Variational Principle in Ablation of Elastic Solid with Thermal Layer

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A method is presented for the analysis of an ablation of elastic solid with thermal layer. The method is in part an application of Biot's variational theory of thermoelasticity. An extension of heat balance integral introduced by Goodman for thermoelasticity is tried. Both mathematical method together with a quadratic approximation of temperature, provide a system of differential equations of the position of melting line $s(t)$, and the thickness of thermal layer $q(t)$. The solution of heat conduction equation provides the initial condition of $q(t)$. With the aid of Adams-Bashforth's method, one can reach the series solution of $s(t)$ and $q(t)$.

§ 1. Introduction

In the previous papers, the auther analysed the melting elastic solid with the aid of Biot and Boley's earlier works.[1)~7)]

Biot introduced the variational invariants in his theory of thermoelasticity:

$$V = \iiint_V \left(\frac{c}{2T} \theta^2 + W \right) dV. \tag{1.1}$$

$$D = p \iiint_V \frac{1}{2\lambda T} H^2 \, dV, \qquad \text{with} \qquad p = \frac{d}{dt}. \tag{1.2}$$

c......heat capacity per unit volume, T......absolute temperature, θ......temperature change, W......strain energy function, λ......coefficient of heat conduction, H......quantity which represents the rate of heat flow by $\dot{H}$.

And he derived the variational equation.

Boley investigated the ablation of solid by use of the heat conduction equation with phase change.[6)]

Goodman introduced the quantity $I = \int \theta dx$, where the integral was taken over the region he considered. He led the heat balance integral and established his theory of heat conduction.[8), 9)]

Duhamel-Neuman's form of Hooke's law is[10)]

$$\sigma_{ij} = C_{ijkl} \, e_{kl} - \beta_{ij} \, \theta. \tag{1.3}$$

The energy equation of thermoelasticity is

$$-\ (h_i)_i = c\dot{\theta} + T \beta_{ij} \, \dot{e}_{ij}. \tag{1.4}$$

σ_{ij}.....stress tensor, e_{ij}.....strain tensor, β_{ij}, c_{ijkl}.....numerical constants, h_i.....heat flow rate for x_i direction, $(\ )_{,i}$.....differentiation with respect to x_i direction.

The aim of this paper is to investigete the ablation of elastic solid by use of Biot's variational principle and other works mentioned above:

1. The variational invariants are introduced by virtue of the formulas (1.1) and (1.2), and the variational equation is formulated.
2. If we take the origin of time as the time when the ablation begins, the initial condition of the thickness of thermal layer $q(0)$ is calculated from the solution of heat conduction equation.
3. Reffering to Goodman's work, the new extension of the heat balance integral for thermoelasticity is found.
4. The series solution of the position of melting line $s(t)$ and the thickness of thermal layer $q(t)$ are found from the variational equation and the heat balance integral.

§2. Initial Condition of Thermal Layer

The problem of melting slab may be concidered. Consider a slab occupying the finite or infinite region of x axis, exposed to a prescribed heat input $Q(t)$ at $x=0$. Let $T+\theta$ denote the temperature of unit element, θ being the temperature change. It will be assumed here that the melted portion is immedeately removed. When the face $x=0$ is suddenly exposed, at $t=0$, to a heat input, a temperature change occures in a small region, called thermal layer. In this paper, we assume the heat input is constant before melting begins.

The solution of heat conduction equation

$$\frac{\partial\theta}{\partial t} = \kappa^2 \frac{\partial^2\theta}{\partial x^2}, \quad (0 < x < \infty) \tag{2.1}$$

with the condition

$$-\lambda \frac{\partial\theta(0,t)}{\partial x} = Q_0 \tag{2.2}$$

is[11]

$$\theta = \frac{\kappa}{\lambda} Q_0 \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{x^2}{4\kappa^2(t-\tau)}\right\} d\tau, \tag{2.3}$$

where κ^2 is λ/c .

