REMARKS ON THE ASYMPTOTIC PROPERTIES FOR SEMILINEAR HEAT EQUATIONS

Kusuo KOBAYASHI

1. Introduction

In this paper we consider the Cauchy problem for the following semilinear heat equation

\[
\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)^{1+\alpha}, \quad t > 0, x \in \mathbb{R}^N,
\]

with the initial condition

\[
u(0,x) = a(x), x \in \mathbb{R}^N.
\]

It is assumed that the initial value \(a(x)\) is a bounded nonnegative continuous function in \(\mathbb{R}^N\), not vanishing identically. It is well-known that there exists a positive local solution \(u(t,x)\) of (1) with (2). In case \(\alpha \leq 2/N\), all positive solutions \(u(t,x)\) of (1) blow up to \(\infty\) in finite time. On the other hand, in case \(\alpha > 2/N\), for sufficiently small nonnegative initial value \(a(x)\) (not identically vanishing), the solution \(u(t,x)\) of (1) with (2) converges to 0 uniformly in \(x\) as \(t \to \infty\), and for sufficiently large initial value \(a(x)\), the solution \(u(t,x)\) of (1) with (2) blows up to infinity in finite time. We have studied the effect by the initial values \(a(x)\) on the asymptotic behavior of the solutions.
u(t, x) of \( \partial u / \partial t = \Delta u + f(u) \) for some special nonlinear terms \( f(u) \) (see [3],[4]). Roughly speaking, it is shown that:

If the initial value \( a(x) \) is greater than the stationary solution \( u_o(x) \) of the above equation then the solution \( u(t, x) \) of the equation blows up or grows up to \( \infty \) as \( t \to \infty \), on the other hand, if the initial value \( a(x) \) is less than the stationary solution \( u_o(x) \) then the solution \( u(t, x) \) converges to 0 as \( t \to \infty \).

In [4] we treated the equation (1) with \( \alpha = 4 / (N - 2) \). In this paper we show that the same assertion holds for the equation (1) with \( \alpha > 2 / (N - 2) \) using the similar direct estimates. These results have been shown by Gui, Ni and Wang[2] and Wang[6] in a different way.

For \( \alpha > 2 / (N - 2) \) there exist positive solutions \( u_\lambda(x) \) of the following equation

\[
(3) \quad \Delta u(t, x) + u(t, x)^{1+\alpha} = 0, \quad x \in \mathbb{R}^N.
\]

For each \( \lambda > 0 \) the positive solution \( u_\lambda(x) \) of (3) has the following properties.

(S_1) \( u_\lambda(x) \) is radially symmetric and strictly decreasing in \( |x| \).

(S_2) \( u_\lambda(0) = \lambda > 0 \).

(S_3) \( \lim_{|x| \to \infty} |x|^{\frac{2}{\alpha}} u_\lambda(x) = \left( \frac{2}{\alpha} (N - 2 - \frac{2}{\alpha}) \right)^{\frac{1}{\alpha}}. \)

By using these stationary solutions \( u_\lambda(x), \lambda > 0 \), our theorem is stated as follows.

**Theorem.** For fixed \( \lambda > 0 \) and the stationary solution \( u_\lambda(x) \) of the equation (3), the followings hold.

(i) If \( a(x) \geq \gamma u_\lambda(x) \) for some \( \gamma (\gamma > 1) \), then the solution \( u(t, x) \) of the equation (1) with (2) blows up in a finite time.

(ii) If \( a(x) \leq \gamma u_\lambda(x) \) for some \( \gamma (0 < \gamma < 1) \), then the solution \( u(t, x) \) of the equation (1) with (2) converges to 0 uniformly in \( x \) as \( t \to \infty \).
2. Proof of Theorem

The heat equation (1) with (2) is transformed into the integral equation

\[(4) \quad u(t, x) = H_t a(x) + \int_0^t H_{t-s} u(s, \cdot)^{1+\alpha}(x) ds, x \in \mathbb{R}^N,\]

where

\[\alpha > \frac{4}{(N - 2)}, \quad N \geq 3,\]

\[H_t a(x) = \int_{\mathbb{R}^N} H(t, x, y) a(y) dy,\]

\[H(t, x, y) = (4\pi t)^{-\frac{N}{2}} \exp(-\frac{|x-y|^2}{4t}).\]

Let a positive constant \(\lambda\) be fixed. We assume that the solution \(u(t, x)\) of the integral equation (4) does not blow up in a finite time. For proving Theorem we prepare several lemmas.

