The Journal of Liberal Arts and Sciences (Toyama Medical and Pharmaceutical University) vol.30, 2003

UNIVERSALITY OF SPLITTABLE GENERALIZED LORENZ-LIKE TEMPLATES

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1. Introduction

A template is a kind of branched surface with boundary equipped with smooth expansive semiflow. Birman and Williams [BW1] [BW2] showed that, if a hyperbolic chain recurrent set of a flow on a three-manifold is given, there exists a template embedded in the manifold such that the link of periodic orbits in the chain reccurent set of the flow is in bijective correspondence with the link of periodic orbits on the template. Furthermore on any finite sublink, this correspondence is via ambient isotopy. This implies that, to investigate knot types of periodic orbits in a hyperbolic chain recurrent set of the flow, it is sufficient to investigate those of periodic orbits on the corresponding template.

After their work, for more than ten years, it is believed that a link type represented as a set of periodic orbits on a template is strongly affected by the structure of the template and there must be some restriction on link types on templates. But in 1997, Ghrist [G] obtained a surprising result. He showed that there exists a template such taht any link type is represented by a set of periodic orbits on it. Such a template is called *universal.* Furthermore he gave several sufficient conditions for a template to be universal, and he proved that infinitely many Lorenz-like templates are universal. But his interest was restricted on Lorenz-like templates whose arms are not knotted. What will happen when arms are knotted?

In this paper we will show that, if a Lorenz-like template has a knotted arm, then it cannot be universal. We have already calculated knot groups of knots on templates in [S], and they were thought to be useful to obtain this result. But as a matter of fact, it is proved without using our previous results. In §2 we prepare some basic notations and definitions about templates. In §3 we state our main theorem and give its proof.

2. Templates and skeletons

In [BW2], Birman and Williams proved the existence of knot holders for hyperbolic chain recurrent sets of flows on three dimensional manifolds. Later knot holders were renamed templates. Thus we will call them templates in this paper.

Definition 2.1. A *template* is a compact branched surface with boundary in a three manifold build locally from two types of charts: *joining* and *splitting*. Each chart, as illustrated in Figure 1, carries a semiflow, endowing the template with an expanding semiflow, and the gluing maps between charts must respect the semiflow and act linearly on the edges.

Date: October 10, 2003.

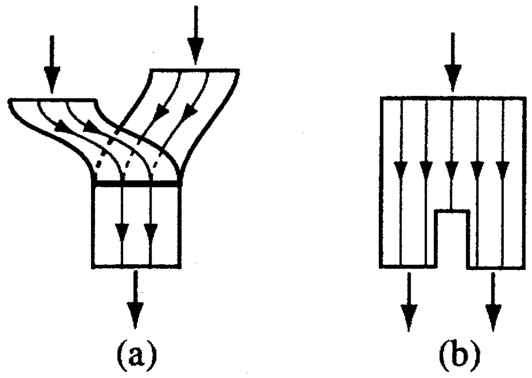

FIGURE 1. (a) a joining chart; (b) a splitting chart.

Theorem 2.2. [BW2] *Given a flow ϕ_t on a three dimensional manifold M having a hyperbolic chain recurrent set, the link L_ϕ of periodic orbits in the chain recurrent set is in bijective correspondence with the link of periodic orbits L_T on a particular embedded template $T \subset M$ (with L_T containing at most two extraneous orbits). On any finite sublink, this correspondence is via ambient isotopy.*

In this paper, we will consider flows and templates in three dimensional sphere S^3. The simplest sort of templates is that build from one joining chart and one splitting chart. For example, the Lorenz template [BW1] is obtained from the Lorenz attractor, and the horseshoe template [BW2] is obtained from the index one chain recurrent set of the suspension flow of Smale's horseshoe map. See Figure 2.

