}{\pi}\int_{\Gamma_1} \phi_{12}(\lambda,\ x)\ \phi_{12}(\lambda,\ y)\ \triangle\rho_1(\lambda)\ \lambda^n \mathrm{d}\lambda$$

$$+ \frac{m}{\pi}\{\int_{\Gamma_1} \phi_{22}(\lambda,\ x)\ \phi_{22}(\lambda,\ y)\ \triangle\rho_1(\lambda)\ \lambda^n \mathrm{d}\lambda\}^*, \qquad (n = 0,2)$$

$$f_1(x,\ y) = -\frac{1}{\pi}\int_{\Gamma_1} \phi_{12}(\lambda,\ y)\ \phi_{22}(\lambda,\ y)\ \triangle\rho_1(\lambda)\ \lambda \mathrm{d}\lambda$$

$$-\frac{1}{\pi}\{\int_{\Gamma_1} \phi_{22}(\lambda,\ x)\ \phi_{12}(\lambda,\ y)\ \triangle\rho_1(\lambda)\ \lambda \mathrm{d}\lambda\}^*. \quad \cdots\cdots(4.\ 10)$$

The relation (4. 3) becomes to

$$2M_2(x,\ x) = \tilde{q}(x)\exp\{mi\int_x^{\infty}[\,|\,\tilde{q}(y)^2|] - |\,q(y)\,|^2]\,\mathrm{d}y\} - q(x). \quad \cdots\cdots(4.\ 11)$$

§ 5. Solutions of Associated Linear Problems

If $\|\triangle\rho_j\|$ is sufficiently small, the Gel'fand-Levitan equation (4. 1) can be linearized and we can get the variation $\delta\boldsymbol{w}(x)$,[11)]

$$\delta\boldsymbol{w}(x) = -\frac{1}{2\pi}\sigma_2\int_{\Gamma^+}\frac{\mathrm{d}\lambda}{\lambda^2}\partial_x\boldsymbol{\Phi}_2(\lambda,\ x)\ \delta\rho_1(\lambda)$$

$$+\frac{1}{2\pi}\sigma_2\int_{\Gamma^-}\frac{\mathrm{d}\lambda}{\lambda^2}\partial_x\boldsymbol{\Phi}_1(\lambda,\ x)\ \delta\rho_2(\lambda), \quad \cdots\cdots(5.\ 1)$$

where $\boldsymbol{\Phi}_j$ is a squared (vector) eigenfunction without a superscript (+). From eqs. (2. 7) and (5. 1) we find that $\sigma_2\partial_x\boldsymbol{\Phi}_2\delta\rho_1$ and $\sigma_2\partial_x\boldsymbol{\Phi}_1\delta\rho_2$ satisfy eq. (2. 9). Using eqs. (3. 5), (3. 7) and (3. 9), we can really show

$$\boldsymbol{L}(x,\ t)\,[\sigma_2\partial_x\boldsymbol{\Phi}_2\mathrm{e}^{2A_\infty t}] = \boldsymbol{L}(x,\ t)\,[\sigma_2\partial_x\boldsymbol{\Phi}_1\mathrm{e}^{-2A_\infty t}] = 0. \quad \cdots\cdots(5.\ 2)$$

Now we consider the following integral,

$$\int_{t_1}^{t_2}\mathrm{d}t\int_{-\infty}^{\infty} < \boldsymbol{u}(x,\ t)\,|\,\boldsymbol{L}(x,\ t)\ \boldsymbol{v}(x,\ t) > \mathrm{d}x. \quad \cdots\cdots(5.\ 3)$$

Integrating by part, we get

$$\int_{t_1}^{t_2}\mathrm{d}t\int_{-\infty}^{\infty}\{< \boldsymbol{u}|\,\boldsymbol{L}\boldsymbol{v} > - < \boldsymbol{L}^\dagger\boldsymbol{u}\,|\,\boldsymbol{v} >\}\ \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} < \boldsymbol{u}\,|\,\boldsymbol{v} > \Big|_{t=t_1}^{t=t_2}\mathrm{d}x + \frac{a}{2}\int_{t_1}^{t_2}\{< \boldsymbol{u}_x\,|\,\sigma_3\boldsymbol{v} > - < \boldsymbol{u}\,|\,\sigma_3\boldsymbol{v}_x >\}\Big|_{x=-\infty}^{x=+\infty}\mathrm{d}t, \quad \cdots\cdots(5.\ 4)$$

where $\boldsymbol{L}^\dagger$ is an adjoint operator of $\boldsymbol{L}$,

$$\boldsymbol{L}^\dagger(x,\ t) = -\partial_t + \frac{a}{2}\sigma_3\partial_x^2 - iaW^A(x,\ t)\ \partial_x. \quad \cdots\cdots(5.\ 5)$$

If we substitute $\boldsymbol{u} = U\boldsymbol{c}_1$ and $\boldsymbol{v} = V\boldsymbol{c}_2$ ($\boldsymbol{c}_{1,2}$ are constants), equation (5. 4) is changed to

$$\int_{t_1}^{t_2}\mathrm{d}t\int_{-\infty}^{\infty}\{U^A\cdot\boldsymbol{L}V - (\boldsymbol{L}^\dagger U)^A\cdot V\}\ \mathrm{d}x$$

$$= \int_{-\infty}^{\infty}(U^A V)\Big|_{t=t_1}^{t=t_2}\mathrm{d}x + \frac{a}{2}\int_{t_1}^{t_2}\{\ U_x^A\sigma_3 V - U^A\sigma_3 V_x\}\Big|_{x=-\infty}^{x=+\infty}\mathrm{d}t. \quad \cdots\cdots(5.\ 6)$$

We define a Green function G and its adjoint $G^\dagger$,

$$\boldsymbol{L}(x,\ t)\,G(x,\ t;\xi,\ \eta)=0,\qquad L^{\dagger}(x,\ t)\,G^{\dagger}(x,\ t;\xi,\ \eta)=0, \tag{5. 7a}$$

$$G(x,\ t;\xi,\ \eta),\quad G^{\dagger}(x,\ t;\xi,\ \eta)\to\sigma_1\delta(x-\xi)\quad \text{as}\quad t\to\eta, \tag{5. 7b}$$

$$G(x,\ t;\xi,\ \eta)\to 0\quad \text{as}\quad x\to+\infty,\quad G^{\dagger}(x,\ t;\xi,\ \eta)\to 0\quad \text{as}\ x\to-\infty. \tag{5. 7c}$$