**Lemma 1.** Let \(\beta > 0\). If \(\gamma > 1\), then

\[\frac{\gamma^{1+2\beta} - \gamma^{1+\beta}}{\gamma^{1+2\beta} - 1} > \frac{\beta}{1+2\beta},\]

**Lemma 2.** For the stationary solution \(u_\lambda(x)\) of (1),

\[\sup_{|x| \leq r^k} \int_0^\infty ds \int_{|y| > r^{k+1}} H(s, x, y) u_\lambda(y)^{1+\alpha} dy\]

converges to 0 uniformly in \(k(k \geq 1)\) as \(r \to \infty\).

**Proof.** We first note that the following properties (S4) and (S5) hold, which are derived from (S1) - (S3).

(S4) There exist positive numbers \(c_1, c_2\) such that:

(i) \(u_\lambda(r x) \geq c_1 r^{-\frac{2}{\alpha}} u_\lambda(x)\) for any \(r \geq 1\) and \(|x| \geq 1\).

(ii) \(u_\lambda(r x) \leq c_2 r^{-\frac{2}{\alpha}} u_\lambda(x)\) for any \(r \geq 1\) and \(|x| \geq 1\).

(S5) \(u_\lambda(x) = \int_0^\infty ds \int_{\mathbb{R}^N} H(s, x, y) u_\lambda(y)^{1+\alpha} dy\).
Making the change of variable \( y = r^{k+1}z \), we have

\[
\int_0^\infty ds \int_{|y|>r^{k+1}} H(s, x, y)u_\lambda(y)^{1+\alpha} dy
\]

\[
= \int_0^\infty ds \int_{|z|>1} (4\pi s)^{-\frac{N}{2}} \exp\left(-\frac{|\frac{x}{r^{k+1}} - z|^2}{4s/r^{2(k+1)}}\right) u_\lambda(r^{k+1}z)^{1+\alpha} r^{(k+1)N} dz
\]

\[
\leq c_2^{1+\alpha} r^{(k+1)(-\frac{2}{\alpha})(1+\alpha)+(k+1)N}
\]

\[
\cdot \int_0^\infty ds \int_{|z|>1} (4\pi s)^{-\frac{N}{2}} \exp\left(-\frac{|\frac{x}{r^{k+1}} - z|^2}{4s/r^{2(k+1)}}\right) u_\lambda(z)^{1+\alpha} dz,
\]

where we have used (S_4). Again, making the change of variable \( s = r^{2(k+1)}\tau \), we find that the last line of the above inequality is equal to

\[
c_2^{1+\alpha} r^{-\frac{2}{\alpha}(k+1)} \int_0^\infty d\tau \int_{|z|>1} (4\pi \tau)^{-\frac{N}{2}} \exp\left(-\frac{|\frac{x}{r^{k+1}} - z|^2}{4\tau}\right) u_\lambda(z)^{1+\alpha} dz
\]

\[
\leq c_2^{1+\alpha} r^{-\frac{2}{\alpha}(k+1)} u_\lambda\left(\frac{x}{r^{k+1}}\right)
\]

\[
\leq c_2^{1+\alpha} r^{-\frac{2}{\alpha}(k+1)} u_\lambda(0),
\]

where we have used (S_5) and (S_1). On the other hand, by (S_1) and (S_4), we have

\[ u_\lambda(x) \geq u_\lambda(r^{k+1}x_1) \geq c_1 r^{-\frac{2}{\alpha}k} u_\lambda(x_1) \quad \text{for} \quad |x| \leq r^k, \]

where \(|x_1| = 1|\).