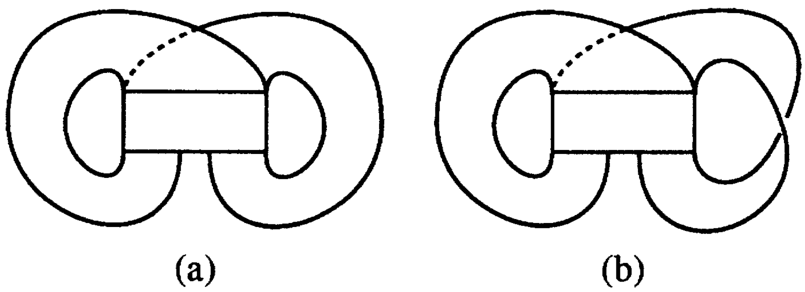

FIGURE 2. (a) the Lorenz template; (b) the horseshoe template.

These templates are thought to be made of three parts: one rectangle, say the *base rectangle*, and two *arms* starting from the bottom edge of the base rectangle and arriving at its top edge. Note that no arm is "knotted" in these templates. The difference is whether the right hand arm is twisted or not. We generalize them by making arms knotted and twisted. To describe our generalization, we need some definitions.

Definition 2.3. A template is called to be *with two arms* if it consists of one joining chart and one splitting chart, like the Lorenz template and the horseshoe template. A template with two arms is called a *generalized Lorenz-like template.*

Definition 2.4. Let T be a generalized Lorenz-like template. It is considered to be made of one base rectangle R and two arms; the *left arm* which start from left half of the bottom edge of R, and the *right arm*. Let A_L be the left arm and A_R be the right arm.

Definition 2.5. We call an arm *untwisted* if the arm together with R is homeomorphic to an annulus. We call an arm *twisted* if the arm with R is homeomorphic to a Möbius band.

For example, both arms of the Lorenz template are untwisted. But, for the horseshoe template, A_L is untwisted and A_R is twisted.

Definition 2.6. If an arm is untwisted, then the orbit of the semiflow through the left edge (resp. right edge) of its base rectangle R and the left (resp. right) arm will be a knot in S^3, say K_L (resp. K_R). If an arm is twisted, then there exists a periodic orbit, say K, of the semiflow which passes through R and the arm only once respectively. We refer to the orbit K as K_L if the arm is A_L, and K_R if the arm is A_R. In this case, we shrink R slightly to the horizontal direction so that the orbit K is the left edge of R when $K = K_L$ or the right edge when $K = K_R$. If A_L is untwisted, let t_L be the linking number of a closed curve on A_L parallel to K_L with K_L. If A_R is untwisted, we can define t_R in the same way. If A_L is twisted, there exits a closed curve K' on $A_L \cup R$ which doubly covers K_L, and let t_L be a half of the linking number of K' with K_L. If the A_R is twisted, we can define t_R in the same way. Note that t_L (resp. t_R) is an integer if and only if A_L (resp. A_R) is untwisted. Let Skel(T) be the figure in S^3 consisting of the base rectangle R and two knots K_L and K_R, accompanied with two numbers t_L and t_R. We call Skel(T) the *skeleton* of T.

Example 2.7. Figure 3 shows an example of a template and its skeleton. In this example, K_L is a trefoil knot and K_R is a trivial knot, and they are linked. Numbers -3 and $\frac{3}{2}$ written near K_L and K_R indicate t_L and t_R respectively.

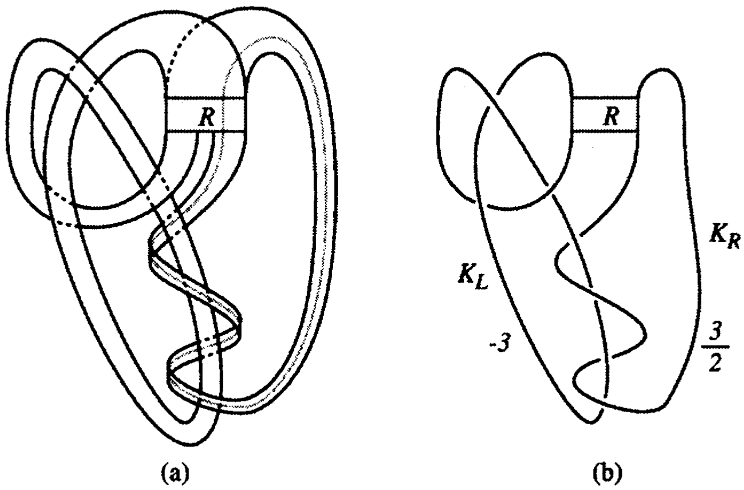

FIGURE 3. (a) a template with two arms, and (b) its skeleton. In (a), the gray curve indicates K_R.

