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Injective envelopes of dynamical systems

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To the memory of my mother Chika and my father Toshio

Abstract. By dynamical systems in the title we mean the objects, called G-modules, of a category $\mathcal{C}_G$ consisting of operator spaces with a certain $L^1(G)$-module structure and the complete contractions commuting with the module operation, where G is a fixed locally compact group. By requiring additional properties we obtain the notion of monotone complete C^*-G-modules, which is a monotone complete C^*-algebra version of W^*-dynamical systems. We show that when injectivity is introduced in $\mathcal{C}_G$, every G-module has a unique injective envelope of the form pAq, where A is a monotone complete C^*-G-module and p, q are invariant projections in A. The G-modules such that $L^1(G) \cdot X = X$ are regarded as a counterpart of C^*-dynamical systems. We relate such a G-module to two Morita equivalent C^*-dynamical systems in the sense of Combes in such a way that the corresponding dynamical systems are the smallest in a certain sense. We formulate a crossed product of a G-module and investigate when the Takesaki type duality holds. We also extend the flow built under a function construction in ergodic theory to the setting of monotone complete C^*-G-modules.

0. Introduction

This is the TeXed version of the preprint with the same title, dated April 1991 (the final draft), which was circulated among some people. Part of Section 4 (the case without the group action) has appeared as the paper [16], but the remaining part with full proofs appears for the first time (see

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also the remark at the end of this section).

A C^*-algebra A is called *monotone complete* if each bounded increasing net in the self-adjoint part A_{sa} of A has a supremum in the partially ordered set A_{sa}. The class of monotone complete C^*-algebras is strictly larger than the class of W^*-algebras, and is contained in the class of AW^*-algebras. Although the existence is known of some sporadic examples of non-trivial monotone complete C^*-algebras (non-W^*, AW^*-factors) (see, for example, [30], [33], [9], [12]), it seems that for a systematic study of monotone complete C^*-algebras we need a generalization of such notions as W^*-dynamical systems, the resulting crossed products, et cetera, which have played a fundamental rôle in the structure theory of W^*-algebras.

In this paper we present such a generalization on the basis of Fubini products (a monotone complete version of W^*-tensor products) of monotone complete C^*-algebras and W^*-algebras introduced in [12]. (Note that besides Fubini products, monotone complete tensor products were treated in [12] also as an extension of W^*-tensor products. But we consider exclusively the former here and use the notation like $A \bar{\otimes} M$, which was used to denote the latter, to denote the former.) Our generalization of a W^*-dynamical system with the acting group G (a locally compact group fixed throughout), called a *monotone complete* C^**-*G*-module*, is defined to be a monotone complete C^*-algebra A together with a unital, normal $*$-monomorphism $\pi : A \to A \bar{\otimes} L^\infty(G)$ such that $(\pi \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ \pi = (\mathrm{id}_A \bar{\otimes} \alpha_G) \circ \pi$, called an *action* of G on A, where $A \bar{\otimes} L^\infty(G)$ is the Fubini product of A and $L^\infty(G)$, id's with some subscripts are the identity maps, and α_G is the comultiplication of the Hopf-von Neumann algebra $L^\infty(G)$ given by $\alpha_G(a)(s,\, t) = a(st)$, $s,\, t \in G$. The action in this sense is of course a simple modification in the W^*-case (see, for example, [23]). But the point here is that in the definition of an action we avoid the use of any topology like the σ-weak or σ-strong topology in the W^*-case, which seems not to be available in our case, since the existence of sufficiently many σ-weakly continuous functionals characterizes W^*-algebras among monotone complete C^*-algebras. We see that monotone complete C^*-G-modules arise naturally through a sort of completion of certain "dynamical systems". More specifically, if injectivity is introduced in a certain category $\mathcal{C}_G$, whose objects and morphisms are called G-modules and G-morphisms, respectively, then every G-module (in particular, every C^*-dynamical system with the acting group G) X has a unique injective envelope, $I_G(X)$, of the form pAq (A itself in the C^*-case), where A is a monotone complete C^*-G-module and p, q are invariant projections in A. Thus $I_G(X)$, or the monotone closure of X in $I_G(X)$ in the C^*-case, may be regarded as a completion of X. The latter is an analogue of the regular monotone completion of a C^*-algebra, [11], and becomes indeed a monotone complete C^*-G-module.

In Section 2 below we prove the existence of $I_G(X)$ in a more general setting, that is, in a category $\mathcal{C}_M$ associated with a certain Hopf-von Neumann algebra M, and in Section 3 we consider the case $\mathcal{C}_G$ in more detail. In Section 4 we treat the G-modules which may be regarded as an abstraction of C^*-dynamical systems, but do not have any algebraic structure, and we show that to each such G-module there correspond two C^*-dynamical systems, which are Morita equivalent in the sense of Combes [5] and are the smallest in a sense. In Section 5 we formulate a crossed product of a G-module so that it becomes a monotone complete C^*-algebra when X is a monotone complete C^*-algebra, and investigate the validity of the Takesaki type duality for crossed products. In Section 6, to provide an example of non-trivial monotone complete C^*-G-modules we extend the flow built under a function construction in ergodic theory to the setting of monotone complete G^*-G-modules. Such an extension was made by Phillips [25] in the W^*-case.

We conclude the introduction with the following remark. Part of the work in this paper was presented at the second international conference on operator algebras and their connection with topology and ergodic theory, August-September, 1989, held at Craiova, Romania. The results in Sections 3-5 were intended to extend the author's previous papers [9], [10], [15], and were announced for the cases $M = \mathbb{C}$ and $M = L^\infty(G)$ in 1985 at the annual meeting of the mathematical society of Japan and at a symposium held at Research Institute for Mathematical Sciences, Kyoto University (Sûrikaisekikenkyûsho Kôkyûroku No. 560, pages 128-141, May 1985). In September 1987 the author received a preprint of [28] by J.-Z. Ruan whose main result is our Theorem 2.7 for $M = \mathbb{C}$.

1. Fubini products of operator spaces

This preliminary section contains an extension of some results in [12] on operator systems and completely positive maps to the case of operator spaces and complete contractions, and related results. Most proofs are omitted, since the corresponding proofs in [12] work quite similarly.

A linear space V is called an *operator space* if it is realized as a linear subspace of some C^*-algebra A and the tensor products $M_n \otimes V$ ($n = 1, 2, \cdots$) with the C^*-algebra M_n of $n \times n$ complex matrices are endowed with the norms induced from the C^*-tensor products $M_n \otimes A$ ($n = 1, 2, \cdots$). We write simply $V \subset A$ to denote this situation. (Note that we write M_n on the left of operator spaces, contrary to the usual convention, since this is relevant to our later considerations.) For operator spaces V and W a linear map $\varphi : V \to W$ is called a *complete contraction* (respectively, *complete isometry*, et cetera) if $\|\varphi\|_{cb} := \sup_n \|\mathrm{id}_n \otimes \varphi\| \le 1$ (respectively, $\mathrm{id}_n \otimes \varphi$ is an isometry, et cetera) for all n, where $\mathrm{id}_n \otimes \varphi : M_n \otimes V \to M_n \otimes W$ and id_n denotes the identity map on M_n. If V and W are completely isometric, we write $V \sim W$, and identify these spaces. An operator space V is called a *C^*-algebra* (respectively, a *monotone complete C^*-algebra*, et cetra) if $V \sim A$ for some C^*-algebra (respectively, some monotone complete C^*-algebra, et cetera) A. Such a C^*-algebra, if it exists, is unique since completely isometric C^*-algebras are *-isomorphic.

Throughout the paper, operator spaces are assumed to be *norm closed*, the spaces to be considered have at least the structure of operator spaces, and $B(H)$ or $B(K)$ denotes the W^*-algebra of all bounded operators on a Hilbert space H or K.

Our interest is in notions which are determined uniquely up to complete isometry. We define the *Fubini product* of operator spaces $V \subset B(H)$ and $W \subset B(K)$ as the following subspace of the W^*-tensor product $B(H) \bar{\otimes} B(K)$:

$$V \bar{\otimes} W = \{x \in B(H) \bar{\otimes} B(K) : \ (\varphi \bar{\otimes} \mathrm{id}_{B(K)})(x) \in W, \\ (\mathrm{id}_{B(H)} \bar{\otimes} \psi)(x) \in V, \quad \forall\, \varphi \in B(H)_*, \ \forall\, \psi \in B(K)_*\},$$

where $\varphi \bar{\otimes} \mathrm{id}_{B(K)} : B(H) \bar{\otimes} B(K) \to B(K)$ is the slice map (a unique σ-weakly continuous extension of the map $\sum a_i \otimes b_i \mapsto \sum \varphi(a_i) b_i$) and similarly for $\mathrm{id}_{B(H)} \bar{\otimes} \psi$. The Fubini products behave well if one of the factors V and W is σ-weakly closed. In what follows (except in 1.4) we assume the second factors or the letters W (with some subscripts) to be *σ-weakly closed.*

For $j = 1, 2$ let $V_j \subset B(H_j)$ and $W_j \subset B(K_j)$ be operator spaces and let $\varphi : V_1 \to V_2$, $\psi : W_1 \to W_2$ be complete contractions with ψ *σ-weakly continuous.* As in [12], 3.5, complete contractions

$$\varphi \bar{\otimes} \mathrm{id}_{B(K_j)} : V_1 \bar{\otimes} B(K_j) \to V_2 \bar{\otimes} B(K_j),$$
$$\mathrm{id}_{B(H_j)} \bar{\otimes} \psi : B(H_j) \bar{\otimes} W_1 \to B(H_j) \bar{\otimes} W_2$$

are defined. In the sense of part (ii) below the Fubini products depend only on the isomorphism classes of the factors.

Proposition 1.1 (cf. [12], 3.8, 3.9)**.** *Keep the above notation.*

(i) *The composites* $(\mathrm{id}_{B(H_2)} \bar{\otimes} \psi) \circ (\varphi \bar{\otimes} \mathrm{id}_{B(K_1)})|V_1 \bar{\otimes} W_1$ *and* $(\varphi \bar{\otimes} \mathrm{id}_{B(K_1)}) \circ (\mathrm{id}_{B(H_1)} \bar{\otimes} \psi)|V_1 \bar{\otimes} W_1$ *coincide, and define a complete contraction into* $V_2 \bar{\otimes} W_2$. *We denote this map by* $\varphi \bar{\otimes} \psi : V_1 \bar{\otimes} W_1 \to V_2 \bar{\otimes} W_2$, *and call it the Fubini product of* φ *and* ψ.

(ii) *If further* φ *and* ψ *are surjective complete isometries, then so is* $\varphi \bar{\otimes} \psi$*; that is,* $V_1 \sim V_2$ *and* $W_1 \sim W_2$ *imply* $V_1 \bar{\otimes} W_1 \sim V_2 \bar{\otimes} W_2$.

(iii) *If* $V_2 \subset V_1$, $W_2 \subset W_1$ *and* φ, ψ *are idempotents onto* V_2, W_2, *respectively, then* $\varphi \bar{\otimes} \psi$ *is an idempotent onto* $V_2 \bar{\otimes} W_2$.

Let $V \subset B(H)$ and $W \subset B(K)$ be as above. As in [12], 3.7, for $f \in V^*$ and $g \in W_* := \{\psi|W : \psi \in B(K)_*\}$ we define the *slice maps* $f \bar{\otimes} \mathrm{id}_W : V \bar{\otimes} W \to W$ and $\mathrm{id}_V \bar{\otimes} g : V \bar{\otimes} W \to V$ with $g \circ (f \bar{\otimes} \mathrm{id}_W) = f \circ (\mathrm{id}_V \bar{\otimes} g)$, written $f \bar{\otimes} g$ and called the *product functional.*

Proposition 1.2 (cf. [12], 4.6, 3.5(iii))**.** (i) *With the notation as above the sets* $\{f \bar{\otimes} \mathrm{id}_W : f \in V^*\}$ *and* $\{\mathrm{id}_V \bar{\otimes} g : g \in W_*\}$ *are separating families on* $V \bar{\otimes} W$, *that is,* $x \in V \bar{\otimes} W$ *and* $(f \bar{\otimes} \mathrm{id}_W)(x) = 0$ *for all* $f \in V^*$ *imply* $x = 0$, *and similarly for the latter set.*

(ii) *If* $\varphi \bar{\otimes} \psi : V_1 \bar{\otimes} W_1 \to V_2 \bar{\otimes} W_2$ *is the Fubini products of* $\varphi : V_1 \to V_2$ *and* $\psi : W_1 \to W_2$, *then*

$$(f \bar{\otimes} \mathrm{id}_{W_2}) \circ (\varphi \bar{\otimes} \psi) = \psi \circ (f \circ \varphi \bar{\otimes} \mathrm{id}_{W_1}),$$
$$(\mathrm{id}_{V_2} \bar{\otimes} g) \circ (\varphi \bar{\otimes} \psi) = \varphi \circ (\mathrm{id}_{V_1} \bar{\otimes} g \circ \psi),$$

where $f \in V_2^*$, $g \in (W_2)_*$ *and* $f \bar{\otimes} \mathrm{id}_{W_2} : V_2 \bar{\otimes} W_2 \to W_2$, $f \circ \varphi \bar{\otimes} \mathrm{id}_{W_1} : V_1 \bar{\otimes} W_1 \to V_1$ *et cetera are slice maps.*

Proposition 1.3 (cf.[12], 3.17, 3.1, 4.3)**.** *Let* V *be a monotone complete* C^**-algebra and* W *a* W^**-subalgebra of some* $B(K)$*.*

(i) $V \bar{\otimes} W$ *is a monotone complete* C^**-algebra.*

(ii) $V \bar{\otimes} W$ *is a* W^**-algebra if and only if* V *is a* W^**-algebra; in this case* $V \bar{\otimes} W$ *is the* W^**-tensor product of* V *and* W*.*

(iii) $V \bar{\otimes} W$ *is an injective* C^**-algebra if and only if* V *and* W *are both injective.*

(iv) *If* V_1 *is a monotone complete* C^**-algebra,* W_1 *is a* W^**-algebra, and* $\varphi : V \to V_1$ *and* $\psi : W \to W_1$ *are normal completely positive maps, then so is* $\varphi \bar{\otimes} \psi : V \bar{\otimes} W \to V_1 \bar{\otimes} W_1$*. In particular, the slice maps* $\mathrm{id}_V \bar{\otimes} g : V \bar{\otimes} W \to V$ *for* $g \in W_*$ *positive are normal, and if* $V \subset V_1$ *is a monotone closed* C^**-subalgebra and* $W \subset W_1$ *is a* W^**-subalgebra, then* $V \bar{\otimes} W$ *is a monotone closed* C^**-subalgebra of* $V_1 \bar{\otimes} W_1$*.*

(v) *For an increasing net* $\{x_i\}$ *in* $V \bar{\otimes} W$ *we have* $x_1 \nearrow x$ *in* $V \bar{\otimes} W$ *(that is,* $\sup x_i = x$*) if and only if* $(\mathrm{id}_V \bar{\otimes} g)(x_i) \nearrow (\mathrm{id}_V \bar{\otimes} g)(x)$ *in* V *for all* $g \in W_*^+$*.*

Proposition 1.4. *Let* $V \subset B(H)$ *and* $W \subset B(K)$ *be operator spaces, which are assumed only to be norm closed. For* $a_j \in B(H)$*,* $b_j \in B(K)$*,* $j = 1, 2$*, we have*

$$(a_1 \otimes b_1)(V \bar{\otimes} W)(a_2 \otimes b_2) \subset (a_1 V b_1) \bar{\otimes} \mathrm{cl}(a_2 W b_2),$$

where the notation cl *denotes the norm closure. In particular, if* a_j *and* b_j *are all unitary, then the both sides coincide.*

Proof. We note that for $x \in B(H) \bar{\otimes} B(K)$ we have $x \in V \bar{\otimes} B(K)$ if and only if $(f \bar{\otimes} \mathrm{id}_{B(K)})(x) = 0$ for all $f \in V^\perp := \{f \in B(H)^* : f|V = 0\}$. Indeed, $x \in V \bar{\otimes} B(K)$ if and only if $(\mathrm{id}_{B(H)} \bar{\otimes} g)(x) \in V$ for all $g \in B(K)_*$, that is,

$$0 = f \circ (\mathrm{id}_{B(H)} \bar{\otimes} g)(x) = g \circ (f \bar{\otimes} \mathrm{id}_{B(K)})(x)$$

for all $f \in V^\perp$ and $g \in B(K)_*$, if and only if $(f \bar{\otimes} \mathrm{id}_{B(K)})(x) = 0$ for all $f \in V^\perp$. Hence, if $x \in V \bar{\otimes} B(K)$, then

$$(f \bar{\otimes} \mathrm{id}_{B(K)})((1 \otimes b_1)x(1 \otimes b_2)) = b_1(f \bar{\otimes} \mathrm{id}_{B(K)})(x)b_2 = 0$$

for all $f \in V^{\perp}$ and so $(1 \otimes b_1)x(1 \otimes b_2) \in V \bar{\otimes} B(K)$. Namely $(1 \otimes b_1)V \bar{\otimes} B(K)(1 \otimes b_2) \subset V \bar{\otimes} B(K)$. On the other hand,

$$(1 \otimes b_1)(B(H) \bar{\otimes} W)(1 \otimes b_2) \subset B(H) \bar{\otimes} b_1 W b_2$$

since for $x \in B(H) \bar{\otimes} W$ and $f \in B(H)_*$,

$$(f \bar{\otimes} \mathrm{id}_{B(K)})((1 \otimes b_1)x(1 \otimes b_2)) = b_1(f \bar{\otimes} \mathrm{id}_{B(K)})(x)b_2 \in b_1 W b_2$$

as above. Thus

$$(1 \otimes b_1)(V \bar{\otimes} W)(1 \otimes b_2) \subset V \bar{\otimes} B(K) \cap B(H) \bar{\otimes} b_1 W b_2 = V \bar{\otimes} b_1 W b_2.$$

Similarly it follows that

$$(a_1 \otimes 1)(V \bar{\otimes} b_1 W b_2)(a_2 \otimes 1) \subset a_1 V a_2 \bar{\otimes} \mathrm{cl}(b_1 W b_2),$$

hence that

$$(a_1 \otimes b_1)(V \bar{\otimes} W)(a_2 \otimes b_2) \subset a_1 V a_2 \bar{\otimes} \mathrm{cl}(b_1 W b_2).$$

If a_j and b_j are unitary, then by replacing W, W, a_j, b_j in the above argument by $a_1 V a_2$, $b_1 W b_2$, a_j^*, b_j^* we obtain the reverse inclusion and hence equality. □

Each element x in the Fubini product $V \bar{\otimes} W$ defines a map $g \mapsto (\mathrm{id}_V \bar{\otimes} g)(x)$ in $B(W_*, V)$, the Banach space of bounded linear maps of W_* into V. When V and W are W^*-algebras, Effros characterized the image of $V \bar{\otimes} W$ in $B(W_*, V)$ under this correspondence (see [22], Theorem 2). Under a stronger hypothesis we obtain a stronger conclusion, which is probably known. (Effros' result and the following will both be used later.)

Proposition 1.5. *Let V be an operator space and W a commutative W^*-algebra. Then the map $l : V \bar{\otimes} W \to B(W_*, V)$ defined by $l(x)(g) = (\mathrm{id}_V \bar{\otimes} g)(x)$ is a surjective isometry.*

Proof. It suffices to consider the case $V = B(H)$. Indeed, if $V \subset B(H)$, then it follows from the definition of $V \bar{\otimes} W$ that the map l defined above is the restriction to $V \bar{\otimes} W$ of a similar map $l : B(H) \bar{\otimes} W \to B(W_*, B(H))$.

If $W = C(\Omega)$ (Ω is the spectrum of W) and δ_t, $t \in \Omega$, is the evaluation at t, that is, $\delta_t(x) = x(t)$, $x \in W$, then as in the proof of [13], 1.1, for $x \in B(H) \bar{\otimes} W$,

$$\|x\| = \sup\{|f \circ (\mathrm{id}_{B(H)} \bar{\otimes} \delta_t)(x)| : \ t \in \Omega, \ f \in B(H)_*, \ \|f\| \leq 1\},$$

which imlpies that

$$\begin{aligned} \|x\| &= \sup\{|(f \bar{\otimes} g)(x)| : \ f \in B(H)_*, \ \|f\| \leq 1, \ g \in W_*, \ \|g\| \leq 1\} \\ &= \|l(x)\|, \end{aligned}$$

that is, l is isometric, since each δ_t is a $\sigma(W^*, W)$-limit of some net $\{g_i\} \subset W_*$, $\|g_i\| \leq 1$, and for each $f \in B(H)_*$,

$$(f \bar{\otimes} g_i)(x) = g_i \circ (f \bar{\otimes} \mathrm{id}_W)(x) \to \delta_t \circ (f \bar{\otimes} \mathrm{id}_W)(x) = f \circ (\mathrm{id}_{B(H)} \bar{\otimes} \delta_t)(x).$$

To see the surjectivity of l let $\{e_{ij}\}_{i,j \in I}$ be a family of matrix units in $B(H)$ so that $x = \sum_{i,j} e_{ij} \otimes x_{ij}$ (the strong limit of finite subsums), $x_{ij} \in W$, for each $x \in B(H) \bar{\otimes} W$ and $l(x)(g) = \sum_{i,j} g(x_{ij}) e_{ij}$. If $T \in B(W_*, B(H))$, then there are $x_{ij} \in W$, $i, j \in I$, such that $e_{ii} T(g) e_{jj} = g(x_{ij}) e_{ij}$ for all $g \in W_*$ and i, j. If $J \subset I$ is finite, $e_J := \sum_{i \in J} e_{ii}$, and $x_J := \sum_{i,j \in J} e_{ij} \otimes x_{ij} \in B(H) \bar{\otimes} W$, then $l(x_J) = e_J T(\cdot) e_J$, and as l is isometric, $\|x_J\| = \|e_J T(\cdot) e_J\| \leq \|T\|$. This shows that $x := \sum e_{ij} \otimes x_{ij}$ defines an element in $B(H) \bar{\otimes} W$ with $l(x) = T$ (see [12], 2.1).

□

2. Category $\mathcal{C}_M$

In this section, for a fixed Hopf-von Neumann algebra M we define a category $\mathcal{C}_M$, and under a mild condition on M we prove the existence and uniqueness of an injective envelope of an object in $\mathcal{C}_M$. Here a Hopf-von Neumann algebra is a W^*-algebra M together with a unital normal *-monomorphism $\Gamma : M \to M \bar{\otimes} M$, called the comultiplication of M, such that $(\Gamma \bar{\otimes} \mathrm{id}_M) \circ \Gamma = (\mathrm{id}_M \bar{\otimes} \Gamma) \circ \Gamma$. The predual M_* of M becomes then a Banach algebra with the product defined by $f \cdot g = (f \bar{\otimes} g) \circ \Gamma$, $f, g \in M_*$ (see, for example, [29], Chapter IV).

Definition 2.1. An *M-comodule* is an operator space X together with a complete isometry $\pi_X : X \to X \bar{\otimes} M$, called the *action of M on X*, such that $(\pi_X \bar{\otimes} \mathrm{id}_M) \circ \pi_X = (\mathrm{id}_X \bar{\otimes} \Gamma) \circ \pi_X$. A (norm closed) linear subspace Y of an M-comodule X is called an *M-subcomodule* of X and written $Y \leq X$ if $\pi_X(Y) \subset Y \bar{\otimes} M$ when regarded as $Y \bar{\otimes} M \subset X \bar{\otimes} M$. In this case Y is indeed an M-comodule with the action $\pi_Y = \pi_X|Y$. An *M-comodule morphism* is a complete contraction $\varphi : X \to Y$ between M-comodules X and Y such that $\pi_Y \circ \varphi = (\mathrm{id}_M \bar{\otimes} \varphi) \circ \pi_X$. An M-comodule morphism is called an *M-comodule monomorphism* (respectively *isomorphism*) if it is also a (respectively surjective) complete isometry. Two M-comodules X and Y are called *isomorphic* and written $X \cong Y$ if there is an M-comodule isomorphism of X onto Y. The Fubini product $V \bar{\otimes} M$ of any operator space V and M is an M-comodule with action $\mathrm{id}_V \bar{\otimes} \Gamma : V \bar{\otimes} M \to V \bar{\otimes} M \bar{\otimes} M$, which is called a *canonical M-comodule*. We write $\mathcal{C}_M$ for the category of M-comodules and M-comodule morphisms.