Substituting $U(x,t)=G^{\dagger}(x,\ t;\xi_2,\ t_2)$ and $V(x,\ t)=G(x,\ t;\xi_1,\ t_1)$ into eq. (5. 6), we get a reciprocal relation of Green functions,

$$G(\xi_2,\ t_2\,;\xi_1,\ t_1)=\sigma_1\,[G^{\dagger}(\xi_1,\ t_1\,;\xi_2,\ t_2)]^{A}\,\sigma_1. \tag{5. 8}$$

We consider the following nonhomogeneous problem which naturally arises in perturbation calculations for the DNLS equation,

$$\boldsymbol{L}(x,\ t)\,\boldsymbol{v}(x,\ t)=\boldsymbol{f}(x,\ t), \tag{5. 9a}$$

$$\boldsymbol{v}(x,\ 0)=\boldsymbol{v}_0(x),\quad \boldsymbol{v}_0(x,\ 0)\to 0\quad \text{as}\quad x\to\pm\infty. \tag{5. 9b}$$

From eqs. (5. 4), (5. 7), (5. 8) and (5. 9) we obtain

$$\sigma_1\boldsymbol{v}(x,\ t)=\int_0^t \mathrm{d}\eta\int_{-\infty}^{\infty}\sigma_1 G(x,\ t;\xi,\ \eta)\,\sigma_1\boldsymbol{f}(\xi,\ \eta)\,\mathrm{d}\xi$$
$$+\int_{-\infty}^{\infty}\sigma_1 G(x,\ t;\xi,\ 0)\,\sigma_1\boldsymbol{v}_0(\xi)\,\mathrm{d}\xi. \tag{5. 10}$$

For this derivation we assumed that $\boldsymbol{v}(x,\ t)$ vanishes as $\boldsymbol{x}\to\pm\infty$ for finite t.

Now we assume the Green function as follows,

$$G(x,\ t;\xi,\ \eta)=\frac{1}{2\pi}\sigma_2\int_{\Gamma^+}|\,\partial_x\boldsymbol{\Phi}_2(\lambda\,;x,\ t)>\mathrm{e}^{2A_\infty t}<\boldsymbol{A}_2(\lambda\,;\xi,\ \eta)\,|\,\mathrm{d}\lambda$$
$$+\frac{1}{2\pi}\sigma_2\int_{\Gamma^-}|\,\partial_x\boldsymbol{\Phi}_1(\lambda\,;x,\ t)>\mathrm{e}^{-2A_\infty t}<\boldsymbol{A}_1(\lambda\,;\xi,\ \eta)\,|\,\mathrm{d}\lambda, \tag{5. 11}$$

where $\boldsymbol{A}_{1,2}$ are unknown vectors. To determine $\boldsymbol{A}_{1,2}$, we impose the condition (5. 7b),

$$\sigma_1\delta(x-\xi)=\frac{1}{2\pi}\sigma_2\int_{\Gamma^+}|\,\partial_x\boldsymbol{\Phi}_2(\lambda\,;x,\ t)>\mathrm{e}^{2A_\infty t}<\boldsymbol{A}_2(\lambda\,;\xi,\ t)\,|\,\mathrm{d}\lambda$$
$$+\frac{1}{2\pi}\sigma_2\int_{\Gamma^-}|\,\partial_x\boldsymbol{\Phi}_1(\lambda\,;x,\ t)>\mathrm{e}^{-2A_\infty t}<\boldsymbol{A}_1(\lambda\,;\xi,\ t)\,|\,\mathrm{d}\lambda. \tag{5. 12}$$

On the other hand, the completeness relation of squared eigenfunctions is given by[11)]

$$\sigma_1\delta(x-\xi)=-\frac{1}{2\pi}\sigma_2\int_{\Gamma^+}|\,\partial_x\boldsymbol{\Phi}_2(\lambda\,;x,\ t)>\frac{\mathrm{d}\lambda}{\lambda s_{11}^2(\lambda)}<\boldsymbol{\Phi}_1^-(\lambda\,;\xi,\ t)\,|$$
$$-\frac{1}{2\pi}\sigma_2\int_{\Gamma^-}|\,\partial_x\boldsymbol{\Phi}_1(\lambda,\ ;x,\ t)>\frac{\mathrm{d}\lambda}{\lambda s_{22}^2(\lambda)}<\boldsymbol{\Phi}_2^-(\lambda\,;\xi,\ t)\,|. \tag{5. 13}$$

Comparing eq. (5. 12) with eq. (5. 13), we obtain

$$<\boldsymbol{A}_2(\lambda\,;\xi,\ \eta)\,|=-\frac{\mathrm{e}^{-2A_\infty(\lambda)\eta}}{\lambda s_{11}^2(\lambda)}<\boldsymbol{\Phi}_1^-(\lambda\,;\xi,\ \eta)\,|,$$
$$<\boldsymbol{A}_1(\lambda\,;\xi,\ \eta)\,|=-\frac{\mathrm{e}^{+2A_\infty(\lambda)\eta}}{\lambda s_{22}^2(\lambda)}<\boldsymbol{\Phi}_2^-(\lambda;\xi,\ \eta)\,|. \tag{5. 14}$$

From eqs. (5. 11) and (5. 14) the Green function is given by

$$G(x, t; \xi, \eta) = -\frac{1}{2\pi}\sigma_2 \int_{\Gamma^+} |\partial_x \Phi_2(\lambda; x, t) > \frac{e^{2A_\infty \cdot (t-\eta)}}{\lambda s_{11}^2(\lambda)} < \Phi_1^-(\lambda; \xi, \eta)| d\lambda$$
$$-\frac{1}{2\pi}\sigma_2 \int_{\Gamma^-} |\partial_x \Phi_1(\lambda; x, t) > \frac{e^{2A_\infty \cdot (t-\eta)}}{\lambda s_{22}^2(\lambda)} < \Phi_2^-(\lambda; \xi, \eta)| d\lambda. \quad \cdots\cdots (5.\ 15)$$

This formula surely satisfies eqs. (5. 7a) and (5. 7b), while the last condition (5. 7c) is also satisfied because of eq. (3. 6).