By transforming the variable as

$$x^2/4\kappa^2(t-\tau) = \xi^2, \tag{2.4}$$

we see from eq. (2.3)

$$\theta = \frac{Q_0 x}{\lambda\sqrt{\pi}} \int_{x/2\kappa\sqrt{t}}^{\infty} \frac{1}{\xi^2} \exp(-\xi^2)\, d\xi. \tag{2.5}$$

By use of integration by part and Taylor's expansion, we see from eq. (2.5)

$$\theta \doteqdot \frac{Q_0}{\lambda}\left(\frac{2\kappa}{\sqrt{\pi}}\sqrt{t} - x + \frac{x}{2\kappa\sqrt{\pi t}}\right). \tag{2.6}$$

This equation is almost nearly correct within the region

$$O < x < \frac{2\kappa}{\sqrt{\pi}} \sqrt{t} \, .$$

We assume the thickness of thermal layer as

$$q(t) = 2\kappa \sqrt{\frac{t}{\pi}} \, . \tag{2.7}$$

The relation of the melting temperature θ_m and the time t_m when melting begins, is found from eq. (2.6) as

$$\theta_m = \frac{Q_0}{\lambda} \frac{2\kappa}{\sqrt{\pi}} \sqrt{t_m} \, . \tag{2.8}$$

The equation (2.7) and (2.8) provide the thermal layer thickness when melting begins:

$$q\ (t_m) = \frac{\lambda}{Q_0} \theta_m . \tag{2.9}$$

If we take the origin of time at $t = t_m$, we see the initial condition of $q(t)$ as

$$q(0) = \frac{\lambda}{Q_0} \theta_m \, . \tag{2.10}$$

§3. Quadratic Approximation

A region $s < x < s+q$ is now defined to be the thermal layer; for $x > s+q$, the slab is at an equilibrium temperature and there is no heat transfer beyond this point.

Let $s = s(t)$ denote the thickness of the portion of the material which has melted and take the test function of $\theta(x, t)$ as [8), 9)]

$$\theta(x, t) = a(t) + b(t)\ (x - s) + c(t)\ (x - s)^2 \tag{3.1}$$

From the assumption made above, we may set the condition:

$$\theta(s, t) = \theta_m , \ \theta(s + q, t) = 0, \quad \frac{\partial\theta(s+q, t)}{\partial x} = 0. \tag{3.2}$$

From the formula (3.1) and the condition (3.2), we find

$$\theta = \theta_m \left\{ 1 - \frac{2}{q} (x - s) + \frac{1}{q^2} (x - s)^2 \right\} . \tag{3.3}$$

§4. Variational Principle

Refering to eqs. (1.1) and (1.2), we define the variational invariants as

$$V = \int_s^{s+q} \frac{1}{2} \left(\frac{c}{T} + E\alpha^2 \right) \theta^2 \, dx, \tag{4.1}$$

$$D = \mathrm{p} \int_s^{s+q} \frac{1}{2\lambda T} H^2 \, dx. \tag{4.2}$$

E......Young's modulus, α......coefficient of linear thermal expansion.

We consider the variations as the changes due to the virtual displacement of the coordinate of the melting line $s(t)$.

The variation of V is calculated as

$$\delta V = -\frac{1}{2}\left(\frac{c_m}{T} + E_m\alpha_m^2\right)\theta_m^2\,\delta s + \int_s^{s+q}\left(\frac{c}{T} + E\alpha^2\right)\theta\,\delta\theta\,dx, \quad (4.3)$$

suffix m........melting state.

The law of heat conduction and the law of thermal expansion are expressed as

$$-\lambda\frac{\partial\theta}{\partial x} = \dot{H}, \qquad c\theta = -\frac{\partial H}{\partial x}, \quad (4.4)$$

$$\alpha\theta = -\frac{\partial\xi}{\partial x}. \quad (4.5)$$

ξ........displacement.

The boundary condition at $x = s(t)$ is expressed as [6),7)]

$$\dot{H} = Q(t) - \rho l\dot{s}.$$

ρl........latent heat per unit volume.

Therefore, we reach

$$H(s, t) = \int_0^t Q(t)\,dt + \Delta - \rho ls. \quad (4.6)$$

Δ ....heat transported to the right across a unit cross-sectional area at $x = 0$, before melting degins.