Remarks. (i) Every M-comodule is an M-subcomodule of some canonical M-comodule, and the canonical M-comodule may be taken to be of the form $B(H) \bar{\otimes} M$. Indeed, for every M-comodule X the image $\pi_X(X)$ under the action π_X is an M-subcomodule of $X \bar{\otimes} M$, since $(\mathrm{id}_X \bar{\otimes} \Gamma)(\pi_X(X)) = (\pi_X \bar{\otimes} \mathrm{id}_M) \circ \pi_X(X) \subset \pi_X(X) \bar{\otimes} M$, and π_X is an M-comodule isomorphism of X onto $\pi_X(X)$, that is, $X \cong \pi_X(X) \leq X \bar{\otimes} M$. Moreover, if $X \subset B(H)$ as an operator space for some $B(H)$, then $X \bar{\otimes} M \leq B(H) \bar{\otimes} M$.

(ii) For any M-comodule X the operator space $M_n \otimes X$ is an M-comodule with action $\mathrm{id}_n \otimes \pi_X : M_n \otimes X \to M_n \otimes X \bar{\otimes} M$.

An M-comodule X is made into a left M_*-module by the operation $f \cdot x = (\mathrm{id}_X \bar{\otimes} f) \circ \pi_X(x)$, $f \in M_*$, $x \in X$ (see 3.3 (i) below). Note that if $X = V \bar{\otimes} M$ is canonical, then $f \cdot (a \otimes b) = a \otimes f \cdot b$ for $f \in M_*$, $a \in V$ and $b \in M$, where $f \cdot b = (\mathrm{id}_M \bar{\otimes} f) \circ \Gamma(b)$, and that $f \cdot (a \otimes 1) = f(1)(a \otimes 1)$ since $f \cdot 1 = (\mathrm{id}_M \bar{\otimes} f)(1 \otimes 1) = f(1)1$. This module operation can be used to give an alternative description of M-comodules and M-comodule morphisms as follows.

Proposition 2.2. *Regard M-comodules as M_*-modules as above.*

(i) *A complete contraction between M-comodules is an M-comodule morphism if and only if it is an M_*-module homomorphism.*

(ii) *An operator space is an M-comodule if and only it is an M_*-submodule of some canonical M-comodule.*

Proof. (i) This follows from 2.3 (iii) below.

(ii) The action $\pi_X : X \to X \bar{\otimes} M$ of any M-comodule X is a completely isometric M_*-module homomorphism (by 2.3 (iii)) onto the M_*-submodule $\pi_X(X)$ of the canonical M-comodule $X \bar{\otimes} M$. Conversely, if X is an M_*-submodule of some canonical M-comodule $V \bar{\otimes} M$, then $(\mathrm{id}_X \bar{\otimes} \Gamma)(X) \subset X \bar{\otimes} M$ by 2.3 (ii), that is, $X \leq V \bar{\otimes} M$. $\square$

Lemma 2.3. (i) *Let X be an operator space and $\varphi : X \to X \bar{\otimes} M$ a complete contraction. Putting $f \cdot x = (\mathrm{id}_X \bar{\otimes} f)(\varphi(x)) \in X$, $f \in M_*$, $x \in X$, we have $f \cdot (g \cdot x) = (f \cdot g) \cdot x$ for all f, $g \in M_*$ and $x \in X$ if and only if $(\varphi \bar{\otimes} \mathrm{id}_M) \circ \varphi = (\mathrm{id}_X \bar{\otimes} \Gamma) \circ \varphi$.*

(ii) *Let X, φ and $f \cdot x$ be as above. For a linear subspace Y of X we have $f \cdot Y \subset Y$ for all $f \in M_*$ if and only if $\varphi(Y) \subset Y \bar{\otimes} M$.*

(iii) *For operator spaces X, Y and complete contractions $\pi_1 : X \to X \bar{\otimes} M$, $\pi_2 : Y \to Y \bar{\otimes} M$ and $\varphi : X \to Y$ we have $(\varphi \bar{\otimes} \mathrm{id}_M) \circ \pi_1 = \pi_2 \circ \varphi$ if and only if*

$$\varphi \circ (\mathrm{id}_X \bar{\otimes} f) \circ \pi_1 = (\mathrm{id}_Y \bar{\otimes} f) \circ \pi_2 \circ \varphi$$

for all $f \in M_*$.

Proof. (i) The first equality is rewritten as

$$(\mathrm{id} \bar{\otimes} f) \circ \varphi \circ (\mathrm{id}_X \bar{\otimes} g) \circ \varphi = (\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g) \circ \Gamma) \circ \varphi$$

for all f, $g \in M_*$, and further as

$$(\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g)) \circ (\varphi \bar{\otimes} \mathrm{id}_M) \circ \varphi = (\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g)) \circ (\mathrm{id}_X \bar{\otimes} \Gamma) \circ \varphi,$$

since

$$\begin{aligned}(\mathrm{id}_X \bar{\otimes} f) \circ \varphi \circ (\mathrm{id}_X \bar{\otimes} g) &= (\mathrm{id}_X \bar{\otimes} f) \circ (\mathrm{id}_{X \bar{\otimes} M} \bar{\otimes} g) \circ (\varphi \bar{\otimes} \mathrm{id}_M) \quad \text{(by 1.2(ii))} \\ &= (\mathrm{id}_X \bar{\otimes} f \circ (\mathrm{id}_M \bar{\otimes} g)) \circ (\varphi \bar{\otimes} \mathrm{id}_M) \\ &= (\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g)) \circ (\varphi \bar{\otimes} \mathrm{id}_M)\end{aligned}$$

and $\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g) \circ \Gamma = (\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g)) \circ (\mathrm{id}_X \bar{\otimes} \Gamma)$. By 1.2 (i) this is equivalent to the second equality.

(ii) This is obvious from the definition of Fubini products.

(iii) By 1.2 (ii) we have $\varphi \circ (\mathrm{id}_X \bar{\otimes} f) = (\mathrm{id}_Y \bar{\otimes} f) \circ (\varphi \bar{\otimes} \mathrm{id}_M)$; hence the assertion follows as in (i). □

We adopt the following as a monotone complete version of a W^*-dynamical system.

Definition 2.4. A *monotone complete C^*-M-comodule* is an M-comodule A such that the underlying operator space A is a monotone complete C^*-algebra and the action $\pi_A : A \to A \bar{\otimes} M$ is a unital, normal *-monomorphism, where normality means that $x_i \nearrow x$ in A ($\{x_i\}$ is an increasing net in A_{sa} with supremum $x \in A_{sa}$) implies $\pi_A(x_i) \nearrow \pi_A(x)$ in $A \bar{\otimes} M$.

Remarks. (i) By 1.3 (i), $A \bar{\otimes} M$ is a monotone complete C^*-algebra containing $\pi_A(A)$ as a monotone closed C^*-subalgebra.

(ii) By 1.2 (iv) the map $x \mapsto f \cdot x = (\mathrm{id}_A \bar{\otimes} f) \circ \pi_A(x)$ on A for $f \in M_*^+$ is a normal completely positive map.

(iii) A W^*-dynamical system with the acting group G is precisely a monotone complete C^*-$L^\infty(G)$-comodule whose underlying operator space is a W^*-algebra (see [23]).

Definition 2.5. An M-comodule is called *M-injective* or *injective in $\mathcal{C}_M$* if for any M-comodules $Y \leq Z$ every M-comodule morphism $\varphi : Y \to X$ extends to an M-comodule morphism $\hat{\varphi} : Z \to X$. An *injective envelope* of an M-comodule X *in $\mathcal{C}_M$* is an M-injective M-comodule containing X as an M-subcomodule, which is minimal under the relation $\leq$.

We show later that under a certain condition on M there are enough injectives in $\mathcal{C}_M$ and that each object in $\mathcal{C}_M$ has a unique injective envelope.

For $X \in \mathcal{C}_M$ we write $X^M = \pi_X^{-1}(\pi_X(X) \cap (X \otimes 1))$, where 1 denotes the unit of M, and call it the *fixed point subspace* of X. As $\pi_X(X) \leq X \bar{\otimes} M$, $X \bar{\otimes} 1 \leq X \bar{\otimes} M$ and so $\pi_X(X) \cap (X \otimes 1) \leq X \bar{\otimes} M$, we have $X^M \leq X$. Note also that for an M-comodule morphism $\varphi : X \to Y$ we have $\varphi(X^M) \subset Y^M$. If further X is a monotone complete C^*-M-comodule, then X^M is a monotone closed C^*-subalgebra of X (that is, the suprema of increasing nets in $(X^M)_{sa}$ as calculated in $(X^M)_{sa}$ and in X_{sa} coincide) since π_X is normal, and $1 \in X^M$ since π_X is unital.

Lemma 2.6. (i) *For $x \in X \in \mathcal{C}_M$ the following are equivalent:*

(1) $x \in X^M$*;*

(2) $\pi_X(x) = x \otimes 1$*;*

(3) $f \cdot x = f(1)x$ *for all* $f \in M_*$*.*

(ii) *If $X \in \mathcal{C}_M$ is also a unital C^*-subalgebra of some monotone complete C^*-algebra A and $\pi_X : X \to X \bar{\otimes} M \leq A \bar{\otimes} M$ is a unital *-monomorphism into $A \bar{\otimes} M$, then $f \cdot (axb) = a(f \cdot x)b$ for all $f \in M_*$, a, $b \in X^M$ and $x \in X$. Hence aXb with a, $b \in X^M$ is an M_*-submodule of X and its norm closure is an M-subcomodule of X.*

Proof. (i) Clearly (2) $\Rightarrow$ (1), and (2) $\iff$ (3), since (2) $\iff$ $(\mathrm{id}_X \bar{\otimes} f)(\pi_X(x)) = (\mathrm{id}_X \bar{\otimes} f)(x \otimes 1)$ for all $f \in M_*$, that is, $f \cdot x = f(1)x$ for all $f \in M_*$. Finally (1) $\Rightarrow$ (3) since π_X is an M_*-module homomorphism and so $x \in X^M$, that is, $\pi_X(x) = y \otimes 1$ for some $y \in X$ implies $\pi_X(f \cdot x) = f \cdot \pi_X(x) = f \cdot (y \otimes 1) = f(1)(y \otimes 1) = \pi_X(f(1)x)$ and $f \cdot x = f(1)x$ for all $f \in M_*$.

(ii) With the notation as above we have

$$\begin{aligned} f \cdot (axb) &= (\mathrm{id}_X \bar{\otimes} f) \circ \pi_X(axb) \\ &= (\mathrm{id}_X \bar{\otimes} f)((a \otimes 1)\pi_X(x)(b \otimes 1)) \\ &= a(\mathrm{id}_X \bar{\otimes} f) \circ \pi_X(x)b \quad \text{(by [12], 4.6(ii))} \\ &= a(f \cdot x)b, \end{aligned}$$

and the final assertion follows from 2.2 (ii). Note that this follows also from 1.4, since $\pi_X(aXb) = (a \otimes 1)\pi_X(X)(b \otimes 1) \subset (a \otimes 1)(X \bar{\otimes} M)(b \otimes 1) \subset aXb \bar{\otimes} M$.

□

Now we can state the main theorem of this section.

Theorem 2.7. *Assume the following condition on M:*

$$(*) \quad \begin{cases} \text{The Banach algebra } M_* \text{ has an approximate unit } \{u_i\} \text{ such that} \\ \lim \|u_i\| = 1. \end{cases}$$

(i) *Every X in $\mathcal{C}_M$ has a unique injective envelope in $\mathcal{C}_M$, written $I_M(X)$; that is, if Y is another injective envelope of X, then the identity map on X extends to an M-comodule morphism of $I_M(X)$ onto Y.*

(ii) *The above $I_M(X)$ is of the form pAq, where A is an M-injective, monotone complete C^*-M-comodule and p, q are projections in A^M.*

(iii) *Assume further that X is a unital C^*-algebra satisfying the following condition:*

$$(**)\quad \begin{cases} \textit{There is a monotone complete } C^*\textit{-algebra } B \textit{ such that } X \le B \bar{\otimes} M \\ \textit{and } X \textit{ is a } C^*\textit{-subalgebra, containing the unit, of } B \bar{\otimes} M. \end{cases}$$

Then $I_M(X)$ is itself a monotone complete C^*-M-comodule containing X as a C^*-subalgbra.

If X is a monotone complete C^*-M-comodule, then it is a monotone closed C^*-subalgebra of $I_M(X)$.

Remarks. (i) The condition $(*)$ is satisfied for $M = L^\infty(G)$ with comultiplication α_G, since the Banach algebra M_* is then $L^1(G)$ with the convolution as the product. But $(*)$ need not be true for general M. Indeed the Hopf-von Neumann algebra $R(G)$ generated by the right regular representation of G on $L^2(G)$ satisfies $(*)$ if and only if G is amenable (see [21]).

(ii) If A, p and q are as in (ii), then M-injectivity of pAq is implied by that of A, since the map $x \mapsto pxq$ is an idempotent M-comodule morphism of A onto pAq by 2.6 (ii). Hence injective objects in $\mathcal{C}_M$ are precisely the M-comodules of the form pAq, where A are M-injcetive, monotone complete C^*-M-comodules and p, q are projections in A^M. The choice of A, p and q in the expression pAq of $I_M(X)$ need not be unique, though we may take q to be $1 - p$ (see the proof below). We see in 2.13 when the M-comodule of the form pAq is itself (isomorphic to) a monotone complete C^*-M-comodule.

The proof is preceded by several lemmas. We observe first (2.8 (ii) below) that the condition $(*)$ assures the existence of enough injectives in $\mathcal{C}_M$. (In the remaining arguments we do not need $(*)$.) We denote by $\mathcal{C}$ the category $\mathcal{C}_M$ for $M = \mathbb{C}$; that is, it is the category of operator spaces and complete contractions.

Lemma 2.8. (i) *Let $B(H)\bar{\otimes} M$ and $B(K)\bar{\otimes} M$ be canonical M-comodules. If $\rho : B(K)\bar{\otimes} M \to B(H)$ is a complete contraction, then there is a unique complete contraction $\omega : B(K)\bar{\otimes} M \to B(H)\bar{\otimes} M$ such that for all $f \in M_*$ and $x \in B(K)\bar{\otimes} M$,*

$$(***) \quad (\mathrm{id}_{B(H)} \bar{\otimes} f)(\omega(x)) = \rho(f \cdot x).$$

If further ρ is completely positive (respectively unital), then so is ω.

(ii) *If an operator space V is injective in $\mathcal{C}$, then the canonical M-comodule $V\bar{\otimes} M$ is M-injective. Hence every M-comodule is an M-subcomodule of some M-injective M-comodule.*

Proof. (i) We use the following result of Effros [22], Theorem 2. As in 1.5, for a W^*-algebra N and $y \in N\bar{\otimes} M$ define $l(y) \in B(M_*, N)$ by $l(y)(f) = (\mathrm{id}_N \bar{\otimes} f)(y)$, $f \in M_*$. Then l gives an order-isomorphism between $(N\bar{\otimes} M)_1^+$, the set of positive elements of norm ≤ 1, and the set of all completely positive maps $\tau : M_* \to N$ with $\tau \leq_{cp} l(1 \otimes 1)$, where $\leq_{cp}$ denotes the partial order induced from complete positivity.

We assume first that ρ is completely positive, and take an $x \in (B(K)\bar{\otimes} M)^+$. Since $f \cdot x = (\mathrm{id}_{B(K)\bar{\otimes} M} \bar{\otimes} f) \circ (\mathrm{id}_{B(K)} \bar{\otimes} \Gamma)(x)$, $(\mathrm{id}_{B(K)} \bar{\otimes} \Gamma)(x) \leq \|x\|(1 \otimes 1)$ ($1 \otimes 1$ is the unit of $(B(K)\bar{\otimes} M)\bar{\otimes} M$), and $\rho(1) \leq \|\rho(1)\|1$, Effros' result (applied to $N = B(K)\bar{\otimes} M$ and then to $N = B(H)$) implies that

$$[M_* \ni f \mapsto f \cdot x \in B(K)\bar{\otimes} M] \leq_{cp} [M_* \ni f \mapsto \|x\| f(1)1 \in B(K)\bar{\otimes} M]$$

in $B(M_*, B(K)\bar{\otimes} M)$, and so

$$\begin{aligned}[M_* \ni f \mapsto \rho(f \cdot x) \in B(H)] &\leq_{cp} [M_* \ni f \mapsto \|x\| f(1)\rho(1) \in B(H)] \\ &\leq_{cp} \|x\|\|\rho(1)\| l(1 \otimes 1)\end{aligned}$$

in $B(M_*, B(H))$, which in turn implies that there is a unique $y \in (B(H)\bar{\otimes} M)^+$ such that

$$(\mathrm{id}_{B(H)} \bar{\otimes} f)(y) = l(y)(f) = \rho(f \cdot x)$$

for all $f \in M_*$. Since $B(K)\bar{\otimes} M$ is the linear span of positive elements, this equality defines a unique map $\omega : B(K)\bar{\otimes} M \to B(H)\bar{\otimes} M$ satisfying $(***)$.

The map ω is positive by construction, and its complete positivity follows from the preceding argument applied to $\mathrm{id}_n \otimes \rho : M_n \otimes (B(K) \bar{\otimes} M) \to M_n \otimes B(H)$ instead of ρ, where $M_n \otimes (B(K) \bar{\otimes} M)$ is identified with the canonical M-comodule $(M_n \otimes B(K)) \bar{\otimes} M$ and so the map corresponding to $\mathrm{id}_n \otimes \rho$ is $\mathrm{id}_n \otimes \omega$.

If further ρ is unital, then

$$\begin{aligned}(\mathrm{id}_{B(H)} \bar{\otimes} f)(\omega(1_{B(K)} \otimes 1_M)) &= \rho(f \cdot (1_{B(K)} \otimes 1_M)) \\ &= f(1_M)\rho(1_{B(K)}) = f(1_M)1_{B(H)} \\ &= (\mathrm{id}_{B(H)} \bar{\otimes} f)(1_{B(H)} \otimes 1_M)\end{aligned}$$

for all $f \in M_*$; hence ω is unital.

If ρ is assumed only to be completely contractive, then by [24], 7.3, there are unital completely positive maps ρ_1, $\rho_2 : B(K) \bar{\otimes} M \to B(H)$ such that the map

$$P = \begin{bmatrix} \rho_1 & \rho \\ \rho^* & \rho_2 \end{bmatrix} : M_2 \otimes (B(K) \bar{\otimes} M) \to M_2 \otimes B(H)$$

is unital completely positive. Applying the above argument to P we obtain a unital completely positive map $\Omega : M_2 \otimes (B(K) \bar{\otimes} M) \to M_2 \otimes (B(H) \bar{\otimes} M)$ such that

$$(\mathrm{id}_{M_2 \otimes B(H)} \bar{\otimes} f)(\Omega([x_{ij}])) = P(([f \cdot x_{ij}])$$

for $f \in M_*$ and $[x_{ij}] \in M_2 \otimes (B(K) \bar{\otimes} M)$. That $\Omega\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\Omega\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ follows from similar equalities for P, and this implies (see the proof of [24], 7.3) that Ω is written in the form

$$\Omega([x_{ij}]) = \begin{bmatrix} \omega_1(x_{11}) & \omega(x_{12}) \\ \omega^*(x_{21}) & \omega_2(x_{22}) \end{bmatrix}, \quad [x_{ij}] \in M_2 \otimes (B(K) \bar{\otimes} M)$$

for some ω_1, ω_2, $\omega : B(K) \bar{\otimes} M \to B(H) \bar{\otimes} M$ with ω_1, ω_2 unital. Then ω is the desired complete contraction satisfying $(***)$.

(ii) We may assume that V is of the form $B(H)$. Indeed, if $V \subset B(H)$, then V being injective, there is a completely contractive projection φ of

$B(H)$ onto V, and the map $\varphi\mathbin{\bar\otimes}\mathrm{id}_M : B(H)\mathbin{\bar\otimes} M \to B(H)\mathbin{\bar\otimes} M$ is an idempotent M-comodule morphism onto $V\mathbin{\bar\otimes} M$ by 1.1 (iii). Hence if $B(H)\mathbin{\bar\otimes} M$ is M-injective, then so is $V\mathbin{\bar\otimes} M$. We also note that every M-comodule X is an M-subcomodule of an M-comodule of the form $B(K)\mathbin{\bar\otimes} M$ (the remark (i) after 2.1).

These facts show that all the assertions of (ii) follow from the proof of the following: For any M-comodule $Y \le Z = B(K)\mathbin{\bar\otimes} M$ an M-comodule morphism $\varphi : Y \to B(H)\mathbin{\bar\otimes} M$ extends to an M-comodule morphism $\hat\varphi : Z \to B(H)\mathbin{\bar\otimes} M$.

For the approximate unit $\{u_i\}$ for M_* satisfying $(*)$ the maps $\psi_i := (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} u_i)\circ\varphi : Y \to B(H)$ are completely bounded, and so by the Arveson-Paulsen-Wittstock theorem [24], 7.2, they extend to completely bounded maps $\hat\psi_i : Z \to B(H)$ with $\|\hat\psi_i\|_{cb} = \|\psi_i\|_{cb} \le \|u_i\|\|\varphi\|_{cb} \le \|u_i\|$. Since the unit ball of $B(Z,\, B(H))$ is compact in the point-σ-weak topology, we may assume by passing to a subnet that there is a completely bounded map $\psi_0 : Z \to B(H)$ such that $\psi_i(x) \to \psi_0(x)$ σ-weakly for all $x \in Z$ and $\|\psi_0\| \le \liminf\|\hat\psi_\mathrm{i}\|_\mathrm{cb} \le \lim\|u_i\| = 1$. By (i) there is a complete contraction $\hat\varphi : Z \to B(H)\mathbin{\bar\otimes} M$ such that $(\mathrm{id}_{B(H)}\mathbin{\bar\otimes} f)(\hat\varphi(x)) = \psi_0(f\cdot x)$ for all $f \in M_*$ and $x \in Z$. This $\hat\varphi$ is an M_*-module homomorphism and so an M-comodule morphism by 2.2 (i). Indeed, for $x \in Z$ and $f,\, g \in M_*$ we have

$$\begin{aligned}(\mathrm{id}_{B(H)}\mathbin{\bar\otimes} f)(\hat\varphi(g\cdot x)) &= \psi_0(f\cdot(g\cdot x)) = \psi_0((f\cdot g)\cdot x)\\ &= (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} f\cdot g)(\hat\psi(x))\\ &= (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} f)(g\cdot\hat\psi(x)),\end{aligned}$$

where the last equality holds, since $f\mathbin{\bar\otimes} g = f\circ(\mathrm{id}_M\mathbin{\bar\otimes} g)$ on $M\mathbin{\bar\otimes} M$ implies

$$\mathrm{id}_{B(H)}\mathbin{\bar\otimes} f\cdot g = (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} f)\circ(\mathrm{id}_{B(H)\bar\otimes M}\mathbin{\bar\otimes} g)\circ(\mathrm{id}_{B(H)}\mathbin{\bar\otimes}\Gamma),$$

and further $(\mathrm{id}_{B(H)\bar\otimes M}\mathbin{\bar\otimes} g)\circ(\mathrm{id}_{B(H)}\mathbin{\bar\otimes}\Gamma)(\hat\varphi(x)) = g\cdot\hat\varphi(x)$.