Therefore we obtain

\[
\sup_{|x| \leq r^k} \frac{\int_0^\infty ds \int_{|y|>r^{k+1}} H(s, x, y)u_\lambda(y)^{1+\alpha} dy}{u_\lambda(x)} \leq \frac{c_2^{1+\alpha} u_\lambda(0)}{c_1 u_\lambda(x_1)} r^{-\frac{2}{\alpha}},
\]

which completes the proof of Lemma 2.

**Lemma 3.** Let \( \gamma > 1 \). If the initial value \( a(x) \geq \gamma u_\lambda(x) \) for any \( x \in \mathbb{R}^N \), then also the following same inequality about the global solution \( u(t, x) \) of (4) holds:

\[ u(t, x) \geq \gamma u_\lambda(x) \quad \text{for} \quad t > 0, \; x \in \mathbb{R}^N. \]
**Lemma 4.** Let $\gamma > 1$ and $\beta = \alpha/2$. Suppose that $u(x) \geq \gamma u_\lambda(x)$ for any $x \in \mathbb{R}^N$. Then, for any integer $m \geq 1$ and any $r_0 > 0$, there exists $T > 0$ such that

$$u(t, x) \geq \gamma^{(1+\beta)^m} u_\lambda(x) \quad \text{for any } t > mT \text{ and } |x| \leq r_0.$$  \hfill (5)

**Proof.** Let a positive integer $m$ be fixed.

Let

$$f(u) = u^{(1+\alpha)},$$

$$J_0(r, x) = \frac{\beta}{2(1+2\beta)} - \frac{\int_0^\infty ds \int_{|y|>r} H(s, x, y)f(u_\lambda(y))dy}{u_\lambda(x)},$$

$$K_0(t, x) = \frac{\beta}{2(1+2\beta)} - \frac{H_1u_\lambda(x)}{u_\lambda(x)}.$$

First, choose $r \geq \max(r_0, 1)$ such that $|x| \leq r^k$ and $k \geq 1$ imply $J_0(r^{k+1}, x) > 0$. Such an $r$ exists by Lemma 2. Next, pick $T > 0$ such that $|x| \leq r^m$ and $t \geq T$ imply $K_0(t, x) > 0$.

We prove by induction on $n$ that

$$u(t, x) \geq \gamma^{(1+\beta)^n} u_\lambda(x)$$  \hfill (6)

for any $t > nT$, $|x| \leq r^{m-n+1}$ and $n = 0, 1, \ldots, m$.

When $n = 0$, we already have this fact for any $t > 0$ and $x \in \mathbb{R}^N$. Assuming that (6) holds for $n$, we shall prove (6) holds also for $n + 1$.

We denote the solution $u(t, x)$ of the equation (1) with the initial condition (2) by $u(t, x; a, f)$. Since

$$u(t + nT, x; a, f) = u(t, x; u_{nT}, f), \quad u_{nT}(x) = u(nT, x; a, f),$$

we have, using the induction hypothesis,

$$u(t + nT, x)$$

$$= H_t u(nT, x) + \int_0^t ds \int_{\mathbb{R}^N} H(t - s, x, y)f(u(nT + s, y))dy$$

$$\geq H_t u_\lambda(x) + \gamma^{(1+\beta)^n(1+2\beta)} \int_0^t ds \int_{|y| \leq r^{m-n+1}} H(t - s, x, y)f(u_\lambda(y))dy$$

$$+ \int_0^t ds \int_{|y| > r^{m-n+1}} H(t - s, x, y)f(u_\lambda(y))dy.$$
On the other hand, the stationary solution $u_\lambda(x)$ of (1) satisfies the following equation

$$
(7) \quad u_\lambda(x) = H_tu_\lambda(x) + \int_0^t ds \int_{\mathbb{R}^N} H(s, x, y)f(u_\lambda(y))dy.
$$