From eq. (4.6), we see

$$\delta H = -\rho l\delta s \quad at \quad x = s(t). \quad (4.7)$$

Using eqs. (4.4), (4.5) and (4.7), we can transform the right side of eq. (4.3) as follows:

$$\begin{aligned}\int_s^{s+q} c\theta\,\delta\theta\,dx &= -\int_s^{s+q}\theta\frac{\partial}{\partial x}(\delta H)\,dx \\ &= -\left\{[\theta\,\delta H]_s^{s+q} - \int_s^{s+q}\frac{\partial\theta}{\partial x}\delta H\,dx\right\} \\ &= \theta_m(\delta H)_{x=s} - \int_s^{s+q}\frac{1}{\lambda}\dot{H}\,\delta H\,dx \\ &= -\rho l\theta_m\delta s - \int_s^{s+q}\frac{1}{\lambda}\dot{H}\,\delta H\,dx. \end{aligned} \quad (4.8a)$$

$$\begin{aligned}\int_s^{s+q} E\alpha\theta\alpha\delta\theta\,dx &= \int_s^{s+q} E\alpha\frac{\partial}{\partial x}(\delta\xi)\,dx \\ &= [E\alpha\theta\delta\xi]_s^{s+q} - \int_s^{s+q}\frac{\partial}{\partial x}(E\alpha\theta)\,\delta\xi\,dx \\ &= -E_m\alpha_m\theta_m(\delta\xi)_{x+s} - \int_s^{s+q}\frac{\partial}{\partial x}(E\alpha\theta)\,\delta\xi\,dx. \end{aligned} \quad (4.8b)$$

Also, from eq. (4.2), we have

$$\begin{aligned}\delta D &= \mathrm{p}\left[-\frac{1}{2\lambda_m T}H(s, t)^2\,\delta s + \int_s^{s+q}\frac{1}{\lambda T}H\,\delta H\,dx\right] \\ &= -\frac{1}{2\lambda_m T}\frac{d}{dt}\{H(s, t)\}^2\,\delta s + \int_s^{s+q}\frac{1}{\lambda T}\dot{H}\,\delta H\,dx. \end{aligned} \quad (4.9)$$

The equations (4.3), (4.8) and (4.9) provide the variational equation

$$\delta V + \delta D = -\frac{1}{2}\left(\frac{c_m}{T} + E_m \alpha_m^2\right)\theta_m^2 \delta s - \frac{\rho l \theta_m}{T}\delta s$$
$$- E_m \alpha_m \theta_m (\delta\xi)_{x=s} - \int_s^{s+q} \frac{\partial}{\partial x}(E\alpha\theta)\, \delta\xi\, dx$$
$$- \frac{1}{2\lambda_m T}\frac{d}{dt}\left\{H(s, t)\right\}^2 \delta s. \tag{4.10}$$

This principle does not require the evaluation of temperature field, so we can insert the test function (3.3).

For the sake of implicity, we take c, E, α, λ etc. as constants.

Substitution of eq. (3.3) into eq. (4.1) provides

$$V = \frac{1}{10}\theta_m^2\left(\frac{}{T} + E\alpha^2\right) q(t). \tag{4.11}$$

This gives

$$\delta V = \frac{1}{10}\theta_m^2\left(\frac{c}{T} + E\alpha^2\right)\delta q. \tag{4.12}$$

The definition of thermal layer provides

$$H(s + q, t) = 0. \tag{4.13}$$

Substituting eq. (3.3) into eq. (4.4$_2$) and integrating by use of the condition (4.13), we have

$$H = -c\theta_m\left\{(x - s) - \frac{1}{q}(x - s)^2 + \frac{1}{3q^2}(x - s)^3 - \frac{1}{3}q\right\}. \tag{4.14}$$

Substitution of eq. (4.14) into eq. (4.2) provides

$$D = \mathrm{p}\left[\frac{1}{2\lambda T \times 63} c^2 \theta_m^2 q^3\right]. \tag{4.15}$$