We consider a perturbed DNLS equation,

$$i\tilde{q}_t + \tilde{q}_{xx} - mi(|\tilde{q}|^2 \tilde{q})_x = if(x, t), \quad \cdots\cdots (5.\ 16)$$

with an initial condition $\tilde{q}(x, 0) = \tilde{q}_0(x)$. We want to get the variation $\delta q(x, t) = \tilde{q}(x, t) - q(x, t)$, where $q(x, t)$ is a solution of eq. (2. 1) with $q(x, 0) = q_0(x)$. If $\| f(x, t) \|$ and $\| \delta q(x, t) \|$ are sufficiently small, its solution is obtained by solving the nonhomogeneous problem,

$$\boldsymbol{L}(x, t) \cdot \delta \boldsymbol{w}(x, t) = \boldsymbol{f}(x, t), \qquad \boldsymbol{f}(x, t) = \begin{pmatrix} f(x, t) \\ mf^*(x, t) \end{pmatrix},$$

$$\delta \boldsymbol{w}(x, 0) = \tilde{\boldsymbol{w}}_0(x) - \boldsymbol{w}_0(x). \quad \cdots\cdots (5.\ 17)$$

There are some symmetries, $\boldsymbol{L}(x, t) = \sigma_1 \boldsymbol{L}^*(x, t)\sigma_1$, $\boldsymbol{f}(x, t) = m\sigma_1 \boldsymbol{f}^*(x, t)$, $\delta \boldsymbol{w}(x, t) = m\sigma_1 \delta \boldsymbol{w}^*(x, t)$ and $G(x, t; \xi, \eta) = \sigma_1 G^*(x, t; \xi, \eta)\sigma_1$. The Green function is reduced to

$$G(x, t; \xi, \eta) = G_1(x, t; \xi, \eta) + \sigma_1 G_1^*(x, t; \xi, \eta)\sigma_1,$$

$$G_1(x, t; \xi, \eta) = -\frac{1}{\pi}\sigma_2 \int_{\Gamma_1} |\partial_x \Phi_2(\lambda; x, t) > \frac{e^{2A_\infty \cdot (t-\eta)}}{\lambda s_{11}^2(\lambda)} < \Phi_1^-(\lambda; \xi, \eta)| d\lambda. \quad \cdots\cdots (5.\ 18)$$

From eqs. (5. 9), (5. 10), (5. 17) and (5. 18), we can obtain

$$\delta \boldsymbol{w}(x, t) = \delta \boldsymbol{w}_1(x, t) + m\sigma_1 \delta \boldsymbol{w}_1^*(x, t), \quad \cdots\cdots (5.\ 19a)$$

$$\delta \boldsymbol{w}_1(x, t) = \int_0^t d\eta \int_{-\infty}^{\infty} G_1(x, t; \xi, \eta)\sigma_1 \boldsymbol{f}(\xi, \eta) d\xi$$
$$+ \int_{-\infty}^{\infty} G_1(x, t; \xi, 0)\sigma_1 \delta \boldsymbol{w}_0(\xi) d\xi. \quad \cdots\cdots (5.\ 19b)$$

§ 6. Considerations for Soliton Perturbation

In this section we analyse the solution (5. 19) regarded as the first correction from a perturbation effect. Our case is limited to the case of one-soliton state, that is, $q(x, t)$ in the linear operator $\boldsymbol{L}(x, t)$ is constructed from only a discrete spectral point $\lambda = \lambda_1$. In Appendix-A such a case is analysed by the method of Zakharov and Shabat[12] and Jost functions and the potential are detailed with explicit forms. We prefer to study the existence of secular terms (proportional to the time) rather than to estimate the lowest term. If we consider a stationary system, any secular term must not arise in that system. From this nonsecularity condition we may expect that the soliton parameter would be specified.

In the same way as eq. (3. 4) we introduce modified squared vectors $\Psi_{1,2}$ from which we get

$$\Psi_1(\lambda, x) = \Phi_1^-(\lambda, x)\, e^{2i\lambda^2 x}, \qquad \Psi_2(\lambda, x) = \Phi_2(\lambda, x)\, e^{-2i\lambda^2 x}. \qquad (6.1)$$

From eqs. (3. 5) and (5. 18) we obtain

$$G_1 = \frac{2}{\pi}\int_{\Gamma_1} \{-\sigma_1 | \Psi_2 > + i\sigma_3 \left(\frac{\Psi_{12}}{\lambda}\Psi_{22}\right) | \boldsymbol{w}(x) > \}\ \frac{e^{2i\lambda^2(\triangle x - 2\lambda^2 \triangle t)}}{s_{11}^2} < \Psi_1 | \lambda d\lambda, \qquad (6.2)$$

where $\triangle x = x - \xi$ and $\triangle t = t - \eta$. We may change the variable λ (and λ_1) to ζ (and ζ_1) by $\zeta = \lambda^2$ (and $\zeta_1 = \lambda_1^2$), where ζ (and ζ_1) is defined on the upper ζ-plane. In the following all functions of λ are regarded as the functions of ζ and instead of Γ_1 we use an integral path Γ_ζ^+ ($C_\zeta^+ = L_\zeta^+ + \Gamma_\zeta^+$) on the upper ζ-plane. Since $s_{11}(\zeta) = \zeta_1^*(\zeta - \zeta_1)/\zeta_1(\zeta - \zeta_1^*)$ obtained from eq. (A. 6) and $e^{i\beta_0} = \lim_{x\to+\infty} \exp[i\beta^-(x)] = \zeta_1^*/\zeta_1$, equation (6. 2) is arranged to

$$G_1 = \frac{1}{\pi}\cdot\frac{(\zeta_1 - \zeta_1^*)^4}{(\zeta_1^*/\zeta_1)^2}\int_{\Gamma_\zeta^+}\frac{e^{2i\zeta(\triangle x - 2\zeta\triangle t)}}{(\zeta - \zeta_1)^2(\zeta - \zeta_1^*)^2}\,\tilde{G}_1(\zeta)\, d\zeta, \qquad (6.3)$$

where $\tilde{G}_1(\zeta) = \tilde{G}_1(\zeta; x, t; \xi, \eta)$ is an entire function of ζ,

$$\tilde{G}_1(\zeta) = | -\sigma_1 \tilde{\Psi}_2(\zeta) + i\tilde{\Omega}(\zeta)\,\sigma_3\,\boldsymbol{w}(x, t) > < \tilde{\Psi}_1(\zeta) |. \qquad (6.4a)$$