Finally we have $\hat\varphi|Y = \varphi$ since for $f \in M_*$ and $x \in Y$,

$$\begin{aligned}\hat\psi_i(f\cdot x) &= \psi_i(f\cdot x) = (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} u_i)\circ\varphi(f\cdot x)\\ &= (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} u_i)(f\cdot\varphi(x)) = (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} u_i\cdot f)(\varphi(x))\\ &\to (\mathrm{id}_{B(H)}\mathbin{\bar\otimes} f)(\varphi(x)) \quad \text{in norm,}\end{aligned}$$

and

$$\hat{\psi}_i(f \cdot x) \to \psi_0(f \cdot x) = (\mathrm{id}_{B(H)} \bar{\otimes} f)(\hat{\varphi}(x)) \quad \sigma\text{-weakly.}$$

Thus the proof is complete. □

The following is an M-comodule version of Paulsen's result [24], 7.3.

Lemma 2.9. *Let B be both a unital C^*-algebra and an M-comodule with $1 \in B^M$ and let $\varphi : B \to B(H) \bar{\otimes} M$ be an M-comodule morphism. Then there are completely positive M-comodule morphisms $\varphi_i : B \to B(H) \bar{\otimes} M$, $\varphi_i(1) = 1 \otimes 1$, $i = 1, 2$, and $\Phi : M_2 \otimes B \to M_2 \otimes (B(H) \bar{\otimes} M)$ such that*

$$\Phi([x_{ij}]) = \begin{bmatrix} \varphi_1(x_{11}) & \varphi(x_{12}) \\ \varphi^*(x_{21}) & \varphi_2(x_{22}) \end{bmatrix}, \quad [x_{ij}] \in M_2 \otimes B.$$

Proof. We modify the argument in the proof of [24], 7.3 as follows. The operator system

$$S := \left\{ \begin{bmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{bmatrix} : \ \lambda, \mu \in \mathbb{C}, \ \ a, b \in B \right\} \subset M_2 \otimes B$$

is an M-subcomodule of $M_2 \otimes B$ since $1 \in B^M$, and the map

$$\Phi : S \to M_2 \otimes (B(H) \bar{\otimes} M), \quad \Phi\left(\begin{bmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{bmatrix} \right) = \begin{bmatrix} \lambda 1 & \varphi(a) \\ \varphi(b)^* & \mu 1 \end{bmatrix},$$

being a unital M_*-module homomorphism, is a completely positive M-comodule morphism, [24], 7.1. Since $M_2 \otimes (B(H) \bar{\otimes} M)$ is M-injective by 2.8 (ii), Φ extends to an M-comodule morphism, written again, $\Phi : M_2 \otimes B \to M_2 \otimes (B(H) \bar{\otimes} M)$. Then Φ, being unital, is completely positive and written in the form above. □

To state the next lemma we need some notation and terminology. Let X be any M-comodule and N an M-injective M-comodule with $X \leq N$ (see 2.8). We say that an M-comodule morphism $\varphi : N \to N$ (respectively a seminorm p on N) is an X-*projection* (respectively an X-*seminorm*) on N if $\varphi^2 = \varphi$ and $\varphi | X = \mathrm{id}_X$ (respectively $p = \|\varphi(\cdot)\|$ for some M-comodule morphism $\varphi : N \to N$ with $\varphi | X = \mathrm{id}_X$). Define a partial order $\prec$ (respectively $\leq$) in the set of all X-projections (respectively X-seminorms) on N by putting $\varphi \prec \psi$ (respectively $p \leq q$) if $\varphi \circ \psi = \psi \circ \varphi = \varphi$ (respectively $p(x) \leq q(x)$ for all $x \in N$).

Lemma 2.10. (i) *Any decreasing net of X-seminorms on N has a lower bound. Hence there is a minimal X-seminorm on N by Zorn's lemma, and each X-seminorm majorizes a minimal one.*

(ii) *If p is a minimal X-seminorm on N with $p = \|\varphi(\cdot)\|$, then φ is a minimal X-projection.*

(iii) *Conversely to* (ii)*, if φ is a minimal X-projection on N, then $\|\varphi(\cdot)\|$ is a minimal X-seminorm on N.*

Proof. For (i) and (ii) confer [10], 3.4, 3.5. (As in 2.8 we have $N \leq B(H) \bar{\otimes} M$. In the proof of [10], 3.4, replace V, W, $B(H)$ by X, N and $B(H) \bar{\otimes} M$; the reasoning there works also here, since the point-σ-weak limit of M-comodule morphisms in $B(N, B(H) \bar{\otimes} M)$ is also an M-comodule morphism because of the σ-weak continuity of the module operation $x \mapsto f \cdot x$ in $B(H) \bar{\otimes} M$.)

(iii) If φ is a minimal X-projection on N, then by (i), $\|\varphi(\cdot)\|$ majorizes a minimal X-seminorm $\|\varphi'(\cdot)\|$. As $\|\varphi \circ \varphi'(\cdot)\| \leq \|\varphi'(\cdot)\|$, the minimality of $\|\varphi'(\cdot)\|$ implies that $\|\varphi \circ \varphi'(\cdot)\| = \|\varphi'(\cdot)\|$. Then by (ii), $\varphi \circ \varphi'$ is a minimal X-projection. Clearly $\operatorname{Im} \varphi \circ \varphi' := \varphi \circ \varphi'(N) \subset \operatorname{Im} \varphi = \varphi(N)$, and $\operatorname{Ker} \varphi \circ \varphi' \supset \operatorname{Ker} \varphi$ since $\|\varphi \circ \varphi'(\cdot)\| = \|\varphi'(\cdot)\| \leq \|\varphi(\cdot)\|$. These inclusions mean that $\varphi \circ \varphi' \prec \varphi$ and so $\varphi \circ \varphi' = \varphi$ by the minimality of φ. Hence the X-seminorm $\|\varphi(\cdot)\| = \|\varphi \circ \varphi'(\cdot)\| = \|\varphi'(\cdot)\|$ is minimal. □

Let X be in $\mathcal{C}_M$. As in [10] we say that $Y \in \mathcal{C}_M$ with $X \leq Y$ is an *essential* (respectively *rigid*) *extension of X in $\mathcal{C}_M$* if for each $Z \in \mathcal{C}_M$ any M-comodule morphism $\varphi : Y \to Z$ is a monomorphism whenever $\varphi|X$ is (respectively if for each M-comodule morphism $\varphi : Y \to Y$, $\varphi|X = \mathrm{id}_X$ implies $\varphi = \mathrm{id}_Y$).

Lemma 2.11. *Let $X \leq Y$ and suppose that Y is M-injective. Then the following are equivalent:*

(i) *Y is an injective envelope of X in $\mathcal{C}_M$.*

(ii) *Y is an essential extension of X in $\mathcal{C}_M$.*

(iii) *Y is a rigid extension of X in $\mathcal{C}_M$.*

Proof. The same as the proof of [10], 3.6, 3.7.

□

Lemma 2.12. *If B is a monotone complete C^*-M-comodule and $\Phi : B \to B$ is a unital idempotent M-comodule morphism, then the image $\operatorname{Im}\Phi$ is a monotone complete C^*-M-comodule (with a product possibly different from that of B).*

Proof. Since Φ, being unital, is completely positive, the results of Choi-Effros [4], 3.1 and Tomiyama [31], 7.1 (see also [12], 3.2) show that $A := \operatorname{Im}\Phi$ is a monotone complete C^*-algebra with product $x \circ y := \Phi(xy)$, where xy is the product in B. Moreover $A \leq B$. Similarly $A \bar{\otimes} M = \operatorname{Im}(\Phi \bar{\otimes} \operatorname{id}_M)$ is a monotone complete C^*-algebra with product $x \circ y := (\Phi \bar{\otimes} \operatorname{id}_M)(xy)$.

It remains to show that the action π_A on A is a normal *-homomorphism. We have $\pi_A \circ \Phi = (\Phi \bar{\otimes} \operatorname{id}_M) \circ \pi_B$ since Φ is an M-comodule morphism and $\pi_A = \pi_B|A$. If $x, y \in A$, then

$$\begin{aligned}\pi_A(x \circ y) &= \pi_A(\Phi(xy)) = (\Phi \bar{\otimes} \operatorname{id}_M) \circ \pi_B(xy) \\ &= (\Phi \bar{\otimes} \operatorname{id}_M)(\pi_A(x)\pi_A(y)) = \pi_A(x) \circ \pi_A(y).\end{aligned}$$

Further, if $x_i \nearrow x$ in A, then $x_i \nearrow x'$ in B for some $x' \in B_{sa}$ and $\Phi(x') = x$ ([31], 7.1 or [12], 3.2). As π_B is normal, it follows that $\pi_A(x_i) = \pi_B(x_i) \nearrow \pi_B(x')$ in $B \bar{\otimes} M$ and so $\pi_A(x_i) \nearrow (\Phi \bar{\otimes} \operatorname{id}_M) \circ \pi_B(x') = \pi_A \circ \Phi(x') = \pi_A(x)$ in $A \bar{\otimes} M$, completing the proof.

□

Proof of Theorem 2.7. (i) Let $X \leq N$ be as in 2.10. Then the image $\operatorname{Im}\varphi$ of a minimal X-projection φ on N is an injective envelope of X in $\mathcal{C}_M$, and its uniqueness follows from the equivalence of 2.11, (i) and (ii) (see the proof of [10], 4.1).

(ii) The argument below is inspired from that of [32]. As in the proof of 2.9 consider the M-comodule

$$S := \left\{ \begin{bmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{bmatrix} : \ \lambda, \mu \in \mathbb{C}, \ \ a, b \in X \right\} \leq M_2 \otimes N,$$

where 1 is the unit of N and note that $1 \in N^M$. Lemma 2.10 applied to $S \leq M_2 \otimes N$ implies the existence of a minimal S-projection Φ on $M_2 \otimes N$. Putting $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S$ we have $\Phi(p) = p$ and

$\Phi(q) = q$; hence as in the proof of 2.8 (i), Φ is written in the form $\Phi([x_{ij}]) = [\Phi_{ij}(x_{ij})]$, where $\Phi_{ij} : N \to N$ are M-comodule morphisms. These maps are idempotent since so is Φ, and $\varphi := \Phi_{12}$ is an X-projection on N.

We show that $\operatorname{Im}\varphi \leq N$ is the injective envelope of X. To see this it suffices by 2.11 to show that if $\psi : \operatorname{Im}\varphi \to \operatorname{Im}\varphi$ is an M-comodule morphism with $\psi|X = \mathrm{id}_X$, then $\psi = \mathrm{id}_{\operatorname{Im}\varphi}$. Lemma 2.9 applied to $\psi \circ \varphi : N \to \operatorname{Im}\varphi \leq N$ yields a completely positive M-comodule morphism $\Psi : M_2 \otimes N \to M_2 \otimes N$, $\Psi([x_{ij}]) = [\Psi_{ij}(x_{ij})]$ with Ψ_{11}, $\Psi_{22} : N \to N$ unital completely positive M-comodule morphisms, $\Psi_{12} = \psi \circ \varphi$, and $\Psi_{21} = (\Psi_{12})^*$. Then $\Psi|S = \mathrm{id}_S$ and so $\Phi \circ \Psi \circ \Phi|S = \mathrm{id}_S$. Hence $\|\Phi \circ \Psi \circ \Phi(\cdot)\|$ is an S-seminorm on $M_2 \otimes N$ with $\|\Phi \circ \Psi \circ \Phi(\cdot)\| \leq \|\Phi(\cdot)\|$. Lemma 2.10 (iii), (ii) applied to $S \leq M_2 \otimes N$ and Φ implies that $\|\Phi \circ \Psi \circ \Phi(\cdot)\| = \|\Phi(\cdot)\|$ and $\Phi \circ \Phi \circ \Phi$ is a minimal S-projection on $M_2 \otimes N$. Then $\Phi \circ \Psi \circ \Phi = \Phi$ as in the proof of 3.10 (iii), and for each $x \in N$,

$$\begin{bmatrix} 0 & \psi \circ \varphi(x) \\ 0 & 0 \end{bmatrix} = \Phi \circ \Psi \circ \Phi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \varphi(x) \\ 0 & 0 \end{bmatrix};$$

that is, $\psi = \mathrm{id}_{\operatorname{Im}\varphi}$ as desired.

By 2.12, $A := \operatorname{Im}\Phi$ is a monotone complete C^*-M-comodule with product $x \circ y = \Phi(xy)$. As $\Phi(p) = p$ and $\Phi(q) = q$, we have $p \circ x \circ q = \Phi(pxq) = p\Phi(x)q = pxq$ for $x \in A$ (see [3], 3.1), and further p, $q \in A \cap (M_2 \otimes N)^M = A^M$. Hence $p \circ A \circ q = pAq \leq A$ and

$$pAq = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [\operatorname{Im}\Phi_{ij}] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{Im}\varphi \\ 0 & 0 \end{bmatrix} \cong \operatorname{Im}\varphi = I_M(X).$$

(iii) Let X satisfy $(**)$. We may assume that B in $(**)$ is an injective C^*-algebra, since the monotone complete C^*-algebra B is a monotone closed C^*-subalgebra of its injective envelope $I(B)$ (see [11]) and $B \bar{\otimes} M \leq I(B) \bar{\otimes} M$ is also a monotone closed C^*-subalgebra of $I(B) \bar{\otimes} M$. Then $B \bar{\otimes} M$ is M-injective and there is a minimal X-projection φ on $B \bar{\otimes} M$ so that $\operatorname{Im}\varphi = I_M(X)$. Since X is a C^*-subalgebra of $B \bar{\otimes} M$ and the product in $I_M(X)$ is given by $x \circ y = \varphi(xy)$, X is a C^*-subalgebra of $I_M(X)$.

Finally let X be a monotone complete C^*-M-comodule. Then, as above, $\pi_X : X \to X \bar{\otimes} M \leq I(X) \bar{\otimes} M$ is a normal *-monomorphism, and we may take $I_M(X)$ so that $\pi_X(X) \le I_M(X) \le I(X) \mathbin{\bar{\otimes}} M$. Hence $\pi_X : X \to \pi_X(X) \le I_M(X)$ is a normal *-monomorphism. $\square$

Proposition 2.13. *For a monotone complete C^*-M-comodule A and projections p, q in A^M the M-comodule pAq is isomorphic to a monotone complete C^*-M-comodule if and only if there are projections p_1, q_1 in A^M such that $p_1 \le p$, $q_1 \le q$, $pAq = p_1 A q_1$, and p_1 and q_1 are equivalent as projections in A^M, that is, $p_1 = uu^*$ and $q_1 = u^*u$ for some partial isometry u in A^M.*

Proof. We need the following facts (see Section 4 for the terminology and proof). An operator space of the form pAq, where A is a C^*-algebra and p, q are its projections, is a triple system with triple product $[x, y, z] := xy^*z$. A surjective linear map between triple systems is a complete isometry if and only if it is a triple isomorphism (see 4.1 (i)).

Necessity: If B is a monotone complete C^*-M-comodule and $\varphi : B \to pAq$ is an M-comodule isomorphism, then it follows from the fact $1 \in B^M$ and the foregoing that $u := \varphi(1) \in (pAq)^M \subset A^M$ and $u = \varphi(1 \cdot 1^* \cdot 1) = \varphi(1)\varphi(1)^*\varphi(1) = uu^*u$, that is, u is a partial isometry in A^M. As $u \in pAq$, $p_1 := uu^* \le p$ and $q_1 := u^*u \le q$; hence $p_1 A q_1 \subset pAq$. Further $pAq = \varphi(1 \cdot B \cdot 1) = \varphi(1)\varphi(B^*)^*\varphi(1) = uqApu = p_1 uqApuq_1 \subset p_1 A q_1$ and so $pAq = p_1 A q_1$. Note also that the map $\pi : B \to q_1 A q_1$ given by $\pi(x) = u^*\varphi(x)$ is simultaneously an M-comodule isomorphism (by 2.2 (ii) and 2.6 (ii)) and a *-isomorphism (since it is a unital complete isometry).

Sufficiency: Conversely, if there are p_1, q_1, u in A^M as above, then the map $\varphi : q_1 A q_1 \to p_1 A q_1 = pAq$, $\varphi(x) = ux$, is clearly an M-comodule isomorphism of the monotone complete C^*-M-comodule $q_1 A q_1$ onto pAq. $\square$

Remarks. The above proof shows also the following:

(i) If A is a C^*-algebra and p, q are projections in A, then pAq is completely isometric to a unital C^*-algebra if and only if there are projections p_1, q_1 in A such that $p_1 \le p$, $q_1 \le q$, $pAq = p_1 A q_1$, and p_1 and q_1 are equivalent in A.

(ii) If B and C are M-comodules which are also unital C^*-algebras with $1 \in B^M$ and $1 \in C^M$, then $B \cong C$ (isomorphic as M-comodules) implies the existence of an M-comodule isomorphism between them which is also a *-isomorphism.

3. Category $\mathcal{C}_G$

For the rest of the paper (except for part of Section 5) we consider the case $M = L^\infty(G)$ with comultiplication α_G, where G is a fixed locally compact group with left invariant Haar measure dt and $\alpha_G : L^\infty(G) \to L^\infty(G) \bar{\otimes} L^\infty(G)$ is given by $\alpha_G(s, t) = x(st)$. In this case we write $\mathcal{C}_G$, $I_G(X)$, X^G et cetera for $\mathcal{C}_M$, $I_M(X)$, X^M et cetera, and use the terminology *G-module, G-morphism, monotone complete C^*-G-module, G-injective,* et cetera, instead of M-comodule, M-comodule morphism, monotone complete C^*-M-comodule, M-injective, et cetera. Namely, a G-module is an operator space X together with a complete isometry $\pi_X : X \to X \bar{\otimes} L^\infty(G)$, called the *action of G on X* for short (rather than the action of $L^\infty(G)$ on X), such that $(\pi_X \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ \pi_X = (\mathrm{id}_X \bar{\otimes} \alpha_G) \circ \pi_X$; a G-morphism is a complete contraction $\varphi : X \to Y$ between G-modules X and Y such that $\pi_Y \circ \varphi = (\varphi \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ \pi_X$; a monotone complete C^*-G-module is a monotone complete C^*-algebra A together with the action $\pi_A : A \to A \bar{\otimes} L^\infty(G)$, which is also a unital normal *-monomorphism; et cetera.

A G-module X becomes a Banach $L^1(G)$-module by the module operation $f \cdot x = (\mathrm{id}_X \bar{\otimes} f) \circ \pi_X(x)$, $f \in L^1(G) = L^\infty(G)_*$, $x \in X$, such that $\|f \cdot x\| \le \|f\|\|x\|$, and by 2.2, G-modules and G-morphisms are characterized as (norm closed) $L^1(G)$-submodules of canonical G-modules and completely contractive $L^1(G)$-module homomorphisms, respectively.

Note also that Theorem 2.7 applies in $\mathcal{C}_G$ by the remark (i) after it and that a G-injective G-module is also injective in the category $\mathcal{C}$ (see 2.8), since it is the image of a G-module of the form $B(H) \bar{\otimes} L^\infty(G)$ under a completely contractive projection (remark (i) after 2.1) and this $B(H) \bar{\otimes} L^\infty(G)$ is injective in $\mathcal{C}$ by 1.3 (iii).

The following examples were the motivation for introducing $\mathcal{C}_G$ and its generalization $\mathcal{C}_M$. For more abstract characterizations of the situations below, see 3.6 (ii), (iii), 3.4 (iii) and the remark (ii) after 3.7.

Examples 3.1. (i) A C^*- or W^*-dynamical system (A, G, α) gives rise to a G-module, called a C^*- or W^*-*G-module*, as follows. The map $\pi_\alpha : A \to A \bar{\otimes} L^\infty(G)$ given by $\pi_\alpha(x)(t) = \alpha_t(x)$, $x \in A$, $t \in G$, defines an action of G on A in the above sense, where the function $\pi_\alpha(x)$ of G into A, which is norm continuous in the C^*-case or is σ-weakly continuous in the W^*-case, is naturally viewed as an element of $A \bar{\otimes} L^\infty(G)$.

(ii) The C^*-case above is slightly generalized as follows. Let X be an operator space and denote by $\operatorname{Aut} X$ the group of all complete isometries of X onto itself, called *automorphisms* of X from now on. If $\alpha : G \to \operatorname{Aut} X$ is strongly continuous group homomorphism, that is, $t \mapsto \alpha_t(x)$ is norm continuous for each $x \in X$, then the map π_α as above defines a G-module structure on X, and the resulting $L^1(G)$-module operation is given by $f \cdot x = \int f(t)\alpha_t(x)dt$, $f \in L^1(G)$, $x \in X$.

(iii) If (u, H) is a strongly continuous unitary representation of G on a Hilbert space H and H is regarded as an operator space canonically (identify H with the subspace $B(H)p \subset B(H)$, where p is a rank-1 projection), then we obtain the situation in (ii) and so a G-module H. (That a unitary operator on H is a complete isometry follows from 4.1 (i) or a direct computation.)

In these examples, the G-module structure of an operator space is induced from an action of G as automorphisms. We now examine to what extent this is true for general G-modules.

Definition 3.2. (i) By a (right) *translation* on a canonical G-module $V \bar{\otimes} L^\infty(G)$ we mean an automorphism on it of the form $\rho_t := \mathrm{id}_V \bar{\otimes} \rho(t)$, $t \in G$, where $\rho(t) : L^\infty(G) \to L^\infty(G)$ is defined by $(\rho(t)a)(s) = a(st)$. A G-submodule Y of $V \bar{\otimes} L^\infty(G)$ is called *translation invariant* if $\rho_t(Y) = Y$ for all t. A G-module X is called *translation invariant* if so is $\pi_X(X) \leq X \bar{\otimes} L^\infty(G)$ in $X \bar{\otimes} L^\infty(G)$. (As is easily seen, if X is already a G-submodule of some $V \bar{\otimes} L^\infty(G)$, then the translation invariance of X in this sense amounts to that of X as a G-submodule of $V \bar{\otimes} L^\infty(G)$.) In this case a group homomorphism $\alpha^X : G \to \operatorname{Aut} X$ is defined by $\alpha^X_t = \pi_X^{-1} \circ \rho \circ \pi_X$, $t \in G$. This α^X is called the *pointwise action* of G on X to distinguish it from the action π_X.

(ii) For a G-module X we write X^c for the set $L^1(G) \cdot X$ consisting of elements of the form $f \cdot x$, $f \in L^1(G)$, $x \in X$, and call it the *continuous part* of X. The G-module X itself is called *continuous* if $X^c = X$. The continuous part X^c of X is indeed a (norm closed) G-submodule of X by the Hewitt-Ross theorem [19], page 268, (32.22), since $L^1(G)$ has a bounded approximate unit.

Remarks. (i) A general G-module need not be translation invariant. Indeed, any (norm closed) linear subspace X of the canonical G-module $L^\infty(G)$ ($\cong \mathbb{C} \bar{\otimes} L^\infty(G)$) containing $L^\infty(G)^c = L^1(G) \cdot L^\infty(G)$ is a G-submodule of $L^\infty(G)$. But, since $L^\infty(G)^c = C^{blu}(G)$, the space of all bounded left uniformly continuous complex-valued functions on G (see 3.3), such an X need not be translation invariant.

We see in 5.6 (iii), (vi) that every G-injective G-module is translation invariant.

(ii) In 3.1 (ii), with π_α and $\rho_t = \mathrm{id}_X \bar{\otimes} \rho(t)$ as above we have $\pi_\alpha \circ \alpha_t = \rho_t \circ \pi_\alpha$ and the G-module X is translation invariant.

(iii) For two G-modules $X \le Y$ we have $X^c = X \cap Y^c$. Indeed, if $x \in X \cap Y^c$, then $x = f \cdot y$ for some $f \in L^1(G)$ and $y \in Y$, and for a bounded approximate unit $\{u_i\}$ for $L^1(G)$, $u_i \cdot x = (u_i f) \cdot y \to f \cdot y = x$ in norm. As X^c is norm closed and $u_i \cdot x \in X^c$, this shows that $X \cap Y^c \subset X^c$, and the reverse inclusion is clear.

Now we identify the continuous part of a canonical G-module $V \bar{\otimes} L^\infty(G)$. Let $C^b(G, V)$ (respectively $C^{blu}(G, V)$) denote the Banach space of all bounded continuous (respectively bounded left uniformly continuous) functions of G into V, where the left uniform continuity of a bounded function $x : G \to V$ means that $\sup\{\|x(sr) - x(s)\| : s \in G, r \in U\} \to 0$ as U runs through the neighborhood system of the unit element of G (see, for example, [26], page 11).