Therefore we have

$$
u(t + nT, x) \geq \gamma^{(1+\beta)^{n+1}} u_\lambda(x) + (\gamma^{(1+\beta)^n(1+2\beta)} - \gamma^{(1+\beta)^{n+1}})u_\lambda(x) + (1 - \gamma^{(1+\beta)^n(1+2\beta)})H_tu_\lambda(x)
$$

$$
+ (1 - \gamma^{(1+\beta)^n(1+2\beta)}) \int_0^t ds \int_{|y|>r^{m-n+1}} H(s, x, y)f(u_\lambda(y))dy
$$

$$
= \gamma^{(1+\beta)^{n+1}} u_\lambda(x) + K(t, x, M) + J(t, x, r^{m-n+1}, M),
$$

where

$$
M = \gamma^{(1+\beta)^n},
$$

$$
K(t, x, M) = \frac{1}{2}(M^{1+2\beta} - M^{1+\beta})u_\lambda(x) - (M^{1+2\beta} - 1)H_tu_\lambda(x),
$$

$$
J(t, x, r, M) = \frac{1}{2}(M^{1+2\beta} - M^{1+\beta})u_\lambda(x)
$$

$$
- (M^{1+2\beta} - 1) \int_0^t ds \int_{|y|>r} H(s, x, y)f(u_\lambda(y))dy.
$$

Using Lemma 1 and the definition of $r$, we have

$$
J(t, x, r^{m-n+1}, M)
$$

$$
= (M^{1+2\beta} - 1)u_\lambda(x) \left( \frac{1}{2} \cdot \frac{M^{1+2\beta} - M^{1+\beta}}{M^{1+2\beta} - 1} - \int_0^t ds \int_{|y|>r^{m-n+1}} H(s, x, y)f(u_\lambda(y))dy \right)
$$

$$
\geq (M^{1+2\beta} - 1)u_\lambda(x) \left( \frac{\beta}{2(1 + 2\beta)} - \int_0^\infty ds \int_{|y|>r^{m-n+1}} H(s, x, y)f(u_\lambda(y))dy \right)
$$

$$
\geq (M^{1+2\beta} - 1)u_\lambda(x) J_0(r^{m-n+1}, x) > 0,
$$

where

- $\gamma$ is the growth exponent.
- $H_t$ is the heat operator.
- $f$ is a given function.
- $u_\lambda(x)$ is the solution to the given partial differential equation.
- $M$ is a constant related to the growth exponent.
- $K(t, x, M)$ and $J(t, x, r, M)$ are auxiliary terms.
- $r$ is a parameter related to the size of the domain.
- $J_0(r^{m-n+1}, x)$ is a specific integral term.
for \(|x| \leq r^{m-n}\), and
\[
K(t, x, M) = (M^{1+2\beta} - 1)u_\lambda(x) \left( \frac{1}{2} \cdot \frac{M^{1+2\beta} - M^{1+\beta}}{M^{1+2\beta} - 1} - \frac{H_t u_\lambda(x)}{u_\lambda(x)} \right)
\]
\[
\geq (M^{1+2\beta} - 1)u_\lambda(x) \left( \frac{\beta}{2(1+2\beta)} - \frac{H_t u_\lambda(x)}{u_\lambda(x)} \right)
\]
\[
\geq (M^{1+2\beta} - 1)u_\lambda(x) K_0(t, x) > 0
\]
for \(|x| \leq r^{m-n}(\leq r^m)\) and \(t \geq T\). Therefore we obtain
\[
u(t + nT, x) \geq \gamma^{(1+\beta)^{n+1}} u_\lambda(x)
\]
for \(|x| \leq r^{m-n}\) and \(t \geq T\), namely
\[
u(t, x) \geq \gamma^{(1+\beta)^{n+1}} u_\lambda(x)
\]
for \(|x| \leq r^{m-n}\) and \(t \geq (n+1)T\), which implies (6), that is, Lemma 4.