The variation of D is calculated as follows:

$$\delta D = \mathrm{p}\left[\frac{1}{2\lambda T \times 21} c^2 \theta_m^2 q^2 \delta q\right]$$
$$= \frac{1}{2\lambda T \times 21} c^2 \theta_m^2\, 2q\dot{q}\ \delta q$$
$$\therefore \delta D = \frac{1}{21\lambda T} c^2 \theta_m^2\ q\dot{q}\ \delta q. \tag{4.16}$$

From eqs. (3.3) and (4.5), we have

$$\xi = \alpha\theta_m\left\{(x - s) - \frac{1}{q}(x - s)^2 + \frac{1}{3q^2}(x - s)^3 - \frac{1}{3}q\right\}. \tag{4.17}$$

We assume that the virtual displacement of $s(t)$ causes the increase of temperature and the increase of temperature causes the variation of $q(t)$. From this assumption we see from eq. (4.17) as

$$\delta\xi = -\alpha\theta_m\left\{1 - \frac{2}{q}(x - s) + \frac{1}{q^2}(x - s)^2\right\}\delta s, \tag{4.18}$$

therefore, we have

$$(\delta\xi)_{x=s} = -\alpha\theta_m\,\delta s. \tag{4.19}$$

From eqs. (3.3) and (4.18), we have

$$\int_s^{s+q} \frac{\partial\theta}{\partial x}\,\delta\xi\,dx = \frac{1}{2}\alpha\,\theta_m^2\,\delta s. \tag{4.20}$$

And from eq. (4.14), we have

$$H(s,t) = \frac{1}{3}\,c\,\theta_m\,q(t). \tag{4.21}$$

Substituting eqs. (4.12), (4,16), (4,19), (4,20) and (4.21) into the variational equation (4.10), we have

$$\left\{\frac{1}{10}\theta_m^2\left(\frac{c}{T}+E\alpha^2\right)+\frac{c^2\theta_m^2}{21\lambda T}\,q\,\dot{q}\right\}\delta q$$

$$=\left\{-\frac{1}{2}\left(\frac{c_m}{T}+E_m\alpha_m^2\right)\theta_m^2-\frac{\rho l\theta_m}{T}+E_m\alpha_m\alpha\,\theta_m^2\right.$$

$$\left.-\frac{1}{2}E\alpha^2\theta_m^2-\frac{1}{9\lambda_m T}\,c^2\theta_m^2\,q\,\dot{q}\right\}\delta s. \tag{4.22}$$

§5. Heat Balance Integral

We shall try to extend the heat balance integral to thermoelasticity.[8),9)] We define this refering to eq. (1.4):

$$I = \int_s^{s+q}(c+TE\alpha^2)\,\theta\,dx. \tag{5.1}$$

Differentiation with respect to time provides

$$\frac{dI}{dt} = -(c_m+TE_m\alpha_m^2)\,\theta_m\dot{s}+\int_s^{s+q}(c+TE\alpha^2)\,\dot{\theta}\,dx. \tag{5.2}$$

Using eq. (1.4) and the definition of thermal layer and the boundary condition at $x=s$ (t),

$$-\lambda_m\left(\frac{\partial\theta}{\partial x}\right)_{x=s} = Q(t)-\rho l\dot{s}, \tag{5.3}$$

we can evaluate the second term of the right side of ep. (5.2) as follows:

$$\int_s^{s+q}(c+TE\alpha^2)\,\theta\,dx = \int_s^{s+q}\frac{\partial}{\partial x}\left(\lambda\frac{\partial\theta}{\partial x}\right)dx$$

$$= Q(t)-\rho l\dot{s}.$$

Therefore eq. (5.2) becomes

$$\frac{dI}{dt} = -\left\{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l\right\}\dot{s}+Q(t). \tag{5.4}$$

Inserrting eq. (3.3) into eq. (5.4), we have

$$\frac{1}{3}\,\theta_m(c+TE\alpha^2)\,\dot{q}$$

$$= -\left\{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l\right\}\dot{s}+Q(t). \tag{5.5}$$