Proposition 3.3. (i) *With the notation as above, the spaces $C^{blu}(G, V)$ and $C^b(G, V)$, viewed as subspaces of $V \bar{\otimes} L^\infty(G)$ canonically, are translation invariant G-submodules of $V \bar{\otimes} L^\infty(G)$: $C^{blu}(G, V) \le C^b(G, V) \le V \bar{\otimes} L^\infty(G)$, where the translation and the module operation in $C^b(G, V)$*

are given by

$$\rho_t(x)(s) = x(st), \tag{3.1}$$

$$(f \cdot x)(s) = \int f(t)x(st)dt. \tag{3.2}$$

(ii) *We have*

$$\begin{aligned}(V \mathbin{\bar{\otimes}} L^\infty(G))^c &= C^{blu}(G, V)\\ &= \{x \in V \mathbin{\bar{\otimes}} L^\infty(G) :\ t \mapsto \rho_t(x) \text{ is norm continuous}\},\end{aligned}$$

and the value of $x \in (V \mathbin{\bar{\otimes}} L^\infty(G))^c$ *at* t *as a function of* G *into* V *is given by*

$$x(t) = \lim(\mathrm{id}_V \mathbin{\bar{\otimes}} \lambda(t)u_i)(x) \quad \text{(norm convergent)}, \tag{3.3}$$

where $(\lambda(t)a)(s) = a(t^{-1}s)$ *and* $\{u_i\}$ *denotes henceforth a fixed bounded approximate unit for* $L^1(G)$.

Proof. (i) Let $V \subset B(H)$ and $L^\infty(G) \subset B(L^2(G))$ (let each element of $L^\infty(G)$ act on $L^2(G)$ by multiplication), and identify each $x \in C^b(G, V)$ with the operator $x \in B(H) \mathbin{\bar{\otimes}} L^\infty(G) \subset B(L^2(G, H))$ ($L^2(G, H)$ is the Hilbert space of H-valued, square-summable, measurable functions on G) defined by $(x\xi)(s) = x(s)\xi(s)$. For $f \in L^1(G)$ and $y \in B(H) \mathbin{\bar{\otimes}} L^\infty(G)$ we have

$$f \cdot y = \int f(t)\rho_t(y)\, dt, \tag{3.4}$$

$$\rho_t(y) = \sigma\text{-weak } \operatorname{limit}(\lambda(t)u_i) \cdot y, \tag{3.5}$$

where $\rho_t = \mathrm{id}_{B(H)} \mathbin{\bar{\otimes}} \rho(t)$. Indeed, ρ_t is σ-weakly continuous, and these hold for elementary tensors $y = a \otimes b$, since

$$\begin{aligned}&f \cdot (a \otimes b) = a \otimes f \cdot b,\\ &f \cdot b = (\mathrm{id}_{L^\infty(G)} \mathbin{\bar{\otimes}} f)(\alpha_G(b)) = \int f(t)b(\cdot\, t)\, dt = \int f(t)(\rho(t)b)(\cdot)\, dt,\\ &(\lambda(t)u_i) \cdot b = \rho(t)(u_i \cdot b),\end{aligned}$$

and the σ-weak limit of $u_i \cdot b$ is b. These imply (3.1), (3.2) and

$$(\mathrm{id}_{B(H)} \mathbin{\bar{\otimes}} f)(x) = \int f(s)x(s)\, ds \tag{3.6}$$

for $f \in L^1(G)$ and $x \in C^b(G, V)$. Then (3.6) shows that $(\mathrm{id}_{B(H)} \bar{\otimes} f)(x) \in V$, hence that $C^b(G, V) \subset V \bar{\otimes} L^\infty(G)$. Further (3.1) and (3.2) show that $C^{blu}(G, V)$ and $C^b(G, V)$ are translation invariant G-submodules of $V \bar{\otimes} L^\infty(G)$.

(ii) In view of $(g \bar{\otimes} f) \circ \alpha_G = gf = \int g(s)(\lambda(s)f)(\cdot)\, ds$ for $f, g \in L^1(G)$ and (3.5), it follows that for $y \in B(H) \bar{\otimes} L^\infty(G)$ and $t \in G$,

$$\begin{aligned}(\mathrm{id}_{B(H)} \bar{\otimes} g)(f \cdot y) &= (\mathrm{id}_{B(H)} \bar{\otimes} gf)(y) \\ &= \int g(s)(\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)f)(y)\, ds,\end{aligned} \tag{3.7}$$

$$\rho_t(f \cdot y) = (\lambda(t)f) \cdot y. \tag{3.8}$$

Here the function $s \mapsto (\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)f)(y)$ of G into $B(H)$ is left uniformly continuous, as follows from the estimate

$$\begin{aligned}&\|(\mathrm{id}_{B(H)} \bar{\otimes} \lambda(sr))f)(y) - (\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)f)(y)\| \\ &\leq \|\lambda(sr)f - \lambda(s)f\|\|y\| = \|\lambda(r)f - f\|\|y\|;\end{aligned}$$

and by (3.8), similarly for the function $t \mapsto \rho_t(f \cdot y)$ of G into $B(H) \bar{\otimes} L^\infty(G)$.

If $x = f \cdot y$ for $f \in L^1(G)$ and $y \in V \bar{\otimes} L^\infty(G)$, then the comparison of (3.7) and (3.6) shows that x is identified with the function $s \mapsto (\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)f)(y)$ in $C^{blu}(G, V)$, that is, $x(s) = (\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)f)(y)$. Further (3.3) follows, since

$$\begin{aligned}&(\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)u_i)(f \cdot y) = (\mathrm{id}_{B(H)} \bar{\otimes} (\lambda(s)u_i)f)(y) \\ &= (\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)(u_i f))(y) \to (\mathrm{id}_{B(H)} \bar{\otimes} \lambda(s)f)(y) \quad \text{in norm.}\end{aligned}$$

Conversely, if $x \in C^{blu}(G, V)$, then the left uniform continuity of x and (3.2) imply that $u_i \cdot x \to x$ in norm and so $x \in (V \bar{\otimes} L^\infty(G))^c$, since $(V \bar{\otimes} L^\infty(G))^c$ is norm closed as noted in 3.2. Hence $(V \bar{\otimes} L^\infty(G))^c = C^{blu}(G, V)$.

For each $x \in (V \bar{\otimes} L^\infty(G))^c$ the function $t \mapsto \rho_t(x)$ is norm continuous by (3.8). Conversely, if $x \in V \bar{\otimes} L^\infty(G)$ and $t \mapsto \rho_t(x)$ is norm continuous, then it follows from (3.4) that as above, $u_i \cdot x \to x$ in norm and $x \in (V \bar{\otimes} L^\infty(G))^c$. $\square$

As for general continuous G-modules we obtain the following:

Proposition 3.4. (i) *The continuous part X^c of any G-module X is a translation invariant G-module with the strongly continuous pointwise action α given by $\alpha_t(x) = \lim(\lambda(t)u_i) \cdot x$ (norm convergent) and the module operation $f \cdot x = \int f(t)\alpha_t(x)\,dt$. An $x \in X$ is in X^c if and only if $\pi_X^{-1} \circ \rho_t \circ \pi_X(x) \in X$ for all t and $t \mapsto \pi_X^{-1} \circ \rho_t \circ \pi_X(x)$ is norm continuous.*

(ii) *The continuous part of a monotone complete C^*-G-module is a C^*-G-module.*

(iii) *A G-module is continuous if and only if it arises as in 3.1 (ii) via a strongly continuous group homomorphism $\alpha : G \to \operatorname{Aut} X$. In particular, every C^*-G-module is continuous, and every continuous G-module is a G-submodule of some C^*-G-module.*

(iv) *A (norm closed) linear subspace of a continuous G-module is a G-submodule if and only if it is translation invariant.*

Proof. (i) Applying (3.8) to $X \mathbin{\bar{\otimes}} L^\infty(G)$ we have

$$\rho_t \circ \pi_X(f \cdot x) = \rho_t(f \cdot \pi_X(x)) = (\lambda(t)f) \cdot \pi_X(x) = \pi_X((\lambda(t)f) \cdot x)$$

for $f \in L^1(G)$ and $x \in X$. Hence $\pi_X(X^c)$ and X^c also is translation invariant, and the pointwise action α on X^c is given by

$$\begin{aligned}\alpha_t(f \cdot x) &= \pi_X^{-1} \circ \rho_t \circ \pi_X(f \cdot x) = (\lambda(t)f) \cdot x \\ &= \lim \lambda(t)(u_i f) \cdot x = \lim(\lambda(t)u_i) \cdot (f \cdot x).\end{aligned}$$

The action α is strongly continuous, since so is $t \mapsto \rho_t$ on $(X \mathbin{\bar{\otimes}} L^\infty(G))^c$ by 3.3 (ii), $\alpha_t = \pi_X^{-1} \circ \rho_t \circ \pi_X$, and $\pi_X(X^c) \leq (X \mathbin{\bar{\otimes}} L^\infty(G))^c$. For $f \in L^1(G)$ and $x \in X^c$ it follows from (3.1) that

$$\begin{aligned}f \cdot x = \pi_X^{-1}(f \cdot \pi_X(x)) &= \pi_X^{-1}\left(\int f(t)\rho_t(\pi_X(x))dt\right) \\ &= \int f(t)\pi_X^{-1} \circ \rho_t \circ \pi_X(x)\,dt = \int f(t)\alpha_t(x)\,dt,\end{aligned}$$

since $t \mapsto \rho_t \circ \pi_X(x)$ is norm continuous. As $\pi_X(X^c) = \pi_X(X) \cap (X \mathbin{\bar{\otimes}} L^\infty(G))^c$ (the remark (iii) after 3.2), the second assertion follows from 3.3 (ii).

Parts (iii) (except for the last assertion) and (iv) follow immediately from (i).

(ii) If X is a monotone complete C^*-G-module, then π_X is a unital *-monomorphism and so, by 3.3 (ii), $X^c = \pi_X^{-1}(\pi_X(X) \cap (X \bar{\otimes} L^\infty(G))^c)$ is a C^*-subalgebra of X containing the unit. The pointwise action α on X^c is strongly continuous, and each $\alpha_t = \pi_X^{-1} \circ \rho_t \circ \pi_X$, $t \in G$, is a unital surjective complete isometry, hence a *-automorphism of X^c.

Finally, if X is a continuous G-module, then we have $X \subset B(H)$ for some $B(H)$ and $X \cong \pi_X(X) = \pi_X(X)^c \leq (X \bar{\otimes} L^\infty(G))^c \leq (B(H) \bar{\otimes} L^\infty(G))^c$. This and (ii) imply the last assertion of (iii). □

We say that a complete contraction φ between translation invariant G-modules X_j, $j = 1, 2$, is *equivariant* if $\varphi \circ \alpha_t^1 = \alpha_t^2 \circ \varphi$ for all $t \in G$, where α^j is the pointwise action on X_j. The study of G-morphisms is reduced, to some extent, to that of equivariant ones as follows.

Proposition 3.5. *Let X_j, $j = 1, 2$, be any G-modules and denote by* $\mathrm{Hom}_G(X_1, X_2)$ *the space of all G-morphisms from X_1 into X_2.*

(i) *The restriction to X_1^c defines an injection of* $\mathrm{Hom}_G(X_1, X_2)$ *into* $\mathrm{Hom}_G(X_1^c, X_2^c)$. *If further X_2 is G-injective, then this map is also a bijection.*

(ii) *If X_j are also translation invariant, then every element of* $\mathrm{Hom}_G(X_1, X_2)$ *is equivariant. In particular,* $\mathrm{Hom}_G(X_1^c, X_2^c)$ *for any G-modules X_j is identified with the set of all equivariant complete contractions of X_1^c into X_2^c.*

(iii) *If $X_2 = V \bar{\otimes} L^\infty(G)$ is canonical, then a bijection between* $\mathrm{Hom}_G(X_1^c, X_2^c)$ *and the space of all complete contractions from X_1^c into V, written $CC(X_1^c, V)$, is defined as follows. For $\varphi \in$* $\mathrm{Hom}_G(X_1^c, X_2^c)$ *define $\psi \in CC(X_1^c, V)$ by $\psi(x) = \varphi(x)(e)$, $x \in X_1^c$, where the right hand side is the value of $\varphi(x) \in X_2^c = C^{blu}(G, V)$ at $e \in G$.*

Proof. We need the following fact. For a bounded approximate unit $\{u_i\}$ for $L^1(G)$,

$$u_i \cdot x = 0 \text{ in any } G\text{-module for all } i \text{ implies } x = 0. \tag{3.9}$$

Indeed, (3.5) shows the validity of (3.9) in a canonical G-module, but every G-module is a G-submodule of some canonical one.

(i) Each $\varphi \in \mathrm{Hom}_G(X_1, X_2)$, being an $L^1(G)$-module homomorphism, maps X_1^c into X_2^c. If $\varphi|X_1^c = 0$ for $\varphi \in \mathrm{Hom}_G(X_1, X_2)$, then for all $x \in X$ and i we have $u_i \cdot \varphi(x) = \varphi(u_i \cdot x) = 0$ and so $\varphi(x) = 0$ by (3.9). If X_2 is G-injective, then each element of $\mathrm{Hom}_G(X_1^c, X_2^c)$ extends to an element of $\mathrm{Hom}_G(X_1, X_2)$, which is unique from the foregoing.

(ii) By (3.9), φ is equivariant if and only if $u_i \cdot (\varphi \circ \alpha_t^1)(x) = u_i \cdot (\alpha_t^2 \circ \varphi)(x)$ for all $x \in X_1$, $t \in G$ and i. Further, in a canonical G-module of the form $B(H) \bar{\otimes} L^\infty(G)$,

$$f \cdot \rho_t(x) = \Delta(t)^{-1}(\rho(t^{-1})f)) \cdot x, \tag{3.10}$$

where Δ is the modular function of G, since for $a \otimes b$ in $B(H) \bar{\otimes} L^\infty(G)$,

$$\begin{aligned} f \cdot \rho_t(a \otimes b) &= f \cdot (a \otimes \rho(t)b) = a \otimes f \cdot (\rho(t)b), \\ f \cdot (\rho(t)b) &= \int f(s)(\rho(t)b)(\cdot s)\, ds = \int f(s)b(\cdot st), ds \\ &= \Delta(t)^{-1} \int f(st^{-1})b(\cdot s)\, ds = \Delta(t)^{-1}(\rho(t^{-1})f) \cdot b; \end{aligned}$$

and every translation invariant G-module is embedded in a canonical one as a ρ-invariant G-submodule. Hence, if φ is a G-morphism, then

$$\begin{aligned} u_i \cdot (\varphi \circ \alpha_t^1)(x) &= \varphi(u_i \cdot \alpha_t^1(x)) = \varphi(\Delta(t)^{-1}(\rho(t^{-1}u_i) \cdot x) \\ &= \Delta(t)^{-1}(\rho(t^{-1})u_i) \cdot \varphi(x) = u_i \cdot (\alpha_t^2 \circ \varphi)(x), \end{aligned}$$

and so φ is equivariant.

The second assertion is clear, since in a continuous G-module with pointwise action α an element $f \cdot x = \int f(t)\alpha_t(x)dt$ is approximated in norm by linear combinations of $\alpha_t(x)$, $t \in G$.

(iii) By (ii) each element of $\mathrm{Hom}_G(X_1^c, X_2^c)$ is viewed as an equivariant complete contraction of X_1^c into X_2^c. With α^j the pointwise actions of X_j^c and φ, ψ as above, we have

$$\varphi(x)(t) = \alpha_t^2(\varphi(x))(e) = \varphi(\alpha_t^1(x))(e) = \psi(\alpha_t^1(x))$$

for all $x \in X_1^c$ and $t \in G$. This shows that the map $\varphi \mapsto \psi$ is injective. Further, for $\psi \in CC(X_1^c, V)$ and $x \in X_1^c$ the function $\varphi(x) : t \mapsto \psi(\alpha_t^1(x))$ belongs to $C^{blu}(G, V) = X_2^c$ and $\varphi : X_1^c \to X_2^c$ is equivariant. $\square$

Remark. The converse to the former assertion of (ii) is false. Indeed, let $X_1 = L^\infty(G)$ be the canonical G-module (that is, the action is given by $(f \cdot x)(s) = \int f(t)x(st)dt$) and let $X_2 = \mathbb{C}$ with the trivial action $f \cdot x = (\int f(t)dt)x$. For a unital (complete) contraction $\varphi : X_1 \to X_2$, φ is a G-morphism (respectively equivariant) if and only if it is a topologically right invariant (respectively right invariant) mean on $L^\infty(G)$. But, in general, there is a right invariant mean on $L^\infty(G)$ which is not topologically right invariant (see [26], page 265, 22.3).

We characterize G-modules as abstract $L^1(G)$-modules.

Proposition 3.6. *Let X be an operator space with a left $L^1(G)$-module structure.*

(i) *Then X is a G-module if and only if*

$$\|x\| = \sup\{\|f \cdot x\| : f \in L^1(G), \|f\| \le 1\} \tag{1}$$

for all $x \in M_n \otimes X$, $n = 1, 2, \cdots$, where $f \cdot x = [f \cdot x_{ij}]$ for $x = [x_{ij}]$.

(ii) *If X is also a unital C^*-algebra, then X is a C^*-G-module if and only if it satisfies (1) and the following*

$$f \cdot 1 = \left(\int f(t)\, dt\right) 1, \quad \forall f \in L^1(G), \tag{2}$$

$$X = L^1(G) \cdot X. \tag{3}$$

Moreover (1), (2) and (3) can be replaced by (2), (3) and the following:

$$\|f \cdot x\| \le \|f\|\|x\|, \quad \forall f \in L^1(G),\ \forall x \in M_n \otimes X,\ n = 1, 2, \cdots . \tag{1'}$$

(iii) *If X is also a monotone complete C^*-algebra such that*

$$(*) \begin{cases} L^1(G) \cdot X \text{ is a } C^*\text{-subalgebra of } X \text{ and generates } X \text{ as a monotone} \\ \text{complete } C^*\text{-algebra,} \end{cases}$$

then X is a monotone complete C^*-G-module if and only if it satisfies (1), (2) and the following:

$$\text{the map } x \mapsto f \cdot x \text{ in } X \text{ is positive and normal for each } f \in L^1(G)^+. \tag{4}$$

In particular, a W^*-G-module is characterized as a W^*-algebra with the $L^1(G)$-module structure satisfying (1), (2) and (4).

Proof. (i) If X is a G-module, then so is $M_n \otimes X$ with the action $\mathrm{id}_n \otimes \pi_X : M_n \otimes X \to M_n \otimes X \bar{\otimes} L^\infty(G)$. As $f \cdot x = (\mathrm{id}_{M_n \otimes X} \bar{\otimes} f) \circ (\mathrm{id}_n \otimes \pi_X)(x)$ for $f \in L^1(G)$ and $x \in M_n \otimes X$ and π_X is completely isometric, the validity of (1) follows from 1.5. Conversely, if X satisfies (1), then 1.5 again implies that for each $n = 1, 2, \dots$ there is an isometry $\pi_n : M_n \otimes X \to (M_n \otimes X) \bar{\otimes} L^\infty(G)$ such that $(\mathrm{id}_{M_n \otimes X} \bar{\otimes} f) \circ \pi_n(x) = f \cdot x$ for all $f \in L^1(G)$ and $x \in M_n \otimes X$. Then $\pi_n = \mathrm{id}_n \otimes \pi_1$ since for $x = [x_{ij}]$,

$$\begin{aligned}(\mathrm{id}_{M_n \otimes X} \bar{\otimes} f) \circ \pi_n(x) &= f \cdot x = [f \cdot x_{ij}] \\ &= [(\mathrm{id}_X \bar{\otimes} f) \circ \pi_1(x_{ij})] = (\mathrm{id}_{M_n \otimes X} \bar{\otimes} f) \circ (\mathrm{id}_n \otimes \pi_1)(x);\end{aligned}$$

hence by 2.3 (i), π_1 is an action of G on X.

(ii) The necessity is clear, since (2) follows, for example, from the identity in 3.1 (ii).

Sufficiency: By (1) and (2), X is a continuous G-module and it has the strongly continuous pointwise action α. Then, as in the proof of 3.4 (ii), the condition (2) assures that each α_t, $t \in G$, is a *-automorphism of X.

For the second assertion it suffices to show that (1′) and (3) imply (1). But, if $\{u_i\}$ is an approximate unit for $L^1(G)$ with $\|u_i\| = 1$, then $u_i \cdot x \to x$ in norm for each $x \in X$, since $x = f \cdot y$ for some $f \in L^1(G)$ and $y \in X$ by (3) and $\|u_i \cdot x - x\| \le \|u_i f - f\| \|y\| \to 0$ by (1′). Hence $u_i \cdot x \to x$ in norm for each $x \in M_n \otimes X$ and $\|u_i \cdot x\| \le \|x\|$ by (1′).

(iii) The necessity follows from the preceding reasoning and the remark (ii) after 2.4 (without the condition $(*)$).

Sufficiency: By (i), X is a G-module and by (ii) and $(*)$, its continuos part X^c is a C^*-G-module with $1 \in X^G \le X^c$. Hence the restriction $\pi_X|X^c$ to X^c of the acton π_X on X, being the action of X^c, is a unital *-monomorphism and so $\pi_X(xy) = \pi_X(x)\pi_X(y)$ for all $x \in X^c$ and $y \in X$ by [3], 3.1. By 1.3 (v), (4) shows that π_X is normal, since $f \cdot x = (\mathrm{id}_X \bar{\otimes} f) \circ \pi_X(x)$, and hence by $(*)$ the above equality holds for all $x, y \in X$. Thus π_X is a unital normal *-monomorphism and X is a monotone complete C^*-G-module.

Since a W^*-G-module is a monotone complete C^*-G-module whose underlying operator space is a W^*-algebra, to see the last asseertion it suffices to show that a W^*-G-module satisfies $(*)$ and that for a W^*-algebra X, $(*)$

follows from (1), (2) and (4). The first claim follows from 3.4 (ii) and the fact that with the notation as in 3.1 (i) and 3.3, $u_i \cdot x = \pi_\alpha^{-1}(u_i \cdot \pi_\alpha(x)) \to x$ σ-weakly for each $x \in X$ by (3.5). (Take $B(H)$ in the proof of 3.3 so that X is a W^*-subalgebra of $B(H)$.) Further, if a left $L^1(G)$-module X is a W^*-algebra satisfying (1), (2) and (4), then $\pi_X : X \to X \bar{\otimes} L^\infty(G)$, which exists by (1), is normal and so σ-weakly continuous. As π_X is also isometric, this together with [8], page 42, Theorem 1, (iv) implies that $\pi_X(X)$ is σ-weakly closed in $X \bar{\otimes} L^\infty(G)$. Hence

$$\begin{aligned}\rho_t(\pi_X(x)) &= \sigma\text{-weak}\lim(\lambda(t)u_i) \cdot \pi_X(x) \\ &= \sigma\text{-weak}\lim \pi_X((\lambda(t)u_i) \cdot x) \in \pi_X(X)\end{aligned}$$

for all $t \in G$ and $x \in X$ by (3.5), and X is translation invariant and it has the pointwise action α^X. Note that this shows also that $L^1(G) \cdot X$ is σ-weakly dense in X or it generates X as a monotone complete C^*-algebra. Each $\alpha_t^X = \pi_X^{-1} \circ \rho_t \circ \pi_X$, being a unital complete isometry of X onto itself, is a *-automorphism of X. Since $X^c = \{x \in X : t \mapsto \alpha_t(x) \text{ is norm continuous}\}$ by 3.4 (i), X^c is a C^*-subalgebra of X. $\square$

Remarks. (i) "For all $x \in M_n \otimes X$, $n = 1, 2, \ldots$" in the condition (1) cannot be replaced by "for all $x \in X$". Indeed, consider the case where G is discrete. Then, as is readily seen, (1) is equivalent to the existence of a group homomorphism $\alpha : G \to \operatorname{Aut} X$ with which the module operation is given by $f \cdot x = \sum f(t)\alpha_t(x)$ for $f \in l^1(G)$ and $x \in X$, and the weaker condition is equivalent to the existence of a similar group homomorphism $\alpha : G \to \operatorname{Isom} X$, where $\operatorname{Isom} X$ is the group of all isometries of X onto itself. In general, $\operatorname{Aut} X \subsetneq \operatorname{Isom} X$ and so an example of an $l^1(G)$-module satisfying the weaker condition, but not (1) can be constructed.