The functions $\tilde{\Psi}_2$, $\tilde{\Omega}$ and $\tilde{\Psi}_1$ must be taken as

$$\tilde{\Psi}_2(\zeta; x, t) = \begin{pmatrix} \tilde{\psi}_{12}^2(\zeta; x, t) \\ \tilde{\psi}_{22}^2(\zeta; x, t) \end{pmatrix}, \quad \tilde{\Omega}(\zeta; x, t) = \frac{1}{\lambda}\,\tilde{\psi}_{12}(\zeta; x, t)\,\tilde{\psi}_{22}(\zeta; x, t),$$

$$\tilde{\Psi}_1(\zeta; \xi, \eta) = \begin{pmatrix} \tilde{\psi}_{11}^2(\zeta; \xi, \eta) \\ \tilde{\psi}_{21}^2(\zeta; \xi, \eta) \end{pmatrix}, \qquad (6.4b)$$

where $\tilde{\psi}_{ij}$ is defined in eqs. (A. 15). Substituting eq. (6. 3) into eq. (5. 19b), we get $\delta\boldsymbol{w}_1 = \delta\boldsymbol{w}_f + \delta\boldsymbol{w}_0$ and

$$\delta\boldsymbol{w}_{f,0}(x, t) = \frac{1}{\pi}\cdot\frac{(\zeta_1 - \zeta_1^*)^4}{(\zeta_1^*/\zeta_1)^2}\int_{\Gamma_\zeta^+}\frac{e^{2i\zeta(x - 2\zeta t)}}{(\zeta - \zeta_1)^2(\zeta - \zeta_1^*)^2}$$

$$\cdot\{-\sigma_1 \tilde{\Psi}_2(\zeta; x, t) + i\tilde{\Omega}(\zeta; x, t)\,\sigma_3\,\boldsymbol{w}(x, t)\}\, C_{f,0}(\zeta)\, d\zeta, \qquad (6.5)$$

where the order of integration was exchanged and expansion coefficients $C_{f,0}$ are given by

$$C_f(\zeta; t) = \int_0^t d\eta \int_{-\infty}^{\infty} < \tilde{\Psi}_1(\zeta; \xi, \eta) | \sigma_1 \boldsymbol{f}(\xi, \eta) > e^{-2i\zeta(\xi - 2\zeta\eta)}\, d\xi, \qquad (6.6a)$$

$$C_0(\zeta) = \int_{-\infty}^{\infty} < \tilde{\Psi}_1(\zeta; \xi, 0) | \sigma_1 \delta\boldsymbol{w}_0(\xi) > e^{-2i\zeta\xi} d\xi. \qquad (6.6b)$$

We remark that C_f is dependent on the time. The integral (6. 6) is divided into a continuous part and a discrete part which are contributed from the paths $-L_\zeta^+$, and C_ζ^+, respectively. The lowest correction $\delta\boldsymbol{w}_1$ is obtained as a sum of the transitional term $\delta\boldsymbol{w}_0$ and the stationary term $\delta\boldsymbol{w}_f$ which are excited by the initial variation $\delta\boldsymbol{w}_0(x)$ and the forced term $\boldsymbol{f}(x, t)$, respectively.

To analyse the integral (6. 5), we introduce convenient variables (y, s) and (η, ν),

$$y=\alpha(x-vt),\qquad s=\beta(x-ut)\,, \qquad\cdots\cdots(6.\ 7a)$$

$$\mu=\alpha(\xi-v\eta),\qquad \nu=\beta(\xi-u\eta)\,, \qquad\cdots\cdots(6.\ 7b)$$

where $\zeta_1=\gamma e^{i\theta}(0<\gamma,\ 0<\theta<\pi)$ and

$$\alpha=2\gamma\sin\theta,\quad \beta=2\gamma\cos\theta,\quad v=4\gamma\cos\theta,\quad u=2\gamma\cos2\theta/\cos\theta. \qquad\cdots\cdots(6.\ 8)$$

The inverse relation of eq. (6. 7b) is

$$\mu-\nu\tan\theta=-4\gamma^2\tan\theta\cdot\eta,\quad \eta-\nu\tan2\theta=-\gamma(\tan2\theta/\cos\theta)\,\xi. \qquad\cdots\cdots(6.\ 9)$$

From eqs. (6. 7) we see

$$2i\zeta_1(x-2\zeta_1 t)=-y+is,\qquad 2i\zeta_1(\xi-2\zeta_1\eta)=-\mu+i\nu, \qquad\cdots\cdots(6.\ 10)$$

and

$$2i\zeta(x-2\zeta t)=i\varkappa(\zeta)\,y+i\omega(\zeta)\,s, \qquad\cdots\cdots(6.\ 11a)$$

where

$$\varkappa(\zeta)=(\zeta/\gamma)^2\cot\theta-(\zeta/\gamma)\cos2\theta/\sin\theta,\qquad i\varkappa\ (\zeta_1)=-1,$$

$$\omega(\zeta)=2(\zeta/\gamma)\cos\theta-(\zeta/\gamma)^2,\qquad \omega(\zeta_1)=1. \qquad\cdots\cdots(6.\ 11b)$$