Remark 2.8. We always assume that K_L and K_R are oriented by the direction of the semiflow. We also assume that the total space S^3 is oriented by left-hand system.

Proposition 2.9. *Suppose we are given a figure in S^3, consisting of one rectangle R and a two-component link $K_L \cup K_R$ such that $K_L \cap R =$ "the left edge of R" and $K_R \cap R =$ "the right edge of R", with two numbers t_L and t_R attached to K_L and K_R respectively. Then we can make a template such that its skeleton is the given figure.*

Definition 2.10. The template as in Proposition 2.9 is denoted by Temp$(K_L \cup K_R; t_L, t_R)$.

Proof of Proposition 2.9. Replace a curve $K_L \setminus R$ by a narrow band B_L such that its long edges consist of K_L and a curve parallel to K_L in a Seifert surface of K_L, and one of its short edges are glued to the top edge of R and the other to the bottom edge of R. Then expand linearly the short edge of B_L attached to the top edge of R to the same length as the top edge of R. Finally cut R and B_L along the top edge of R and twist B_L t_L-times and glue again. Apply similar operations to K_R. Then glue B_L to the top edge of R from the front, and glue B_R to the top edge of R from the rear. Then we obtain the required template Temp$(K_L \cup K_R; t_L, t_R)$. Note that, if t_L (resp. t_R) is an integer then the left (resp. right) arm is untwisted. Otherwise the arm is twisted. □

Although templates generally admit linking of K_L and K_R, we need to restrict our attention to a special case for our main result.

Definition 2.11. A generalized Lorenz-like template is called *splittable* if there exists a three ball B^3 in S^3 such that K_L is included in the interior of B^3, and K_R is included in the interior of $S^3 \setminus B^3$, and $\partial B^3 \cap R =$ a "vertial line" of R.

For example, the Lorenz template and the horseshoe template are splittable, but the template shown in Figure 3 is not splittable since K_L and K_R are linked.

Definition 2.12. A template T is called *universal* if every finite link type is represented by a set of periodic orbits of its semiflow.

3. Universality of splittable generalized Lorenz-like template

Lorenz-like template is a template with two arms both of whose arms are unknotted. In [G] (see also [GHS]), Ghrist showed that, among Lorenz-like templates, there exist infinitely many examples of universal templates. He also showed examples of non-universal templates. But he did not consider about the generalized Lorenz-line templates, that is, Lorenz-like templates whose arms are knotted. Our result is on universality of generalized Lorenz-like template as follows:

Theorem 3.1. *Let $T =$ Temp$(K_L \cup K_R; t_L, t_R)$ be an splittable generalized Lorenz-like template. If T is universal, then $K_L \cup K_R$ should be a two-component unlink.*

Proof. We prove this theorem by proving that, if at least one of K_L and K_R is knotted, then T can not hold every knot.

First suppose that K_L is knotted and K_R is an unknot. Since T is splittable, we can shrink the right arm A_R so small that it can be contained in a small tubular neighborhood U of $A_L \cup R$. See Figure 4. Let K be an orbit on T. If K passes through A_L at least once, then K is regarded as a satellite knot of K_L. This implies that, since any satellite of any non-trivial knot should be non-trivial and K_L is knotted, K cannot be a trivial knot. Suppose K does not pass through A_L. Then K should pass through A_R only once, because the semi flow is expanding. Thus K must be an unknot. Summarizing these results, K

can represent only a satellite knot of K_L or a trivial knot. But it is easy to show that there exists a knot type which is not one of them. This implies that T is not universal. If K_R is knotted and K_L is an unknot, non-universality of T can be proved similarly.