(ii) The author does not know whether every monotone complete C^*-G-module satisfies (the latter half of) $(*)$ or not. (The former half holds by 3.4 (ii).) At any rate, in every monotone complete C^*-G-module X, the monotone closure of X^c is an $L^1(G)$-submodule of X (by (4)) and hence a monotone complete C^*-G-module satisfying $(*)$ trivially.

We characterize the pointwise actions on translation invariant G-modules as follows.

Proposition 3.7. *Let X be an operator space with a group homomorphism $\alpha : G \to \operatorname{Aut} X$. Then X is a translation invariant G-module with the pointwise action α if and only if there is a linear subspace F of X^* such that*

(1) *when $M_n \otimes F$ is regarded as a subspace of $(M_n \otimes X)^*$, $\|x\| = \sup\{|\varphi(x)| : \varphi \in M_n \otimes F,\ \|\varphi\| \leq 1\}$ for all $x \in M_n \otimes X$, $n = 1, 2, \ldots$;*

(2) *$\varphi \circ \alpha_t \in F$ for all $\varphi \in F$ and $t \in G$;*

(3) *the function $t \mapsto \varphi(\alpha_t(x))$ of G into $\mathbb{C}$ is continuous for all $\varphi \in F$ and $x \in X$;*

(4) *for all $f \in L^1(G)$ and $x \in X$ there is an $f \cdot x \in X$ for which $\varphi(f \cdot x) = \int f(t)\varphi(\alpha_t(x))dt$ for all $\varphi \in F$.*

In this case the $L^1(G)$-module structure of X is given by (4).

Proof. Necessity: If X is a translation invariant G-module, that is, $X \leq V \bar{\otimes} L^\infty(G)$ for some canonical G-module $V \bar{\otimes} L^\infty(G)$, $\rho_t(X) = X$ and $\alpha_t = \rho_t|X$ for all t, then we may take as the space F the linear span of the restrictions to X of the product functionals $\psi \bar{\otimes} g$, $\psi \in V^*$, $g \in L^1(G) = L^\infty(G)_*$, on $V \bar{\otimes} L^\infty(G)$ (see Section 1). To see this we may assume that $X = V \bar{\otimes} L^\infty(G) = B(H) \bar{\otimes} L^\infty(G)$. Then (1) is clear. For $\varphi = \psi \bar{\otimes} g$ as above, $t \in G$, $f \in L^1(G)$ and $x \in X$,

$$\begin{aligned}\varphi \circ \alpha_t &= (\psi \bar{\otimes} g) \circ (\mathrm{id}_{B(H)} \bar{\otimes} \rho(t)) \\ &= \psi \bar{\otimes} g \circ \rho(t) = \psi \bar{\otimes} \Delta(t)^{-1}(\rho(t^{-1})g) \in F,\end{aligned}$$

as in the proof of (3.10); $(\psi \bar{\otimes} \mathrm{id}_{L^\infty(G)})(x) \in L^\infty(G)$ and 1.2 (ii) imply

$$\begin{aligned}\varphi(\alpha_t(x)) &= g \circ (\psi \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ (\mathrm{id}_{B(H)} \bar{\otimes} \rho(t))(x) \\ &= g \circ \rho(t) \circ (\psi \bar{\otimes} \mathrm{id}_{L^\infty(G)})(x) \\ &= \int g(s)(\psi \bar{\otimes} \mathrm{id}_{L^\infty(G)})(x)(st)ds;\end{aligned}$$

and further,

$$\begin{aligned}\int f(t)\varphi(\alpha_t(x))dt &= \int\int g(s)f(t)(\psi \bar{\otimes} \mathrm{id}_{L^\infty(G)})(x)(st)dsdt \\ &= \int (gf)(u)(\psi \bar{\otimes} \mathrm{id}_{L^\infty(G)})(x)(u)du \\ &= (\psi \bar{\otimes} gf)(x) = (\psi \bar{\otimes} g)(f \cdot x) = \varphi(f \cdot x),\end{aligned}$$

showing respectively (2), (3) and (4).

Sufficency: By (1), $f \cdot x$ in (4) is uniquely determined by the equation in (4), and with this operation X is an $L^1(G)$-module. Indeed, for $\varphi \in F$, $s \in G$, f, $g \in L^1(G)$ and $x \in X$, $\varphi \circ \alpha_s \in F$ by (2), and so by (4),

$$\varphi(\alpha_s(g \cdot x)) = \int g(t)\varphi(\alpha_s(\alpha_t(x))dt = \int g(s^{-1}t)\varphi(\alpha_t(x))dt = \varphi((\lambda(s)g) \cdot x).$$

Hence

$$\begin{aligned} \varphi(f \cdot (g \cdot x)) &= \int f(s)\varphi(\alpha_s(g \cdot x))ds = \int f(s)\varphi((\lambda(s)g) \cdot x)ds \\ &= \int\int f(s)(\lambda(s)g)(t)\varphi(\alpha_t(x))dsdt \\ &= \int (fg)(t)\varphi(\alpha_t(x))dt = \varphi((fg) \cdot x) \end{aligned}$$

and $f \cdot (g \cdot x) = (fg) \cdot x$ by (1). Further, for $x \in M_n \otimes X$, $n = 1, 2, \ldots$, we have by (1), (2) and (4),

$$\begin{aligned} &\sup\{\|f \cdot x\| : f \in L^1(G), \|f\| \le 1\} \\ &= \sup\{|\varphi(f \cdot x)| : f \in L^1(G), \|f\| \le 1, \varphi \in M_n \otimes F, \|\varphi\| \le 1\} \\ &= \sup\{\left|\int f(t)\varphi((\mathrm{id}_n \otimes \alpha_t)(x))dt\right| : f \in L^1(G), \|f\| \le 1, \\ &\qquad\qquad\qquad\qquad \varphi \in M_n \otimes F, \|\varphi\| \le 1\} \\ &= \sup\{|\varphi(x)| : \varphi \in M_n \otimes F, \|\varphi\| \le 1\} = \|x\|. \end{aligned}$$

It follows from 3.6 that X is a G-module. $\square$

Remarks. (i) For any (not necessarily translation invariant) G-module X there is a linear subspace F of X^* satisfying the above condition (1) and the condition $u_i \cdot x \to x$ $\sigma(X, F)$ for all $x \in X$. Indeed, if $X \le B(H) \bar{\otimes} L^\infty(G)$, then the space F defined as above satisfies (1), and for $\psi \bar{\otimes} g \in F$ with $\psi \in B(H)_*$ and $g \in L^1(G)$, $(\psi \bar{\otimes} g)(u_i \cdot x) = (\psi \bar{\otimes} gu_i)(x) \to (\psi \bar{\otimes} g)(x)$.

(ii) If X is a G-module and is reflexive as a Banach space, then $X^c = X$. In particular, G-modules whose underlying operator spaces are Hilbert spaces are continuous, and so such G-modules are exactly the G-modules considered in 3.1 (iii). Indeed, (1) shows that X^c is $\sigma(X, F)$-dense in X.

Further (1) for $n = 1$ implies that the unit ball of F is $\sigma(X^*, X)$-dense in that of X^*, which in turn implies that F is norm dense in X^*, since $X^{**} = X$ and $\sigma(X^*, X) = \sigma(X^*, X^{**})$. Thus X^c, being norm closed, coincides with X.

(iii) If a G-module X has a pointwise action α, then the fixed point subspace X^G equals $X^\alpha := \{x \in X\ \alpha_t(x) = x,\ \forall\, t \in G\}$. Indeed, by (4) we have $f \cdot x = f(1)x$ for all $f \in L^1(G)$, where $f(1) = \int f(t)dt$, if and only if $\alpha_t(x) = x$ for all $t \in G$.

4. Triple envelopes of continuous G-modules

By 3.4 we may and shall think of continuous G-modules and G-morphisms between them as operator spaces X together with strongly continuous group homomorphisms $\alpha^X : G \to \operatorname{Aut} X$ and complete contractions $\varphi : X \to Y$ such that $\varphi \circ \alpha_t^X = \alpha_t^Y \circ \varphi$ for all $t \in G$, respectively. In this section we associate with each continuous G-module another continuous G-module, called its triple envelope, which defines the Morita equivalence in the sense of Combes [5], page 294, Definition 1, of two C^*-G-modules minimally generated by the original G-module (see 4.3).

We establish some terminology and notation. For a while we treat the case without the action of G. As in [32] an operator space is called a *triple system* if it is realized as a (norm closed) linear subspace T of some C^*-algebra A which is closed under the *triple product* $[x, y, z] := xy^*z$, that is, $[x, y, z] \in T$ for all $x, y, z \in T$. (A synonym of "triple system" is "ternary ring of operators", [35].) In this case T is called a *triple subsystem* of A. A linear subspace I of a triple system T is called a *two-sided triple ideal* of T if $TT^*I + IT^*T \subset I$. A linear map φ between triple systems is called a *triple homomorphism* if $\varphi([x, y, z]) = [\varphi(x), \varphi(y), \varphi(z)]$ for all x, y, z. Clearly the kernel $\operatorname{Ker}\varphi$ of of φ is a two-sided triple ideal.

As is well known (see, for example, [2], [34], [6]), triple systems are exactly the triple subsystems of C^*-algebras of the form pAq, where A is a C^*-algebra and p, q are projections in the multiplier algebra $M(A)$ of A, and they are essentially the same as the inner product modules over C^*-algebras. Indeed, the subspace pAq as above is clearly a triple subsystem of A. Moreover, with every triple subsystem T of a C^*-algebra A we can associate the following C^*-algebras:

$$K_l(T) := \overline{\text{lin}}(TT^*) \subset A, \quad K_r(T) := \overline{\text{lin}}(T^*T) \subset A,$$

and

$$L(T) := \begin{bmatrix} K_l(T) & T \\ T^* & K_r(T) \end{bmatrix} \subset M_2 \otimes A,$$

where $\overline{\text{lin}}$ denotes the closed linear span, which are C^*-subalgebras of the right hand sides. These $K_l(T)$, $K_r(T)$ and $L(T)$ depend only on the triple isomorphism class of T, and not on the choice of the C^*-algebras A containing T as a triple subsystem (see 4.1 (iv)). Then the map $\tau : T \to L(T)$ given by $\tau(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ defines an isometric triple isomorphism (bijective triple homomorphism) of T onto the triple subsystem $eL(T)(1-e)$ of $L(T)$, where $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $1 - e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the 1's are the units of $M(K_l(T))$ and $M(K_r(T))$. Hence T has the $K_l(T)$-$K_r(T)$-valued inner products $\langle \cdot, \cdot \rangle_l$ and $\langle \cdot, \cdot \rangle_r$ given by $\langle x, y \rangle_l := xy^*$ and $\langle x, y \rangle_r := x^*y$, and it is an equivalence bimodule which defines the strong Morita equivalence of $K_l(T)$ and $K_r(T)$ in the sense of Rieffel.

We state the next, essentially known result (see [34]) in the form suitable for later use. We deduce it from Harris' result [17], [18] on J^*-algebras. Here a *J^*-algebra* is a (norm closed) linear subspace T of a C^*-algebra such that $xx^*x \in T$ for all $x \in T$, and a linear map φ between J^*-algebras is called a *J^*-homomorphism* if $\varphi(xx^*x) = \varphi(x)\varphi(x)^*\varphi(x)$ for all x.

Proposition 4.1. (i) *If T and U are triple systems and $\varphi : T \to U$ is a surjective linear map, then the following are equivalent:*

(1) *φ is 2-isometry;*

(2) *φ is a triple isomorphism;*

(3) *φ is a complete isometry.*

(ii) *If T is a triple system and I is its closed two-sided triple ideal, then the quotient space T/I is a triple system with the triple product induced from T in the following sense. The C^*-algebra $L(I)$ defined for I instead of T is* *a closed two-sided ideal of* $L(T)$*, and the composite of the triple homomorphism* $\tau : T \to L(T)$ *and the quotient *-homomorphism* $L(T) \to L(T)/L(I)$ *induces an isometric triple isomorphism of* T/I *onto the triple subsystem* $\bar{e}(L(T)/L(I))(1 - \bar{e})$ *of* $L(T)/L(I)$*, where* $\bar{e} = e + L(I) \in L(T)/L(I)$*.*

(iii) *For a triple homomorphism* $\varphi : T \to U$ *between triple systems* T *and* U *the image* $\varphi(T)$ *is a triple subsystem of* U*.*

(iv) *A triple homomorphism* $\varphi : T \to U$ *between triple systems* T *and* U *induces a *-homomorphism* $\tilde{\varphi} : L(T) \to L(U)$*, where* $\tilde{\varphi}([x_{ij}]) = [\varphi_{ij}(x_{ij})]$*,* $\varphi_{12} := \varphi$*,* $\varphi_{21} := \varphi^* : T^* \to U^*$ *is defined by* $\varphi^*(x) = \varphi(x^*)^*$*, and* $\varphi_l := \varphi_{11} : K_l(T) \to K_l(U)$ *and* $\varphi_r := \varphi_{22} : K_r(T) \to K_r(U)$ *are *-homomorphisms obtained as unique extensions of the maps* $\sum x_i y_i^* \mapsto \sum \varphi(x_i)\varphi(y_i)^*$ *and* $\sum x_i^* y_i \mapsto \sum \varphi(x_i)^*\varphi(y_i)$ *on the linear spans* $\mathrm{lin}(TT^*) \subset K_l(T)$ *and* $\mathrm{lin}(T^*T) \subset K_r(T)$*, respectively.*

The proof is based on the following lemma.

Lemma 4.2. (i) *Let* T *be a linear subspace of a* C^**-algebra* A*. If* T *is a triple subsystem of* A*, then* $M_n \otimes T$ $(n = 1, 2, \dots)$ *is also a triple subsystem and hence a* J^**-subalgebra of* $M_n \otimes A$ $(n = 1, 2, \dots)$*. Conversely, if* $M_2 \otimes T$ *is a* J^**-subalgebra of* $M_2 \otimes A$*, then* T *is a triple subsystem of* A*.*

(ii) *Let* $\varphi : T \to U$ *be a linear map between triple systems* T *and* U*. If* φ *is a triple homomorphism, then* $\mathrm{id}_n \otimes \varphi : M_n \otimes T \to M_n \otimes U$ $(n = 1, 2, \dots)$ *is also a triple homomorphism and hence a* J^**-homomorphism. Conversely, if* $\mathrm{id}_2 \otimes \varphi$ *is a* J^**-homomorphism, then* φ *is a triple homomorphism.*

(iii) *If* T *is a triple system, then* $K_l(T)T = T = TK_r(T)$*.*

(iv) *If* T *is a triple system, then*

$$\|x\| = \sup\{\|xy\| : y \in T, \|y\| \leq 1\}$$

for all $x \in K_l(T)$*, and similarly for elements of* $K_l(T)$*.*

Proof. The first parts of (i) and (ii) are obvious. The second part of (i) follows from the next identity:

$$\begin{bmatrix} 0 & x \\ z & y \end{bmatrix} \begin{bmatrix} 0 & x \\ z & y \end{bmatrix}^* \begin{bmatrix} 0 & x \\ z & y \end{bmatrix} = \begin{bmatrix} xy^*z & * \\ * & * \end{bmatrix}.$$

If $\mathrm{id}_2 \otimes \varphi$ in (ii) is a J^*-homomorphism, then application of $\mathrm{id}_2 \otimes \varphi$ to this identity implies

$$\begin{bmatrix} \varphi(xy^*z) & * \\ * & * \end{bmatrix} = (\mathrm{id}_2 \otimes \varphi)\left(\begin{bmatrix} 0 & x \\ z & y \end{bmatrix}\right)(\mathrm{id}_2 \otimes \varphi)\left(\begin{bmatrix} 0 & x \\ z & y \end{bmatrix}\right)^* \times (\mathrm{id}_2 \otimes \varphi)\left(\begin{bmatrix} 0 & x \\ z & y \end{bmatrix}\right) = \begin{bmatrix} \varphi(x)\varphi(y)^*\varphi(z) & * \\ * & * \end{bmatrix},$$

establishing (ii).

(iii) See, for example, [34], the proof of 2.4.

(iv) If T is regarded as an inner product module over $K_r(T)$ with inner product $\langle \cdot, \cdot \rangle_r$, then the space, $B(T)$, of all bounded $K_r(T)$-module endomorphisms with adjoint is a C^*-algebra, and the map sending each $x \in K_l(T)$ to an element $y \mapsto yx$ in $B(T)$ defines a $*$-homomorphism, which is 1 to 1 since $xT = 0$ implies $xK_l(T) = 0$. □

Proof of Proposition 4.1. (i) By [17], page 19 and Theorem 4, a surjective linear map between J^*-algebras is an isometry if and only if it is a J^*-isomorphism.

(1) ⇒ (2): Hence if $\mathrm{id}_2 \otimes \varphi : M_2 \otimes T \to M_2 \otimes U$ is an isometry, then by 4.2 (ii), φ is a triple isomorphism.

(2) ⇒ (3): If φ is a triple isomorphism, then $\mathrm{id}_n \otimes \varphi : M_n \otimes T \to M_n \otimes U$ ($n = 1, 2, \dots$) is also a triple and hence J^*-homomorphism. Hence, again by the foregoing, φ is a complete isometry.

That (3) ⇒ (1) is clear.

(ii) That $L(T)$ is a closed two-sided ideal of $L(T)$ follows from a direct computation. For example,

$$L(T)L(I) = \begin{bmatrix} K_l(T)K_l(I) + TI^* & K_l(T)I + TK_r(I) \\ T^*K_l(T) + K_r(T)I^* & T^*I + K_r(T)K_r(I) \end{bmatrix}$$

and the (1, 1) entry $K_l(T)K_l(I) + TI^*$ is contained in $K_l(I)$. Indeed, $TT^*I \subset I$ implies $TT^*II^* \subset II^*$ and $K_l(T)K_l(I) \subset K_l(I)$. As $\overline{\mathrm{lin}}(II^*I) = I$ by 4.2 (iii) and $TI^*I \subset TT^*I \subset I$, we have

$$TI^* = T\overline{\mathrm{lin}}(I^*II^*) \subset \overline{\mathrm{lin}}(TI^*I)I^* \subset \overline{\mathrm{lin}}(II^*) = K_l(I).$$

Reasoning similarly we see $L(T)L(I) \subset L(I)$ and $L(I)L(T) \subset L(I)$. Now the remaining assertions are clear.

(iii) First note that a J^*-homomorphism is contractive and it is isometric if and only if it is one-to-one, [18], 3.4. Hence the kernel $I = \operatorname{Ker}\varphi$ is a closed two-sided triple ideal of T, and in view of (ii), φ induces a one-to-one, hence an isometric triple homomorphism $\bar{\varphi} : T/I \to U$ with the norm closed image $\bar{\varphi}(T/I) = \varphi(T)$ in U.

(iv) It suffices to show that the maps $\varphi_l|\mathrm{lin}(TT^*)$ and $\varphi_r|\mathrm{lin}(T^*T)$ are contractive (and hence well-defined), since they extend then to the whole of $K_l(T)$ and $K_r(T)$, and further, φ being a triple homomorphism, it follows that the extended maps and $\tilde{\varphi}$ are *-homomorphisms. But, by (iii), φ maps the unit ball of T onto that of $\varphi(T)$. Since φ is contractive, by 4.2 (iv),

$$\begin{aligned}
\|\sum \varphi(x_i)\varphi(y_i)^*\| &= \sup\{\|\sum \varphi(x_i)\varphi(y_i)^*\varphi(z)\| : z \in T,\ \|z\| \le 1\} \\
&= \sup\{\|\varphi(\sum x_i y_i^* z)\| : z \in T,\ \|z\| \le 1\} \\
&\le \sup\{\|\sum x_i y_i^* z\| : z \in T,\ \|z\| \le 1\} \\
&= \|\sum x_i y_i^*\|,
\end{aligned}$$

so $\varphi_l|\mathrm{lin}(TT^*)$ is contractive, and similarly for $\varphi_r|\mathrm{lin}(T^*T)$. □

From now on we deal with continuous G-modules. We call a continuous G-module (respectively a G-morphism) *triple* if it is also a triple system (respectively a triple homomorphism). Let X be a continuous G-module. Then we have $X \le A$ for some C^*-G-module A by 3.4 (iii). (If X is triple, then A can be taken to contain X as a triple subsystem.) As G acts on A as *-automorphisms, the triple subsystem, written $T = T(X;\, A)$, of A generated by X (the smallest triple subsystem containing X) is a continuous triple G-module, and the C^*-algebras $K_l(X;\, A) := K_l(T)$, $K_r(X;\, A) := K_r(T)$ and $L(X;\, A) := L(T)$ defined as before are C^*-G-modules with $K_l(X;\, A)$ and $K_r(X;\, A)$ Morita equivalent in the sense of Combes. Further a continuous G-module version of 4.1 can be formulated naturally.

These continuous triple G-modules $T(X;\, A)$ obtained when X is fixed and A varies are related as follows:

Theorem 4.3. *Let X be a continuous G-module and consider continuous* *triple G-modules U such that*

$$(*) \quad X \le U \text{ and } U \text{ is generated by } X \text{ as a triple system.}$$

Then there is a unique continuous triple G-module, minimal with respect to $(*)$*, which is written* $T(X)$ *and called the triple envelope of* X*. Namely,* $T(X)$ *satisfies* $(*)$*, and for each* U *satisfying* $(*)$ *there are surjective maps* $(\varphi_l, \varphi, \varphi_r) : (K_l(U), U, K_r(U)) \to (K_l(X), T(X), K_r(X))$*, where* $\varphi : U \to T(X)$ *is a surjective triple G-morphism,* $K_j(X) := K_j(T(X))$*, and* $\varphi_j : K_j(U) \to K_j(X)$ $(j = l, r)$ *defined for* φ *as in 4.1 (iv) are both surjective *-homomorphisms and G-morphisms.*

Proof. If $I_G(X) = eBf$, with B, e, f as in 2.7, is the G-injective envelope of X, then the triple subsystem, $T(X)$, of eBf generated by X is a continuous triple G-module, since so is $eB^cf = (eBf)^c$ and $X = X^c \le eB^cf$. For U as above there is a G-injective, monotone complete C^*-G-module N such that $X \le U \le N$ and U is the triple subsystem of N generated by X. Indeed, U is identified with the continuous triple G-module $eL(U)(1-e)$, where $L(U)$ and e are defined as above and e is G-invariant. Then we may take as N the G-injective envelope of $L(U)$.

As in the proof of 2.7 (ii) define $S \le M_2 \otimes N$, $\Phi = [\Phi_{ij}]$, p, q and $A := \operatorname{Im}\Phi$. Then A is a G-injective, monotone complete C^*-G-module, $\operatorname{Im}\Phi_{12} = I_G(X)$, and

$$\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \le \begin{bmatrix} 0 & I_G(X) \\ 0 & 0 \end{bmatrix} \le A = \operatorname{Im}\Phi \le M_2 \otimes N.$$

Further, the C^*-subalgebra, $C^*(A^c)$, of $M_2 \otimes N$ generated by $A^c \le (M_2 \otimes N)^c$ is a C^*-G-module with $C^*(A^c) \le (M_2 \otimes N)^c$, since $(M_2 \otimes N)^c$ is a C^*-G-module and $A^c \le (M_2 \otimes N)^c$ is translation invariant. Since the map $\tau : N \to M_2 \otimes N$ defined by $\tau(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ is a triple G-morphism, $\tau(X) = \tau(X^c) \le A^c \le C^*(A^c)$, and so $\tau(U) \le C^*(A^c)$. The restriction $\Phi|C^*(A^c) : C^*(A^c) \to A^c$ is both a surjective *-homomorphism and a G-morphism, since by [9], 2.3, the restriction of Φ to the C^*-subalgebra of $M_2 \otimes N$ generated by A is a *-homomorphism onto A and Φ maps $(M_2 \otimes N)^c$ into A^c. Hence the map $\varphi : U \to C^*(A^c) \to A^c \le A \to pAq = I_G(X)$

defined by $\varphi(x) = p\,\Phi \circ \tau(x)q$ is a triple G-morphism with $\varphi|X = \mathrm{id}_X$ and so it maps U onto $T(X)$. $\square$

Remark. This together with 2.7 is an extension of [10], 4.1, in which operator systems (self-adjoint operator spaces containing the unit) are considered in place of continuous G-modules, and shows that triple systems play in the category $\mathcal{C}$ the rôle of unital C^*-algebras in the category of operator systems and unital complete contractions.