For above exchange of variables, we still use the same notations as $\tilde{\varPsi}_{ij}(\zeta;x,\ t)=\tilde{\varPsi}_{ij}(\zeta;y,\ s)$ and $q(x,\ t)=q(y,\ s)$ etc.. Considering eq. (A. 16), we obtain

$$\tilde{\varPsi}_{12}(\zeta;y,\ s)=\frac{2m\lambda}{\zeta_1-\zeta_1^*}\cdot\frac{b_+^*(\lambda_1)\,e^{-y}}{\triangle_+^*(y)}e^{-is}, \qquad\cdots\cdots(6.\ 12a)$$

$$\tilde{\varPsi}_{22}(\zeta;y)\ =\frac{1}{\triangle_+(y)}+\frac{\zeta-\zeta_1}{\zeta_1-\zeta_1^*}\,e^{-i\beta^+(y)}, \qquad\cdots\cdots(6.\ 12b)$$

$$r(y,\ s)=-4ie^{-i\beta^+(y)}\frac{b_+(\lambda_1)\,e^{-y}}{\triangle_+(y)}e^{is}, \qquad\cdots\cdots(6.\ 12c)$$

$$\tilde{\varPsi}_{21}(\zeta;\mu,\ \nu)=-\frac{2m\lambda}{\zeta_1-\zeta_1^*}\cdot\frac{b_-(\lambda_1)\,e^{-\mu}}{\triangle_-(\eta)}e^{-i\nu}, \qquad\cdots\cdots(6.\ 13a)$$

$$\tilde{\varPsi}_{11}(\zeta;\mu)\ =\frac{1}{\triangle_-(\mu)}+\frac{\zeta-\zeta_1}{\zeta_1-\zeta_1^*}\,e^{i\beta^-(\mu)}, \qquad\cdots\cdots(6.\ 13b)$$

$$q(\mu,\ \nu)=-4ie^{+i\beta^-(\mu)}\frac{b_-(\lambda_1)\,e^{\mu}}{\triangle_-(\mu)}e^{-i\nu}, \qquad\cdots\cdots(6.\ 13c)$$

where

$$\triangle_\pm(y)=1+\frac{4m\zeta_1}{(\zeta_1-\zeta_1^*)^2}|\,b_\pm(\lambda_1)|^2/e^{\pm2y},\qquad e^{\pm i\beta^\pm(y)}=\triangle_\pm(y)/\triangle_\pm^*(y).$$

From eqs. (6. 7-9) the integral (6. 6a) is rewritten by

$$C_f(\zeta,\ t)=\frac{1}{8\gamma^3\sin\theta}\int_{-\infty}^{\infty}e^{-i\varkappa(\zeta)\mu}d\mu\int_{\nu_1}^{\nu_2}<\tilde{\varPsi}_1(\zeta;\mu,\ \nu)|\,\sigma_1 \boldsymbol{f}(\mu,\ \nu)>e^{-i\omega(\zeta)\nu}d\nu, \qquad\cdots\cdots(6.\ 14a)$$

where

$$\nu_2=\nu_1+4\gamma^2t,\qquad \nu_1=\mu\cot\theta. \qquad\cdots\cdots(6.\ 14b)$$

We consider the discrete component $\delta \boldsymbol{w}_{fd}(x, t)$ of stationary term. The contribution from a double pole $\zeta = \zeta_1$ is evaluated as

$$\delta \boldsymbol{w}_{fd}(x, t) \propto \{ -2 \frac{i(\zeta_1 - \zeta_1^*)(x - 4\zeta_1 t) - 1}{\zeta_1 - \zeta_1^*} \tilde{\Lambda}(\zeta_1 ; x, t)\, C_f(\zeta_1, t) + \frac{\mathrm{d}}{\mathrm{d}\zeta} \tilde{\Lambda}(\zeta ; x, t)\, C_f(\zeta, t)\Big|_{\zeta=\zeta_1} \} \, \mathrm{e}^{2i\zeta_1(x-2\zeta_1 t)}, \quad \text{(6. 15)}$$

where $\tilde{\Lambda} = -\sigma_1 \tilde{\Psi}_2 + i\tilde{\Omega}\sigma_3 \boldsymbol{w}$. By this relation we can examine the presence of secular terms. For the briefness of discussions only consider a simple case that the forced term is equal to the potential. We find that from eps. (6. 13) the integrand $< \tilde{\Psi}_1 | \sigma_1 \boldsymbol{f} > \mathrm{e}^{-i\nu}$ in eq. (6. 14a) is a simple periodic function of ν. If the average value of that integrand dose not vanish, the function $C_f(\zeta_1, t)$ results in a linear term of t. For eq. (6. 15) it is better to use a new frame (y, t) where y is the same one defined in eq. (6. 7a). Under this frame we see $2i\zeta_1 x - 4i\zeta_1^2 t = -y + i(y\cot\theta + 4\gamma^2 t)$ etc.. Along the direction $y =$ const. there appear two kinds of secular terms in eq. (6. 15) that is, one is linear about t while another is square. For the case of transitional term $\delta \boldsymbol{w}_{od}$ there also appears a secular term which is linear about t and propagates along $y =$ const..

Since it is rather difficult to evaluate the continuous part exactly, we approximately estimate it under the large t limit. We take the stationary term,

$$\delta \boldsymbol{w}_{fc}(y, t) \propto \int_{-\infty}^{\infty} \mathrm{e}^{i\chi(\zeta)t} \frac{\mathrm{e}^{2i\zeta y/\alpha}}{|\zeta - \zeta_1|^4} \tilde{\Lambda}(\zeta ; y, t)\, C_f(\zeta, t)\, \mathrm{d}\zeta, \quad \text{(6. 16)}$$

where $\chi(\zeta) = 2\zeta(v - 2\zeta)$ and C_f is still defined by eq. (6. 14). Since $\omega(\zeta)\,[= \pm 2\cos\theta - 1]$ is real and independent on ζ from eq. (6. 11b), the integrand $< \tilde{\Psi}_1 | \sigma_1 \boldsymbol{f} > \mathrm{e}^{-i\omega\nu}$ of eq. (6. 14a) is always a periodic function. Clearly the function $C_f(\zeta, t)$ is periodic as to t, that is, nonsecular. From above considerations, equation (6. 16) is treated by the method of stationary phase,

$$\delta \boldsymbol{w}_{fc}(y, t) \propto \left(\frac{\pi}{4t}\right)^{1/2} \frac{\mathrm{e}^{2i\zeta_0 y/\alpha}}{|\zeta_0 - \zeta_1|^4} \tilde{\Lambda}(\zeta_0 ; y, t)\, C_f(\zeta_0, t) \exp\left[\frac{i}{4}(t \pm \pi)\right], \quad \text{(6. 17)}$$

where $\zeta_2\,(= v/4)$ is a stationary point. The case of transitional term is treated by the same way and the result is obtained from the replacement of C_f by C_0. Consequently we can say that the continuous part results in a decaying oscillation.