Recalling $s(0)=0$, we have; by integrating eq. (5.5),

$$\frac{1}{3}\theta_m\,(c+TE\alpha^2)\,(q(t)-q(0))$$

$$=-\{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l\}\,s+\int_0^t Q(t)\,dt. \qquad (5.6)$$

This equation provides the relation between δs and δq:

$$\delta q=-\frac{3\,\{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l\}}{\theta_m\,(c+TE\alpha^2)}\,\delta s. \qquad (5.7)$$

Inserting eq. (5.7) into eq. (4.22), we have the equation of $s(t)$ and $q(t)$:

$$-\frac{3\,\{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l\}}{\theta_m(c+TE\alpha^2)}\left\{\frac{1}{10}\theta_m^2\left(\frac{c}{T}+E\alpha^2\right)\right.$$

$$\left.+\frac{c^2\theta_m^2}{21\lambda T}\,q\,\dot{q}\right\}=-\frac{1}{2}\left(\frac{c_m}{T}+E_m\alpha_m^2\right)\theta_m^2-\frac{\rho l\theta_m}{T}$$

$$+E_m\alpha_m\alpha\,\theta_m^2-\frac{1}{2}E\alpha^2\theta_m^2-\frac{1}{9\lambda_m T}\,c^2\theta_m^2\,q\,\dot{q}. \qquad (5.8)$$

We reached the simultaneous equation for $s(t)$ and $q(t)$ as eqs. (5.5) and (5.8). In the next paragraph, we shall calculate the series solution for $s(t)$ and $q(t)$ from these equations.

§6. Series Solution

We can evaluate the series solutions

$$s(t)=\dot{s}(0)t+\frac{1}{2!}\ddot{s}(0)\,t^2+\cdots\cdots\cdots\cdots\cdots$$

$$q(t)=q(0)+\dot{q}(0)t+\frac{1}{2!}\ddot{q}(0)t^2+\cdots\cdots\cdots\cdots$$

from eqs. (5.5) and (5.8), recalling $s(0)=0$, and eq. (2.10), i.e.

$$q(0)=\frac{\lambda}{Q_0}\theta_m, \qquad (6.1)$$

with the aid of Adams-Bashforth's method:

$$\dot{q}(0)=\frac{1}{q(0)}\left[\frac{c^2}{7\lambda T}\,\frac{\{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l\}}{c+TE\alpha^2}-\frac{c^2\theta_m}{9\lambda_m T}\right]^{-1}$$

$$\times\left\{\frac{1}{5}\left(\frac{c_m}{T}+E_m\alpha_m^2\right)\theta_m+\frac{7\rho l}{10T}-E_m\alpha_m\alpha\,\theta_m\right.$$

$$\left.+\frac{1}{2}E\alpha^2\,\theta_m\right\}, \qquad (6.2)$$

$$\dot{s}(0)=\frac{Q_0-\frac{1}{3}\theta_m(c+TE\alpha^2)\,\dot{q}(0)}{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l}, \qquad (6.3)$$

$$\ddot{q}(0)=-\{\dot{q}(0)\}^2/q(0), \qquad (6.4)$$

$$\ddot{s}(0)=\frac{\dot{Q}(0)-\frac{1}{3}\theta_m(c+TE\alpha^2)\,\ddot{q}(0)}{(c_m+TE_m\alpha_m^2)\,\theta_m+\rho l}, \qquad (6.5)$$

etc.

§7. Conclusion

The formulation of the ablation of elastic solid with thermal layer has been done by the variational equation and the heat balance integral for the thermoelastic solid. By use of the variational invariants (4.1) and (4.2), the variational equation has been formulated. This equation has two parameters, $s(t)$ and $q(t)$. With the aid of the heat balance integral for thermoelasticity, this equation has been formulated as a varitional equation with subsidary condition. This system of equations does not require the evaluation of temperature field, therefore the quadratic form of temperature field can be used as a test function. By avoiding the need for highly ingenious guesses and the complicated mathematical structure of differential equation, this formulation provides the series solution, which reveal the feature of $s(t)$ and $q(t)$.

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