5. Crossed products and the Takesaki duality

Let G as before be a fixed locally compact group with left invariant Haar measure dt and let us denote by $R(G)$ the Hopf-von Neumann algebra generated by the right regular representation ρ of G on $L^2(G)$ with comultiplication δ_G given by $\delta_G(\rho(t)) = \rho(t) \otimes \rho(t)$, $t \in G$ (see [23]). We call an $R(G)$-comodule and the action of $R(G)$ on it in the sense of 2.1 a *G-comodule* and the *coaction* of G, respectively. In this section, for a G-module X (G-comodule Z) we define a G-comodule $X \rtimes G$ (G-module $Z \ltimes G$), called the *crossed product*, so that the double crossed product $(X \rtimes G) \ltimes G$ makes sense, and we investigate when the Takesaki type duality $(X \rtimes G) \ltimes G \cong X \mathbin{\bar{\otimes}} B(L^2(G))$ (as G-modules) holds. If X is a monotone complete C^*-G-module, then $X \rtimes G$ becomes by construction a monotone complete C^*-algebra and it is a generalization of the crossed product for a W^*-dynamical system and that for the case where G is discrete, [13].

Our construction of the crossed product is based on the monograph [23] of Nakagami-Takesaki, and most notation is adopted from it, but, since we are working with left invariant Haar measure rather than right one in [23], a slight modification (for example, the definitions of V_G, V'_G, W_G in [23], page VII, et cetera) is needed. In accordance with the usage in [23], actions and coactions are denoted by letters α, β, $\cdots$ and δ, ε, $\cdots$, respectively; a G-module X with action α is denoted as a pair (X, α) and the fixed point subspace of X is denoted by X^α; and similarly for G-comodules.

Although we need the following definition in a very special case, we give it in full generality.

Definition 5.1 (Commutativity of actions)**.** On the same operator space X consider two actions α_j of (possibly diffrent) Hopf-von Neumann algebras M_j, $j = 1, 2$; that is, X becomes both an M_1-comodule and an M_2-comodule. We say that α_1 and α_2 *commute* if

$$(\alpha_1 \bar{\otimes} \mathrm{id}_{M_2}) \circ \alpha_2 = (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\alpha_2 \bar{\otimes} \mathrm{id}_{M_1}) \circ \alpha_1,$$

where $(\alpha_1 \bar{\otimes} \mathrm{id}_{M_2}) \circ \alpha_2 : X \to X \bar{\otimes} M_2 \to X \bar{\otimes} M_1 \bar{\otimes} M_2$ and $(\mathrm{id}_X \bar{\otimes} \sigma) \circ (\alpha_2 \bar{\otimes} \mathrm{id}_{M_1}) \circ \alpha_1 : X \to X \bar{\otimes} M_1 \to X \bar{\otimes} M_2 \bar{\otimes} M_1 \to X \bar{\otimes} M_1 \bar{\otimes} M_2$. Here and henceforth σ denotes a *-isomorphism between W^*-tensor products which sends $x \otimes y$ to $y \otimes x$.

Lemma 5.2. *With the notation as above the fixed point subspace X^{α_1} with respect to α_1 is an M_2-subcomodule of (X, α_2), and so $\alpha_2|X^{\alpha_1}$ becomes an action of M_2 on X^{α_1}.*

Proof. We must show that $\alpha_2(X^{\alpha_1}) \subset X^{\alpha_1} \bar{\otimes} M_2$. For $x \in X^{\alpha_1}$ we have $\alpha_1(x) = x \otimes 1$ and it follows that

$$\begin{aligned}(\alpha_1 \bar{\otimes} \mathrm{id}_{M_2}) \circ \alpha_2(x) &= (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\alpha_2 \bar{\otimes} \mathrm{id}_{M_1})(x \otimes 1) \\ &= (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\alpha_2(x) \otimes 1).\end{aligned}$$

Applying the slice map $\mathrm{id}_{X \bar{\otimes} M_1} \bar{\otimes} f$, $f \in (M_2)_*$, to both sides and noting that $(\mathrm{id}_{X \bar{\otimes} M_1} \bar{\otimes} f) \circ (\alpha_1 \bar{\otimes} \mathrm{id}_{M_2}) = \alpha_1 \circ (\mathrm{id}_X \bar{\otimes} f)$ (see 1.2 (ii)), it follows that

$$\alpha_1 \circ (\mathrm{id}_X \bar{\otimes} f) \circ \alpha_2(x) = (\mathrm{id}_{X \bar{\otimes} M_1} \bar{\otimes} f) \circ (\mathrm{id}_X \bar{\otimes} \sigma)(\alpha_2(x) \otimes 1).$$

Denote the right hand side by y. Then for each $g \in (M_1)_*$,

$$\begin{aligned}(\mathrm{id}_X \bar{\otimes} g)(y) &= (\mathrm{id}_X \bar{\otimes} (g \bar{\otimes} f)) \circ (\mathrm{id}_X \bar{\otimes} \sigma)(\alpha_2(x) \otimes 1) \\ &= (\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g))(\alpha_2(x) \otimes 1) \\ &= ((\mathrm{id}_X \bar{\otimes} f) \bar{\otimes} g)(\alpha_2(x) \otimes 1) \\ &= g(1)(\mathrm{id}_X \bar{\otimes} f)(\alpha_2(x)) \\ &= (\mathrm{id}_X \bar{\otimes} g)((\mathrm{id}_X \bar{\otimes} f)(\alpha_2(x)) \otimes 1),\end{aligned}$$

and so $y = (\mathrm{id}_X \bar{\otimes} f)(\alpha_2(x)) \otimes 1$. This shows (see 2.6 (i)) that $(\mathrm{id}_X \bar{\otimes} f) \circ \alpha_2(x) \in X^{\alpha_1}$ for all $f \in (M_2)_*$, that is, $\alpha_2(x) \in X^{\alpha_1} \bar{\otimes} M_2$. $\square$

As in [23], let us introduce unitary operators V_G, V'_G, W_G on $L^2(G \times G)$ by

$$\begin{aligned}
(V_G\xi)(s, t) &= \Delta(t)^{1/2}\xi(st, t),\\
(V'_G\xi)(s, t) &= \xi(t^{-1}s, t),\\
(W_G\xi)(s, t) &= \Delta(s)^{1/2}\xi(s, ts), \quad \xi \in L^2(G \times G), \ \ s, t \in G.
\end{aligned}$$

Then $V_G, V'_G \in B(L^2(G)) \mathbin{\bar{\otimes}} L^\infty(G)$ and $W_G \in B(L^2(G)) \mathbin{\bar{\otimes}} R(G)$, and actions ρ, λ, λ' and a coaction ε are defined on $B(L^2(G))$ as follows:

$$\begin{aligned}
&\rho, \lambda, \lambda' : B(L^2(G)) \to B(L^2(G)) \mathbin{\bar{\otimes}} L^\infty(G),\\
&\rho(x) = (\operatorname{Ad} V_G)(x \otimes 1), \ \lambda(x) = (\operatorname{Ad} V'_G)(x \otimes 1), \ \lambda'(x) = (\operatorname{Ad} {V'_G}^*)(x \otimes 1),\\
&\varepsilon : B(L^2(G)) \to B(L^2(G)) \mathbin{\bar{\otimes}} R(G), \quad \varepsilon(x) = (\operatorname{Ad} W_G^*)(x \otimes 1).
\end{aligned}$$

Note that ρ and ε are extensions of the action α_G on $L^\infty(G)$ to $B(L^2(G))$ and the coaction δ_G on $R(G)$ to $B(L^2(G))$, respectively.

For any operator spaces V, actions $\mathrm{id}_V \mathbin{\bar{\otimes}} \rho, \cdots$ and a coaction $\mathrm{id}_V \mathbin{\bar{\otimes}} \varepsilon$ are defined on the Fubini product $V \mathbin{\bar{\otimes}} B(L^2(G))$. Indeed, by 1.1 (i),

$$(\mathrm{id}_V \mathbin{\bar{\otimes}} \rho)(V \mathbin{\bar{\otimes}} B(L^2(G)) \subset V \mathbin{\bar{\otimes}} B(L^2(G)) \mathbin{\bar{\otimes}} L^\infty(G),$$

and $\mathrm{id}_V \mathbin{\bar{\otimes}} \rho$ is an action on $V \mathbin{\bar{\otimes}} B(L^2(G))$ since so is ρ on $B(L^2(G))$, and similarly for other maps. Here we used the associativity for Fubini products of complete contractions, which follows, for example, from 1.2 (ii).

Lemma 5.3. (i) *For a G-module (X, α) an action $\widetilde{\alpha}$ is defined on $X \mathbin{\bar{\otimes}} B(L^2(G))$ by*

$$\widetilde{\alpha} = (\mathrm{id}_X \mathbin{\bar{\otimes}} \operatorname{Ad} V'_G) \circ (\mathrm{id}_X \mathbin{\bar{\otimes}} \sigma) \circ (\alpha \mathbin{\bar{\otimes}} \mathrm{id}_{B(L^2(G))}).$$

This action $\widetilde{\alpha}$ commutes with the coaction $\mathrm{id}_X \mathbin{\bar{\otimes}} \varepsilon$ on $X \mathbin{\bar{\otimes}} B(L^2(G))$.

(ii) *For a G-comodule (Z, δ) a coaction $\widetilde{\delta}$ is defined on $Z \mathbin{\bar{\otimes}} B(L^2(G))$ by*

$$\widetilde{\delta} = (\mathrm{id}_Z \mathbin{\bar{\otimes}} \operatorname{Ad} W_G) \circ (\mathrm{id}_Z \mathbin{\bar{\otimes}} \sigma) \circ (\delta \mathbin{\bar{\otimes}} \mathrm{id}_{B(L^2(G))}).$$

This coaction $\widetilde{\delta}$ commutes with the action $\mathrm{id}_Z \mathbin{\bar{\otimes}} \lambda$ on $Z \mathbin{\bar{\otimes}} B(L^2(G))$.

Proof. We prove only (i) since the proof of (ii) is similar. By the remark (i) after 2.1 we may assume that $(X, \alpha) \leq (Y, \beta)$, where $Y = B(H) \bar{\otimes} L^\infty(G)$ and $\beta = \mathrm{id}_{B(H)} \bar{\otimes} \alpha_G$, that is, $X \subset Y$, $\beta(X) \subset X \bar{\otimes} L^\infty(G)$ and $\beta|X = \alpha$. If $\widetilde{\beta}$ is defined for β as above, then $\widetilde{\beta}$ is an action on Y by [23] since β is the usual action of G on the W^*-algebra Y. As $\widetilde{\alpha} = \widetilde{\beta}|X \bar{\otimes} B(L^2(G))$, to see that $\tilde{\alpha}$ is an action on $X \bar{\otimes} B(L^2(G))$ it suffices to show that $\widetilde{\beta}(X \bar{\otimes} B(L^2(G)) \subset X \bar{\otimes} B(L^2(G)) \bar{\otimes} L^\infty(G)$. But this is true since by 1.1 (i) and 1.4,

$$(\beta \bar{\otimes} \mathrm{id}_{B(L^2(G))})(X \bar{\otimes} B(L^2(G)) \subset X \bar{\otimes} L^\infty(G) \bar{\otimes} B(L^2(G)),$$
$$(\mathrm{id}_Y \bar{\otimes} \sigma) \circ (\beta \bar{\otimes} \mathrm{id}_{B(L^2(G))})(X \bar{\otimes} B(L^2(G)) \subset X \bar{\otimes} B(L^2(G)) \bar{\otimes} L^\infty(G),$$

and

$$(\mathrm{id}_Y \bar{\otimes} \mathrm{Ad}\, V_G') \circ (\mathrm{id}_Y \bar{\otimes} \sigma) \circ (\beta \bar{\otimes} \mathrm{id}_{B(L^2(G))})(X \bar{\otimes} B(L^2(G))) \\ \subset X \bar{\otimes} B(L^2(G)) \bar{\otimes} L^\infty(G).$$

Since $\widetilde{\alpha} = \widetilde{\beta}|X \bar{\otimes} B(L^2(G))$ and $\mathrm{id}_X \bar{\otimes} \varepsilon = (\mathrm{id}_Y \bar{\otimes} \varepsilon)|X \bar{\otimes} B(L^2(G))$, the commutativity of $\tilde{\alpha}$ and $\mathrm{id}_X \bar{\otimes} \varepsilon$ follows from that of $\widetilde{\beta}$ and $\mathrm{id}_Y \bar{\otimes} \varepsilon$. The latter fact, which seems to be well-known, is verified directly as follows. Since $\widetilde{\beta}$ and $\mathrm{id}_Y \bar{\otimes} \varepsilon$ act as identities on the first factor $B(H)$ of $Y \bar{\otimes} B(L^2(G)) = B(H) \bar{\otimes} L^\infty(G) \bar{\otimes} B(L^2(G))$, it suffices to show the identity in 5.1 for $\widetilde{\alpha_G}$, the action on $L^\infty(G) \bar{\otimes} B(L^2(G))$ defined as above with α replaced by α_G, and $\mathrm{id}_{L^\infty(G)} \bar{\otimes} \varepsilon$, that is,

$$\begin{aligned}&(\widetilde{\alpha_G} \bar{\otimes} \mathrm{id}_{R(G)}) \circ (\mathrm{id}_{L^\infty(G)} \bar{\otimes} \varepsilon) \\ &= (\mathrm{id}_{L^\infty(G) \bar{\otimes} B(L^2(G))} \bar{\otimes} \sigma) \circ (\mathrm{id}_{L^\infty(G)} \bar{\otimes} \varepsilon \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ \widetilde{\alpha_G}\end{aligned} \qquad (*)$$

on $L^\infty(G) \bar{\otimes} B(L^2(G))$. If S denotes the unitary $S : L^2(G) \otimes L^2(G) \to L^2(G) \otimes L^2(G)$, $S(\xi \otimes \eta) = \eta \otimes \xi$, then $\sigma : B(L^2(G)) \bar{\otimes} B(L^2(G)) \to B(L^2(G)) \bar{\otimes} B(L^2(G))$, $\sigma(a \otimes b) = b \otimes a$, is written as $\sigma = \mathrm{Ad}\, S$. Applications of the left and right hand sides of $(*)$ to $a \otimes b \in L^\infty(G) \bar{\otimes} B(L^2(G))$ yield respectively

$$\begin{aligned}&\mathrm{Ad}[(1 \otimes V_G' \otimes 1)(1 \otimes S \otimes 1)(V_G \otimes 1 \otimes 1)(1 \otimes S \otimes 1)(1 \otimes 1 \otimes S) \\ &\quad \times (1 \otimes W_G^* \otimes 1)](a \otimes b \otimes 1 \otimes 1)\end{aligned} \qquad (**)$$

and

$$\begin{aligned}&\mathrm{Ad}[(1\otimes 1\otimes S)(1\otimes W_G^*\otimes 1)(1\otimes 1\otimes S)(1\otimes V_G'\otimes 1)(1\otimes S\otimes 1)\\&\quad\times (V_G\otimes 1\otimes 1)(1\otimes S\otimes 1)](a\otimes b\otimes 1\otimes 1).\end{aligned}\tag{$***$}$$

Indeed,

$$\begin{aligned}\widetilde{\alpha_G}(x\otimes y)&=\mathrm{Ad}(1\otimes V_G')\circ\mathrm{Ad}(1\otimes S)[(\mathrm{Ad}\,V_G)(x\otimes 1)\otimes y]\\&=[\mathrm{Ad}((1\otimes V_G')(1\otimes S)(V_G\otimes 1)](x\otimes 1\otimes y)\\&=[\mathrm{Ad}((1\otimes V_G')(1\otimes S)(V_G\otimes 1)(1\otimes S)](x\otimes y\otimes 1),\\(\mathrm{id}_{L^\infty(G)}\,\bar{\otimes}\,\varepsilon\,\bar{\otimes}\,\mathrm{id}_{L^\infty(G)})(x\otimes y\otimes z)&=x\otimes W_G^*(y\otimes 1)W_G\otimes z\\&=\mathrm{Ad}(1\otimes W_G^*\otimes 1)(x\otimes y\otimes 1\otimes z)\\&=[\mathrm{Ad}(1\otimes W_G^*\otimes 1)(1\otimes 1\otimes S)](x\otimes y\otimes z\otimes 1),\\&\quad x,\ z\in L^\infty(G),\ y\in B(L^2(G)),\end{aligned}$$

imply

$$\begin{aligned}\widetilde{\alpha_G}(x)&=[\mathrm{Ad}((1\otimes V_G')(1\otimes S)(V_G\otimes 1)(1\otimes S)](x\otimes 1),\\&\quad x\in L^\infty(G)\,\bar{\otimes}\,B(L^2(G)),\\(\mathrm{id}_{L^\infty(G)}\,\bar{\otimes}\,\varepsilon\,\bar{\otimes}\,\mathrm{id}_{L^\infty(G)})(x)&=[\mathrm{Ad}(1\otimes W_G^*\otimes 1)(1\otimes 1\otimes S)](x\otimes 1),\\&\quad x\in L^\infty(G)\,\bar{\otimes}\,B(L^2(G))\,\bar{\otimes}\,L^\infty(G),\end{aligned}$$

which in turn imply the expression $(***)$. Similarly, the expression $(**)$ follows, since

$$\begin{aligned}(\mathrm{id}_{L^\infty(G)}\,\bar{\otimes}\,\varepsilon)(x)&=[\mathrm{Ad}(1\otimes W_G^*)](x\otimes 1),\quad x\in L^\infty(G)\,\bar{\otimes}\,B(L^2(G)),\\(\widetilde{\alpha_G}\,\bar{\otimes}\,\mathrm{id}_{R(G)})(x)&=[\mathrm{Ad}((1\otimes V_G'\otimes 1)(1\otimes S\otimes 1)(V_G\otimes 1\otimes 1)\\&\qquad(1\otimes S\otimes 1)(1\otimes 1\otimes S)](x\otimes 1),\\&\qquad x\in L^\infty(G)\,\bar{\otimes}\,B(L^2(G))\,\bar{\otimes}\,R(G).\end{aligned}$$

If X and Y denote the unitaries on $L^2(G\times G\times G\times G)$ in the square brackets in $(**)$ and $(***)$, respectively, then computation shows that $X^*Y=1\otimes 1\otimes V_G^*S$, hence that $(**)$ and $(***)$ coincide, establishing $(*)$. $\square$

Definition 5.4 (Crossed products)**.** (i) The *crossed product* of a G-module (X, α) is the G-comodule $(X \rtimes_\alpha G, \widehat{\alpha})$, where $X \rtimes_\alpha G = (X \bar{\otimes} B(L^2(G)))^{\widetilde{\alpha}}$, $\widehat{\alpha} = (\mathrm{id}_X \bar{\otimes} \varepsilon)|X \rtimes_\alpha G$ and $(X \rtimes_\alpha G, \widehat{\alpha}) \leq (X \bar{\otimes} B(L^2(G)), \mathrm{id}_X \bar{\otimes} \varepsilon)$ by 5.1, 5.3 (i).

(ii) The *crossed product* of a G-comodule (Z, δ) is the G-module $(Z \ltimes_\delta G, \widehat{\delta})$, where $Z \ltimes_\delta G = (Z \bar{\otimes} B(L^2(G)))^{\widetilde{\delta}}$, $\widehat{\delta} = (\mathrm{id}_Z \bar{\otimes} \lambda)|Z \ltimes_\delta G$ and $(Z \ltimes_\delta G, \widehat{\delta}) \leq (Z \bar{\otimes} B(L^2(G)), \mathrm{id}_Z \bar{\otimes} \lambda)$ by 5.1, 5.3 (ii).

Remark. This definition was suggested by [23], page 23, Theorem 1.2, which shows that these crossed products coincide with the usual ones in the W^*-case.

The following is relevant to the formulation of the Takesaki duality.

Definition 5.5 (G-completion)**.** Given two G-modules $X \leq Y$, X is said to be *G-closed in Y* if $y \in Y$ and $f \cdot y \in X$ for all $f \in L^1(G)$ imply $y \in X$. For any G-modules $X \leq Y$ the smallest G-closed G-submodule of Y containing X is called the *G-closure of X in Y*, and written $G\text{-cl}_Y X$. A G-module X is called *G-complete* if for any G-module Y with $X \leq Y$, X is G-closed in Y. The smallest G-complete G-module containing a G-module X is called the *G-completion* of X, and written $\widetilde{X}$; that is, $\widetilde{X}$ is G-complete and $X \leq \widetilde{X}$, and if $X \leq Y$ and Y is G-complete, then there are a G-module Y_1 with $X \leq Y_1 \leq Y$ and a G-isomorphism $\psi : \widetilde{X} \to Y_1$ with $\psi|X = \mathrm{id}_X$. (Hence $\widetilde{X}$ is unique, and its existence is shown below.)

These notions clarify the relation between a G-module and its continuous part as follows.

Proposition 5.6. (i) *For any G-modules $X \leq Y$ we have $G\text{-cl}_Y X = \{y \in Y : f \cdot y \in X, \forall f \in L^1(G)\}$, and this depends only on the continuous part X^c of X in the sense that $G\text{-cl}_Y X$ is the largest G-submodule Z of Y such that $X \leq Z$ and $X^c = Z^c$.*

(ii) *For any G-module X the G-closure of X in its G-injective envelope $I_G(X)$ is the G-completion $\widetilde{X}$ of X. The G-completion $\widetilde{X}$ is the largest G-closure of X in the following sense. For any G-module Y with $X \leq Y$ the identity map on X extends to a G-isomorphism of $G\text{-cl}_Y X$ onto a G-*

submodule of $\widetilde{X}$*. If* Y *is further* G*-complete, then this* G*-isomorphism is onto* $\widetilde{X}$*.*

(iii) *A* G*-injective* G*-module is* G*-complete.*

(iv) *A* G*-submodule* X *of a* G*-complete* G*-module* Y *is* G*-complete if and only if* X *is* G*-closed in* Y*.*

(v) *A canonical* G*-module is* G*-complete.*

(vi) *A* G*-complete* G*-module is translation invariant (see* 3.2 (i)*) and so it has the pointwise action of* G*.*

Proof. (i) Denote by X_1 the right hand side of the equality. Clearly X_1 is a G-submodule of Y and $X_1 \leq G\text{-cl}_Y X$. The G-closedness of X_1 in Y follows from the fact that $L^1(G) \cdot L^1(G)$ is norm dense in $L^1(G)$. Hence $X_1 = G\text{-cl}_Y X$. If $y \in G\text{-cl}_Y X$ and $f \cdot y \in X \cap Y^c = X^c$ (the remark (iii) after 3.2); hence $(G\text{-cl}_Y X)^c \leq X^c$ and $(G\text{-cl}_Y X)^c = X^c$. If $X \leq Z \leq Y$ and $X^c = Z^c$, then for all $z \in Z$ and $f \in L^1(G)$ we have $f \cdot z \in Z^c = X^c \leq X$ and so $Z \leq G\text{-cl}_Y X$.