§ 7. Concluding Remarks and Discussions

Linear problems associated with the DNLS equation are studied from the point of view of inverse scattering technique. By the generalized Gel'fand-Levitan equation we found the solution of a linear homogeneous equation corresponding to the first variational system of DNLS equation. A certain integral formula as to the linear partial-differential operator $\boldsymbol{L}(x, t)$ was derived for the treatment of a nonhomogeneous linear problem which naturally arises in perturbations of the DNLS equation. This integral formula corresponds to the classical Green formula. A Green function was defined self-consistently and was constructed explicitly by using the completeness of squared eigenfunctions. The solution of nonhomogeneous problem was obtained by this Green function.

For actual applications of generalized inverse theory and Green function method for perturbations, it is important to give Jost functions and potentials explicitly with specified scattering data. For this purpose we used the method by Zakharov and Shabat and listed these functions for the case of a discrete spectral point (corresponding to the pure one-soliton state). As a simple but important case, we studied the nonhomogeneous solution which should give the lowest correction of a soliton. This correction consists of "stationary" and "transitional" parts excited by the forced term and the initial value of correction, respectively. These both parts are classified into "continuous" and "discrete" components. The transitional term is interesting from another point of view, that is, the linear stability problem of soliton. We note that the nonlinear stability problem should be analysed by the generalized Gel'fand-Levitan equation.

For both cases of stationary and transitional terms, the continuous component results in the decaying oscillation while the discrete one mostly yields secular terms. If the physical system in question is considered in a sufficiently long time scale, we cannot allow the existence of any secular terms. Then we expect the presence of nonsecularity condition from which the soliton parameter will be determined.

If the initial variation is absent, from eqs. (6. 14) the non-secularity condition is given by

$$C_f(\zeta_1,\ t) \propto \int_{-\infty}^{\infty} e^{+\mu} d\mu \int_{\nu_1}^{\nu_2} < \Psi_1(\zeta_1;\mu,\ \nu) | \sigma_1 \boldsymbol{f}(\mu,\ \nu) > e^{-i\nu} d\nu = 0, \qquad \text{(7. 1)}$$

where $\Psi_1(\zeta_1;\mu,\ \nu) = \tilde{\Psi}_1(\zeta_1;\mu,\ \nu)$. We remark that the transitional term also contains a secular term as the partial contribution from the double pole $\zeta = \zeta_1$ in eq. (6. 5). However the non-soliton part of solutions has been usually regarded as the decaying oscillation. In this paper we can not make clear this point, but we comment that the linearized solution (5. 1) of generalized Gel'fand-Levitan equation is equivalent to the transitional term. Especially in eq. (5. 13) of ref. 11 we had given an explicit representation by the scattering data. By this point we may give a conclusion for the above problem.

Acknowledgment

The authors would like to thank Professor Y. H. Ichikawa for his valuable discussions and advices. This work was carried out under the collaborating research program at the Institute of Plasma Physics of Nagoya University.

Added Note

This paper had been published in part.[15]

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Appendix-A. Explicit Representations of Jost Functions by the Scattering Data

For actual applications of eq. (4. 9) or (5. 19), it becomes necessary to give the Jost vectors $\phi_1^{\pm}$ and $\phi_2^{\pm}$ explicitly. In this Appendix we treat this problem according to Zakharov and Shabat,[12] because their method is more profitable for solving the Jost functions than the usual method[8,13,14] using the Gel'fand-Levitan integral equation.

In the same way as used in ref. **13**, we get the following relations between Jost vectors,

$$\psi_1^+(\lambda,\ x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{+i\beta^+(x)} - \frac{1}{2\pi i}\int_{\Gamma^+} \frac{d\lambda'}{\lambda'-\lambda}\rho_+(\lambda')\,\psi_2^+(\lambda',\ x)\, e^{+2i\lambda'^2 x}$$

$$(\mathrm{Im}\,(\lambda^2) \leqq 0), \quad \cdots\cdots\cdots\cdots\cdots\text{(A. 1a)}$$

$$\psi_2^-(\lambda,\ x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\beta^-(x)} + \frac{1}{2\pi i}\int_{\Gamma^+} \frac{d\lambda'}{\lambda'-\lambda}\rho_-(\lambda')\,\psi_1^-(\lambda',\ x)\, e^{-2i\lambda'^2 x}$$

$$(\mathrm{Im}\,(\lambda^2) \leqq 0), \quad \cdots\cdots\cdots\cdots\cdots\text{(A. 1b)}$$

where $\rho_+ = \rho_1 = s_{21}/s_{11}$ and $\rho_- = s_{12}/s_{11}$. Modified Jost vectors $\psi_1^{\pm}$, $\psi_2^{\pm}$ and functions $\beta^{\pm}(x)$ are defined by

$$\psi_1^{\pm}(\lambda,\ x) = \phi_1^{\pm}(\lambda,\ x)\, e^{+i\lambda^2 x}, \qquad \psi_2^{\pm}(\lambda,\ x) = \phi_2^{\pm}(\lambda,\ x)\, e^{-i\lambda^2 x}, \cdots\cdots\cdots\cdots\text{(A. 2a)}$$

$$\beta^{\pm}(x) = \frac{1}{2}\int_{\pm\infty}^{x} q(y)\, r(y)\, dy, \qquad \beta_0 = \beta^-(x) - \beta^+(x). \cdots\cdots\cdots\cdots\text{(A. 2b)}$$

By eqs. (A. 1) Jost vectors are expanded to λ-inverse series, with which we compare another type of λ-inverse expansions shown in eq. (2. 5) of ref. **11**. Then we get

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} r(x)\, e^{+i\beta^+(x)} = -\frac{1}{\pi}\int_{\Gamma^+} d\lambda \rho_+(\lambda)\, \psi_2^+(\lambda, x)\, e^{+2i\lambda^2 x}, \quad \cdots\cdots (A.\ 3a)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} q(x)\, e^{-i\beta^-(x)} = -\frac{1}{\pi}\int_{\Gamma^+} d\lambda \rho_-(\lambda)\, \psi_1^-(\lambda, x)\, e^{-2i\lambda^2 x}. \quad \cdots\cdots (A.\ 3b)$$