**Lemma 5.** Let \(0 < \gamma < 1\) and \(\beta = \alpha/2\). Suppose that \(0 \leq a(x) \leq \gamma u_\lambda(x)\) for any \(x \in \mathbb{R}^N\). Then, for any integer \(m \geq 1\) and any \(r_0 > 0\), there exists \(T > 0\) such that
\[
u(t, x) \leq \gamma^{(1+\beta)^m} u_\lambda(x)\]
for any \(t > mT\) and \(|x| \leq r_0\).

**Proof.** Let a positive integer \(m\) be fixed.

Let
\[
f(u) = u^{1+\alpha},
\]
\[
J_1(r, x, M) = \frac{1}{2} \cdot \frac{M^{1+2\beta} - M^{1+\beta}}{M^{1+2\beta} - 1}
\]
\[
- \frac{\int_0^\infty ds \int_{|y| > r} H(s, x, y) f(u_\lambda(y)) dy}{u_\lambda(x)},
\]
\[
K_1(t, x, M) = \frac{1}{2} \cdot \frac{M^{1+2\beta} - M^{1+\beta}}{M^{1+2\beta} - 1} - \frac{H_t u_\lambda(x)}{u_\lambda(x)}.
\]

First, choose \(r \geq \max(r_0, 1)\) such that \(|x| \leq r^k\) and \(k \geq 1\) imply
\[
J_1(r^{k+1}, x, M_m) > 0,
\]
where $M_m = \gamma^{(1+\beta)^m}$. Such an $r$ exists by Lemma 2. Next, pick $T > 0$ such that $|x| \leq r^m$ and $t \geq T$ imply $K_1(t, x, \gamma^{(1+\beta)^m}) > 0$. We can also prove by induction on $n$ that

$$u(t, x) \leq \gamma^{(1+\beta)^n} u_\lambda(x)$$

for any $t > nT$, $|x| \leq r^{m-n+1}$ and $n = 0, 1, \ldots, m$.

Indeed, we have

$$u(t + nT, x)$$

$$\leq H_t u_\lambda(x) + \gamma^{(1+\beta)^n(1+2\beta)} \int_0^t ds \int_{|y| \leq r^{m-n+1}} H(t - s, x, y) f(u_\lambda(y)) dy$$

$$+ \int_0^t ds \int_{|y| \geq r^{m-n+1}} H(t - s, x, y) f(u_\lambda(y)) dy$$

$$= \gamma^{(1+\beta)^{n+1}} u_\lambda(x) + K(t, x, M_n) + J(t, x, r^{m-n+1}, M_n),$$

where $M_n \equiv \gamma^{(1+\beta)^n}$. Since $1 > M_n \geq M_m = \gamma^{(1+\beta)^m}$, we have

$$J(t, x, r^{m-n+1}, M_n)$$

$$\leq (M_n^{1+2\beta} - 1) u_\lambda(x) J_1(r^{m-n+1}, x, M_m) < 0,$$

for $|x| \leq r^{m-n}$, and

$$K(t, x, M_n)$$

$$\leq (M_n^{1+2\beta} - 1) u_\lambda(x) K_1(t, x, M_m) < 0,$$

for $|x| \leq r^{m-n}(\leq r^m)$ and $t \geq T$, which implies (9), that is, Lemma 5.

**Proof of Theorem.** Since the function $f(u)$ is non-decreasing and satisfies the following condition

$$\int_\varepsilon^\infty \frac{1}{f(u)} du < \infty \text{ for some } \varepsilon > 0,$$

Lemma 4 contradicts that $u(t, x)$ does not blow up (see [5]). This completes the proof of (i).
Since $\gamma u_\lambda(x)$ is radially symmetric and decreasing in $|x|$, by the comparison theorem, it is sufficient to prove (ii) for radially symmetric and decreasing initial values $a(x)$. Then $u(t, x; a, f)$ is also radially symmetric and decreasing in $|x|$. Lemma 5 implies that, for any integer $m$,

$$u(t, x) \leq \gamma^{(1+\beta)m} u_\lambda(0) \text{ for any } t \geq nT.$$  

Thus the proof of Theorem is completed.

References


Department of Mathematics  
Faculty of Science  
Toyama University  
Toyama 930, JAPAN  
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