(ii) Let X_1 be the G-closure of X in $I_G(X)$ and suppose $X \leq Y$. Then $X \leq Y \leq Y_1$ for some G-injective Y_1 (see the remark (i) after 2.1 and note that $B(H) \bar{\otimes} L^\infty(G)$ is G-injective), and there are an idempotent G-morphism $\varphi : Y_1 \to Y_1$ and a G-isomorphism $\psi : I_G(X) \to \varphi(Y_1)$ such that $\varphi|X = \mathrm{id}_X = \psi|X$ (see the proof of 2.7). We have $G\text{-cl}_{Y_1} X \leq \varphi(Y_1)$. Indeed, if $y \in G\text{-cl}_{Y_1} X$, then $f \cdot y \in X$ for all f and $f \cdot y = \varphi(f \cdot y) = f \cdot \varphi(y)$ in Y_1 for all f; hence $y = \varphi(y) \in \varphi(Y_1)$ by (3.9). Thus

$$G\text{-cl}_{\varphi(Y_1)} X = (G\text{-cl}_{Y_1} X) \cap \varphi(Y_1)) = G\text{-cl}_{Y_1} X. \tag{$*$}$$

Further, since ψ is a G-isomorphism and $\psi|X = \mathrm{id}_X$, we have $\psi(X_1) = G\text{-cl}_{\varphi(Y_1)} X$ and so

$$\psi(X_1) = G\text{-cl}_{Y_1} X. \tag{$**$}$$

Suppose first that $X_1 = X$, that is, X is G-closed in $I_G(X)$. Then so is X in $\varphi(Y_1)$, and $X = G\text{-cl}_{Y_1} X$ by ($*$). Hence $X = G\text{-cl}_Y X$, that is, X is G-closed in Y, since $X \leq Y \leq Y_1$ and so $G\text{-cl}_Y X \leq G\text{-cl}_{Y_1} X$. As Y is arbitrary, this means that X is G-complete.

Suppose next that X is arbitrary, but Y is G-complete. Since $I_G(X_1) = I_G(X)$ and X_1 is G-closed in $I_G(X)$, it follows from the foregoing that X_1

is G-complete. As Y is G-complete, $G\text{-cl}_{Y_1}X \le G\text{-cl}_{Y_1}Y = Y$, and by $(**)$, $\psi(X_1) = G\text{-cl}_Y X \le Y$ with $\psi(X_1) \cong X_1$. Therefore X_1 is the G-completion of X.

Finally suppose only that $X \le Y$. Applying the above argument to $X \le Y \le \widetilde{Y}$ (the G-completion of Y) we see that there is a G-isomorphism ψ of X_1 onto $G\text{-cl}_{\widetilde{Y}}X$ with $\psi|X = \mathrm{id}_X$, hence that as $X \le G\text{-cl}_Y X \le G\text{-cl}_{\widetilde{Y}}X$, $G\text{-cl}_Y X$ is isomorphic to the G-submodule $\psi^{-1}(G\text{-cl}_Y X)$ of X_1.

Parts (iii) and (iv) are clear from (ii).

(v) Let $V \bar{\otimes} L^\infty(G)$ be a canonical G-module. If $V \subset B(H)$, then $V \bar{\otimes} L^\infty(G) \le B(H) \bar{\otimes} L^\infty(G)$, and so by (iii) and (iv) it suffices to show that $V \bar{\otimes} L^\infty(G)$ is G-closed in $B(H) \bar{\otimes} L^\infty(G)$. Suppose that $y \in B(H) \bar{\otimes} L^\infty(G)$ and $f \cdot y \in V \bar{\otimes} L^\infty(G)$ for all $f \in L^1(G)$. Hence $(\mathrm{id}_{B(H)} \bar{\otimes} u_i \cdot f)(y) = (\mathrm{id}_{B(H)} \bar{\otimes} u_i)(f \cdot y) \in V$ for a bounded approximate unit $\{u_i\}$ for $L^1(G)$, and so $(\mathrm{id}_{B(H)} \bar{\otimes} f)(y) \in V$ for all $f \in L^1(G)$. This shows by the definition of Fubini products that $y \in V \bar{\otimes} L^\infty(G)$.

(vi) If X is G-complete, then we may assume by the remark (i) after 2.1 that X is a G-closed G-submodule of $B(H) \bar{\otimes} L^\infty(G)$. For $x \in X$ and $t \in G$ we have by (3.10), $f \cdot \rho_t(x) = \Delta(t)^{-1}(\rho(t^{-1})f) \cdot x \in X$ for all $f \in L^1(G)$, and it follows that $\rho_t(x) \in X$. $\square$

Remark. Let M be a Hopf-von Neumann algebra satisfying the condition $(*)$ in 2.7. Then we can extend Definitions 5.5 and 3.2 (ii) to the setting of M-comodules, and prove the assertion corresponding to 5.6.

We formulate the Takesaki duality as follows.

Proposition 5.7 (The Takesaki duality)**.** *For any G-module (X, α) let $(\widetilde{X}, \alpha_1)$ be its G-completion. Then the map $\pi = (\mathrm{id}_{\widetilde{X}} \bar{\otimes} \mathrm{Ad}\, V_G^*) \circ (\alpha_1 \bar{\otimes} \mathrm{id}_{B(L^2(G))}) : \widetilde{X} \bar{\otimes} B(L^2(G)) \to \widetilde{X} \bar{\otimes} B(L^2(G)) \bar{\otimes} B(L^2(G))$ gives a G-isomorphism of $(\widetilde{X} \bar{\otimes} B(L^2(G)), \widetilde{\alpha_1})$ onto the double crossed product $((X \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G, \widehat{\widehat{\alpha}})$, where $\widetilde{\alpha_1}$ is defined for $(\widetilde{X}, \alpha_1)$ as in* 5.3(i)*. Hence the Takesaki duality holds if and only if X is G-complete.*

Remark. If (X, α) is a W^*-G-module (a W^*-dynamical system with the acting group G), then the Takesaki duality holds, and so (X, α) is G-complete. (A direct proof of this fact follows from 5.6 (iv), (v).) By 5.6 (iii) the Takesaki duality holds for a G-injective G-module. But the author does not know whether that is true for every monotone complete C^*-G-module, that is, whether such a G-module is always G-complete or not.

Proof. As in the proof of 5.3 take a canonical G-module (Y, β) so that $Y = B(H) \bar{\otimes} L^\infty(G)$, $\beta = \mathrm{id}_{B(H)} \bar{\otimes} \alpha_G$, $X \leq Y$ and $\alpha = \beta|X$, and define $\widetilde{\beta}$, $\mathrm{id}_Y \bar{\otimes} \varepsilon$ and $Y \rtimes_\beta G = (Y \bar{\otimes} B(L^2(G)))^{\widetilde{\beta}}$. Then

$$\begin{aligned} X \rtimes_\alpha G = (X \bar{\otimes} B(L^2(G)))^{\widetilde{\alpha}} &= X \bar{\otimes} B(L^2(G)) \cap (Y \bar{\otimes} B(L^2(G)))^{\widetilde{\beta}} \\ &= X \bar{\otimes} B(L^2(G)) \cap (Y \rtimes_\beta G), \end{aligned}$$

$\widehat{\beta} = (\mathrm{id}_Y \bar{\otimes} \varepsilon)|Y \rtimes_\beta G$ is a coaction on $Y \rtimes_\beta G$ with $\widehat{\beta}|X \rtimes_\alpha G = \widehat{\alpha}$, and further by 5.6,

$$\widetilde{X} = G\text{-cl}_Y X. \tag{5.1}$$

For G-comodules $(X \rtimes_\alpha G, \widehat{\alpha}) \leq (Y \rtimes_\beta G, \widehat{\beta})$ define the coactions $(\widehat{\alpha})^\sim$, $(\widehat{\beta})^\sim$ as in 5.3 (ii), and construct the double crossed products $(X \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G = [(X \rtimes_\alpha G) \bar{\otimes} B(L^2(G))]^{(\widehat{\alpha})^\sim}$ and $(Y \rtimes_\beta G) \ltimes_{\widehat{\beta}} G = [(Y \rtimes_\beta G) \bar{\otimes} B(L^2(G))]^{(\widehat{\beta})^\sim}$. Then $(\widehat{\alpha})^\sim = (\widehat{\beta})^\sim|(X \rtimes_\alpha G) \bar{\otimes} B(L^2(G))$ and

$$\begin{aligned} (X \rtimes_\alpha G) \bar{\otimes} B(L^2(G)) &= [(X \bar{\otimes} B(L^2(G)) \cap (Y \rtimes_\beta G)] \bar{\otimes} B(L^2(G)) \\ &= X \bar{\otimes} B(L^2(G)) \bar{\otimes} B(L^2(G)) \cap (Y \rtimes_\beta G) \bar{\otimes} B(L^2(G)), \end{aligned}$$

$$\begin{aligned} (X \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G &= [(X \rtimes_\alpha G) \bar{\otimes} B(L^2(G))]^{(\widehat{\alpha})^\sim} \\ &= X \bar{\otimes} B(L^2(G)) \bar{\otimes} B(L^2(G)) \cap [(Y \rtimes_\beta G) \bar{\otimes} B(L^2(G))]^{(\widehat{\beta})^\sim} \\ &= X \bar{\otimes} B(L^2(G)) \bar{\otimes} B(L^2(G)) \cap ((Y \rtimes_\beta G) \ltimes_{\widehat{\beta}} G). \end{aligned} \tag{5.2}$$

Here and henceforth we use the following identity on Fubini products, which is an immediate consequence of the definition:

$$\bigcap_{j,k} (V_j \bar{\otimes} W_k) = (\bigcap_j V_j) \bar{\otimes} (\bigcap_k W_k).$$

Since the Takesaki duality holds for the W^*-G-module (Y, β), the map $\pi_1 = (\mathrm{id}_Y \bar{\otimes} \mathrm{Ad}\, V_G^*) \circ (\beta \bar{\otimes} \mathrm{id}_{B(L^2(G))})$ is a G-isomorphism (an equivariant *-isomorphism) of $(Y \bar{\otimes} B(L^2(G)), \widetilde{\beta})$ onto $((Y \rtimes_\beta G) \ltimes_{\widehat{\beta}} G, \widehat{\widehat{\beta}})$ (see [23], page 8), that is,

$$\pi_1(Y \bar{\otimes} B(L^2(G))) = (Y \rtimes_\beta G) \ltimes_{\widehat{\beta}} G. \tag{5.3}$$

Since $\pi_1|\widetilde{X}\,\bar{\otimes}\,B(L^2(G)) = \pi$ and $\widehat{\widehat{\beta}}|(X \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G = \widehat{\widehat{\alpha}}$, to prove the proposition it suffices to show that π_1 maps $\widetilde{X}\,\bar{\otimes}\,B(L^2(G))$ onto $(X \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G$, that is,

$$\begin{aligned}\pi_1(\widetilde{X}\,\bar{\otimes}\,B(L^2(G))) &= (X \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G \\ &= X\,\bar{\otimes}\,B(L^2(G))\,\bar{\otimes}\,B(L^2(G)) \cap (Y \rtimes_\beta G) \ltimes_{\widehat{\beta}} G \\ &= X\,\bar{\otimes}\,B(L^2(G))\,\bar{\otimes}\,B(L^2(G)) \cap \pi_1(Y\,\bar{\otimes}\,B(L^2(G)))\end{aligned} \tag{5.4}$$

by (5.2), (5.3). Since $(\beta\,\bar{\otimes}\,\mathrm{id}_{B(L^2(G))})(Y\,\bar{\otimes}\,B(L^2(G))) = \beta(Y)\,\bar{\otimes}\,B(L^2(G))$, we have

$$\pi_1(Y\,\bar{\otimes}\,B(L^2(G))) = (1 \otimes 1 \otimes V_G^*)(\beta(Y)\,\bar{\otimes}\,B(L^2(G)))(1 \otimes 1 \otimes V_G)$$

and similarly

$$\pi_1(\widetilde{X}\,\bar{\otimes}\,B(L^2(G))) = (1 \otimes 1 \otimes V_G^*)(\beta(\widetilde{X})\,\bar{\otimes}\,B(L^2(G)))(1 \otimes 1 \otimes V_G).$$

Substituting these into (5.4) and then multiplying it by $1 \otimes 1 \otimes V_G$ on the left and by $1 \otimes 1 \otimes V_G^*$ on the right, we see by 1.4 that (5.4) is equivalent to

$$\begin{aligned}\beta(\widetilde{X})\,\bar{\otimes}\,B(L^2(G)) &= X\,\bar{\otimes}\,V_G[B(L^2(G))\,\bar{\otimes}\,B(L^2(G))]V_G^* \cap \beta(Y)\,\bar{\otimes}\,B(L^2(G)) \\ &= X\,\bar{\otimes}\,B(L^2(G))\,\bar{\otimes}\,B(L^2(G)) \cap \beta(Y)\,\bar{\otimes}\,B(L^2(G)) \\ &= [X\,\bar{\otimes}\,B(L^2(G)) \cap \beta(Y)]\,\bar{\otimes}\,B(L^2(G)),\end{aligned}$$

which in turn is equivalent to

$$\beta(\widetilde{X}) = X\,\bar{\otimes}\,B(L^2(G)) \cap \beta(Y). \tag{5.5}$$

But the truth of (5.5) is seen as follows. For $y \in Y$ we have $\beta(y) \in X\,\bar{\otimes}\,B(L^2(G))$ if and only if $\beta(y) \in X\,\bar{\otimes}\,L^\infty(G)$, since $\beta(y) \in Y\,\bar{\otimes}\,L^\infty(G)$ and $X\,\bar{\otimes}\,B(L^2(G)) \cap Y\,\bar{\otimes}\,L^\infty(G) = X\,\bar{\otimes}\,L^\infty(G)$, or equivalently, by the definition of Fubini products,

$$f \cdot y = (\mathrm{id}_Y\,\bar{\otimes}\,f) \circ \beta(y) \in X, \quad \forall f \in L^1(G),$$

that is, $y \in G\text{-}\mathrm{cl}_Y X = \widetilde{X}$ by (5.1). This completes the proof. $\square$

The following result provides a condition for a crossed product to be injective. It is well known for the W^*-case (see also [1], 4.2, [13], 3.1).

Proposition 5.8. *If a G-module (X, α) is G-injective, then the crossed product $X \rtimes G$ is injective in $\mathcal{C}$. The reverse implication is true if G is discrete, but is not true in the general case.*

Proof. As noted in Section 3, the G-injectivity of X implies the injectivity of X in $\mathcal{C}$ and hence that of $X \bar{\otimes} B(L^2(G))$ (1.3 (iii)). Further $X \rtimes_\alpha G = (X \bar{\otimes} B(L^2(G)))^{\widetilde{\alpha}} \subset X \bar{\otimes} B(L^2(G))$. Therefore $X \rtimes_\alpha G$ is injective if and only if there is a completely contractive projection of $X \bar{\otimes} B(L^2(G))$ onto $X \rtimes_\alpha G$.

If X is G-injective, then the G-isomorphism $\alpha^{-1} : \alpha(X)(\leq X \bar{\otimes} L^\infty(G)) \to X$ extends to a G-morphism $\varphi : X \bar{\otimes} L^\infty(G) \to X$, and so $\varphi \circ \alpha = \mathrm{id}_X$. We show that the complete contraction

$$\begin{aligned}\psi =&(\varphi \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\mathrm{id}_X \bar{\otimes} \lambda') : X \bar{\otimes} B(L^2(G)) \to \\ &X \bar{\otimes} B(L^2(G)) \bar{\otimes} L^\infty(G) \to X \bar{\otimes} L^\infty(G) \bar{\otimes} B(L^2(G)) \to X \bar{\otimes} B(L^2(G))\end{aligned}$$

is a projection onto $X \rtimes_\alpha G$, where λ' is as before. First, we see that if $x \in X \rtimes_\alpha G$, that is, $x \in X \bar{\otimes} B(L^2(G))$ and $\widetilde{\alpha}(x) = x \otimes 1$, then $\psi(x) = x$. Indeed, it follows from

$$(\mathrm{id}_X \bar{\otimes} \mathrm{Ad}\, V'_G) \circ (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\alpha \bar{\otimes} \mathrm{id}_{B(L^2(G))})(x) = \widetilde{\alpha}(x) = x \otimes 1$$

that

$$\begin{aligned}(\alpha \bar{\otimes} \mathrm{id}_{B(L^2(G))})(x) &= (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\mathrm{id}_X \bar{\otimes} \mathrm{Ad}\, {V'_G}^*)(x \otimes 1) \\ &= (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\mathrm{id}_X \bar{\otimes} \lambda')(x).\end{aligned}$$

Hence, by applying $\varphi \bar{\otimes} \mathrm{id}_{B(L^2(G))}$ to both sides and noting $\varphi \circ \alpha = \mathrm{id}_X$, we have $\psi(x) = x$. Next, we show that for all $x \in X \bar{\otimes} B(L^2(G))$,

$$\widetilde{\alpha}(\psi(x)) = \psi(x) \otimes 1,$$

that is, $\psi(x) \in (X \bar{\otimes} B(L^2(G)))^{\widetilde{\alpha}} = X \rtimes_\alpha G$. Applying $(\mathrm{id}_X \bar{\otimes} \sigma) \circ (\mathrm{id}_X \bar{\otimes} \mathrm{Ad}\, {V'_G}^*)$ to both sides, this equality becomes

$$(\alpha \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ \psi(x) = (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\mathrm{id}_X \bar{\otimes} \lambda') \circ \psi(x). \tag{5.6}$$

Since $\varphi : X \bar{\otimes} L^\infty(G) \to X$ is a G-morphism and so $\alpha \circ \varphi = (\varphi \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ (\mathrm{id}_X \bar{\otimes} \alpha_G)$ (see 2.1), (5.6) is rewritten as

$$\begin{aligned}
&(\varphi \bar{\otimes} \mathrm{id}_{L^\infty(G)} \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ (\mathrm{id}_X \bar{\otimes} \alpha_G \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ (\mathrm{id}_X \bar{\otimes} \sigma) \\
&\circ (\mathrm{id}_X \bar{\otimes} \lambda')(x) \\
&= (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\mathrm{id}_X \bar{\otimes} \lambda') \circ (\varphi \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ (\mathrm{id}_X \bar{\otimes} \sigma) \circ (\mathrm{id}_X \bar{\otimes} \lambda')(x).
\end{aligned}$$

Applying the slice map $\mathrm{id}_X \bar{\otimes} (f \bar{\otimes} g) : X \bar{\otimes} L^\infty(G) \bar{\otimes} B(L^2(G)) \to X$, $f \in L^1(G)$, $g \in B(L^2(G))_*$, to both sides and using 1.2 (ii), the left hand side becomes

$$\varphi([\mathrm{id}_X \bar{\otimes} (\mathrm{id}_{L^\infty(G)} \bar{\otimes} (f \bar{\otimes} g) \circ (\alpha_G \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ \sigma \circ \lambda'](x))$$

and the right hand side becomes

$$\varphi([\mathrm{id}_X \bar{\otimes} ((g \bar{\otimes} f) \circ \lambda' \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ \lambda'](x)).$$

But we have

$$\begin{aligned}
&(\mathrm{id}_{L^\infty(G)} \bar{\otimes} (f \bar{\otimes} g)) \circ (\alpha_G \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ \sigma \circ \lambda' \\
&= ((g \bar{\otimes} f) \circ \lambda' \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ \lambda' \qquad \text{on } B(L^2(G)),
\end{aligned} \tag{5.7}$$

and by 1.2 (i), (5.6) follows. To see (5.7) we need only show it on the continuous part $B(L^2(G))^c$ of the G-module $(B(L^2(G)), \lambda')$, since both sides of (5.7) is σ-weakly continuous and $B(L^2(G))^c$ is σ-weakly dense in $B(L^2(G))$. If $y \in B(L^2(G))^c$, then

$$\lambda'(y) \in (B(L^2(G)) \bar{\otimes} L^\infty(G))^c = C^{blu}(G, B(L^2(G)))$$

is identified with the function

$$s \mapsto (\mathrm{Ad}\,\lambda(s^{-1}))y$$

from G to $B(L^2(G))$, where λ is the left regular representation of G on $L^2(G)$, and

$$(\alpha_G \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ \sigma \circ \lambda'(y)$$

is identified with the function

$$(s, t) \mapsto (\mathrm{Ad}\,\lambda(t^{-1}s^{-1}))y$$

from $G \times G$ into $B(L^2(G))$. Hence

$$(\mathrm{id}_{L^\infty(G)} \bar{\otimes} (f \bar{\otimes} g)) \circ (\alpha_G \bar{\otimes} \mathrm{id}_{B(L^2(G))}) \circ \sigma \circ \lambda'(y) \in L^\infty(G)$$

is the function

$$s \mapsto g\left(\int f(t)(\mathrm{Ad}\,\lambda(t^{-1}s^{-1}))y\,dt\right)$$

on G. On the other hand, the functional

$$(g \bar{\otimes} f) \circ \lambda' : B(L^2(G)) \to B(L^2(G)) \bar{\otimes} L^\infty(G) \to \mathbb{C}$$

is given by

$$(g \bar{\otimes} f) \circ \lambda'(z) = g\left(\int f(t)(\mathrm{Ad}\,\lambda(t^{-1}))z\,dt\right), \quad z \in B(L^2(G)),$$

and so

$$((g \bar{\otimes} f) \circ \lambda' \bar{\otimes} \mathrm{id}_{L^\infty(G)}) \circ \lambda'(y) \in L^\infty(G)$$

is the function

$$s \mapsto g\left(\int f(t)(\mathrm{Ad}\,\lambda(t^{-1}))\mathrm{Ad}\,\lambda(s^{-1}))y\,dt\right)$$

on G. Thus (5.7) holds, and consequently ψ is a completely contractive projection onto $X \rtimes_\alpha G$.

Suppose now that G is discrete, and let us show that the injectivity of $X \rtimes_\alpha G$ implies the G-injectivity of X. The proof is essentially the same as that of the only if part of [13], 3.1 (ii) except for the notation, and so we only sketch it. The action $\alpha : X \to X \bar{\otimes} l^\infty(G) = l^\infty(G, X)$ is given by $\alpha(x)(t) = \alpha_t(x)$, $x \in X$, $t \in G$, where $t \mapsto \alpha_t \in \mathrm{Aut}\,X$ is the pointwise action of G on X. The Fubini product $X \bar{\otimes} B(l^2(G))$ is regarded as a space of matrices $x = [x_{t,u}]$ $(t, u \in G)$ over X, and the crossed product $X \rtimes_\alpha G$ is its subspace consisting of $x = [x_{t,u}]$ such that $\alpha_s(x_{t,u}) = x_{st,su}$ for all $s, t, u \in G$. If $X \rtimes_\alpha G$ is injective, then there is a completely contractive projection ψ of $X \bar{\otimes} B(l^2(G))$ onto $X \rtimes_\alpha G$. If we define a complete contraction $\tau : X \bar{\otimes} l^\infty(G) \to X \bar{\otimes} B(l^2(G))$ by $\tau(x) = [\delta_{t,u}x(t)]$, then $\varphi : X \bar{\otimes} l^\infty(G) \to X$, $\varphi(x) = (\psi \circ \tau(x))_{e,e}$ (the (e, e) entry of $\psi \circ \tau(x)$) is shown to be a G-morphism with $\varphi \circ \alpha = \mathrm{id}_X$. Thus X is G-injective.

Finally we give an example of a G-module X for which $X \rtimes G$ is injective, but X is not G-injective. Let $X = \mathbb{C}$ with the trivial action ι of G, that is, $\iota : \mathbb{C} \to L^\infty(G)$, $\iota(x) = x$. Then, since a G-morphism (an $L^1(G)$-module homomorphism) $\varphi : L^\infty(G) \to \mathbb{C}$ with $\varphi \circ \iota = \mathrm{id}_{\mathbb{C}}$ is precisely a topologically right invariant mean on $L^\infty(G)$, X is G-injective if and only if G is amenable. Moreover $X \rtimes_\iota G = \rho(G)''$ is the von Neumann algebra on $L^2(G)$ generated by the right regular representation ρ of G. If we take $SL(2, \mathbb{C})$ as G, then G is of type I, but non-amenable, and hence $X \rtimes_\iota G = \rho(G)''$ is injective, but X is not G-injective. This example was suggested by [1], 4.4. □

6. Flow built under a function

In [25] Phillips extended the notion of a flow built under a function in ergodic theory, which may be regarded as an action of $\mathbb{R}$ on a commutative W^*-algebra, to the case of noncommutative W^*-algebras. In this section we extend that notion further to the case of monotone complete C^*-algebras, and we use it to give an example of a non-W^*, ergodic, monotone complete C^*-$\mathbb{R}$-module. Note that the "flow" meant originally the action of $\mathbb{R}$ both in ergodic theory and in [25], but the action of any locally compact group arises in our case.