We assume that $s_{11}(\lambda)$ has only simple zeros $\lambda_j\,(j=1,\ldots, N)$ on the first quadrant of λ-plane. Considering $s_{11}(\lambda) = s_{11}(-\lambda) = s_{11}^*(\lambda^*)$, we obtain the following expression,[8)]

$$s_{11}(\lambda) = \exp\left\{ i\beta_0 + \frac{1}{2\pi i}\int_{L_1} \frac{\mu d\mu}{\mu^2-\lambda^2} \log|\, s_{11}^2(\mu)|\right\} \cdot \prod_{j=1}^{N} \frac{\lambda^2-\lambda_j^2}{\lambda^2-\lambda_j^{*2}}, \quad \cdots\cdots (A.\ 4)$$

where an integral path $L_1\,(C_1 = \Gamma_1 + \Gamma_1)$ is taken along the real and imaginary axes. Since $\det[S(\lambda)] = 1$, we also get

$$|\, s_{11}(\mu)|^2 \pm m|\, s_{21}(\mu)|^2 = 1, \qquad (\mu\varepsilon L_1) \quad \cdots\cdots (A.\ 5)$$

where the sign "$-$" (or "$+$") is the case that μ lies on the real (nr imaginary) axis. We note that for $m = 1$ (or -1) the zero of s_{11} may be on the real (or imaginary) axis. This results in the algebraic soliton. In the following we only consider the case of discrete spectrums, that is, $\rho_\pm(\mu) = 0$ for $\mu\varepsilon L_1$. From eqs. (A. 4) and (A. 5) we get

$$s_{11}(\lambda) = \exp(i\beta_0) \cdot \prod_{j=1}^{N} \frac{\lambda^2-\lambda_j^2}{\lambda^2-\lambda_j^{*2}}. \quad \cdots\cdots (A.\ 6)$$

Considering similar symmetries as eq. (3. 10a), we reduce eqs. (A. 1) and (A. 3) to

$$\psi_{12}(\lambda) = -\sum_{j=1}^{N} \frac{2m\lambda}{\lambda_j^{*2}-\lambda^2}\, b_+^*(\lambda_j)\, \psi_{22}^*(\lambda_j)\, e^{-2i\lambda_j^{*2}x}, \qquad (\mathrm{Im}(\lambda^2) \geqq 0) \quad \cdots\cdots (A.\ 7a)$$

$$\psi_{22}(\lambda) = e^{-i\beta^+(x)} - \sum_{j=1}^{N} \frac{2\lambda_j^*}{\lambda_j^{*2}-\lambda^2}\, b_+^*(\lambda_j)\, \psi_{12}^*(\lambda_j)\, e^{-2i\lambda_j^{*2}x}, \quad \cdots\cdots (A.\ 7b)$$

$$r(x)\, e^{+i\beta^+(x)} = -4i\sum_{j=1}^{N} b_+(\lambda_j)\, \psi_{22}(\lambda_j)\, e^{+2i\lambda_j^2 x}, \quad \cdots\cdots (A.\ 7c)$$

$$\psi_{21}(\lambda) = +\sum_{j=1}^{N} \frac{2m\lambda}{\lambda_j^{*2}-\lambda^2}\, b_-^*(\lambda_j)\, \psi_{11}^*(\lambda_j)\, e^{+2i\lambda_j^{*2}x}, \qquad (\mathrm{Im}(\lambda^2) \geqq 0) \quad \cdots\cdots (A.\ 8a)$$

$$\psi_{11}(\lambda) = e^{+\beta^-(x)} + \sum_{j=1}^{N} \frac{2\lambda_j^*}{\lambda_j^{*2}-\lambda^2}\, b_-^*(\lambda_j)\, \psi_{21}^*(\lambda_j)\, e^{+2i\lambda_j^{*2}x}, \quad \cdots\cdots (A.\ 8b)$$

$$q(x)\, e^{-i\beta^-(x)} = -4i\sum_{j=1}^{N} b_-(\lambda_j)\, \psi_{11}(\lambda_j)\, e^{-2i\lambda_j^2 x}, \quad \cdots\cdots (A.\ 8c)$$

where we used the following notations,

$$b_+(\lambda) = \frac{s_{21}(\lambda)}{\dot{s}_{11}(\lambda)}, \qquad b_-(\lambda) = \frac{s_{12}(\lambda)}{\dot{s}_{11}(\lambda)}, \qquad (\dot{s}_{11} = ds_{11}/d\lambda)$$

$$\psi_1^-(\lambda, x) = \begin{pmatrix} \psi_{11}(\lambda) \\ \psi_{21}(\lambda) \end{pmatrix}, \qquad \psi_2^+(\lambda, x) = \begin{pmatrix} \psi_{12}(\lambda) \\ \psi_{22}(\lambda) \end{pmatrix}. \quad \cdots\cdots (A.\ 9)$$

If we substitute $\lambda = \lambda_k\,(k = 1.,,.N)$ into eqs. (A. 7) and (A. 8), the resultant equations with eq. (A. 2b) are closed as to Jost functions and potentials. The one-soliton case ($\lambda = \lambda_1$) is important and we list the results as follows,

$$\psi_{22}(\lambda_1) = \frac{1}{\Delta_+(x)}, \qquad \psi_{12}(\lambda_1) = \frac{2m\lambda_1}{\lambda_1^2 - \lambda_1^{*2}} \left[\frac{b_+(\lambda_1)}{\Delta_+(x)} e^{2i\lambda_1^2 x}\right]^*, \qquad \text{(A. 10a)}$$

$$r(x) = -4ie^{-i\beta^+(x)} \frac{b_+(\lambda_1)}{\Delta_+(x)} e^{2i\lambda_1^2 x}. \qquad e^{i\beta^+(x)} = \frac{\Delta_+(x)}{\Delta_+^*(x)}, \qquad \text{(A. 10b)}$$

$$\Delta_+(x) = 1 + \frac{4m\lambda_1^2}{(\lambda_1^2 - \lambda_1^{*2})^2} |b_+(\lambda_1)|^2 e^{2i(\lambda_1^2 - \lambda_1^{*2})x}, \qquad \text{(A. 10c)}$$