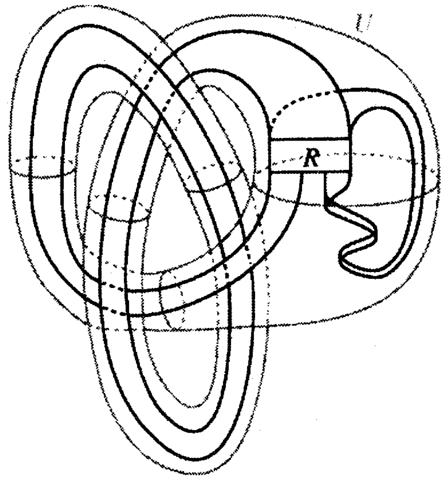

FIGURE 4

Next suppose that both of K_L and K_R are knotted. Then, like the above case, we can shrink A_R so small that it can be contained in a small tubular neighborhood U of $A_L \cup R$. See Figure 5. This figure is thought to be obtained by embedding an standard torus U' containing $T' = R \cup A_R \cup$ (an unknotted arm A'_R) into U. Thus any orbit K on T is a satellite knot of some orbit K' on T' with the companion K_L. Furthermore as in the above case K' is a satellite knot with the companion K_R. Since K_R and K_L are non-trivial knots, K' and K should be non-trivial. This implies that T is not universal.

The proof is completed.

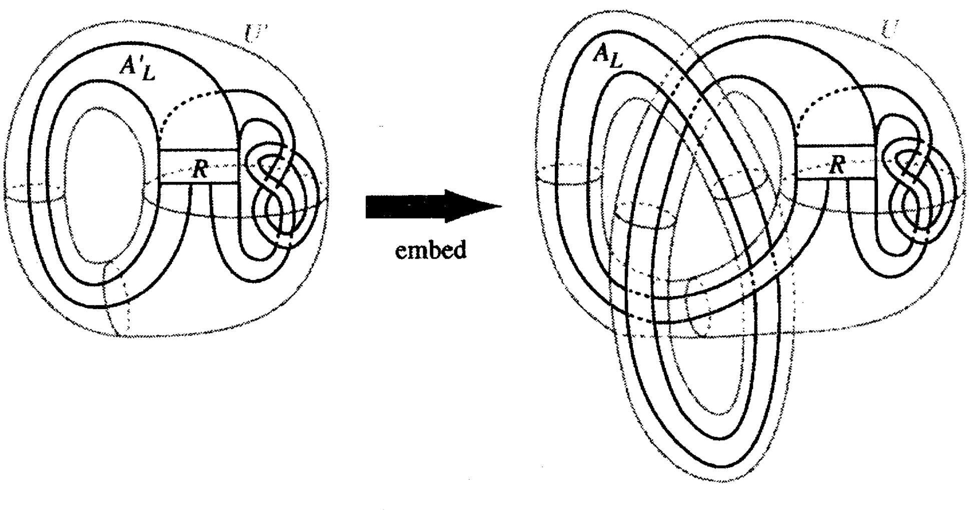

FIGURE 5

4. Concluding remarks

We have considered "splittable" Lorenz-like knot. If it is not splittable, does our theorem still remain true? In the proof above it was proved that, if both of K_L and K_R are non-trivial, then any orbit is non-trivial. But if we permit the linking of K_L and K_R, this is not true. For example the template whose skeleton is as in Figure 6 can hold an unknot even though both of K_R and K_L are non-trivial. But we conjecture that our theorem is still true for non-splittable templates. This may be proved that the unlink with 2 components can not be represented as orbits.

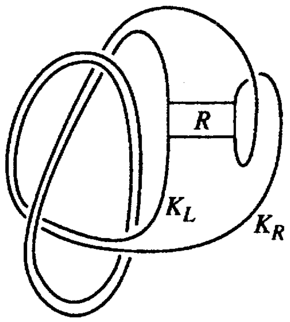

Figure 6

On the other hand, we have treated templates with only 2 arms. But this restriction is only for simplicity, and is not essential. Thus, by the same argument as in our proof, we can prove our theorem for splittable templates with many arms.

References

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