For a C^*-algebra X we denote by $*$-$\mathrm{Aut}\, X$ the group of all $*$-autmorphisms of X. (Recall that $\mathrm{Aut}\, X$ was used before to denote the group of all complete isometries of X onto itself.) If further X is a G-module, we denote by $*$-$\mathrm{Aut}_G(X)$ the subgroup of $*$-$\mathrm{Aut}\, X$ consisting of elements which are also G-morphisms.

Let B be a monotone complete C^*-algebra with center $Z(B) = C(\Omega)$ (Ω is the spectrum of $Z(B)$) and consider the monotone complete C^*-G-module $(B \bar{\otimes} L^\infty(G), \mathrm{id}_B \bar{\otimes} \alpha_G)$. Define subgroups $\mathcal{G}_0 \subset \mathcal{G}$ of $*$-$\mathrm{Aut}_G(B \bar{\otimes} L^\infty(G))$ as

$$\mathcal{G} = \{\gamma \in *\text{-}\mathrm{Aut}_G(B \bar{\otimes} L^\infty(G)) :\ \gamma(B \otimes 1) = B \otimes 1\},$$
$$\mathcal{G}_0 = \{\gamma \in *\text{-}\mathrm{Aut}_G(B \bar{\otimes} L^\infty(G)) :\ \gamma | B \otimes 1 = \mathrm{id}_{B \otimes 1}\}.$$

Clearly $\beta \mathbin{\bar{\otimes}} \mathrm{id}_{L^\infty(G)} \in \mathcal{G}$ for all $\beta \in$ *-Aut B and

$$\mathcal{G} = \{\gamma \circ (\beta \mathbin{\bar{\otimes}} \mathrm{id}) : \ \gamma \in \mathcal{G}_0 \ \text{ and } \ \beta \in {}^*\text{-Aut}\, B\}. \tag{6.1}$$

We construct another group $\mathcal{F}$ as follows. In the set of all continuous functions $f : \Omega_f \to G$ with $\Omega_f \subset \Omega$ open dense in Ω depending on f, define an equivalence relation $\sim$ by writing $f \sim g$ if and only if f and g coincide on some open dense subset of Ω. If we denote by $\mathcal{F}$ the set of equivalence classes $[f]$ of all such functions f, then we can make $\mathcal{F}$ into a group by setting

$$[f] \cdot [g] = [f \cdot g], \quad [f]^{-1} = [\tilde{f}],$$

where $(f \cdot g)(\omega) = f(\omega)g(\omega)$ for $\omega \in \Omega_f \cap \Omega_g$ and $\tilde{f}(\omega) = f(\omega)^{-1}$ for $\omega \in \Omega_f$.

Definition 6.1. The fixed point subspace $(B \mathbin{\bar{\otimes}} L^\infty(G))^\gamma$ for some $\gamma \in \mathcal{G}$ is a monotone complete C^*-G-module with $(B \mathbin{\bar{\otimes}} L^\infty(G))^\gamma \leq B \mathbin{\bar{\otimes}} L^\infty(G)$. We call such a G-module a *flow built under a function.*

In view of (6.1) and the flow built under a function construction in ergodic theory (see, for example, [25]), the following result justifies the term flow built under a function and show that the "function" suggesting an element of $\mathcal{F}$ corresponds to an element of $\mathcal{G}_0$.

Proposition 6.2. (i) *With the notation as above, there corresponds to each* $[f] \in \mathcal{F}$ *a unique element* $\xi_{[f]} \in \mathcal{G}_0$ *such that*

$$\xi_{[f]}(c)(\omega, t) = c(\omega, f(\omega)^{-1}t) \tag{6.2}$$

for all $c \in C_0(\Omega_f \times G) \subset Z(B) \mathbin{\bar{\otimes}} L^\infty(G)$,*and this correspondence is a bijectition between* $\mathcal{F}$ *and* $\mathcal{G}_0$.

(ii) *If we denote by* β' *the self-homeomorphism of* Ω *induced by* $\beta|Z(B)$ *for* $\beta \in$ *-Aut B, *that is,* $\beta(a) = a \circ \beta'$ *for* $a \in Z(B) = C(\Omega)$, *then we have*

$$(\beta \mathbin{\bar{\otimes}} \mathrm{id}_{L^\infty(G)}) \circ \xi_{[f]} = \xi_{[f \circ \beta']} \circ (\beta \mathbin{\bar{\otimes}} \mathrm{id}_{L^\infty(G)}) \tag{6.3}$$

for $[f] \in \mathcal{F}$ *and* $\beta \in$ *-Aut B, *and* $\mathcal{G}$ *is isomorphic to the semidirect product of* $\mathcal{F}$ *and* *-Aut B *with the group operations given by*

$$\begin{gathered} ([f_1], \beta_1) \cdot ([f_2], \beta_2) = ([f_1 \cdot (f_2 \cdot \beta_1')], \beta_1\beta_2), \\ ([f], \beta)^{-1} = ([(f \circ (\beta^{-1})')\tilde{}\,], \beta^{-1}) \end{gathered} \tag{6.4}$$

for $[f_1], [f_2], [f] \in \mathcal{F}$ *and* $\beta_1, \beta_2, \beta \in$ *-Aut B.

We begin with two lemmas. The first one is probably well known.

Lemma 6.3. *Let B and C be C^*-algebras with C commutative, and let $\pi : C \to Z(B)$ be a *-homomorphism. Then there is a *-homomorphism $\bar{\pi} : B \otimes C \to B$ such that $\bar{\pi}(b \otimes c) = b\pi(c)$ for all $b \in B$ and $c \in C$, where $B \otimes C$ denotes the minimal C^*-tensor product.*

Proof. Replacing B by its universal enveloping von Neumann algebra, we may assume that B and $Z(B)$ also is a W^*-algebra. Further, since there is a *-homomorphism $\mathrm{id}_B \otimes \pi : B \otimes C \to B \otimes Z(B)$, it suffices to consider the case where $C = Z(B)$ and $\pi : Z(B) \to B$ is the inclusion map, and show that $\| \sum b_j c_j \| \le \| \sum b_j \otimes c_j \|$ for $b_j \in B$ and $c_j \in Z(B)$. Since the linear combinations of projections are norm dense in $Z(B)$, we may assume that the c_j's are all such elements. Then $\sum b_j c_j$ and $\sum b_j \otimes c_j$ are rewritten as $\sum b'_k e_k$ and $\sum b'_k \otimes e_k$, where the e_k's are an orthogonal sequence of nonzero projections in $Z(B)$. Hence

$$\| \sum b'_k e_k \| = \max \|b'_k e_k\| \le \max \|b'_k\| = \| \sum b'_k \otimes e_k \|,$$

as desired. □

Remark. The referee suggested the following, shorter, alternative proof of Lemma 6.3. The *-homomorphism $\bar{\pi}$ is well-defined on the maximal C^*-tensor product of B and C by definition, and it suffices to note that the maximal C^*-tensor product coincides with the minimal C^*-tensor product, since C is commutative.

Lemma 6.4. *For a monotone complete C^*-algebra B the restriction to the continuous part $(B \bar{\otimes} L^\infty(G))^c$ gives a bijection between *-$\mathrm{Aut}_G(B \bar{\otimes} L^\infty(G))$ and *-$\mathrm{Aut}_G((B \bar{\otimes} L^\infty(G))^c)$.*

Proof. In view of 3.5 (i) it suffices to show the surjectivity of the restriction. By the uniqueness of the G-injective envelope, C, of $(B \bar{\otimes} L^\infty(G))^c$ each element γ of *-$\mathrm{Aut}_G((B \bar{\otimes} L^\infty(G))^c)$ extends uniquely to an element $\widehat{\gamma}$ of *-$\mathrm{Aut}_G(C)$. By 5.6 (v), $B \bar{\otimes} L^\infty(G)$ is the G-completion of $(B \bar{\otimes} L^\infty(G))^c$ and so, by 5.6 (ii), it is identified with the G-closure of $(B \bar{\otimes} L^\infty(G))^c$ in C. Hence the restriction of $\widehat{\gamma}$ to $B \bar{\otimes} L^\infty(G)$ gives an element of *-$\mathrm{Aut}_G(B \bar{\otimes} L^\infty(G))$ extending γ. □

Proof of Proposition 6.2. (i) Each function $f : \Omega_f \to G$ as above induces a *-homomorphism $h : C_0(G) \to C^b(\Omega_f) = C(\Omega) = Z(B)$, $h(b) = b \circ \tilde{f}$ with $\tilde{f}$ as above, where by [27], [20], $C^b(\Omega_f)$ is identified with $C(\Omega)$ since $C^b(\Omega_f)$ is the multiplier algebra of $C_0(\Omega_f)$ and $C_0(\Omega_f)$ is an essential ideal of the commutative AW^*-algebra $C(\Omega)$. By 6.3 there is a *-homomorphism $\pi_0 : B \otimes C_0(G) \to B$ such that $\pi_0(a \otimes b) = a(b \circ \tilde{f})$ for $a \in B$ and $b \in C_0(G)$. The surjective *-homomorphism $\pi_0 : B \otimes C_0(G) \to \pi_0(B \otimes C_0(G))$ extends to a unital *-homomorphism between the multiplier algebras $M(B \otimes C_0(G))$ and $M(\pi_0(B \otimes C_0(G)))$. (The unique normal extension of π_0 to the enveloping von Neumann algebras of $B \otimes C_0(G)$ and $\pi_0(B \otimes C_0(G))$ maps $M(B \otimes C_0(G))$ into $M(\pi_0(B \otimes C_0(G)))$.) The subset $\pi_0(B \otimes C_0(G))$ of B has annilator $\{0\}$ in B since so does $h(C_0(G))$ in B. Hence, again by [27], [20], $M(\pi_0(B \otimes C_0(G)))$ is identified with the set of multipliers of $\pi_0(B \otimes C_0(G))$ in B. Moreover we have

$$(B \bar{\otimes} L^\infty(G))^c = C^{blu}(G, B) \subset C^b(G, B) = M(B \otimes C_0(G)).$$

Thus we obtain a *-homomorphism $\pi : (B \bar{\otimes} L^\infty(G))^c \to B$ extending π_0. Then the map $\xi_0 : C^{blu}(G, B) \to C^{blu}(G, B)$ defined by $\xi_0(x)(t) = \pi(\rho_t(x))$, $x \in C^{blu}(G, B)$, $t \in G$, is an equivariant *-homomorphism, where $\rho_t(x)(s) = x(st)$ as in 3.2. For $a \in C_0(\Omega_f)$ and $b \in C_0(G)$ we have

$$\xi_0(a \otimes b)(t) = \pi(\rho_t(a \otimes b)) = \pi_0(a \otimes \rho(t)b) = a(\rho(t)b) \circ \tilde{f},$$

and writing $\xi_0(a \otimes b)(\omega, t)$ for the value at $\omega \in \Omega_f$ of $\xi_0(a \otimes b)(t) \in C_0(\Omega_f)$, we see that ξ_0 satisfies the condition (6.2). This ξ_0 is indeed a *-automorphism, since if $\tilde{\xi_0}$ is defined as above with f replaced by by $\tilde{f}$, then ξ_0 and $\tilde{\xi_0}$ are shown to be the inverses to each other. Therefore, by 3.5 (ii), ξ_0 is in *-$\mathrm{Aut}_G(C^{blu}(G, B))$, and by 6.4, it extends uniquely to an element, ξ_f, of *-$\mathrm{Aut}_G(B \bar{\otimes} L^\infty(G))$. By the construction another function g gives the same *-homomorphism h defined for f (and hence $\xi_f = \xi_g$) if and only if $f \sim g$, and in view of (6.2) it follows that $\xi_f \circ \xi_g = \xi_{f \cdot g}$ for any f and g. Hence we may write ξ_f as $\xi_{[f]}$, and we obtain an injective group homomorphism $[f] \mapsto \xi_{[f]}$ of $\mathcal{F}$ into $\mathcal{G}_0$.

Now we show the surjectivity of the map $[f] \mapsto \xi_{[f]}$. We have $1 \otimes C^{blu}(G) \subset$ the center of $C^{blu}(G, B)$. Each $\xi \in \mathcal{G}_0$ restricted to $C^{blu}(G, B)$

is an equivariant *-automorphism. Hence the unital *-homomorphism $h: x \mapsto \xi(x)(e)$ from $C^{blu}(G, B)$ into B maps $1 \otimes C^{blu}(G)$ into $Z(B)$. Since $C_0(G)$ is an essential ideal of $C^{blu}(G)$, the spectrum, $\tilde{G}$, of $C^{blu}(G)$ contains G as an open subset. The restriction $h|1 \otimes C^{blu}(G) : 1 \otimes C^{blu}(G) = 1 \otimes C(\tilde{G}) \to C(\Omega)$ is induced by a surjective continuous map $g : \Omega \to \tilde{G}$, that is, $\xi(1 \otimes b)(e) = b \circ g$, $b \in C(\tilde{G})$. Then $\Omega_0 := g^{-1}(G)$ is open dense in Ω and $f := \tilde{g}|\Omega_0 : \Omega_0 \to G$ is continuous. Here $\tilde{g}(\omega) = g(\omega)^{-1}$ and so $g = \tilde{f}$ on Ω_0. As ξ is equivariant in $C^{blu}(G, B)$, for $b \in C_0(G)$ and $t \in G$ we have $(\rho(t)b) \circ \tilde{f} \in C^b(\Omega_0)$ and

$$\begin{aligned}\xi(1 \otimes b)(t) &= \rho_t(\xi(1 \otimes b))(e) = \xi(\rho_t(1 \otimes b))(e)\\ &= \xi(1 \otimes \rho(t)b)(e) = (\rho(t)b) \circ \tilde{f},\end{aligned}$$

where the last term is regarded as a function on Ω_0. If $\xi_{[f]} \in \mathcal{G}_0$ is defined for f as above, then this identity shows that $\xi = \xi_{[f]}$ on $1 \otimes C_0(G)$, and since $\xi = \mathrm{id}_{B\otimes 1} = \xi_{[f]}$ on $B \otimes 1$, it follows that $\xi = \xi_{[f]}$.

(ii) In view of (6.2) we easily check that both sides of (6.3) coincide on $B \otimes 1$ and $1 \otimes L^\infty(G)$, and hence on the whole of $B \bar{\otimes} L^\infty(G)$. The other assertions are immediate consequences of (6.3). □

Remark. By 6.2 each $\gamma \in \mathcal{G}$ is written as $\gamma = \xi_{[f]} \circ (\beta \bar{\otimes} \mathrm{id}_{L^\infty(G)})$ for some $\beta \in$ *-Aut B and some continuous function $f : \Omega_f \to G$. Then the flow built under a function $A := (B \bar{\otimes} L^\infty(G))^\gamma$ with action $\alpha = (\mathrm{id}_B \bar{\otimes} \alpha_G)|A$ is ergodic, that is, $A^\alpha = \mathbb{C}1$ if and only if β is ergodic in B. Indeed, since $\xi_{[f]}$ commutes with the translations ρ_t, $t \in G$, on $B \bar{\otimes} L^\infty(G)$ and so does γ, we have $\rho_t(A) = A$, $t \in G$, and since $\rho : t \mapsto \rho_t$ is the pointwise action of G on the G-module $(B \bar{\otimes} L^\infty(G),\, \mathrm{id}_B \bar{\otimes} \alpha_G)$ (see 3.2), the restriction $\rho|A$ is the pointwise action of the G-module (A, α). As $(B \bar{\otimes} L^\infty(G))^\rho = B \otimes 1$, it follows from the remark (iii) after 3.7 that

$$A^\alpha = [(B \bar{\otimes} L^\infty(G))^\gamma]^\rho = [(B \bar{\otimes} L^\infty(G))^\rho]^\gamma = (B \otimes 1)^\gamma = B^\beta \otimes 1,$$

and our assertion follows.

In the following we retain the above notation and consider the case $G = \mathbb{R}$.

Proposition 6.5. *Let B be a non-W^*, monotone complete C^*-algebra with an ergodic *-automorphism β and let $\gamma = \xi_{[f]} \circ (\beta \bar{\otimes} \mathrm{id}_{L^\infty(\mathbb{R})}) \in \mathcal{G}$, where $[f] \in \mathcal{F}$ and $f : \Omega_f \to \mathbb{R}$ satisfies $f(\omega) > 0$ for all $\omega \in \Omega_f$. Then the flow built under a function $A := (B \bar{\otimes} L^\infty(\mathbb{R}))^\gamma$ is a non-W^*, ergodic, monotone complete C^*-$\mathbb{R}$-module.*

The ergodicity of A follows from the above remark and the non-W^*-ness of A is a consequence of the following:

Lemma 6.6. (i) *With the notation and assumption as above, there is a central projection e of $B \bar{\otimes} L^\infty(\mathbb{R})$ such that $\{\gamma^n(e) : n \in \mathbb{Z}\}$ is an orthogonal sequence with $\sum_{n\in\mathbb{Z}} \gamma^n(e) = 1$.*

(ii) *Let C be a monotone complete C^*-algebra and let γ be a *-automorphism of C for which there is a central projection e of C such that $\{\gamma^n(e) : n \in \mathbb{Z}\}$ is an orthogonal sequence with $\sum_{n\in\mathbb{Z}} \gamma^n(e) = 1$. Then C is *-isomorphic to $C^\gamma \bar{\otimes} l^\infty(\mathbb{Z})$, and so C^γ is non-W^* if and only if C is.*

Proof. (i) Let $F = 1 \otimes \chi_{(-\infty,\,0]} \in B \bar{\otimes} L^\infty(\mathbb{R})$, where $\chi_{(-\infty,\,0]}$ is the characteristic function of $(-\infty, 0]$. Then $\gamma^n(F)$, $n \in \mathbb{Z}$, is a central projection of $B \bar{\otimes} L^\infty(\mathbb{R})$ and we have

$$\gamma^n(F)(\omega,\, t) = \chi_{(-\infty,\, nf(\omega)]}(t), \quad \omega \in \Omega_f, \;\; t \in \mathbb{R}, \tag{6.5}$$

since

$$\gamma(F)(\omega,\, t) = \xi_{[f]}(1 \otimes \chi_{(-\infty,\,0]})(\omega,\, t) = \chi_{(-\infty,\,0]}(t - f(\omega))$$

by (6.2). Hence $\{\gamma^n(F) : n \in \mathbb{Z}\}$ is an increasing sequence.

Now we show that $\sup \gamma^n(F) = 1$ and $\inf \gamma^n(F) = 0$. Since Ω is stonean and f is continuous and positive on the open dense subset Ω_f of Ω, there are pairwise disjoint closed and open subsets E_i, $i \in I$, of Ω_f with $\overline{\bigcup_i E_i} = \Omega$ and positive numbers ε_i, δ_i, $i \in I$, such that

$$\varepsilon_i \le f(\omega) \le \delta_i, \quad \forall\, \omega \in E_i, \;\; \forall\, i \in I. \tag{6.6}$$

By (6.5) and (6.6), $\gamma^n(F)(\omega,\, t) \ge \chi_{(-\infty,\, n\varepsilon_i]}(t)$ for all $\omega \in E_i$ and $t \in \mathbb{R}$, and so

$$\gamma^n(F)(\chi_{E_i} \otimes 1) \ge \chi_{E_i} \otimes \chi_{(-\infty,\, n\varepsilon_i]}.$$

Hence

$$\sup_n \gamma^n(F) = \sup_{n,i} \gamma^n(F)(\chi_{E_i} \otimes 1) \geq \sup_i [\sup_n (\chi_{E_i} \otimes \chi_{(-\infty, n\varepsilon_i]})]$$
$$= \sup_i (\chi_{E_i} \otimes 1) = 1 \otimes 1$$

and $\sup \gamma^n(F) = 1 \otimes 1$. Similarly we have $\inf \gamma^n(F) = 0$. Thus $e = \gamma(F) - F$ is the desired central projection.

(ii) For $x \in C$ set $\pi_0(x) = \sum_{n\in\mathbb{Z}} \gamma^n(xe)$. By the assumption on e the right hand side defines an element of C, which is also in C^γ, and $\pi_0 : C \to C^\gamma$ gives a *-homomorphism with $\pi_0|C^\gamma = \mathrm{id}$. We identify the Fubini product $C^\gamma \mathbin{\bar{\otimes}} l^\infty(\mathbb{Z})$ with the C^*-algebra of all bounded functions from $\mathbb{Z}$ to C^γ, and define a map $\pi : C \to C^\gamma \mathbin{\bar{\otimes}} l^\infty(\mathbb{Z})$ by setting $\pi(x)(m) = \pi_0(\gamma^m(x)) \in C^\gamma$, $m \in \mathbb{Z}$. Then π is clearly a *-homomorphism. It is injective, since $\pi(x) = 0$, that is, $\pi_0(\gamma^m(x)) = 0$ for all m implies $x\gamma^{-m}(e) = 0$ and so $x = 0$. It is surjective, since for $y \in C^\gamma \mathbin{\bar{\otimes}} l^\infty(\mathbb{Z})$, $x = \sum_{n\in\mathbb{Z}} y(n)\gamma^{-n}(e) \in C$ satisfies $\pi(x) = y$. $\square$

Remarks. (i) In the commutative case, the concrete examples of B and β as in 6.5 are constructed from the following, probably well known observations. Let $C = C(\Omega)$ be a unital commutative C^*-algebra with Ω any compact Hausdorff space and let $\gamma \in$ *-Aut C be induced by a self-homeomorphism h of Ω so that $\gamma(a) = a \circ h$, $a \in C$. The regular monotone completion, $\bar{C}$, of C, [11], is a commutative AW^*-algebra and γ extends uniquely to $\beta := \bar{\gamma} \in$ *-Aut $\bar{C}$. (This $\bar{C}$ is non-W^*, for example, if Ω is metric and perfect, and it is identified with the quotient C^*-algebra of all bounded Borel functions on Ω by its ideal of Borel functions vanishing off a meager set, which was used by Dixmier [7] to give the first example of non-W^*, commutative AW^*-algebra.) Then β is ergodic if and only if h is topologically ergodic, that is, each non-empty open subset of Ω invariant under h is dense in Ω. Indeed, this follows from the fact that the projections in $\bar{C}$ correspond to regular open subsets (open sets which are the interiors of their closures) of Ω bijectively (see, for exmple, [14]). Finally, note that if we take as h an irrational rotation $t + \mathbb{Z} \mapsto (t+\theta) + \mathbb{Z}$ on the 1-dimensional torus $\mathbb{R}/\mathbb{Z}$ with $\theta \in \mathbb{R}$ irrational, then we obtain the situation considered by Takenouchi [30].

(ii) We note that an ergodic, non-W^*, monotone complete C^*-G-module as in 6.5 arises only when G is not compact. Indeed, let G be compact and let (A, α) be an ergodic, monotone complete C^*-G-module. Then for the constant function $1 \in L^1(G)$ the map $x \mapsto 1 \cdot x$ on A is positive, normal (the remark (ii) after 2.4) and faithful, since $1 \cdot x = (\mathrm{id}_A \bar{\otimes} \int \cdot\, dt) \circ \alpha(x)$ and $\int \cdot\, dt : a \mapsto \int a(t)dt$ is faithful on $L^\infty(G)$. Moreover $1 \cdot x \in A^\alpha = \mathbb{C}1$ since

$$f \cdot (1 \cdot x) = (f \cdot 1) \cdot x = \left(\int f(t)dt \right) \cdot x = \left(\int f(t)dt \right) (1 \cdot x)$$

for all $f \in L^1(G)$ (see 2.6 (i)). Thus the monotone complete C^*-algebra A has a faithful normal state and so it is a W^*-algebra.

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