$$\psi_{11}(\lambda_1) = \frac{1}{\Delta_-(x)}, \qquad \psi_{21}(\lambda_1) = -\frac{2m\lambda_1}{\lambda_1^2 - \lambda_1^{*2}} \left[\frac{b_-(\lambda_1)}{\Delta_-(x)} e^{-2i\lambda_1^2 x}\right]^*, \qquad \text{(A. 11a)}$$

$$q(x) = -4ie^{+i\beta^-(x)} \frac{b_-(\lambda_1)}{\Delta_-(x)} e^{-2i\lambda_1^2 x}, \qquad e^{+i\beta^-(x)} = \frac{\Delta_-^*(x)}{\Delta_-(x)}, \qquad \text{(A. 11b)}$$

$$\Delta_-(x) = 1 + \frac{4m\lambda_1^2}{(\lambda_1^2 - \lambda_1^{*2})^2} |b_-(\lambda_1)|^2 e^{2i(\lambda_1^{*2} - \lambda_1^2)x}. \qquad \text{(A. 11c)}$$

From eqs. (A. 2a), (A. 10b) and (A. 10c) we find

$$e^{i\beta_0} = (\lambda_1^*/\lambda_1)^2. \qquad \text{(A. 12a)}$$

Since $s_{11}(\lambda) = e^{i\beta_0}(\lambda^2 - \lambda_1^2)/(\lambda^2 - \lambda_1^{*2})$ and $\det S = 1$, we get

$$\dot{s}_{11}(\lambda_1) = \frac{2\lambda_1}{\lambda_1^2 - \lambda_1^{*2}} e^{i\beta_0}, \qquad s_{12}(\lambda_1)\, s_{21}(\lambda_1) = -1. \qquad \text{(A. 12b)}$$

Furthermore from eqs. (3. 2) and (A. 2a) we obtain

$$\psi_{12}(\lambda_1) = -s_{12}(\lambda_1)\,\psi_{11}(\lambda_1), \qquad \psi_{22}(\lambda_1) = -s_{21}(\lambda_1)\,\psi_{21}(\lambda_1). \qquad \text{(A. 12a)}$$

Using eqs. (A. 12), we can show that two sets of relations (A. 10) and (A. 11) are equivalent each other. We note that the Jost vector is completely determined from the substitution of eqs. (A. 10) and (A. 11) into eqs. (A. 7) and (A. 8),

$$\tilde{\psi}_{12}(\lambda) = \frac{\lambda}{\lambda_1}\psi_{12}(\lambda_1), \qquad \tilde{\psi}_{22}(\lambda) = \psi_{22}(\lambda_1) + \frac{\lambda^2 - \lambda_1^2}{\lambda_1^2 - \lambda_1^{*2}} e^{-i\beta^+(x)}, \qquad \text{(A. 13a)}$$

$$\tilde{\psi}_{21}(\lambda) = \frac{\lambda}{\lambda_1}\psi_{21}(\lambda_1), \qquad \tilde{\psi}_{11}(\lambda) = \psi_{11}(\lambda_1) + \frac{\lambda^2 - \lambda_1^2}{\lambda_1^2 - \lambda_1^{*2}} e^{+i\beta^-(x)}, \qquad \text{(A. 13b)}$$

where we denoted $\psi_{ij}(\lambda) = [(\lambda_1^2 - \lambda_1^{*2})/(\lambda^2 - \lambda_1^{*2})]\,\tilde{\psi}_{ij}(\lambda)$.

To obtain the time dependence of Jost functions and the potential, we only replace $b_\pm(\lambda_1)\, e^{\pm 2i\lambda_1^2 x}$ of eqs. (A. 10) and (A. 11) by

$$b_\pm(\lambda_1, t)\, e^{\pm 2i\lambda_1^2 x} = b_\pm(\lambda_1)\, e^{\pm 2i\lambda_1^2 (x - 2\lambda_1^2 t)}. \qquad \text{(A. 14)}$$

For the briefness of notations it is convenient to use such a simplified quantity as $\zeta = \lambda^2$ ($\zeta_1 = \lambda_1^2$), then we get then we get

$$\begin{cases} \tilde{\psi}_{12}(\zeta; x, t) = \dfrac{2m\lambda}{\zeta_1 - \zeta_1^*} \left[\dfrac{b_+(\lambda_1)\, e^{2i\zeta_1(x - 2\zeta_1 t)}}{\Delta_+(x, t)} \right]^*, \\ \\ \tilde{\psi}_{22}(\zeta; x, t) = \dfrac{1}{\Delta_+(x, t)} + \dfrac{\zeta - \zeta_1}{\zeta_1 - \zeta_1^*}\, e^{-i\beta^+(x, t)}, \end{cases} \qquad \cdots\cdots (A.\ 15a)$$

$$\begin{cases} \tilde{\psi}_{21}(\zeta; \xi, \eta) = -\dfrac{\zeta - \zeta_1}{\zeta_1 - \zeta_1^*} \left[\dfrac{b_-(\lambda_1)\, e^{2i\zeta_1(\xi - 2\zeta_1 \eta)}}{\Delta_-(\xi, \eta)} \right]^*, \\ \\ \tilde{\psi}_{11}(\zeta; \xi, \eta) = \dfrac{1}{\Delta_-(\xi, \eta)} + \dfrac{\zeta - \zeta_1}{\zeta_1 - \zeta_1^*}\, e^{+i\beta^-(\xi, \eta)}, \end{cases} \qquad \cdots\cdots (A.\ 15b)$$

where λ is still used for $\lambda = \sqrt{\zeta}$ and

$$\Delta_\pm(x, t) = 1 + \frac{4m\zeta_1}{(\zeta_1 - \zeta_1^*)^2} |b_\pm(\lambda_1)|^2\, e^{\pm 2i(\zeta_1 - \zeta_1^*)[x - 2(\zeta_1 + \zeta_1^*)t]},$$

$$e^{\pm i\beta^\pm(x, t)} = \Delta_\pm(x, t) / \Delta_\pm^*(x, t). \qquad \cdots\cdots (A.\ 16)$$

We note $\tilde{\psi}_{ij}(\zeta_1; x, t) = \psi_{ij}(\zeta_1; x, t)$.

(Received October. 31. 1983)